Chapter Six

Beliefs and Knowledge of Expert Mathematics Teachers

6.1 Introduction

This chapter includes three parts. The first reports common beliefs held by the three expert mathematics teachers, including their beliefs about mathematics, mathematics learning, and mathematics teaching. The second reports common characteristics of the three expert mathematics teachers' knowledge, which are categorized into mathematics subject knowledge, pedagogy content knowledge, curriculum knowledge, and knowledge of learners. The third reports teaching strategies identified among the three expert mathematics teachers.

6.2 Beliefs

6.2.1 Beliefs about mathematics

To explore the three expert mathematics teachers' beliefs about mathematics, they were asked questions related to the nature of mathematics. Table 6.1 summarizes the main aspects related to what mathematics is in their views. Although their answers vary, there exist some commonalities as follow:

6.2.1.1 A vehicle for developing students' thinking and ability

As shown in Table 6.1, the first common statement made by the three expert mathematics teachers is that mathematics is a vehicle for developing students' thinking and abilities. They all mentioned that studying mathematics can change people's thinking style; for example, Ms. Qian stated "I think the essential function of it (mathematics) is to develop students' thinking", while Ms. Sun said mathematics can "develop some methods to think about problems". Mr. Zhao commented that:

> *the main task of mathematics is to develop students' logical thinking system, and the ability to learn [by themselves], and the ability to analyze and solve problems.*

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Table 6.1 Expert mathematics meachers' beliefs about mathematics

	Mr. Zhao	Ms. Qian	Ms. Sun
A vehicle to develop students thinking and ability	+	+	+
Being from solving problems in real life and in turn, being able to be applied in real life	+	+	+
Basics for other subjects and science	+	+	+
A school learning and examination subject	+	+	+
Developing students' views of mathematics		+	+
An accomplishment of human being	+		
An instrument to explore relationships in space and quantity relationships		+	

Note. "+" indicates that the teacher made a corresponding statement.

6.2.1.2 Application in real life

The second common statement is that mathematics can be applied in real life. Ms. Sun emphasized that mathematics is tightly linked with real life situations, and the fact that some mathematics theories cannot be applied in real life at certain times does not mean that they cannot be applied at all; some seemingly inapplicable theories might one day be used to solve practical problems. Ms. Qian also mentioned that, even though not every student will become a mathematician in the future, mathematics makes students view the world around them mathematically and will influence their ways to solve problems in daily life. Mr. Zhao pointed out that "mathematics is from real life; mathematics can be applied in real life situations; mathematics can be used to serve our life".

6.2.1.3 Base for other subjects and science

According to the three expert teachers' descriptions, mathematics acts as the base for other fields or subjects, such as physics and information technology. Mr. Zhao noted that "mathematics can be infiltrated into other subjects, for example, physics". Ms. Qian saw mathematics as an instrumental subject that serves as a basis for the development of other subjects. All three teachers emphasized that mathematics is the basis for scientific development; as Ms. Sun argued, "without the development of mathematics, it is quite difficult for science to make any progress".

6.2.1.4 A school learning and examination subject

The last commonly-held view among the three expert mathematics teachers was that mathematics is not only a school subject, but also an important examination subject at different levels, such as the *Zhongkao* and *Gaokao*. According to Mr. Zhao, “mathematics is a necessary subject in examination in secondary school”. In Ms. Qian’s view, mathematics is also a subject students have to learn for their senior secondary, university and life-long learning. Ms. Sun thought that, based on the arrangement of the curriculum in mainland China, mathematics has a high status as a school subject at both the primary and secondary levels, and students have to learn it well, because it is also a very important subject in examinations at both levels.

6.2.2 Beliefs about mathematics learning

Teachers’ beliefs about how their students learn influence how they plan their teaching and how they interact with students while teaching (Calderhead, 1996). The three expert teachers’ beliefs about students’ ability to learn mathematics, the best ways to learn mathematics, and the most important parts of mathematics learning were examined to investigate their beliefs about mathematics learning.

6.2.2.1 Beliefs about the ability to learn mathematics

According to the three teachers, not every student can learn mathematics well under the same standard. In their opinions, giftedness is a very important, factor to determine whether a student can learn mathematics well or not, but it is not the only one. Ms. Sun pointed out that some students “are not sensitive to numbers”, whilst Ms. Qian added that “there exist some differences between students’ ability”. Mr. Zhao thought that learning mathematics is a process, and that a student cannot learn mathematics well at a time does not mean s/he will never learn it well; however, he admitted that “learning mathematics well depends on one’s wisdom”. All three teachers further emphasized that, even if a student is very talented in mathematics, s/he cannot learn mathematics well if s/he does not study hard.

6.2.2.2 Beliefs about the best ways to learn mathematics

Despite the commonalities expressed about ability to learn mathematics well, the best ways to learn mathematics described by the three teachers varied. Ms. Sun thought that students should preview what they will learn and review what they have learned regularly. Ms. Qian summarized the best way to learn mathematics as “more practicing, more questioning, more reflecting”. In other words, students need to practice exercises at their own ability level, ask their teacher and peers questions, think and reflect deeply. Mr. Zhao thought that the best way to learn mathematics is that students should become intellectually involved in the learning process and deeply understand the knowledge development process. In addition, he thought that students need to memorize some definitions and theorems based on understanding, that is, not memorize everything blindingly.

A common characteristic implicitly embodied in the beliefs described above is that students should engage themselves in mathematics learning and become intellectually involved in the process of mathematics learning, rather than superficially receiving or memorizing by rote. This indicates that students cannot learn mathematics well simply by receiving teachers’ direct instruction and rote memorization. In other words, students’ deep understanding is essential to learning mathematics well. For example, Ms. Qian said that students should more often ask themselves how to reason a particular formula and question why it should be reasoned in that way when they study mathematics. Similarly, Mr. Zhao commented that students should learn to summarize and conclude based on their own discovering and understanding.

6.2.2.3 Beliefs about the most important parts of mathematics learning

According to the three teachers, the most important parts of mathematics learning at the junior secondary school level are mathematical thinking, the ability to apply mathematics, problem-type training, and knowledge; of these, all three view mathematical thinking as the most important. In their views, mathematics thinking, not knowledge, influences students’ life forever. The three teachers also commented that the ability to apply mathematics to solve either mathematics problems or problems from real life is a very important part of mathematics learning. Mr. Zhao and Ms. Sun further thought it quite necessary for students to summarize or generalize problem solving strategies they can employ in the future, because it is unrealistic to expect students to finish all mathematics problems. Lastly, Mr. Zhao and Ms. Qian thought that it is quite necessary to lay a solid knowledge foundation for students, which will benefit their future development in mathematics and in other areas.

6.2.3 Beliefs about mathematics teaching

Teachers' beliefs about mathematics teaching are "the key determinant of how mathematics is taught" (Ernest, 1989, p. 22). To understand a teacher's teaching, it is necessary to investigate her/his beliefs about teaching. In this study, beliefs about the goals of mathematics teaching, what constitutes a successful mathematics lesson, and effective mathematics teaching methods were examined to investigate the three expert mathematics teachers' beliefs about mathematics teaching.

6.2.3.1 Beliefs about the goals of mathematics teaching

Common goals of mathematics teaching mentioned by the three teachers include developing students': 1) mathematical thinking, 2) ability to apply mathematics, and 3) ability to learn mathematics independently. The three teachers saw the first of these as the most important. They thought it quite normal that students will forget their knowledge one day; however, the mathematical thinking they develop will influence their life forever. As Ms. Qian said:

> *As a saying goes, mathematics is like gymnastics of thinking to develop students' thinking. After our students' graduation, they will work in different fields. However, those students who have strong mathematics ability, whatever things they work on, they will think logically and finish it logically. Therefore, I think the most important objective of mathematics teaching is to develop students' mathematics thinking.*

Ms. Sun and Mr. Zhao also mentioned that it is very important to develop students' ability to apply mathematical knowledge to solve both mathematics problems and problems in real life. Ms. Qian and Mr. Zhao commented that mathematics teaching should develop students' ability to learn mathematics by themselves to benefit their future study and life.

6.2.3.2 Beliefs about a successful mathematics lesson

To evaluate whether a mathematics lesson is successful or not, the following three aspects were frequently mentioned by the three teachers: 1) students' active participation; 2) students' deep understanding; and, 3) linking mathematics with students' real life situations. All three teachers noted that, for a successful mathematics lesson, a teacher should employ various teaching strategies to inspire students' interests and encourage students to participate in activities. Intellectual engagement was particularly emphasized. Mr. Zhao said:

> *A successful mathematics lesson cannot be judged only according to its active classroom atmosphere. In contrast, it should make students really involve in the logical system of mathematics. They completely involve in mathematics and they solve some problems by themselves. During the whole process, teacher only act as a guider.*

Mr. Zhao further pointed out that students should have the opportunity to construct mathematical knowledge through their own experience and exploration, which was echoed by Ms. Qian and Ms. Sun. In addition, all the teachers mentioned that a successful mathematics lesson should promote students' deep understanding and help them to master knowledge, skills and problem-solving strategies in an appropriate way. Ms. Sun and Ms. Qian stated that a teacher should employ examples or construct situational problems related to students' real life so as to have students experience knowledge development process.

