Construction of Symplectic Quadratic Lie Algebras from Poisson Algebras

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Abstract We introduce the notion of quadratic (resp. symplectic quadratic) Poisson algebras and we show how one can construct new interesting quadratic (resp. symplectic quadratic) Lie algebras from quadratic (resp. symplectic quadratic) Poisson algebras. Finally, we give inductive descriptions of symplectic quadratic Poisson algebras.

1 Introduction

In this paper, we consider finite dimensional algebras over a commutative field K of characteristic zero.

Recall that the Lie algebra $\mathscr{G}$ of a Lie group $\mathfrak{G}$ which admits a bi-invariant pseudo-Riemannian structure is quadratic (i.e. $\mathscr{G}$ is endowed with a symmetric non degenerate invariant (or associative) bilinear form B). Conversely, any connected Lie group whose Lie algebra $\mathscr{G}$ is quadratic, is endowed with bi-invariant pseudo-Riemannian structure [14]. The semisimple Lie algebras are quadratic. Many solvable Lie algebras are also quadratic. Quadratic Lie algebras appear, in particular, in connection with Lie bialgebras and physical models based on Lie algebras. Recall that quadratic Lie algebras are precisely the Lie algebras for which a Sugawara construction exists [9]. Several papers provided interesting results on the structure of quadratic Lie algebras [4, 5, 8–10, 12, 13].

In [13], Medina and Revoy have introduced the concept of double extension in order to give an inductive description of quadratic Lie algebras. This concept is also a tool to construct a new quadratic Lie algebra from a quadratic Lie algebra $(\mathfrak{g}_1, B_1)$ and a Lie algebra $\mathfrak{g}_2$ (not necessarily quadratic) which acts on $\mathfrak{g}_1$ by skew-symmetric derivations with respect to B_1. Let us remark that the non-trivial new quadratic

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Lie algebra will be obtained if $\mathfrak{g}_2$ acts by non-inner skew-symmetric derivations on $(\mathfrak{g}_1, B_1)$. In general, it is difficult to find a Lie algebra $\mathfrak{g}_2$ of dimension upper or equal to 2. In the first part of this paper, we will show how from quadratic Poisson-admissible algebra $(\mathscr{A}, B)$ we can find a Lie algebra $\mathfrak{g}_2$ of dimension upper or equal to 2 acting on a quadratic Lie algebra $(\mathfrak{g}_1, B_1)$ by non-inner skew-symmetric derivations.

In addition, we introduce the concept of symplectic quadratic Poisson algebra and we show how one constructs interesting symplectic quadratic Lie algebras from symplectic quadratic Poisson algebras. Let us recall that the Lie algebra of a Lie group which admits a bi-invariant pseudo-Riemannian metric and also a left-invariant symplectic form is a symplectic quadratic Lie algebra. These Lie groups are nilpotent and their geometry (and, consequently, that of their associated homogeneous spaces) is very rich. In particular, they carry two left-invariant affine structures: one defined by the symplectic form and another which is compatible with a left-invariant pseudo-Riemannian metric. Moreover, if the symplectic form is viewed as a solution r of the classical Yang Baxter equation of Lie algebras (i.e. r is an r-matrix), then the Poisson-Lie tensor $\pi = r^+ - r^-$ and the geometry of double Lie groups $D(r)$ can be nicely described in [7]. In addition, symplectic quadratic Lie algebras were described by methods of double extensions in [1, 2]. Further, in [2], it is proved that every symplectic quadratic Lie algebra $(\mathscr{G}, B, \omega)$, over an algebraically closed field $\mathbb{K}$, may be constructed by T^*-extension of nilpotent Lie algebra which admits an invertible derivation.

In the last section, we study structures of symplectic quadratic Poisson algebras and we give inductive descriptions of symplectic quadratic Poisson algebras over an algebraically closed field with characteristic zero by using some results of [2, 3].

