Epistemological Judgments in Mathematics: An Interview Study Regarding the Certainty of Mathematical Knowledge

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Content

Abstract

Research on personal epistemology is confronted with theoretical issues as there exist conflicting data regarding its coherence, discipline-relation and context-dependence as well as methodological issues regarding the often used questionnaires to measure epistemological beliefs. We claim that it is necessary to distinguish between relatively stable "epistemological beliefs" and situation-specific "epistemological judgments". In a sequence of interviews with regard to the topic of "certainty of mathematical knowledge", we show that the usual categories used in questionnaires to measure epistemological beliefs have to be differentiated. We argue that epistemological judgments provide a promising framework to interpret the statements of the interviewees.

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1 Introduction the Research Project „LeScEd“

Research orientation is a key characteristic of higher education and university education (cf. Tremp & Futter 2012). It is represented in normative frameworks for educational studies such as teacher education (e.g., KMK 2004). Research orientation is characterized as the competence to receive and understand scientific knowledge (“engagement with research”) and in addition to think and work scientifically (“engagement in research”) (cf. Borg 2010). A development of these competencies is seen as essential to prepare pedagogical and educational professions in understanding science and science communication.

The research project “LeScEd” (an acronym for “Learning the Science of Education”), which is funded by the BMBF[4] , is dedicated to examine three facets of research orientation of university students and doctoral candidates:

- Knowledge and mastery of procedures and methods of social sciences;
- Scientific argumentation and communication;
- Epistemological beliefs about the nature of knowledge and knowing.

This article addresses a subproject which studies epistemological beliefs with a special focus on mathematics. The purpose of the subproject is the construction of an instrument to measure beliefs of students with respect to the epistemology of mathematics. In a first step we investigated the assumption that the usual categories used in questionnaires have to be differentiated. We argue that epistemological judgments can be grounded in different beliefs as well as other cognitive arguments (e.g., Stahl 2011), within a considerable range of sophistication.

1.1 Theoretical Background

A person's beliefs are his/her "[p]sychologically held understandings, premises, or propositions about the world that are thought to be true." (Philipp 2007, p. 259) They filter his/her perceptions and direct his/her actions (cf. Philipp 2007). For example, beliefs about mathematics influence the person's mathematical problem solving performance (e.g., Schoenfeld 1992) and his/her acquisition of mathematical knowledge (see Muis 2004, p. 339 ff. for an overview of related studies).

Epistemology is a branch of philosophy dealing with the nature of human knowledge and its justification. Researchers attribute a growing interest in per-

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sonal epistemology development and epistemological beliefs (= beliefs about the nature of knowledge and knowing) (cf. Hofer & Pintrich 1997).

Research on personal epistemology origins in the work of Piaget and Perry and is nowadays part of both psychology and education (cf. Hofer 2000). Whereas early studies modeled personal epistemology as a unidimensional sequence of stages, recent studies consider personal epistemology as "a system of more or less independent epistemological beliefs" (Hofer 2000, p. 379). Hofer and Pintrich (1997) proposed a structure for that system of epistemological beliefs: According to Hofer and Pintrich there are two general areas with two dimensions each. The first area is nature of knowledge (what one believes knowledge is) with the two dimensions certainty of knowledge and simplicity of knowledge; the second area is nature or process of knowing (how one comes to know) with the dimensions source of knowledge and justification of knowledge. Of special interest for this article is the dimension certainty of knowledge which is defined as follows:

> "Certainty of knowledge. The degree to which one sees knowledge as fixed or more fluid appears throughout the research, again with developmentalists likely to see this as a continuum that changes over time, moving from a fixed to a more fluid view. At lower levels, absolute truth exists with certainty. At higher levels, knowledge is tentative and evolving. […]" (Hofer & Pintrich 1997, S. 119 f.)