6.2.3.3 Beliefs about effective mathematics teaching methods

Although there exist variations in the effective teaching methods mentioned by the three teachers, common facets identified from their responses include: 1) encouraging students' intellectual engagement; 2) inspiring students' interests; and, 3) emphasizing learning methods and problem-solving strategies. All three teachers mentioned that, to teach mathematics effectively, a teacher should encourage students to involve themselves intellectually in activities or problems designed by the teacher. For example, a teacher should provide students enough time to explore the relationship between prior knowledge and current knowledge, look for solutions or alternative solutions to a problem, and summarize their own discoveries. Ms. Sun and Ms. Qian further commented that, to teach mathematics effectively, a teacher should be able to build an approachable relationship with students, and inspire students' interests to learn. Ms. Qian used a traditional Chinese saying, "interest is the best teacher", to convey this. Mr. Zhao and Ms. Sun thought that, to teach mathematics effectively, a teacher should stress developing students' problem solving strategies. As Ms. Sun said:

> *Necessary methods, include learning methods, problem-solving methods and methods to apply some basic knowledge, like "two basics" we are talking about, there are "practicing basic knowledge" and "practicing basic skills" in "two basics". We should stress these in teaching.*

6.2.4 Discussion

The three expert mathematics teachers' beliefs were qualitatively investigated through semi-structured interviews. As reported above, when the three teachers answered questions related to the nature of mathematics, they talked about mathematics as a vehicle for developing students' thinking and ability and saw it as a basis for other subjects and the progress of science. In addition, mathematics was thought of as coming from real life problems and as applicable to real life. Their comments on the function of mathematics seem to close to a combination of the instrumentalist and the problem-solving view of mathematics as proposed by Ernest (1991).

The three teachers' beliefs about mathematics are different from some beliefs expressed by Chinese mathematics teachers in previous studies, such as the "more rigid view of mathematics being more a product than as a process" (Leung, 1995, p. 315), or clear statement that mathematics is a type of knowledge that students need to learn (Wong *et al.*, 2002). None of the three teachers mentioned these two factors for several possible reasons. First, previous studies were conducted before or near 2000. As mentioned in Chapter Three, since 2001, there has been a mathematics curriculum reform, which depicts mathematics as an accomplishment of human beings coming from real life (MOE, 2001), and which might have influenced the three teachers' beliefs about the nature of mathematics. Second, the teachers in the two previous studies were not expert mathematics teachers. In Li *et al.*'s (2005) study, they found that non-expert mathematics teachers at elementary school level in China tend to hold a "mastering of knowledge" view of mathematics, whereas expert mathematics teachers tend to hold a "problem solving" view of mathematics.

Viewing mathematics as coming from real life and as a vehicle for developing students' ways of thinking differs from long-held beliefs on mathematics education in China. Under the influence of the former Soviet Union, viewing mathematics as an abstract and rigorous subject with wide application (as suggested by Aleksandrov *et al.*, 1964), was widely accepted in China (Li *et al.*, 2008; Zhang *et al.*, 2004). Though none of the three expert teachers mentioned this, Wang and Cai (2007) recently found that some very experienced elementary mathematics teachers still held the belief that "mathematics itself is an abstract and coherent knowledge system" (p. 290).

The three teachers tended to believe that mathematics teaching and learning should promote the development of students' thinking and abilities. To learn mathematics effectively, students need to explore mathematics knowledge on their own and become intellectually involved in the learning process. Similarly, to teach mathematics effectively, teachers should encourage students to participate intellectually in activities or tasks. Their beliefs about mathematics teaching are similar to the "learner-focused" model described by Kuhs and Ball (1986). Moreover, according to the three teachers' descriptions, the teacher's role in the classroom is similar to the facilitator model proposed by Ernest (1989). In other words, they tended to believe that students should have opportunities to experience the process of knowledge development and to construct their own knowledge through experience and exploration. This view of developing students' mathematical thinking is also found in other studies on Chinese mathematics teachers (*e.g.*, An, 2004; Wang & Cai, 2007; Wong *et al.*, 2002).

The above ideas (such as emphasis on students' experience, participation, and exploration, emphasis on knowledge development process during teaching, and linking teaching content to real life) have been strongly advocated in China since the latest mathematics curriculum reform (MOE, 2001), and the three expert teachers' beliefs about mathematics teaching and learning might be influenced by relevant ideas in this curriculum reform. However, the teachers asserted that they tended to hold these kinds of beliefs even before the curriculum reform. They mentioned that, after several years of working, they started to demonstrate the knowledge development process and to emphasize student experience and participation. This suggests that, even though they worked in a teaching context dominated by traditional beliefs (*e.g.*, Zhang *et al.*, 2004; Zheng, 2006), they still could develop non-traditional beliefs about mathematics, mathematics learning, and mathematics teaching.

This is not to say that their beliefs are not influenced by the social and cultural context in which they are working. Their beliefs are still influenced by the social and cultural context to a certain degree. In the Confucian culture, students' thinking is stressed; for example, Confucius said that "mere reading without thinking causes credulity; mere thinking without reading results in perplexities" (学而不思则罔，思而不学则殆) (*Analects*, translated by Lao, 1992, p. 43). This culture might make them believe that mathematics is a vehicle for developing students' thinking; that mathematics teaching and learning should aim to develop students' mathematics thinking; that to learn mathematics well and teach mathematics effectively, students should be intellectually engaged, and so on. In Chinese learning culture, individual effort and diligence are highly valued; according to Li (2004), resolve, diligence, hardship, perseverance, and concentration are highly recommended as the five most important cultural and learning merits. This might make the three teachers think that students need to work hard to learn mathematics well, regardless of their talent in mathematics.

Moreover, one important characteristic of Chinese mathematics teaching tradition is its emphasis on seeking deep understanding and promoting students' mathematics thinking and mathematical ability (Kang, 2010; Shao *et al.*, 2012; Zhang, 2010; Zhang *et al.*, 2004; Zheng, 2006). For example, the idea of developing students' thinking and abilities has long been emphasized in mathematics teaching syllabi and curriculum standard (Research Institution of Curriculum and Textbooks, 2004).This tradition might have influenced the teachers' beliefs about mathematics, mathematics learning, and mathematics teaching. In addition, they all believe that mathematics is a school subject students need to learn for examination, and that mastering basic knowledge and skills is a necessary objective of mathematics teaching and learning. These suggest that China's examination culture might have influenced their beliefs. In fact, during interviews, the teachers often mentioned that, under the current examination system in China, they have to help students prepare for examinations.

6.3 Knowledge Base

Teachers' knowledge is an important factor influencing their teaching (Shulman, 1986). The characteristics of the three expert mathematics teachers' mathematics knowledge, pedagogical content knowledge (PCK), curriculum knowledge, and knowledge of the characteristics of learners were investigated through observations and interviews to gain a more comprehensive picture. The following sub-sections will report relevant characteristics identified in the three teachers:

6.3.1 Mathematics knowledge

Teachers' knowledge of mathematics is an important factor affecting their teaching practice and students' learning (Ball, 1991; Ball *et al.*, 2001; Blömeke & Kaiser, 2012; Schmidt, Cogan, & Houang, 2011; Hill *et al.*, 2005). According to observations and interviews, it could be safe to say that the three expert mathematics teachers have a solid mathematics knowledge base. Two common characteristics of the teachers' mathematics knowledge were identified: 1) a profound understanding of teaching content; and 2) a connected knowledge structure.

6.3.1.1 Profound understanding of teaching content

The first common characteristic is that all three teachers deeply understand teaching content:

Understanding the essence of concepts. Firstly, the three teachers demonstrated that they understand the essence of relevant concepts. For example, after Ms. Qian presented the definition of "Ratio of Two Line Segments", she further made three statements to this definition: 1) the value of the ratio is independent of the unit for measuring the length of the line segments; 2) when calculating the value, the lengths should be measured under the same unit; and 3) the value of the ratio is a positive real number. These statements indicate that Ms. Qian comprehended the essence of the definition. Moreover, these statements are not clearly stated in the textbook and teacher manual; according to her responses in the post-observation interview, she attained this deep understanding by her own work.

In Mr. Zhao's sixth lesson, he explored the geometric meaning of "k" (named by him) in an algebraic expression of y=k/x. He used several examples similar to the one shown on the left side of Figure 6.1 to explore the relationship between the value of the area of rectangle BPAO and the value of "k" so as to make students realize that the two values are equal. After this, he further explored the value of "k" with the value of the area of a triangle as shown on the right in Figure 6.1. These examples are not found either in textbooks or in teacher manual. It appears safe to judge that Mr. Zhao not only has a thorough understanding of the definition of the inverse proportion function, but also its relevant extensional meaning.