2 Definitions and Preliminary Results

Definition 2.1 Let $\mathscr{A}$ be a vector space endowed with two bilinear operations $[\,,\,]$ and $\circ$. $(\mathscr{A}, [\,,\,], \circ)$ is called a Poisson algebra if $(\mathscr{A}, [\,,\,])$ is a Lie algebra and $(\mathscr{A}, \circ)$ is a commutative associative algbra (not necessarily unital) such that

$$[a, b \circ c] = [a, b] \circ c + b \circ [a, c], \quad \forall a, b, c, \in \mathscr{A} \quad \text{(Leibniz rule)}.$$

Definition 2.2 Let $\mathscr{A}$ be an algebra, we denote by . its multiplication. On the underlying vector space of $\mathscr{A}$ one can defined the two following new products:

$$[x, y] := x.y - y.x; \quad x \circ y := \frac{1}{2}(x.y + y.x), \quad \forall x, y \in \mathscr{A}.$$

$\mathscr{A}$ (or .) is called Poisson-admissible if $(\mathscr{A}, [\,,\,], \circ)$ is a Poisson algebra.

We denote by $\mathscr{A}^-$ (resp. $\mathscr{A}^+$) the algebra $(\mathscr{A}, [\,,\,])$ (resp.$(\mathscr{A}, \circ)$).

Definition 2.3 1. Let $(\mathscr{A}, .)$ be an algebra and $B : \mathscr{A} \times \mathscr{A} \to \mathbb{K}$ be a bilinear form. We say that B is associative (or invariant) if

$$B(a.b, c) = B(a, b.c), \ \forall a, b, c \in \mathscr{A}.$$

2. Let $(\mathfrak{g}, [\ ,\ ])$ be a Lie algebra and $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ be a bilinear form. $(\mathfrak{g}, B)$ is called a quadratic Lie algebra if B is symmetric, non-degenerate and invariant. In this case, B is called an invariant scalar product on $\mathfrak{g}$.
3. Let $(\mathscr{A}, \circ)$ be an associative algebra and $B : \mathscr{A} \times \mathscr{A} \to \mathbb{K}$ be a bilinear form. $(\mathscr{A}, B)$ is called symmetric algebra if B is symmetric, non-degenerate and associative. In this case, B is called an invariant scalar product on $\mathscr{A}$.
4. Let $(\mathscr{A}, .)$ be a Poisson-admissible algebra and $B : \mathscr{A} \times \mathscr{A} \to \mathbb{K}$ be a bilinear form. $(\mathscr{A}, B)$ will be called quadratic if B is symmetric, non-degenerate and associative. In this case, B is called an invariant scalar product on $\mathscr{A}$.
5. Let $(\mathscr{A}, [\ ,\ ], \circ)$ be a Poisson algebra and $B : \mathscr{A} \times \mathscr{A} \to \mathbb{K}$ be a bilinear form. $(\mathscr{A}, B)$ will be called quadratic if B is symmetric, non-degenerate such that:-

$$B([a, b], c) = B(a, [b, c]) \text{ and } B(a \circ b, c) = B(a, b \circ c), \ \forall a, b, c \in \mathscr{A}.$$

Remark 2.1 1. Let $(\mathscr{A}, .)$ be a Poisson-admissible algebra and $B : \mathscr{A} \times \mathscr{A} \to \mathbb{K}$ be a bilinear form. It is clear that $(\mathscr{A}, B)$ is quadratic if and only if $(\mathscr{A}^-, B)$ is a quadratic Lie algebra and $(\mathscr{A}^+, B)$ is a symmetric algebra.
2. $(\mathscr{A}, ., B)$ is a quadratic Poisson-admissible algebra if and only if $(\mathscr{A}, [\ ,\ ], \circ, B)$ is a quadratic Poisson algebra (where $[x, y] := x.y - y.x$ and $x \circ y := \frac{1}{2}(x.y + y.x)$, $\forall x, y \in \mathscr{A}$).

