Chapter 4
Convex Bodies of Constant Width

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§ 1. Introduction, Historical Remarks

Convex bodies of constant width b in Euclidean 3-space (which are different from the round ball) may be regarded as generalizations of such a round ball of diameter b. They have quite surprising properties which make it interesting to study them more closely. In general it will make no difference if we consider the convex body (of constant width) or its boundary (of constant width).

Many results have been known for a long time. Already L. EULER [14] studied in 1778 the two-dimensional analogues. The constant width of such a plane convex curve seems to suggest that it must be a circle. Because of certain similarities with the circle EULER called these curves in Latin "orbiformes" (loc. cit.) which means "circular-like curves". Of course the circle itself is one of them. However, in general these orbiformes are not round circles. About 100 years later in 1875 F. REULEAUX [29] mentioned these curves in his book about kinematics. He was interested in their kinematic aspects and he gave a certain number of examples. Later some special cases received his name (Reuleaux triangle or Reuleaux polygon). In the beginning of this century there was a growing interest in convex bodies of constant width in dimension two and three. This was caused on one hand by the study of convex bodies in general and on the other hand by considering examples of special curves. Here we have to mention in the first place the ingenious mathematician H. MINKOWSKI [25] who paid attention to the convex bodies of constant width among the convex bodies in general. About the same time A. HURWITZ [15] and little later E. MEISSNER [21] have treated two-dimensional objects of that kind.

For better visualization of three-dimensional convex bodies of constant width the Martin-Schilling-Verlag, Leipzig [31] published in 1911 a collection of mathematical models containing a certain number of plane curves of constant width and three plaster models of 3-dimensional bodies of constant width b ($b = 12$cm). The theoretical foundations are due to E. MEISSNER [23] with the collaboration of F. SCHILLING (see also [31]). Certain phenomena and peculiarities are intended to be visualized by these models. Photos of these three models are shown in the present volume. MEISSNER [22] found at that time a more elegant and characteristic property of convex bodies of constant width which uses certain set-theoretic-metric ideas and makes use of the completeness of a convex point-set (see also B. JESSEN [17]). Using MEISSNER's definition of completeness it is immediately clear that there is a generalization of the notion of a convex set of constant width in two different ways: first it is not hard to see that a complete body in Euclidean n-space is an n-dimensional convex body of constant width and vice versa, secondly one can pass over from the Euclidean

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metric to a Minkowski metric together with a suitable gauge body and consider analogously convex bodies of constant width.

Some years later G. TIERCY [32, 33] worked on two- and three-dimensional convex bodies of constant width and called them "spheriformes" – by analogy with Euler's orbiformes. In the books by L. BIEBERBACH [5] and H. RADEMACHER/O. TÖPLITZ [27] there are detailed and elementary descriptions of plane convex curves of constant width, and the classical standard book by T. BONNESEN and W. FENCHEL [8] contains an encyclopedical collection of what was known at that time, together with an extensive list of references. A corresponding presentation of recent date 1982 is due to G. D. CHAKERIAN and H. GROEMER [14]. In particular this makes clear that the subject didn't lose its actuality and that there are still many unsolved problems. Other mathematicians who have made further contributions to this theory are W. BLASCHKE [6, 7], H. LEBESGUE [19], K. REIDEMEISTER [28], and more recently H. G. EGGLESTON [1], I. M. JAGLOM and W. G. BOLTJANSKI [16], compare also the books by R. V. BENSON [2] and S. R. LAY [18].

Although in the following we will consider convex bodies or their boundary in n-dimensional Euclidean space we preferably think of the cases $n = 2$ and $n = 3$. There are even certain phenomena which behave quite different or which are not yet understood in higher dimensions. The convex bodies are assumed to be compact and to have interior points. We start with the two-dimensional case and then treat the three-dimensional case to which this chapter is mainly devoted.