A growing amount of psychological research presents relationships between epistemological beliefs and various aspects of learning. It is generally assumed that more sophisticated epistemological beliefs are related to more adequate learning strategies and therefore better learning outcomes (cf. Hofer & Pintrich 1997; Stahl 2011). However, conflicting data exist that cannot be explained with traditional theories about epistemological beliefs (cf. Bromme, Kienhues, & Stahl 2008). Even though most researchers have conceived this construct as general and rather stable, growing empirical evidence showed that epistemological beliefs are less coherent, more discipline-related and more context-dependent than it was hitherto assumed (cf. Hofer 2000).

For example, Muis, Franco, and Gierus (2011) analyzed the epistemological beliefs of students enrolled in a statistics course. They showed that "slight changes in context influence what epistemic beliefs are activated, which can subsequently influence learning." (ibid., p. 516)

Stahl (2011) claims that it is necessary to distinguish between relatively stable epistemological beliefs and situation specific epistemological judgments when examining this construct in more detail. Epistemological judgments are defined

> "[…] as learners' judgments of knowledge claims in relation to their beliefs about the nature of knowledge and knowing. They are generated in dependency of specific scientific information that is judged within a specific learning context. […] [A]n

> epistemological judgment might be a result of the activation of different cognitive elements (like epistemological beliefs, prior knowledge within the discipline, methodological knowledge, and ontological assumptions) that are combined by a learner to make the judgment." (Stahl 2011, p. 38 f.)

Stahl (2011) elaborates these theoretical considerations of a generative nature of epistemological judgments with fictitious examples. Three persons with different backgrounds (content knowledge, methodological knowledge, ontological assumptions, epistemological beliefs, etc.) in physics each judge the claim that the distance between sun and earth is 149.60 million kilometers. In this article we intend to support this assumption by empirical examples.

In mathematics education the terms "personal epistemology" and "epistemological beliefs" are rarely used. Instead, research on this topic is assessed under the construct of beliefs (cf. Muis 2004, p. 322). Muis summarizes several studies dealing with beliefs about mathematics:

> "The majority of research that has examined students' beliefs about mathematics suggests that students at all levels hold nonavailing[5] beliefs. In general, when asked about the certainty of mathematical knowledge, students believe that knowledge is unchanging. The use and existence of mathematics proofs support this notion, and students believe the goal in mathematics problem solving is to find the right answer. [...]" (Muis 2004, p. 330)

Researchers investigating beliefs about mathematics as a discipline deal with opposing perceptions of mathematics: process-orientation versus rule-orientation, dynamic versus static interpretation, formal versus informal discipline, or its applicability (cf. Muis 2004; Grigutsch, Raatz & Törner 1998).

The global intentions of our research project are (a) to identify epistemological beliefs about mathematics as a science and (b) to develop the instruments to do so economically (cf. Muis 2004, p. 354). The research intentions for this paper are (i) to identify epistemological beliefs about mathematics as a science (especially regarding "certainty of knowledge"), and (ii) to provide empirical evidence that supports the theoretical differentiation between epistemological beliefs and epistemological judgments.

[5] To avoid a negative connotation, Muis (2004, p. 323 f.) does not use the common labels "naïve – sophisticated" or "inappropriate – appropriate" from psychological and educational research. Instead she suggests to use the labels "nonavailing – availing" for beliefs that are associated with better learning outcomes ("availing"), and for beliefs that have no influence or a negative influence on learning outcomes ("nonavailing").

1.2 Development and Implementation of the Interviews

Because of our global intentions, we chose suitable positions from the philosophy of mathematics (e.g., about the ontology of mathematical objects) and started to design a manual for semi-structured interviews with the long-term goal to develop an adaptive, web-based questionnaire to collect data about according beliefs. The aim of the interviews is to examine the idea of a generative nature of beliefs in more detail.

To get more insight into our subjects' beliefs, we did not just ask general questions with philosophical orientation but presented quotes of representatives of opposing epistemological positions and had our subjects relate themselves thereto. Afterwards, we intervened with information contrary to the subjects' positions to further identify their lines of reasoning.