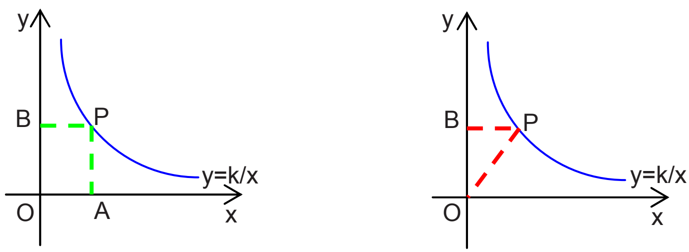

Figure 6.1 Examples used by Mr.Zhao to explore the geometric meaning of "k"

Comprehending critical differences between similar concepts. Secondly, the three teachers could differentiate critical differences between newly learned topics and similar prior knowledge. For example, Mr. Zhao once pointed out that, at the junior secondary school level, graphs of linear functions, direct proportion functions, and quadratic functions are continuous. However, graphs of inverse proportion function are not continuous. This suggests that Mr. Zhao had the ability to identify some distinguishing characteristics of various types of functions. Similarly, when Ms. Sun taught the definition of “bisector of an angle of a triangle”, she pointed out that the bisector of an angle of a triangle is a line segment; however, the bisector of an angle (as taught in primary school) is a ray. This also indicates that Ms. Sun not only understood the critical feature of angle bisector of a triangle and bisector of an angle, but also that, more important, she could discover critical differences between these two concepts.

Comprehending multiple representations. The three teachers were able to understand various representations of a concept or theorem. For example, regarding the definition of the inverse proportion function, Mr. Zhao demonstrated that he understood its algebraic expression, graph, table, and verbal description. In Ms. Sun’ teaching, she could use concrete paper triangles or figures to present a concept related to triangle. Furthermore, she could also give a verbal explanation or definition. For example, when she examined the relationship between lengths of the three sides of a triangle, she clearly stated that “the sum of the lengths of any two sides is greater than the length of the third side” and “the absolute value of the difference of the lengths of any two sides is less than the length of the third side”. After this, she used the following inequality, $|a-b|<c<a+b$, to summarize her statements.

6.3.1.2 Connected knowledge structure

Another characteristic of the teachers’ mathematics knowledge is that all three can interconnect relevant mathematics topics. During data collection, the three teachers were asked to draw a knowledge structure picture related to the teaching topic in the observed lessons. Due to the characteristics of knowledge, Mr. Zhao’s picture was relatively simple. However, as shown in Figure 6.2 and Figure 6.3, Ms. Qian and Ms. Sun developed a very comprehensive and interconnected knowledge structure on “Similar Figure” or “Triangle”. According to the two pictures, first of all, Ms. Qian and Ms. Sun well understood the interconnections among the topics within a certain chapter. For example, as shown in the shadowed frames in Figure 6.2, in the unit of “similarity”, Ms. Qian could link the ratio, golden section, similar polygons, and similar triangles together. In addition, as shown in Figure 6.2 and Figure 6.3, both the teachers clearly knew about students' prior knowledge and what students are going to learn in near future related to "similarity" or "triangle". This suggests that, even though they did not mention much knowledge at the senior secondary school level related to "Similar Figure" or "Triangle", they may still have a very comprehensive knowledge structure network in their mind at the level they are teaching.

The comprehensive knowledge structures further demonstrate that the teachers could organize and re-organize relevant topics to form a spider's web-like knowledge structure. This illustrates that they could discern relationships among similar topics and build proper connections among them. It has been well argued in the literature that "the degree of understanding is determined by the number and the strength of the connections" (Hiebert & Carpenter, 1992, p. 67). The rich connections built by the three teachers also suggest that they deeply understand the teaching content.

6.3.2 Pedagogical content knowledge

Pedagogical content knowledge (PCK), which refers to "the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 9), is another important type of knowledge affecting teachers' teaching practice. From interviews and observations, it could be said that the three expert mathematics teachers have extensive PCK. In particular, the following common aspects were identified:

6.3.2.1 Knowing students' prerequisite knowledge

The first characteristic related to PCK is that the three teachers clearly know how what students learned previously related to the topic in a certain lesson, or what students should or might already grasp before they learn a certain topic, or to what degree students already understood relevant content necessary for learning the new topic. For example, in the first pre-observation interview, Ms. Qian pointed out that what students had already learned was related to what she was going to teach in those two lessons (double lessons):

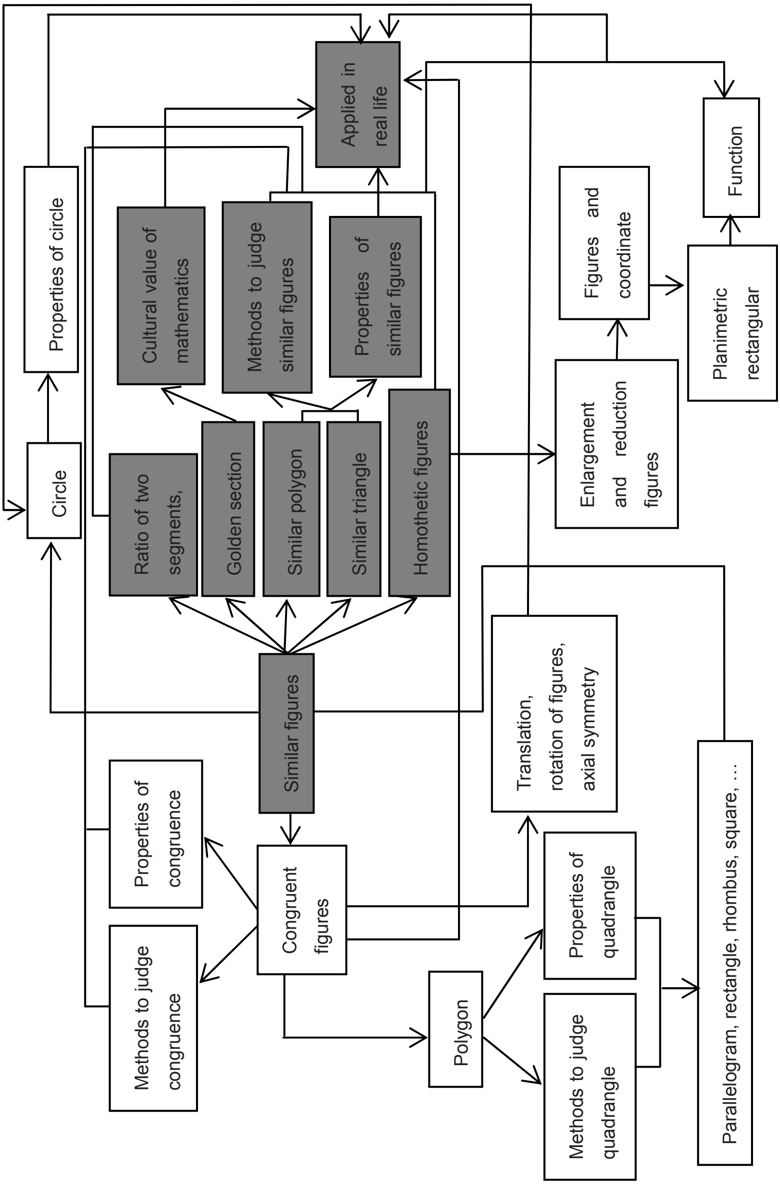

Figure 6. 2. Knowledge structure related to "Similar Figures" developed by Mr. Qian