§ 2. Intuitive Introduction

The following may serve as an intuitive introduction into the problem – starting with the original concern (compare [27]). A circle is defined to be a closed plane curve whose points have the same distance from a well-defined centre M. This property is of practical use for a wheel whose spokes are of equal length: the nave of the wheel will remain in constant distance form the horizontal plane, independently of rotations of the wheel. This makes it possible to move a carriage horizontally. For such a wheel it is necessary that the boundary is a circle and that the nave sits at the centre because this property characterizes the circle. Otherwise the wheel would "wobble", i.e. the horizontal motion would be accompanied with a vertical one. Another possibility to move something like a box is to use cylinders and to roll the box orthogonally to the axes of the cylinders. If the cross cut of these cylinders is a circle then this motion will be purely horizontally, i.e. with constant distance of the box from the plane. This however is not necessary. In fact the requirement is only that the distances between two parallel straight lines touching the cylinders on top and bottom is always the same (compare the cross cut shown in Figure 4.1). For this property a centre is unimportant. Of course the circle has this property, but this does not characterize it.

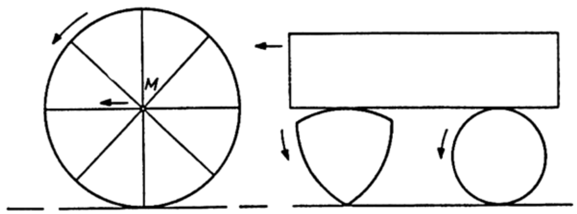

Fig. 4.1

Besides the circle there is a great variety of plane curves leading to such a purely horizontal motion by cylinders. All curves with this property (including the circle) are called *curves of constant width*. It is obvious how the width of a closed plane curve in a given direction **u** can be determined: take the length of the orthogonal projection of the curve on a straight line in direction **u** lying in the same plane. The two extremal projection lines have the property that they contain at least one point of the curve and that the curve lies on one side of the line. Such a straight line is called *supporting line*. A closed plane curve which is the boundary of a two-dimensional body has exactly two supporting lines for each direction. Note that in general the notion "supporting line" is different from "tangent line" because the curve is not necessarily differentiable (compare Figure 4.2).

It follows that the width of a plane (convex) body in a given direction may be defined to be the distance between the two parallel supporting lines which are perpendicular to the given direction. The body will lie between these two supporting lines (see Figure 4.2). Similarly for a three-dimensional convex body of constant width b

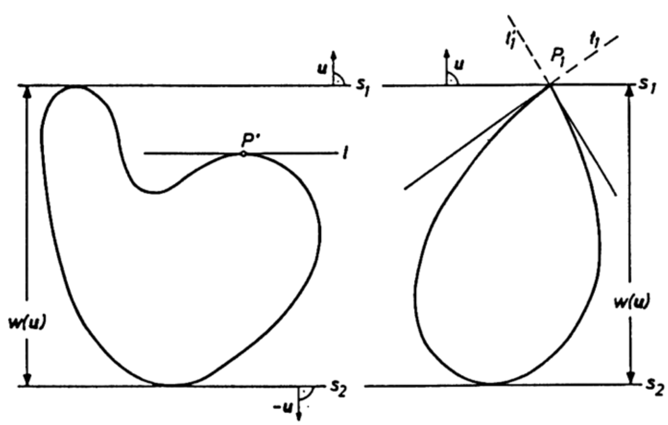

Fig. 4.2

we have two parallel supporting planes of distance b. Fixing these two supporting planes the possible motions of the body still have five degrees of freedom, preserving the contact with the two supporting planes. Looking at Photo 102 one sees a gauging instrument which may be interpreted just as a part of these supporting planes touching the convex body from both sides (compare also Fig. 4.5).

§ 3. Definitions and Basic Facts

To give an analytic definition for the width of a convex body in n-dimensional space $\mathbb{E}^n$ in terms of supporting hyperplanes and associated support functions we start with the Cartesian coordinate system of $\mathbb{E}^n$ and with the standard Euclidean metric.

The support function $\mathrm{h}(K, \mathbf{u})$ of the convex body K in the direction $\mathbf{u} \in \mathbf{S}^{n-1}$ is defined to be the oriented distance between the origin and the supporting hyperplane H which is orthogonal to $\mathbf{u}$. The normal of H has to be chosen such that it points into the half space not containing K. This may be written as

$$\mathrm{h}(K, \mathbf{u}) = \sup(\mathbf{u} \cdot \mathbf{x} : \mathbf{x} \in K).$$

The Hesse normal form of H is $\mathbf{u} \cdot \mathbf{x} = \mathrm{h}(K, \mathbf{u})$. Then the width of K in the direction $\mathbf{u}$ is the distance between the two supporting hyperplanes orthogonal to $\mathbf{u}$, i.e. $\mathrm{w}(K, \mathbf{u}) = \mathrm{h}(K, \mathbf{u}) + \mathrm{h}(K, -\mathbf{u})$. This leads to the following definition:

Let K be a n-dimensional convex body. K is said to have constant width b if and only if $\mathrm{w}(K, \mathbf{u}) = b$ for all directions $\mathbf{u} \in \mathbf{S}^{n-1}$.