During the first phase of data collection, we optimized our selection of quotes as well as our interview questions and developed additional interventions for the various subjects' positions and reasons. This can be seen as an application of Grounded Theory (cf. Strauss & Corbin 1996) which also postulates that the research design can be developed further with respect to successively analyzed data. As topics for the interviews we chose different key questions on the epistemology of mathematics as a science. In the following we chose to present – as an example from the larger body of data we collected – a single setting which deals with the topic of certainty of mathematical knowledge.

2 Sample Setting: Certainty of Mathematical Knowledge

2.1 Theoretical Background in the Philosophy of Mathematics:

Mathematical knowledge is regarded as certain since antiquity, because of formal proofs and deductive reasoning with respect to valid rules and axioms (cf. Heintz 2000, p. 52 ff.; Hoffmann 2011, p. 1 ff.) But this belief was shaken several times during the history of mathematics: (i) It is impossible to justify the axioms that theorems rely on and the discovery of non-Euclidean geometries has shown that different determinations can lead to divergent mathematics. (ii) The finding of contradictory derivations from axioms (Russell's paradox) led to the attempt of establishing formal rules of derivation by D. Hilbert but was doomed to failure because of Gödel's incompleteness theorems in 1931 (cf. Hoffmann 2011, p. 52 ff.). (iii) Proofs of mathematical theorems can be inaccurate or even incorrect and the review process of publishing magazines cannot guarantee identifying all weak spots. Often, mathematical work is so specialized that only a handful of experts is able to comprehend it and the history of mathematics is full of examples of accepted proofs that were discovered to be wrong years after their publication. (iv) Finally, a growing number of mathematical results is achieved with the help of computers and no living mathematician is able to verify them without trusting the machines as well as hoping for error-free hard- and software (cf. Borwein & Devlin 2011, p. 8 ff.).

These aspects all relate to the topic of certainty of mathematics. Interviewees can relate to these aspects in different ways when arguing about their individual judgment on the certainty of mathematics.

Table 1 Starting positions for "Certainty of Mathematical Knowledge".

Mathematical knowledge is certain	**Mathematical knowledge is uncertain**
"In mathematics knowledge is valid forever. A theorem is never incorrect. In contrast to all other sciences, knowledge is accumulated in mathematics. […] It is impossible, that a theorem that was proven correctly will be wrong from a future point of view. Each theorem is for eternity." (Albrecht Beutelspacher) [2001, p. 235; translated by the first author]	"The issue is […] whether mathematicians can always be absolutely confident of the truth of certain complex mathematical results […]. With regard to some very complex issues, truth in mathematics is that for which the vast majority of the community believes it has compelling arguments. And such truth may be fallible. Serious mistakes are relatively rare, of course." (Alan H. Schoenfeld) [1994, p. 58 f.]

2.2 Realization of the Interview:

We confronted to our subjects with two quotes (see Table 1) and invited them to answer the following prompt: "These are two positions of mathematicians regarding the certainty of mathematical knowledge. With which position can you identify yourself? Please give reasons for your answer."

Further questions were: "Can you explain your position on the basis of your mathematical experience?" "Please compare the certainty of mathematical knowledge to that of other sciences, for example to physical, linguistic, or educational knowledge."

If a subject settled on “math knowledge is certain”, we confronted him/her with the story of a false proof of the four color theorem by A. Kempe in 1879 that was accepted by the community of mathematicians and which was shown to be false by P. Heawood not earlier than 11 years later (e.g., Wilson 2002). If a subject thought that “math knowledge is uncertain”, we asked whether a theorem like the Pythagorean one could be uncertain as there are hundreds of proofs, countless validations and practical applications like in masonry.

So far, the first author interviewed 10 pre-service teachers of mathematics (students at the University of Education Freiburg), 2 in-service teachers of mathematics, 2 professional mathematicians and 2 professors of mathematics. Below you’ll find a selected sample of these interviews.