Now, we are going to give some examples of quadratic Poisson-admissible (or Poisson) algebras

1. Let $(\mathscr{A}, [\ ,\ ], \circ)$ be a Poisson algebra and $\mathscr{A}^*$ is the dual vector space of underlying vector space of $\mathscr{A}$. An easy computation prove that the following bracket $[\ ,\ ]_\sim$ and multiplication $\star$ define a Poisson algebra structure on the vector space $\mathscr{A} \oplus \mathscr{A}^*$:

$$[x + f, y + h]_\sim := [x, y] - h \circ \mathrm{ad}x + f \circ \mathrm{ad}y;$$

$$(x + f) \star (y + h) := x \circ y + h \circ L_x + f \circ L_y, \ \forall (x, f), (y, h) \in \mathscr{A} \times \mathscr{A}^*,$$

where L_x is the left multiplication by x in the algebra $(\mathscr{A}, \circ)$.

Moreover, if we consider the bilinear form $B : (\mathscr{A} \oplus \mathscr{A}^*) \times (\mathscr{A} \oplus \mathscr{A}^*) \to \mathbb{K}$ defined by:

$$B(x + f, y + h) := f(y) + h(x), \ \forall (x, f), (y, h) \in \mathscr{A} \times \mathscr{A}^*,$$

then $(\mathscr{A} \oplus \mathscr{A}^*, B)$ is a quadratic Poisson algebra.

Let us remak that if $(\mathscr{A}, .)$ is a Poisson-admissible algebra, then the following multiplication $\bowtie$ on the vector space $\mathscr{A} \oplus \mathscr{A}^*$ define a Poisson-admissible structure on $\mathscr{A} \oplus \mathscr{A}^*$:

$$(x + f) \bowtie (y + h) := x.y + h \circ R_x + f \circ L_y, \ \forall (x, f), (y, h) \in \mathscr{A} \times \mathscr{A}^*,$$

where L_y (resp. R_x) is the left (resp. right) multiplication by y (resp. y) in the algebra $(\mathscr{A}, .)$.

In addition, the bilinear form $B : (\mathscr{A} \oplus \mathscr{A}^*) \times (\mathscr{A} \oplus \mathscr{A}^*) \to \mathbb{K}$ defined by:

$$B(x + f, y + h) := f(y) + h(x), \ \forall (x, f), (y, h) \in \mathscr{A} \times \mathscr{A}^*,$$

is an invariant scalar product on $\mathscr{A} \oplus \mathscr{A}^*$ (ie. $(\mathscr{A}, B)$ is a quadratic Poisson-admissible algebra).

2. Let $(\mathscr{A}, ., B)$ be a quadratic Poisson-admissible algebra and $(\mathfrak{H}, \star, \varphi)$ be a symmetric commutative algebra.

The commutativity and the associativity of $\star$ imply that the vector space $\mathscr{A} \otimes \mathfrak{H}$ with the multiplication:

$$(a \otimes x) \bullet (b \otimes y) := a.b \otimes x \star y, \ \forall (a, x), (b, y) \in \mathscr{A} \times \mathfrak{H},$$

is the Poisson-admissible algebra.

Moreover, the bilinear form $B \otimes \varphi : (\mathscr{A} \otimes \mathfrak{H}) \times (\mathscr{A} \otimes \mathfrak{H}) \to \mathbb{K}$ defined by:

$$B \otimes \varphi(a \otimes x, b \otimes y) := B(a, b)\varphi(x, y), , \ \forall (a, x), (b, y) \in \mathscr{A} \times \mathfrak{H},$$

define a quadratic structure on the Poisson-admissible algebra $(\mathscr{A} \otimes \mathfrak{H}, \bullet)$.

Definition 2.4 Let $(\mathscr{A}, .)$ be an algebra and $\omega : \mathscr{A} \times \mathscr{A} \to \mathbb{K}$ be a bilinear form. We say that $(\mathscr{A}, \omega)$ is a symplectic algebra (or ω is a symplectic structure on $(\mathscr{A}, .)$) if:

1. $\omega(x, y) = -\omega(y, x) \, \forall x, y \in \mathscr{A}$, (ie. ω is skew-symmetric);
2. ω is non-degenerate;
3. $\omega(x.y, z) + \omega(y.z, x) + \omega(z.x, y) = 0, \forall x, y, z \in \mathscr{A}$.