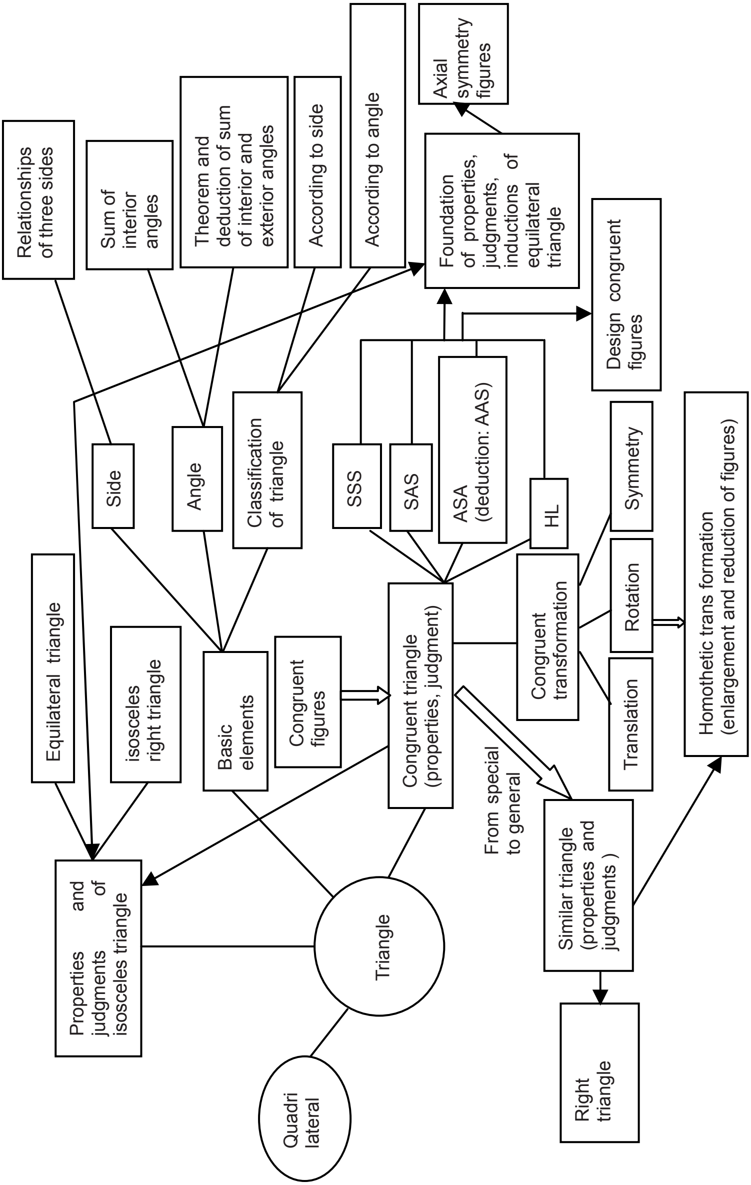

Figure 6.3. Knowledge structure related to "Triangle" developed by Ms.Sun

> *The first part is ratios of numbers, which they learned at Grade 6. This is the basic [for today's topic]. …, another part is what they learned at Grade 7 and in the previous chapter, the properties of equations. Meanwhile, they know well about linear function and fraction equations, they will use some knowledge of linear function and fraction equations to solve some difficult problems today.*

Similarly, Mr. Zhao also stated that what students had learned before related to the definition of the inverse proportion function:

> *Firstly, they should know the relationship between two variables, understand the definition of direct proportion function, and linear function thoroughly …., and the inverse proportion they learned in elementary school.*

6.3.2.2 Anticipating students' difficulty and various ways of working on it

For some new topics, the three teachers knew what parts would be easy and what parts would be difficult for their students. For a particular topic, they could also anticipate the kinds of mistakes students might make. For example, in Ms. Qian's second lesson, she asked students to prove the property of ratio of equality; however, most of her students could not find a way to do so at first. In the post-observation interview, she explained:

> *Like what I thought before the lesson, as to the property of ratio of equality, they did not know how to prove, they could not find the way right after I presented [this property].*

Similarly, Ms. Sun also pointed out the difficulty her students might have in her fourth lesson (introducing the concept of the height of a triangle). She said:

> *The height of a triangle is a difficult part. First of all, students had difficulties when they learned vertical line. They did not know how to draw a vertical line. They might solve the difficulties at that time. However, they did not understand it thoroughly. Therefore,*

> *they should have difficulties on today's topic, how to draw a height of a triangle, especially a height of an obtuse angled triangle, they will make some mistakes. …, I will review how to draw a vertical line first before I present the methods to draw a height of a triangle.*

For difficult parts, the three teachers would prepare various methods to facilitate students' understanding, such as reviewing relevant content to make sure students have necessary foundational knowledge, employing concrete examples to help students visualize abstract concepts, demonstrating some positive examples or counterexamples, discussing in small groups, and breaking the difficulties down into several lessons. For instance, Mr. Zhao thought that the graph of the inverse proportion function would be a difficult point for his students. In view of this, he divided the topic into several lessons, instead of teaching it in a single lesson. In his first lesson, even though the main aim was to teach the definition of the inverse proportion function, Mr. Zhao mentioned information such as how to list a table (which is an important step before drawing the graph of a function), possible locations of graphs of inverse proportion functions in the coordinate plane, and so on. He started to teach how to draw a graph systematically in the third lesson.

The three teachers could also anticipate which individual students might have difficulties with a given topic, and paid special attention to those students during the relevant lessons. Their most common ways of doing so included: 1) asking students with weak academic backgrounds to answer a particular question before moving to another topic, in order to make sure that they understood the current topic; 2) staying close to those students to identify what kind of difficulties they were having or what kind of mistakes they were making, tutoring them individually, and later discussing their difficulties publicly to facilitate and enhance their understanding.

6.3.2.3 Integrating various representations and selecting proper representation in teaching

The three teachers could integrate various representations into one lesson. For example, there are different representations of inverse proportion function, such as its verbal definition, graph, and table. In Mr. Zhao's teaching, he could choose an appropriate representation according to different situations and then translate one presentation to another flexibly. In the first lesson, he mainly emphasized tables, verbal representations, and analytic expressions. He used the tables to explore the relationships between two variables to facilitate students' understanding of the definition. In this lesson, he also mentioned the graph of the inverse proportion function. In his fourth and fifth lessons, he mainly focused on the analytic expression of inverse proportion function.

In Ms. Qian and Ms. Sun's teaching, due to the features of their teaching content, they initially used concrete materials or pictures to let students experience relevant concepts. Then, they used verbal representations and algebraic expressions to enhance students' understanding. For example, in Ms. Sun's third lesson, after she reviewed the concept of an angle bisector, she asked students to find the angle bisectors of three angles of an acute angled triangle by folding a paper triangle. Then, she asked them to explore their intersection and its location. After this, she asked students to fold a right angled triangle and an obtuse angled triangle to find out their angle bisectors and intersections. Through this, students discovered that three angle bisectors of a triangle join at a same point and this point locates inside the triangle. Finally, she presented the verbal definition of an angle bisector of a triangle. The reason that she chose to let students fold paper triangles is as followed:

> *I feel that asking students to fold paper triangles can impress them more deeply. Why the angle bisectors of a triangle join at a same point, is related to the properties of angle bisector and its judgment. This is very difficult to prove at this moment. However, I can let students realize that they can get this [conclusion] through their manipulation.*

Ms. Sun's explanation suggests that she can choose an appropriate representation to encourage students to discover and experience the property, which is difficult to mathematically prove at that moment. By doing so, she can help students to have a deep impression of this property and facilitate their understanding. In other words, she demonstrated the ability to make students more deeply understand relevant knowledge by changing its representation style, rather than mechanically teaching it.

In general, the three teachers all demonstrate the ability to use different representations flexibly in their teaching. More important, they can connect and integrate different representations of a topic to help students build connections among them, which plays “a role in learning mathematics with understanding” (Heibert & Carpenter, 1992, p. 66).

6.3.2.4 Designing appropriate teaching tasks according to the characteristics of the content

Another characteristic related to PCK is that the three teachers can design appropriate teaching tasks according to the characteristics of the teaching content to enrich students’ experience and facilitate students’ understanding. For example, Mr. Zhao adopted several methods to introduce “inverse proportion function”. In the first lesson, Mr. Zhao chose to let students explore several situational problems in which the idea of inverse proportion function was embodied. Based on this, he guided students to discover some common properties and further extract the definition of inverse proportion function from these situational problems. His reason for doing so, according to Mr. Zhao, was to let students experience the thinking behind inverse proportion function and the process of modeling, so that they could understand its definition more easily. In the third lesson, Mr. Zhao let students work in groups to draw graphs of some inverse proportion functions, and then made relevant modifications to those graphs. The main purpose of doing so was to let students explore how to draw the graph of inverse proportion function, based on which they could then discover some characteristics of the graphs. The fourth lesson’s main content was the properties of inverse function, and the teacher chose to explore these properties with students together.

Similarly, in Ms. Sun’s first lesson, she displayed many pictures of applications of triangles to introduce the definition of triangle and make students realize that a triangle is very useful in the real world. In the second lesson, she explored, together with students, some methods of proving that the sum of three interior angles of a triangle is 180^0, because Ms. Sun thought that this part is difficult for most of her students. In the third and fourth lessons, she asked students to find the bisectors of a triangle, median of a triangle, and height of a triangle by folding concrete paper triangles. This, in her view, gave students the opportunity and time to discover that three bisectors of a triangle join at a point, three median lines of a triangle join at a point, and three heights of triangles join at a point. In addition, it gave students opportunities to compare the locations of these three intersections, and more easily understand these characteristics.

6.3.2.5 Choosing and posing appropriate problems or exercises

For most mathematics lessons, the mathematics problem is a main tool to consolidate what students have newly learned and enhance students' understanding. The problems used by the three expert mathematics teachers suggest that they have the ability to choose or pose appropriate problems according to students' background, characteristics of teaching content, and teaching sequence to facilitate students' understanding and challenge students as well.

Various types of problems with different difficulties. Using the categories described in Chapter 4, types of problems were classified and their distribution was listed in Table 6.2. Due to differences in teaching content and students' background, the number and characteristics of problems used in different lessons varied among the teachers. Except for lessons introducing new topics, like Mr. Zhao's first and second lesson, Ms. Qian's first lesson, and Ms. Sun's second lesson, the three teachers used around 10 problems per lesson (most were taken from textbooks, and were very simple and basic for their students). In the other lessons, they used fewer problems, some of which were drawn from other materials. Although as shown in Table 6.2, many problems were routine and closed-ended, when the content was suitable for posing an application problem, the three teachers employed or posed application problems.