For two different points p, q of the boundary of K the straight line segment between them is called *chord* of K.

Then it follows that an equivalent condition for the constancy of the width is the following: For all directions $\mathbf{u} \in \mathbf{S}^{n-1}$ each of the two supporting planes of K with normal vectors $\mathbf{u}$ and $-\mathbf{u}$ contains exactly one point of K, and the chord spanned by these points p_1, p_2 is perpendicular to the two supporting planes.

p_1 and p_2 are called *opposite points* of the convex body of constant width. Consequently for every boundary point p of a convex body of constant width the following holds: Every normal of the boundary at p is also a normal at a suitable opposite point. Such a normal is called *double normal*.

This leads to the following statement for a convex body K:

K is of constant width if and only if every normal is a double normal.

Now we are going to discuss the various possibilities for the set of supporting planes through a fixed boundary point of a convex body. This turns out to be quite useful for the study of convex bodies of constant width.

A boundary point p of K is called *regular* if K has exactly one supporting plane through p, otherwise it is called *singular*.

The boundary (and also K itself) is called *smooth* if all boundary points are regular. Only a singular boundary point can have more than one opposite point (of course, there is only one in a given direction). A singular boundary point of K is said to have order $r-1$ if the normals belonging to supporting planes through p form a r-dimensional cone. As an example a boundary point is called *vertex* if it has order $n-1$. For $n = 3$ a singular boundary point of order 1 is called *edge point*. The direction of the edge is given by the common straight line of the pencil of supporting planes. For a convex body of constant width the set of points opposite to such an edge point will be a circular arc.

For completeness let us shortly discuss the following general methods in convex geometry which originate from H. MINKOWSKI. For the foundations compare

K. LEICHTWEISS [20], MINKOWSKI [24, 26] or BONNESEN/FENCHEL [8].

Using the notions of "Quermassintegral" or "mixed volume" one can formulate the theorem of A. DINGHAS. For $n = 2$ this is known as the theorem of E. BARBIER [1] saying that all plane convex curves of constant width b have constant perimeter πb. For $n = 3$ one gets in particular the equation of W. BLASCHKE [6] which implies that the surface area can be different for convex bodies with the same constant width. Therefore there is no analogue of BARBIER's theorem in dimensions greater than 2. However in the 3-dimensional case it can be shown that the two-dimensional orthogonal projections of a convex body of constant width have constant perimeter. The converse has been shown already by MINKOWSKI [25] whose theorem may be formulated as follows: The class of 3-dimensional convex bodies of constant width b equals the class of 3-dimensional convex bodies of constant perimeter $\pi \cdot b$. The plaster models allow an intuitive verification using a fabric cylinder which is made of flexible but incompressible material. Pulling such a cylinder over a convex body of constant width it will always touch the surface of the body independently of the direction. In this case the generating lines of the cylinder will be supporting lines of the body.

§ 4. The Two-Dimensional Case

Let us consider plane curves of constant width bounding a 2-dimensional convex body of constant width. Particularly simple examples which are different from the circle are the convex arc polygons which necessarily have an odd number of vertices and whose arcs are parts of circles with the same radius b. The vertices of the polygon are the centres of the opposite circular arcs. Each pair of opposite points contains at least one vertex and the supporting line corresponding to the other points is the tangent of the opposite circular arc. Consequently the width in any direction is just b. Such 2-dimensional arc polygons of constant width are called *Reuleaux polygons*. The particular case of the Reuleaux triangle (see Figure 4.3) is remarkable because of different reasons: it is necessarily regular with three arcs of length $\frac{\pi}{3} b$, it has the minimal possible number of vertices and it has certain other extremal properties. By elementary geometric considerations (see [28]) it follows

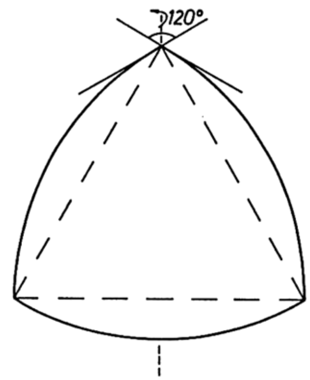

Fig. 4.3

that every interior angle of such a Reuleaux polygon cannot be smaller than 120°. Therefore the Reuleaux triangle with three angles of 120° has the smallest (or "sharpest") possible ones. Furthermore, among all curves of constant width b the Reuleaux triangle bounds the surface with the smallest area $\frac{\pi - \sqrt{3}}{2} b^2$ (This is a theorem of BLASCHKE-LEBESGUE).