3 Initial Results

Our initial results with respect to the area of “certainty of math knowledge” are twofold: Firstly, we present two different lines of reasoning each for “certain” and “uncertain” to point out what arguments we found empirically to support these positions. Secondly, we show that subjects who support the same position and would (and actually did) check the same boxes for according questions in a typical beliefs questionnaire can do so for differing reasons. This supports our argument for the theoretical introduction of *epistemological judgments*.

3.1 Interviewees judging that “mathematical knowledge is certain”

1) T.W. is a pre-service teacher in his second year at the University of Education in Freiburg. For him, mathematical knowledge is certain, “the first quote of Beutelspacher is more likely correct in my view.” He says that he thinks of proofs as inevitable and irrefutable. And he adds: “How can there possibly be errors in mathematics?”

Confronted with the historical episode of the four color theorem, T.W. admits “Of course, there can be errors, […] but it got proven eventually, didn’t it?” When asked why he was so sure about the certainty of mathematical knowledge, T.W. mentioned the Pythagorean Theorem as an example of a theorem which is inevitable for him, which has several hundred proofs, and which will not change in the next 10, 100, or 1000 years. He concludes with: “Hopefully. Otherwise, my fundamental conception would be destroyed.”

2) A.R., a mathematician who just finished his diploma at the University of Oldenburg, considers mathematical knowledge for certain: “I identify myself definitely with Albrecht Beutelspacher.” A.R. adds that errors are possible, but these would be the errors of mathematicians but not of mathematics itself. "Humans are fallible. […] There might be errors in proofs which are accepted by many people. […] But when a theorem is proved correctly from the axioms by formal rules of derivation then it will last for eternity." As an example for human fallibility A.R. refers to Andrew Wiles' proof of Fermat's Last Theorem. This was regarded as proven for a short time, then rejected and republished after some years. It might take another several years until the methods Wiles used pass over to the common mathematical knowledge, but A.R. believes that there will be a time when this theorem and its proof will have been checked thoroughly and will have been finally accepted as certain.

The quote of Schoenfeld might go well with great mathematical puzzles like Riemann's Conjecture but otherwise, it does not describe A.R.'s view of mathematical knowledge. In comparison, other scientific disciplines are dependent on tests and laboratory experiments which results in their knowledge being uncertain. In contrast, mathematical knowledge is reducible to basic elements, the axioms, and to logical conclusions, which makes it certain.

3.2 Interviewees judging that "mathematical knowledge is uncertain"

3) B.G. is a pre-service teacher who just finished her degree at the University of Education in Freiburg. She thinks that mathematical knowledge is uncertain, because "for me, there is always the possibility that someone figures out that something is not quite correct. A theorem might be proven and checked but there is always the possibility of finding an aspect that it may not be correct." She generally would not agree to any statement regarding "ever" or "never".

Asked if there is a counter-example for the Pythagorean Theorem she responds that she is not able to come up with any, but there might be others with a better mathematical background who could. The interviewer wanted to know if she was certain of the consequences of her position. The logical construction of mathematics might collapse if basic findings like the Pythagorean Theorem or Complete Induction were not certain. B.G. responded with "I know of the consequence and I'm fine with it. […] This is no problem for me. […] But the possibility for this to happen is very, very small."

She states that in comparison to other scientific disciplines, knowledge in mathematics is very certain, but some uncertainty remains.

4) S.W. is a mathematics professor at the University of Hanover for several years. In his view, Beutelspacher holds a Platonic view which he cannot agree to. He says that he does not believe in a mathematical realm with eternal conceptions that exists outside the human sphere. S.W. describes in detail that he can think of basically two arguments for mathematical knowledge being uncertain: Firstly, mathematicians are fallible and errors can occur during proving and reviewing. But this is not the main point, because the community is very careful and all but maybe the most complex things are thought through very thoroughly and therefore very certain. Secondly, this is the crucial point according to S.W., mathematical knowledge cannot be definitely certain because that would imply an infallible system of rules with an otherworldly justification. Mathematics would need a justification outside of the human sphere and outside of the mathematical discourse, a realm that could be observed and described. S.W. concludes: "That there is such a realm, such a sphere, I am very skeptical about it."