Moreover, based on the categories described in Chapter Four, the procedure complexity of every problem was analyzed, and the results were listed in Table 6.3. The distribution shows that the complexity of problems varies in each lesson. Normally, problems with moderate or high complexity are comparatively difficult. In addition, as shown in Table 6.2, some problems are combination problems. Prior knowledge and skills are needed to solve them. These problems are relatively difficult for students, especially those with weak mathematics backgrounds, and give students more mathematics challenges. This suggests that the three teachers could choose or pose problems with various levels of difficulty in teaching.

Table 6. 2. Distribution of different types of problems used in the three teachers' lessons

	Mr. Zhou				Ms. Qian						Ms. Sun				
	Lesson				Lesson						Lesson				
	1	2	5	6	1	2	3	4	5	6	1	2	3	4	5
Routine Problem	12	4	9	3	8	3	4	1	6	3	4	11	3	4	6
Non-routine Problem	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0
Open-ended Problem	0	0	0	0	0	0	0	0	0	0	1	0	3	0	0
Closed-ended Problem	12	4	9	3	8	3	4	1	6	4	4	11	0	4	6
Application Problem	2	1	0	0	3	0	2	0	2	1	2	0	0	0	0
Non-application Problem	10	3	9	3	5	3	2	1	4	3	3	11	3	4	6
Combination Problem	2	3	5	1	3	0	3	1	3	1	0	1	1	3	3
Non-combination Problem	10	1	4	2	5	3	1	0	3	3	5	10	2	1	3

Table 6. 3. Distribution of complexity of problems used in the three teachers' lessons

	Mr. Zhou				Ms. Qian						Ms. Sun				
	Lesson				Lesson						Lesson				
	1	2	5	6	1	2	3	4	5	6	1	2	3	4	5
Low complexity	9	0	4	1	5	3	1	0	4	2	3	9	2	1	4
Moderate complexity	1	3	2	0	3	0	3	0	2	1	2	0	1	2	1
High complexity	2	1	3	2	0	0	0	1	0	1	0	2	0	1	1

The varieties of problems in individual lessons indicate that the three teachers consider individual students' differences and needs in their teaching.

Problems with increasing complexity and difficulty. As described above, the three teachers employed problems with different levels of difficulty in their teaching. They were further able to organize the sequence of the problems both within and across lessons in a reasonable manner. They could arrange problems by degree of difficult in a reasonable manner, so as to establish "scaffolding" for students, especially for those with average mathematics ability. Moreover, this further indicates that the three teachers are capable of identifying the complexity and difficulty of problems properly, based on teaching content. As shown in Tables 6.2 and 6.3, when presenting new topics, they employed more problems of low complexity and difficulty to help students consolidate the newly learned knowledge. In the following lessons, the teachers started to use problems with increasing difficulty to promote and reinforce students' understanding. This also suggests that the teachers are able to teach students with different academic backgrounds within the same class.

For particular problems of high complexity and difficulty, the three teachers demonstrated the ability to establish an appropriate ladder of difficulty among sub-problems. The first sub-problem would be relatively easy, and could be solved by most students with the use of newly learned knowledge. For example, the first problem in Mr. Zhao's fifth lesson was as follows:

Let $y=\frac{-1}{x}$,

1) if point A (1, y_1) and point B (2, y_2) are in its graph, which one is bigger? y_1 or y_2 ?

2) if point A (x_1, y_1) and point B (x_2, y_2) are in its graph and $x_1 < x_2 < 0$, which one is bigger? y_1 or y_2 ?

3) if point A (x_1, y_1) and point B (x_2, y_2) are in its graph and $x_1 < x_2$, which one is bigger? y_1 or y_2 ?

As the condition moves from special situation to general situation, the difficulty level of the sub-problems gradually increased. However, though the third sub-problem is difficult, with the preparation of the first two sub-problems, students with average mathematics ability might be able to solve it, or at least know how to begin to approach it. Moreover, as the level of difficulty level increases gradually and the conditions move from special to general, students can also get some experience of reasoning from special to more general situations.

Problems adhering examination requirements. The three teachers can integrate relevant examination requirements, especially *Zhongkao* requirements, into problems to teach students skills they can use in future. For example, in Ms. Sun's fourth lesson, she used the problem shown in Figure 6.4. Her reason for doing so is as followed.

In the figure, ABC is a triangle. The angle bisectors BD and CE meet at I. What is the relations between $\angle BIC$ and $\angle A$? Let α be $\angle A$. Find $\angle BIC$ in terms of α. Using the above relations, calculate:
$\angle BIC$ if $\angle A = 50^0$;
$\angle A$ if $\angle BIC = 130^0$.

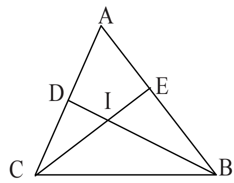

Figure 6. 4. A problem used in Ms. Sun's fourth lesson

Interviewer: *When you solved this problem, you guided students to prove the relationship between $\angle A$ and $\angle CIB$.* Why did you choose *to do so?*

Ms. Sun: *Honestly, for examination; this one is important in different levels of examinations.*

Interviewer: *This is important in examinations?*

Ms. Sun: *Actually, this one is not only an important part in examination, we will learn it again in grade 9 because this point, point I, is the incenter of this triangle. In the second semester of grade 9, we will learn it again. At that time, it [conditions of the problem] will not tell you [students] that the concept of bisector of angle, it will only tell you that point I is the incenter of the triangle. You should know BI is the bisector of $\angle ABC$.*

Interviewer: *So you chose this one here.*

Ms. Sun: *Yes.*

Interviewer: *Don't you think this one is difficult?*

Ms. Sun: *Not so difficult. Or, for students in our school, it is not so difficult, we will use this kind of problems in our regular examinations. Not only in regular examinations, in our unit examinations and other supplementary materials, this problem is very popular. It is a relatively classic problem, and in the second semester of Grade 8, we will use this problem again.*

Ms. Sun's explanation suggests that she knows not only what is important in examinations at different levels, but also knows how to choose important and typical problems with the integration of examination requirements whenever necessary to train students' relevant skills.

6.3.2.6 Making necessary preparation for future teaching

Making preparation for future teaching firstly means that the three teachers are able to prepare for the following teaching content within a lesson; that is, the flow of content within a lesson. In addition, they were able to make necessary preparations for the coming lesson(s). In other words, the three teachers did not consider an activity or a lesson in isolation. Instead, they could consider all the tasks within a lesson or in a unit, a chapter or relevant content at the junior secondary school level together. In the current lesson, the three teachers were laying the foundation for future lessons and connecting to previous lessons.

For example, at the beginning of Ms. Qian's third lesson, she let students view several pictures in which knowledge of golden section is embodied, and posed some problems for their consideration. After she presented the definition of golden section, she referred back to these problems. In Mr. Zhao' first lesson, he presented problem 1.8 to the students. After they finished the table, he guided them to explore the location of points in different quadrants and their relationships, Next, he asked the students to guess the possible location of its graph (graph of the inverse proportion function in this problem). His reason follows the problem:

> *Problem 1.8: Let y be the inverse proportion function of x. In the following table, some values of x and y are given. Answer the following questions:*

x		-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1		3
y	$\frac{2}{3}$		2				-1	

1) *Find the analytical expression of this inverse proportion function;*
2) *According to the analytical expression, finish the table above.*

Mr. Zhao: …. especially when I presented the third exercise (problem 1.8), I made some variation. After I made this change, which aims to make students realize and understand that if the coordinates of two points are given, they are not always in the graph of inverse proportion function. If I have more time, I will dig it deeper. I will let them think about the situation of coordinates of three points are given.

Interviewer: Why did you choose to do so?

Mr. Zhao: It is for the coming lessons. It makes some preparation for the coming lesson (s), the combination of linear function and inverse proportion function.

6.3.3 Knowledge of the characteristics of learners

Teacher's knowledge of learners' characteristics is believed to significantly contribute to teaching practice (Even & Tirosh, 2002). As some information related to knowledge of learners has already been included in PCK reported above, this section reports on two other common characteristics related to knowledge of learners, namely knowledge related to students' personality and family background, and to students' mathematical ability and cognitive development.

6.3.3.1 Knowing learners' personality and family background well

The three teachers know students' personality well, including their interests, habits and learning attitudes. For example, Mr. Zhao once mentioned that boys in his class are very active and talkative. Ms. Sun mentioned that most students in the observed class are very shy and not brave or confident enough to express their opinions, even though, sometimes, they can solve problems successfully. Mr. Zhao mentioned that most of his students are not so self-disciplined and do not spend much time on their homework. Ms. Qian mentioned that most of her students are very self-disciplined and work very hard on mathematics after her teaching.

In addition to knowing current students' personality well, the three teachers were also aware of the differences between present students and those they had taught before. For example, Mr. Zhao and Ms. Qian once mentioned that there exist some differences in beliefs about learning, world, and self between their current and former students. These differences, in the teachers' opinions, make them update their knowledge and adjust how they deal with the teacher-student relationship.