By the property that all normals of the boundary are double normals it follows easily that a convex body K of constant width generates a family of convex bodies with constant width by taking the parallel surfaces of the boundary with the extra condition that these must be convex. These parallel surfaces bound convex bodies of constant width (invariance under Minkowski sum). In the 2-dimensional case this family of parallel curves of constant width has a common evolute as focal curve. By the filar-relationship between evolute and evolvent a curve of constant width can be constructed from a given evolute with certain properties (e.g. odd number of cusps, at least three) by rolling off a tangential straight line segment. After twofold traversing around the evolute the orbit of any point of this segment (which is sufficiently far away from the point of tangency) will be a curve of constant width.

This implies that we have a second principle for the construction of curves of constant width. Particularly suitable examples are the evolute of hypocycloids with an odd number of cusps. The simplest one is the *Steiner hypocycloid* with 3 cusps (compare Figure 4.4, curve C_3) which is the orbit of a point of a circle of radius r rolling inside of a circle of radius $3r$. The member of the corresponding family of convex curves of constant width with the smallest possible width $\frac{16}{3}r$ meets these three cusps (see Figure 4.4, curve C) and is called fundamental orbiform of order 3.

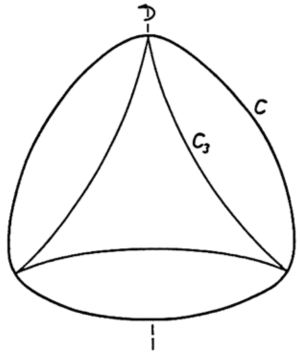

Fig. 4.4

Finally let us mention an analytic parametrization of a plane convex curve in terms of its support function which is supposed to be differentiable. The origin of the Cartesian coordinate system is assumed to lie in the interior of the curve. Now it is possible to describe the curve uniquely by its support function $\bar{\mathrm{h}}(\mathbf{u})$. By convexity the direction $\mathbf{u}$ can be expressed uniquely by the oriented angle φ with the positive x-axis, and therefore we may write $\bar{\mathrm{h}}(\mathbf{u}) = \mathrm{h}(\varphi)$ where h is a continuous and 2π-periodic function. The curve is enveloped by the supporting lines with the equation

$$\mathbf{u}\cdot\mathbf{x} = x\cdot\cos\varphi + y\cdot\sin\varphi = \mathrm{h}(\varphi), \qquad \mathbf{u} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} \in \mathbf{S}^2,$$

leading to the parametrization

$$\begin{aligned} x &= \mathrm{h}(\varphi)\cdot\cos\varphi - \mathrm{h}'(\varphi)\cdot\sin\varphi \\ y &= \mathrm{h}(\varphi)\cdot\sin\varphi + \mathrm{h}'(\varphi)\cdot\cos\varphi \end{aligned} \qquad 0 \leqslant \varphi < 2\pi.$$

$\mathrm{h}(\varphi)$ is the (positive) distance of the corresponding supporting line from the origin and $\mathrm{h}'(\varphi)$ is the oriented distance between the foot of the perpendicular from the origin to the supporting line and the common point of the supporting line and the curve. Because of the convexity the radius of curvature $\rho(\varphi) = \mathrm{h}(\varphi) + \mathrm{h}''(\varphi)$ must be nonnegative.

For the evolute we get the representation

$$\begin{aligned} \zeta &= x - \rho\cos\varphi \\ \eta &= y - \rho\sin\varphi. \end{aligned} \qquad 0 \leqslant \varphi < 2\pi.$$

The above curve C with constant width b and with the Steiner hypocycloid C_3 as evolute has the support function $\mathrm{h}(\varphi) = \frac{b}{2}\left(1 + \frac{1}{8}\cos 3\varphi\right)$ which implies $\rho = \frac{b}{2}(1 - \cos 3\varphi)$. This curve C has 6 points of extremal curvature at $\varphi = s\,\frac{\pi}{3}$ $(s = 0, 1, \ldots, 5)$ and no vertices. At the critical points with $s = 0, 2, 4$ we have $\rho = 0$ and the curve looks locally like $x = t^4$, $y = t^3$ and $t = 0$.