Asked whether theorems like the Pythagorean one are not sure, S.W. answers: "This theorem cannot be certain because it is unclear what certainty means in this context." He says that the Pythagorean Theorem is an innermathematical theorem that the community of mathematicians considers true under certain axiomatic assumptions. But this does not mean that it would be true if there were no humans or the universe came to an end, because of the missing mathematical realm that would justify such eternal truth.

In comparison to other scientific disciplines, S.W. states that mathematical knowledge is more certain, because the other sciences have the same problem of a missing justification as well as additional disturbances in the form of assumptions, hypothesis and doctrines. "Mathematics is more rigorous and so to speak more pure and therefore more certain in a sense."

3.3 Interpretation using the category of epistemological judgments

Both T.W. and A.R. answered the knowledge claim whether mathematical knowledge is certain or uncertain in the same way and both checked the box "mathematics is very certain" in a questionnaire (CAEB, Stahl & Bromme 2007) they completed previously to the interview. Within a questionnaire study this would contribute to positioning them on a belief scale with respect to certainty.

But actually they have shown a substantially different argumentation in those interviews due to their background. Whereas T.W. could only refer to simple examples such as the Pythagorean Theorem, A.R. was able to activate more content knowledge in the form of Andrew Wiles' proof as well as the Riemann Conjecture. Additionally, A.R. did argue with mathematical axioms and rules of derivation, whereas T.W. solely relied on his "fundamental conceptions". A.R. was conscious about possible errors in mathematical proofs but was able to integrate this into his beliefs. T.W., on the other hand, was not aware of this fact up to the intervention and did not use this piece of information for his argumentation ("it got proven eventually, didn't it?").

The same is true for B.G. and S.W. who both supported the position that "mathematical knowledge is uncertain". B.G. could only rely on fundamental conceptions ("I generally do not agree to statements referring to 'ever' or 'never'."). On the other hand, S.W. could not only refer to his content knowledge about the fallibility of the mathematical review process, but also to his ontological knowledge regarding Platonism to support his arguments.

This empirical data supports the theoretical claim of Stahl (2011, p. 49):

> "In a questionnaire with rating scales, [these] persons would give the same answer. However, the conclusion that their responses are an expression for comparable epistemological beliefs would be wrong. Their epistemological judgments are built on different cognitive elements to evaluate the knowledge claim." (Stahl 2011)

4 Discussion

The evaluations of the interviews show the breadth of arguments for the positions of "mathematical knowledge is certain / uncertain". There are more or less reflected representatives of both statements which is somewhat surprising in relation to results from research on epistemological beliefs. For the dimension *certainty of knowledge* more "sophistication" is seen as less belief in truth with certainty (cf. Hofer & Pintrich 1997; Hofer 2000). But the example of A.R. shows that this position can be held in a reflected way (which is revealed by the way he judges the certainty of knowledge of other scientific disciplines).

The evaluations of the interviews also show the gain of the theoretical introduction of *epistemological judgments*. Persons that hold the same position regarding the certainty of mathematical knowledge can do so with differing backgrounds. A traditional beliefs questionnaire would not be able to detect or explain those differences. Therefore it seems doubtful to rely on instruments that measure epistemological beliefs as a locus on a scale. It should be necessary to take into account different strands of argumentation and different backgrounds. The concept of "epistemological judgment" can be a promising starting point for developing instruments that can capture such important differences.

Future prospects include developing an adaptive, web-based questionnaire to measure epistemological judgments and beliefs. A first pilot study was conducted in July 2013 with 45 university students; a second one is scheduled for the 2013/14 winter term.

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