In addition, the three teachers also know their students' family background well. Mr. Zhao mentioned that most of his students come from working-class families of average economic status. The parents do not pay much attention to their children's studies or are not knowledgeable enough to tutor their children's learning. Therefore, he has to try to help students solve their difficulties in his teaching. Ms. Sun and Ms. Qian mentioned that most of their students are from upper-class families of good economic status, and most of their parents are well educated. They heavily stress their children's learning and, sometimes, according to Ms. Sun and Ms. Qian, even like to put their own ideas into their teaching. However, Ms. Sun and Ms. Qian said that they would ignore this and continue on their own way, since they know how to teach students effectively.

6.3.3.2 Knowing learners' mathematical ability and cognitive development

The three teachers knew their students' mathematical abilities and cognitive development well, and could give detailed information about their students' academic backgrounds. For example, Mr. Zhao said there are few differences in mathematics achievement among the students in the observed class, and not many are mathematically gifted. He also mentioned that, even though boys are very talkative during lessons, they do not really think or reason mathematically. Similarly, Ms. Sun mentioned that most of her students are not very good at mathematics and that it will take her some time to improve their mathematics achievement. Ms. Qian mentioned that half of her students have strong mathematics ability and that some are good at logical reasoning.

In addition, they also know individual students' academic backgrounds well. For example, in their teaching they chose particular students to answer particular questions, either very easy or very difficult questions, because they know the student can successfully solve it or would have difficulty in solving it. In Ms. Qian's third lesson, when she presented a problem, she let students work on their own for a while. Then, she asked a boy to demonstrate his method. The reason she chose this boy was as follows:

> *I want to test their [all students] real understanding situation. This student is at the middle level [in the class]. He might have some difficulty in solving this problem. To successfully solve this problem, students need to use knowledge of square root, that is, irrational number. For most students, this is a difficult point and they learned it in last semester, most students might already forget it. Therefore, I asked this student to do some demonstrations. I knew he might make some mistakes. It does not matter. I can make some corrections or asked some students with good mathematics ability to correct it. This can also improve students' understanding.*

Ms. Qian's explanation indicates that, on certain topics or problems, she not only knew individual students' mathematics ability, but also some possible mistakes they might make. More important, it seems that she can make good use of such information in her teaching to correct her students' mistakes and enhance their understanding. Moreover, she used the information to assess her students' understanding.

6.3.4 Curriculum knowledge

Curricular knowledge, as described by Shulman (1986), is an important knowledge influencing teachers' teaching practice. Based on interviews and observations, the three teachers were found to have critical judgments about the latest curriculum reform, know the strengths and weaknesses of textbooks, have well structured vertical mathematics curricular knowledge, and know the connection between mathematics and other subjects or the real world.

6.3.4.1 Critical judgments about the curriculum reform

The first characteristic is that all three teachers have their own opinions about the latest curriculum reform, which has been implemented in mainland China since 2001. First, they seem to know the underlying ideas well. During the interviews, they could use ideas from the new curriculum standard to explain their teaching behaviors, such as “every student attains relevant development in mathematics”, “different student should get different improvement”, “students should have chance to experience the process of mathematics knowledge development”.

However, this does not mean that they blindly, or even entirely accept the ideas in this curriculum reform. Instead, Mr. Zhao and Ms. Sun raised concerns regarding those parts they think are not so reasonable. For example, Mr. Zhao mentioned:

> *In the current mathematics curriculum at the junior secondary school level, the content related to proof has been reduced. Therefore, students’ mathematics thinking cannot be effectively developed. In addition, because of this curriculum reform, students’ computation ability also becomes poorer and poorer. I personally think this is not good for students’ continuous development and learning in future. …., Therefore, if you want to link your teaching with the content they will learn in senior secondary school, you should add some content, not only some extra content, but also some difficult content. You should enhance the difficulty of your teaching content.*

Mr. Zhao’s statement indicates that he not only knows the unreasonable parts of the new mathematics curriculum well, but also has clear ideas about how to make relevant modification to remedy these parts. Similarly, Ms. Sun commented on ideas in the curriculum reform. In the first post-observation interview, when she explained why she stressed heavily the process of exploring the definition of triangle in her teaching, she said:

> *As to the definition of triangle, in our new mathematics curriculum, it does not emphasize too much on this definition. However, I want to spend more time on exploring this concept with students. In my point of view, for those basic methods, which can be employed [as basics] to explore new knowledge in future, we should explore its fundamental elements thoroughly. Although the new mathematics curriculum does not stress heavily this definition, in my teaching, I stressed heavily it because it acts as a foundation to explore some theorems later.*

6.3.4.2 Well aware of strengths and weaknesses of textbooks

As mentioned in Chapter Four, all three teachers used textbooks published by Beijing Normal University Press. However, they also read textbooks published by other publishing houses and made comparisons among them. In the meantime, they also made comparisons between their current textbooks and the ones they used before. After this, they were able to identify the strengths and weaknesses in the current textbooks. All three teachers pointed out that their students are not familiar with some situational problems or tasks in the textbooks because they are set in the context of the northern part of China. They also mentioned that some content is not difficult or challenging enough for their students because their students' academic backgrounds are above the average (they all work in top key middle schools). They also pointed out that the textbook is not as readable for students compared with other textbooks. Therefore, it affects students' self-study. For example, Mr. Zhao mentioned:

> *As to the textbooks published by Beijing Normal University Publishing Press, my personal feeling is that they choose many excellent problems once used in Zhongkao in different cities, or some typical sample exercises. For our teachers, these provide us with some materials you have to deeply study before [you] teach them. However, my first feeling is that the textbooks are not readable for students. Students cannot get much information from it if they study by themselves. For teachers, we can analyze it gradually, but for students, it does not have too much readability. Sometimes, students cannot understand it at all, and*

> *sometimes, there is not much information that students need to read.*

In addition to identifying some of the textbooks' weaknesses, they also realized their strengths. They all felt this set of textbooks gives teachers much freedom in their teaching, even though they sometimes felt very challenging. In addition, this set of textbooks gives students opportunities and space to experience and explore mathematics because many situational problems are used to introduce new concepts. In the meantime, they pointed out that the arrangement of knowledge in this set of textbooks is relatively reasonable. The teachers mentioned that, unlike in former textbooks, in which a "linear type" of knowledge arrangement was adopted, these textbooks adopt a "spiral type" that reduces students learning pressure.

6.3.4.3 Well structured vertical mathematics curricular knowledge

Vertical curricular knowledge, according to Shulman (1986), refers to teacher's familiarity with the topics and issues that have been taught before and will be taught later. The three teachers were found to be able to clearly articulate the relationship between the newly presented knowledge, knowledge students learned before (even at the primary school level), and knowledge students will study later (including at the senior secondary school level). For example, Ms. Qian knew the knowledge structure within a certain topic very well. Before she taught "similar polygons", she clearly described the knowledge structure:

> *"Similar polygons" makes preparation for tomorrow's learning, making knowledge preparation for the learning of "similar triangle". It is a transitional part between the "similar figures" what learned yesterday and the "similar triangle" what will be learned tomorrow.*

She clearly knows the role and position of a given topic in a given unit. That is, she clearly knows the basis on which the new concept develops and what role it plays in students' future mathematics learning. In addition, she knows what knowledge students learned before was related to the topic, and how. For example, she could describe what kind of knowledge was related to the ratio of line segments and its properties:

> *In grade 6, they already learned ratio of numbers and its properties. These are knowledge foundation for this topic [ratio of line segments and its properties]. …., the second foundation is the properties of equality they learned in grade 7 and what they learned in the chapter right before this chapter is fraction and its properties. In addition, they are also very familiar with linear function. All these are the knowledge foundation for today's topic.*

6.3.4.4 Familiarity with the relationship between mathematics and other subjects/real life

The three teachers were quite familiar with the relationships between mathematics and other subjects, as well as real life. That is, they had rich lateral curricular knowledge, as defined by Shulman (1986). For example, Mr. Zhao knew which knowledge in other subjects was relevant to the inverse proportion function quite well. In his first lesson, one of the two situational problems constructed by Mr. Zhao to introduce the inverse proportion function involved the relationship between distance, speed, and time, which related to physics. After Mr. Zhao presented the two problems, he asked students to suggest another example from physics. The reason he chose to do so is that there are many topics in physics related to the inverse proportion function. In the mathematics textbook, there is an example showing the relationship among electric current (I), electric resistance (R) and electric voltage (U), which also relate to the inverse proportion function; however, Mr. Zhao did not adopt this example because students had not learnt that topic when he taught inverse proportion function. This indicates that Mr. Zhao knows not only the relationship between mathematics topics and those in other subjects, but also the teaching sequence of other subjects as well.

Similarly, Ms. Sun employed many examples from architecture and arts to introduce the definition of triangle. At the end of the fourth lesson, she mentioned the intersection of the three bisectors of a triangle could be named its "incenter" and the intersection of the three heights of a triangle could be named its "orthocenter". Her reason was:

> *I mentioned these two concepts here. According to textbook, it is not necessary. ….. However, I mentioned them here because students will start to learn physics in next term. Orthocenter is also an important concept in physics. There exist some relationships among topics in different subjects.*

6.3.5 Discussion

Investigating the characteristics of teachers' knowledge is critical to understanding the complexities of teaching (Schoenfeld, 2000). Similar to previous studies (*e.g.*, Rowland, 2008; Schoenfeld, 2000), this study used observations and interviews to explore the three expert mathematics teachers' knowledge because it is difficult to pose questions to test their knowledge before the classroom observation since the observation topic was decided by the three teachers themselves. Some knowledge, like PCK, knowledge of learners and knowledge of curriculum, is difficult to measure fully with pre-designed questions. The three teachers' knowledge was analyzed mainly based on information from observations, interviews and some other artifacts, such as lesson plans and knowledge structure pictures drawn by the three teachers.