§ 5. The Three-Dimensional Case, Description of the Models

Let us first mention the special case of a convex 3-dimensional body of constant width which is rotationally symmetric. This means that we have to consider convex surfaces of revolution with constant width. The corresponding meridian curves are exactly the plane convex curves of constant width which are axially symmetric. A point p is a vertex if and only if it lies on the axis of revolution and is a vertex of the meridian curve. The set of its opposite points will be a part of a sphere. Every such plane curve of constant width b with axial symmetry gives rise to a convex surface of revolution with the same constant width b. All possible kinds of singular points may occur in this case, however there is at most one vertex. If there are edge points then these are always complete circles (= orbits of points under the revolution).

Examples for such surfaces are the two plaster models shown in Photos 98 and 99. **Photo 98** shows a surface of revolution which is generated by the curve C discussed above which has the Steiner hypocycloid C_3 as evolute (see Figure 4.4). The support function of "half" of this curve (the meridian curve) is given by the formula $\mathrm{h}(\theta) = \frac{b}{2}\left(1 + \frac{1}{8}\cos 3\theta\right), 0 \leqslant \theta \leqslant \pi$.

For the surface of revolution around the axis $\theta = 0$ in the direction

$$\mathbf{u} = \begin{pmatrix}\cos\varphi\,\sin\theta\\ \sin\varphi\,\sin\theta\\ \cos\theta\end{pmatrix}$$

we get the same expression

$$h(\varphi, \theta) = \frac{b}{2}\left(1 + \frac{1}{8}\cos 3\theta\right), \quad 0 \leqslant \varphi < 2\pi,\ 0 \leqslant \theta \leqslant \pi$$

of its support function. Consequently we get the following parametrization in Cartesian coordinates:

$$x = h(\varphi, \theta)\cdot\cos\varphi\sin\theta - h'(\varphi, \theta)\cdot\cos\varphi\cos\theta$$
$$y = h(\varphi, \theta)\cdot\sin\varphi\sin\theta - h'(\varphi, \theta)\cdot\sin\varphi\cos\theta$$
$$z = h(\varphi, \theta)\cdot\cos\theta - h'(\varphi, \theta)\cdot\sin\theta.$$
$$\left(0 \leqslant \varphi < 2\pi;\ 0 \leqslant \theta \leqslant \pi;\ h' := \frac{\partial h}{\partial\theta}\right).$$

The geometric interpretation of φ and θ refers to the direction **u** which is the normal direction of the supporting plane at the point (x, y, z). For every φ but only for $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$ and π this agrees with the spherical coordinates of this point.

This surface has no vertex and no edge point. By the substitution $s = \tan\frac{\varphi}{2}$, $t = \tan\frac{\theta}{2}$ one can see that it is an algebraic surface. For $\theta = 0$ and $\theta = \frac{\pi}{3}$ one principal curvature and therefore the Gaussian curvature tends to infinity. In the photo these points are the top of the body ($\theta = 0$) and a certain striking circle $\left(\theta = \frac{\pi}{3}\right)$ which lies quite below. Approximately the contour of the model in the photo coincides with the curve C itself.

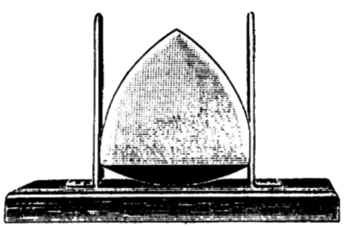

Fig. 4.5

Photo 99 shows the surface which is generated by the rotation of the Reuleaux triangle around an axis through one of its three vertices (in Fig. 4.5 from [31] it is represented with a gauging instrument). This surface has one vertex (the top in the photo) and edge points along a distinguished circle which lies quite below. The edge circle decomposes the surface into a piece of a sphere (below) and a spindle-shaped piece of a torus (above). These are the sets of opposite points for the vertex and for the edge circle, respectively. The extremality property of the angles of the Reuleaux triangle carries over to this vertex and this edge circle of the surface. The measure of this solid angle or edge angle turns out to be $\frac{1}{4}$ or $\frac{1}{3}$ of the entire sphere or circle, respectively. Also the contour of this photo coincides roughly with the Reuleaux triangle itself.

