8 Conclusion

In this thesis, we investigated the complexity and approximability of generalized network improvement and packing problems. In detail, we studied three extensions of the traditional maximum flow and minimum cost flow problem and revealed a strong connection to a novel variant of the bounded knapsack problem. For all of these problems, we both presented exact algorithms and investigated their approximability under involvement of a diverse set of graph classes.

As it became evident, extensions to the formulation of the traditional maximum flow or minimum cost flow problem that seem to be minor at the first glance turn out to have a significant impact on the complexity and approximability of the corresponding problems. Established combinatorial algorithms for well-known network flow problems turn out to be highly specialized to the inherent structure of these problems and cannot be directly applied to more general variants. The integrality assumption as a fundamental property of minimum cost flows could not be applied to any of the considered problems. When it was enforced to hold, it even made the budget-constrained minimum cost flow problem and the maximum flow problem in generalized processing networks $\mathcal{NP}$-hard to solve and approximate. Moreover, while efficient strongly polynomial-time algorithms are known both for the minimum cost and the maximum flow problem, the maximum flow problem in generalized processing networks turned out to be at least as hard to solve as any packing LP, which makes a strongly polynomial-time algorithm unlikely to exist. Finally, although the traditional maximum generalized flow problem can be solved efficiently, the counterpart considered in this thesis becomes strongly $\mathcal{NP}$-hard to solve and approximate.

Nevertheless, using more sophisticated approaches, we were able to adapt several results that are valid for the most fundamental network flow problems. In Chapter 4, we were able to extend the network simplex algorithm for the minimum cost flow problem to the more general case with an additional budget constraint. In addition, we were able to reduce the problem $\text{BCMCFP}_{\mathbb{R}}$ to a sequence of $\widetilde{\mathcal{O}}(\min\{\log M, nm\})$ traditional minimum cost flow computations, so our algorithms benefit from the significant amount of research that lead to more and more advanced algorithms for the minimum cost flow problem in the past decades. Similarly, although the maximum flow problem becomes much harder to solve in the case of generalized processing networks as shown in Chapter 6, we were able to adapt a well-known result for the minimum cost flow problem to the case of our problem on series-parallel graphs. For the discrete versions of the budget-constrained minimum cost flow problem considered in Chapter 5, we observed an interesting connection of the problem on extension-parallel

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graphs to a novel variant of the bounded knapsack problem. This connection made it possible to derive efficient approximation algorithms for the budget-constrained minimum cost flow problem on extension-parallel graphs although the problem turned out to be $\mathcal{NP}$-hard to approximate on series-parallel graphs. This observation made once more obvious that network flow problems — just as knapsack type problems — are packing problems in their core.

In addition to the above results, we were able to show that one of the most fundamental theorems for network flow problems — namely the flow decomposition theorem — remains its validity for each of the considered problems (although the notion of "basic components" each flow decomposes into needs to be adapted). Hence, all of the considered problems can be seen as packing problems in which flows on such basic components are packed subject to a set of capacity constraints. This observation inspired the development of the generalized fractional packing framework in Chapter 3 as an integration of Megiddo's (1979) parametric search technique into the fractional packing framework of Garg and Koenemann (2007). This generalized framework leads to fully polynomial-time approximation schemes for a large class of network flow problems for which the flow decomposition theorem translates into the containment of each flow in a polyhedral cone, whose dual cone can be separated efficiently.

As already mentioned in the corresponding chapters, all of the investigated problems raise several questions for future research. It seems worthwhile to put more effort into further research on exact and approximation algorithms for the two continuous problems $\text{BCMCFP}_{\mathbb{R}}$ and MFGPN. In particular, one might hope to be able to derive new solution methods for these two problems by adapting algorithms for the traditional minimum cost and maximum flow problem (as it was done with the network simplex algorithm for the problem $\text{BCMCFP}_{\mathbb{R}}$) and applying more advanced speed-up techniques such as parameter scaling or the usage of sophisticated data structures like dynamic trees. For both problem, it would be fruitful to compare the empirical performance of the presented combinatorial algorithms to their non-combinatorial counterpart. For the three $\mathcal{NP}$-complete problems $\text{BCMCFP}_{\mathbb{N}}$, $\text{BCMCFP}_{\mathbb{B}}$, and CGMFP, it may be reasonable to develop more advanced approaches in order to speed up the existing algorithms and to identify efficiently solvable and approximable special cases that are of importance in practice.