Even though some knowledge reported above was identified mainly based on the three teachers' descriptions and the researcher's observation, as Schoenfeld (2000) proposed, observation, pre- and post-observation interviews, and artifacts such as lesson plans and journals can be used to attribute knowledge to a teacher. Evidence from these suggests that the three teachers have a wide and profound knowledge base. All of them understand mathematics deeply and can connect different topics into a network structure; they know students' knowledge backgrounds and cognitive development well; for individual topics, they know students' prerequisite knowledge and possible difficulties; they know how to design various teaching tasks and pose various problems to maximize students' engagement and reduce students' difficulties; and they not only know the structure, strengths and weaknesses of textbooks, but what and how topics from other fields relate to topics in mathematics.

These findings support statements made by researchers who proposed a prototype model of expert teachers (*e.g.*, Cowley, 1996; Sternberg & Horvath, 1995) and relevant statements by the 21 interviewees in the present study as well. In the prototypical models for expert teachers, having extensive, accessible knowledge of subject matter, teaching and curriculum is an important component. Consistent with previous research findings such a knowledge base was found to be possessed by the three expert teachers in this study. For example, Smith and Strahan (2004) and Tsui (2003, 2009) found that expert teachers are masters of their content areas, Li *et al.* (2005) found that expert mathematics teachers tend to understand mathematics more deeply than non-experts; Zhu *et al.* (2007) found that the expert mathematics teacher in their study knows students' prior experience and prior knowledge base much better than the novice teacher; Li, Huang and Yang (2011) found that middle school expert mathematics teachers in mainland China have sound subject content knowledge and the ability to identify and deal with students' difficulties appropriately in learning; and, many other studies have found that expert teachers, including expert mathematics teachers, know the students they are teaching well (*e.g.*, Berliner & Carter, 1989; Lin, 1999; Ropo, 1990).

However, most of these studies mainly focused on one or two aspects of expert teachers' knowledge, rather than exploring expert mathematics teachers' knowledge as a whole. Moreover, unlike the findings found in Leinhardt and Smith's (1985) study, in which not all four expert mathematics teachers were able to understand mathematics deeply, the three expert mathematics teachers in this study all demonstrated a deep understanding of the teaching content, even though the sample is very small.

Many social and cultural factors might influence the three expert mathematics teachers' knowledge base, such as their schooling experience. As Ms. Sun and Mr. Zhao recalled, they were very good at mathematics when they were in secondary school. As found in Ma's (1999) research, many secondary school graduates in mainland China already have a profound understanding of mathematics. In addition, Ms. Sun and Mr. Zhao mentioned that the methods used by their secondary school mathematics teachers impacted on their teaching. This indicates that they gained some knowledge about how to teach mathematics from their own learning.

Their pre-service training experience might also have helped them to understand mathematics deeply. As described in Chapter Three, teacher training institutions in mainland China have a "strong academic emphasis" (Williams & Morris, 2000, p. 268). In particular, under the influence of the former Soviet education model, pre-service mathematics teachers in China are required to study many courses related to both basic and advanced mathematics (Li *et al.*, 2008). Indeed, the international comparative study, *Teacher Education and Development Study: Learning to Teach Mathematics (TEDS-M)*, recently found that pre-service mathematics teachers' mathematics content knowledge is influenced by their learning experience and opportunities in training institutions (Blömeke & Kaiser, 2012; Schmidt, Blömeke, & Tatto, 2011; Schmidt, Cogan, & Houang, 2011). Even though the relationship between opportunities to learning and mathematics content knowledge is described as "complex" (Tatto & Senk, 2011, p. 133), pre-service mathematics teachers exposed to advanced or basic university mathematics generally performed significantly better than those who only learned school mathematics (Blömeke & Kaiser, 2012). More specifically, Schmidt, Cogan and Houang (2011) found that pre-service mathematics teachers at the junior secondary level in countries with the highest mathematical content knowledge scores took almost twice as many mathematics courses, and significantly more mathematics methods courses, than their counterparts in lower performing countries. Since, as Ms. Qian and Ms. Sun recalled, they had spent a lot of time learning advanced mathematics in teachers colleges, the three teachers' learning and training experience at the pre-service stage may have helped them to understand mathematics deeply, at least mathematics at the junior secondary school level.

The school-based research system might also have made some contribution to their knowledge base. According to the teachers, in-service teacher models adopted in China –like the apprenticeship model, public lessons, and teaching contests – provide teachers opportunities to develop their knowledge base and, particularly, to learn how to teach (Han, 2012; Yang & Ricks, 2012). Activities organized by lesson preparation groups, such as discussing how to teach a certain topic and planning lessons together, provide opportunities for teachers to learn how to teach from other teachers(Han, 2012; Huang, Li, & Su, 2012; Li, Qi, & Wang, 2012; Li, Tang & Gong, 2011; Yang, 2009). In the meantime, since the activities are closely connected to mathematics content, teachers would enhance their deep understanding of mathematics content gradually as well (Huang *et al.*, 2011; Li, Tang & Gong, 2011; Yang, 2009). At the very beginning of their teaching careers, the teachers said that they learned a great deal of firsthand experience from their mentors and other teachers. They also acknowledged that they benefited from participating in teaching competitions at various levels. Even now, the three teachers stated that they still learn from other teachers, sometimes even from novice teachers.

One thing that needs to be pointed out is that the three teachers mentioned knowledge at primary school level related to certain topics; however, they did not mention knowledge at the senior secondary school level, neither in their knowledge structures diagrams nor in the interviews. One reason might be that there really are not many relationships between the topic and knowledge at the senior secondary school level. Moreover, mathematics teachers in other places, such as Hong Kong, have the opportunity to teach at various grade levels at the junior and senior secondary school levels; in mainland China, junior secondary mathematics teachers only teach students from grade 7 to grade 9. Therefore, they might not be as familiar with senior secondary content. However, as all three teachers taught at the junior secondary school level for about 15 years, such an experience might have enhanced their understanding of teaching content, such as the structure of textbooks and the characteristics of students, especially students' difficulties and mistakes, at the junior secondary school level. During their fifteen years of teaching, they used several sets of textbooks; this kind of experience might have contributed to their ability to compare the strengths and weakness of various sets of textbooks and therefore, more easily discern the strengths and weakness of newly adopted textbooks.

6.4 Teaching Strategy

In addition to having various kinds of knowledge, the three expert mathematics teachers were also found to have rich teaching strategies, which include using their previous teaching experience in current teaching, showing respect to students, using class time effectively, and making effective use of the blackboard.

6.4.1 Using previous teaching experience

All three teachers were skillful at using their previous experience in their current teaching. For example, they could adjust their previous teaching design to suit current students' characteristics rather than designing a completely new lesson plan. They could also employ those effective examples they had used before and make assumptions based on their former students' difficulties. For example, in Ms. Sun's second lesson, she guided students to prove the sum of the three interior angles of a triangle equals to 180^0 using three different methods as shown in Figure 6.5, even though it was not compulsory in the textbook. Her reason is:

> *According to my previous experience, as to how to add auxiliary line, it is very difficult for most students. However, it is very important in proof, especially for students to learn geometry. At junior secondary school level, to solve a problem, we need to add one or two auxiliary lines.*

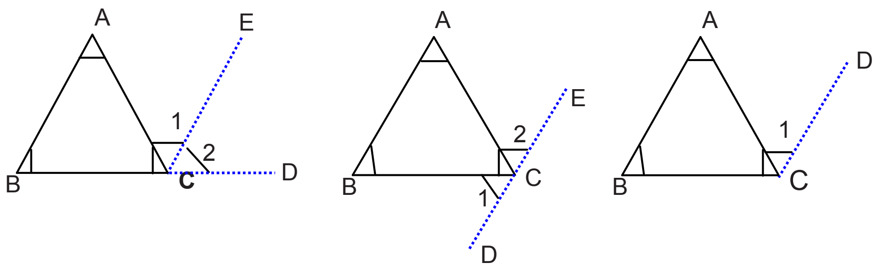

Figure 6. 5. Auxiliary line added by Ms.Sun

Similarly, in Mr. Zhao's first lesson, he used problem 1.8 (see Section 6.3.2.6) to demonstrate the combination of a linear function and an inverse proportion function, for the following reason:

> *this problem, en, actually, when I taught the last cohort of students, it was in a public lesson, I had a brainwave and decide to use this problem in that lesson. It was highly appreciated by the teachers who observed this lesson. They said that it is very good to deal with it in this way. Therefore, I used it again.*

6.4.2 Showing respect to students

The three teachers were very skillful at creating a harmonious classroom atmosphere to inspire students' interests and showed respect to their students during teaching. All were able to use just their tone of voice to attract their students' attention and to emphasize important topics or critical points during their teaching. When students were very tired or not concentrateing, instead of criticizing them, they would make little jokes to relax the students, and then continue with their teaching. According to their explanations in the post-observation interviews, they thought their approach was effective and they should respect students.