The last example of a convex body of constant width is not rotationally symmetric. It turns out to be impossible to build such a convex body by pieces of spheres, like in the case of the Reuleaux polygon. However, already in 1911 in the model catalogue of the M. SCHILLING-Verlag [31] and later in 1912 E. MEISSNER [23] gave a slightly modified example. By analogy with the Reuleaux triangle MEISSNER starts with a regular tetrahedron with edge length b. Following his idea we now describe how to get a tetrahedron of constant width (called Meissner body) with these four vertices and opposite pieces of spheres. First we construct four round balls with radius b around the vertices. The intersection of these four balls is a body D with four vertices and four opposite pieces of spheres and with six curved edges which are intersections of two spheres (small circles of radius $\frac{b}{2}\sqrt{3}$). The corresponding central angle α with $\cos\alpha = \frac{1}{3}$ is the same as the angle between two faces of the Euclidean regular tetrahedron. The edge angle turns out to be $\frac{2}{3}\pi$. There are three pairs of opposite curved edges. It is easily seen that we can have at most one edge for each pair if the convex body has constant width. Now let us choose three such edges belonging to three different pairs. There are essentially two different possibilities for such a choice:

a) the three opposite edges have a common vertex,
b) the three opposite edges form a triangle.

Now we are going to modify those opposite edges by rounding them off and replacing a neighborhood of them by pieces of a spindle torus. We get these pieces by rotation of a piece of a circle of radius b centered at a vertex of the tetrahedron where the rotation is around the axis joining the two vertices of the edge to be rounded off. These pieces of tori come to lie inside D and the angle of rotation is $\arccos\frac{1}{3}$.

In this way a convex body of constant width b has been constructed. This has four vertices, three circle edges, four spherical pieces and three toroidal pieces. At the boundary between spherical and toroidal pieces the surface is continuously differentiable. According to the two different possibilities above there are two noncongruent types of such bodies of constant width. However, they have the same volume and the same area. For type a) we have a plaster model shown in the **Photos 100–102**. The long standing conjecture that this body has the smallest volume among all 3-dimensional convex bodies of constant width b (compare the theorem by BLASCHKE-LEBESGUE) turns out to be false according to a smaller upper bound for the volume due to G. D. CHAKERIAN [9].

Every plane containing two vertices of this body and one point p of the Euclidean tetrahedron different from these two vertices yields a slice of this body which is a planar arc polygon of the same constant width and which in addition coincides with the contour in the direction perpendicular to the plane. If p is a vertex of the Euclidean tetrahedron then the slice is a Reuleaux triangle. We get also a Reuleaux triangle if the two vertices are the endpoints of an edge which has been rounded off. If they belong to an edge of the other kind and if p is an interior point of the tetrahedron then the slice is a polygon with two vertices containing two circular pieces of radius b and two circular pieces (opposite to each other) whose sum of radii is b. This last case can be seen approximately in the contours of Photos 101 and 102.

Photo 100 shows in particular the vertex incident with the three edges rounded off. The notches on the surface give the position of the boundaries between the spherical and toroidal pieces. All of these points are regular ones and the pieces fit together continuously differentiably.

Photo 101 and 102 show the three edges which have not been rounded off and which bound the lower trigonal piece of a sphere. This piece is not a geodesic spherical triangle because the edges are pieces of circles with a radius different from b. This spherical piece is just the set of all points opposite to the fourth vertex. In the edge point there are several supporting planes. As an example consider a point of the edge which is the lowest part of the contour of Photo 101. Then it is possible to see the set of opposite points which is a piece of a circle lying in the corresponding toroidal piece. If the edge point moves along this edge from vertex to vertex then this circular arc moves from one marked boundary to the other. This can be seen quite clearly in Photo 101. To the left lower vertex there corresponds the opposite trigonal spherical piece as set of opposite points. Its marked boundary pieces are great circles with respect to the corresponding spherical piece of radius b but the lower edge is not. Consequently this spherical piece is not a geodesic triangle.

Photo 102 shows the model between the two planar metal pieces of the gauging instrument which verifies and demonstrates the constant width. The right hand metal touches it at an edge point. The corresponding opposite point touching the left hand metal lies on the opposite toroidal piece of the rounded edge.

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