The three teachers demonstrated the wisdom of respecting students during their teaching. For example, in Ms. Qian's fifth lesson, she approached a girl with a weak academic background about a question that the girl did not know how to solve. Ms. Qian did not explicitly ask the girl if she could answer the question (even though she knew the girl could not have solved it, according to her later explanation); she merely asked whether she finish the problem. After the girl told her she had not, Ms. Qian asked her to continue to work on it. Similarly, Ms. Sun used eye contact to urge some of her students to concentrate on her teaching instead of criticizing them publicly for their inattention.

6.4.3 Effective use of lesson time

The three teachers were able to use their lesson time effectively in the observed lessons and seldom spent time on matters irrelevant to teaching. Before the lessons, they would make necessary preparations. For example, before Mr. Zhao began his first lesson, he drew three tables on the blackboard that he wanted to use in the coming teaching. While students worked on their exercises, he wrote down other exercises for followed-up teaching. Similarly, as Ms. Qian and Ms. Sun often used an overhead projector, they would turn it on before their teaching. When they planned to use hands-on activities in their teaching, they would ask students to make necessary preparation in advance, which also indicates that they planned their teaching well and had the ability to maximize lesson time.

6.4.4 Effective use of blackboard

Mr. Zhao and Ms. Qian demonstrated the ability to design their writing on the blackboard (*Banshu*) logically and clearly (Ms. Sun used the overhead projector in every lesson). As shown in Figure 6.6 and Figure

6.7, important points, such as definitions and problem solving procedures, were written down by Mr. Zhao and Ms. Qian on the blackboard and remained there as a written record for the entire lesson. During their teaching, they referred to topics written down previously to make the knowledge relationship more explicit and to help students understand more easily and construct an organized knowledge structure.

6.4.5 Discussion

In this section, some characteristics related to teaching strategies were reported. The three expert mathematics teachers used previous experience to plan their lessons, respected their students, effectively used lesson time, and logically designed *banshu* to facilitate students' understanding. These characteristics were also found in other studies. For example, respecting students was described as a prototypical proposition of expert teachers (Berliner, 2001, 2004). Borko and Livingston (1989) found that expert mathematics teachers use their previous experience to plan their lessons and Leung (1995) found that mathematics teachers in Beijing spend less time on matters un-related to teaching.

The teaching strategies identified among the three expert mathematics teachers might be influenced by social and cultural factors. For example, teaching mathematics is traditionally mainly by the use chalk and blackboard (Li *et al.*, 2008); as such, the ability to design *banshu* clearly and neatly is highly emphasized in both pre- and in-service training, and is an important factor in some teaching evaluation systems (Jiang, 2001). Furthermore, as Paine (1990) argued, the teacher in Mainland China is "virtuoso", that is, teaching is an act of performance. This tradition might have contributed to the three teachers' excellent performance as actors in their teaching, such as using their voices well and making facial expressions. However, the tradition of teachers having and exercising a high degree of authority during teaching, as described by Paine (1990), was not found to influence the three teachers' teaching; instead, they respected their students, built equal teacher-student relationships, and created a harmonious classroom atmosphere.

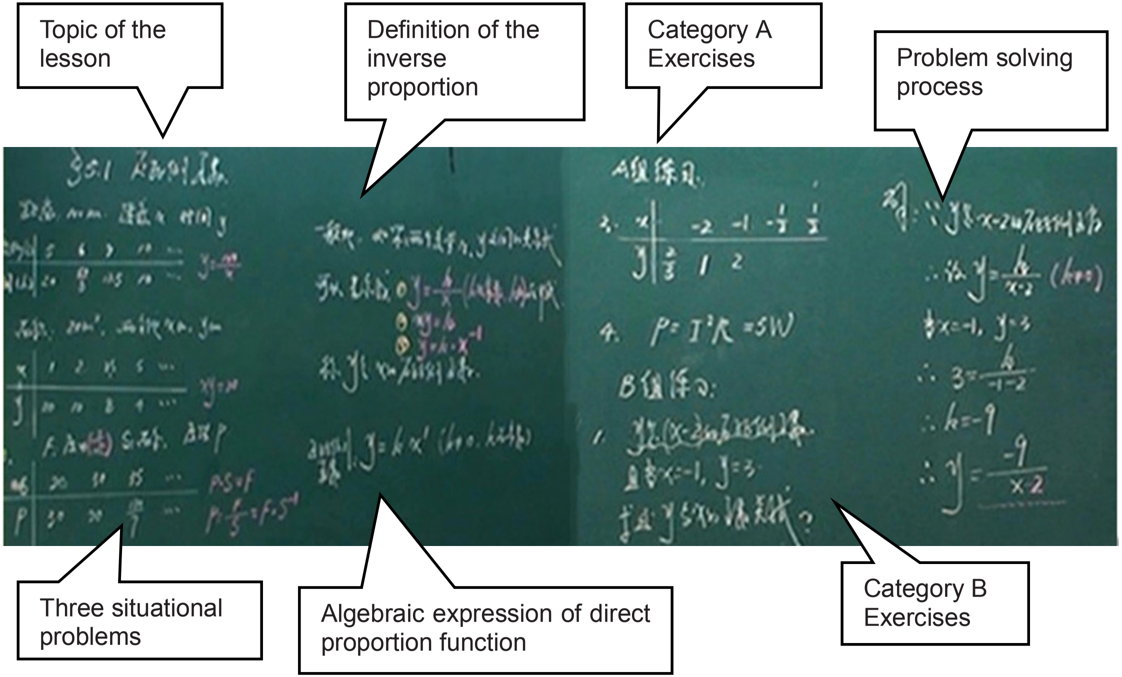

Figure 6. 6. Banshu of Mr. Zhao's first lesson

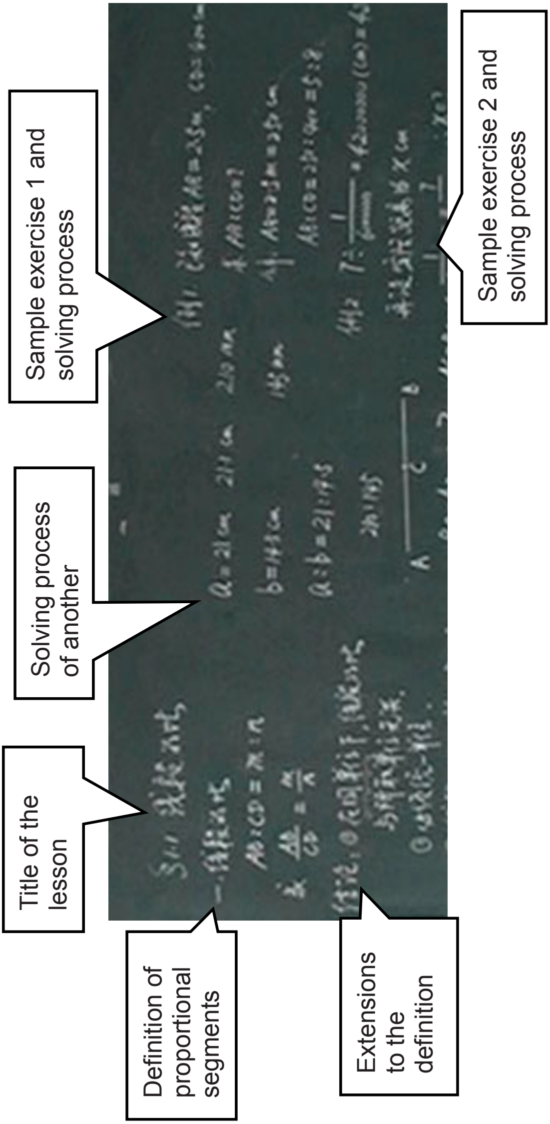

Figure 6. 7. Banshu of Ms. Qian's first lesson

6.5 Summary of the Chapter

This chapter reported the study's findings related to the characteristics of the three expert mathematics teachers' beliefs, knowledge, and teaching wisdom. Their beliefs about mathematics were found to be a combination of the instrumental view of mathematics and the problem solving view of mathematics (Ernest, 1990). They tended to believe that, to learn mathematics well, students need to become intellectually involved in the process of mathematics learning; and that, to teach mathematics effectively, teachers must also involve students intellectually. Moreover, they were found to have a wide and profound knowledge base in mathematics, pedagogy content knowledge, curriculum, and learners. They were also found to use lesson time effectively, design *banshu* clearly and logically, make use of previous teaching experience, and respect their students. The beliefs they hold and the knowledge they possess influence their teaching practice. The next chapter will report how the three expert mathematics teachers plan their lessons, use the curriculum, implement their teaching, and reflect on their teaching.