Definition 2.5 If $(\mathscr{A}, .)$ is an algebra, B an associative scalar product on $\mathscr{A}$ and ω is a symplectic structure on $\mathscr{A}$, we say that $(\mathscr{A}, B, \omega)$ is a symplectic quadratic algebra.

If $(\mathscr{A}, .)$ is an associative algebra, we can also say that $(\mathscr{A}, B, \omega)$ is a symplectic symmetric algebra.

Proposition 2.1 *If $(\mathscr{A}, B)$ is a quadratic algebra, ω is a symplectic structure on $\mathscr{A}$ if and only if there exists a unique skew-symmetric (with respect to B) invertible derivation of $(\mathscr{A}, ., B)$ such that:*

$$\omega(x, y) = B(D(x), y), \ \forall x, y \in \mathscr{A}.$$

Proof It is straightforward calculation considering $\omega(x, y) = B(D(x), y)$, for all $x, y \in \mathscr{A}$.

We finish this section by showing how to construct symplectic quadratic Poisson-admissible algebras from an arbitrary Poisson-admissible algebras.

Let $(\mathscr{P}, .)$ be a Poisson-admissible algebra. Let $\mathscr{O} := X\mathbb{K}[X]$ be the ideal of $\mathbb{K}[X]$ generated by X and $\mathscr{R} := \mathscr{O}/X^n\mathscr{O}$, where $n \in \mathbb{N}^*$. $\mathscr{R}$ is a commutative and associative algebra and $\{\bar{X}, \bar{X}^2, \ldots, \bar{X}^n\}$ is a basis of the underlying vector space of $\mathscr{R}$. The vector space $\tilde{\mathscr{P}} := \mathscr{P} \otimes \mathscr{R}$ endowed with the multiplication defined by:

$$(x \otimes \bar{P}) \bullet (y \otimes \bar{Q}) := x.y \otimes \overline{PQ}, \ \ \forall x, y \in \mathscr{P}, \forall P, Q \in \mathscr{O},$$

is a nilpotent Poisson-admissible algebra. Next, $(\mathscr{A} := \tilde{\mathscr{P}} \oplus \tilde{\mathscr{P}}^*, \bowtie, B)$ is a quadratic Poisson-admissible algebra, where:

$$(x + f) \bowtie (y + h) := x \bullet y + h \circ R_x + f \circ L_y,$$

and

$$B(x + f, y + h) := f(y) + h(x), \quad \forall (x, f), (y, h) \in \tilde{\mathscr{P}} \times \tilde{\mathscr{P}}^*.$$

Now, let us consider the endomorphism D of $\tilde{\mathscr{P}}$ defined by:

$$D(x \otimes \bar{X}^i) := ix \otimes \bar{X}^i, \ \forall x \in \mathscr{P}, \forall i \in \{1, \ldots, n\},$$

is an invertible derivation of $\tilde{\mathscr{P}}$. It easy to verify that the endomorphism $\tilde{D}$ of $\mathscr{A}$ defined by:

$$\tilde{D}(x + f) := D(x) - f \circ D, \quad \forall (x, f) \in \tilde{\mathscr{P}} \times \tilde{\mathscr{P}}^*,$$

is an invertible derivation of $\mathscr{A}$ which is skew-symmetric with respect to B. Consequently, the bilinear form ω on $\mathscr{A}$ defined by:

$$\omega(x + f, y + h) := B(\tilde{D}(x + f), y + h), \quad \forall (x, f), (y, h) \in \tilde{\mathscr{P}} \times \tilde{\mathscr{P}}^*,$$

is a symplectic structure on $\mathscr{A}$. Then, $(\mathscr{A}, B, \omega)$ is symplectic quadratic Poisson-admissible algebra.

3 Construction of Quadratic (Resp. Symplectic Quadratic) Lie Algebras from Quadratic (Resp. Symplectic Quadratic) Poisson-Admissible Algebras

First, let us recall the concept of the double extension in the case of quadratic Lie algebras.

Let $(\mathfrak{g}_1, [\ ,\ ]_1, B_1)$ be a quadratic Lie algebra and $(\mathfrak{g}_2, [\ ,\ ]_2)$ be a Lie algebra which is not necessarily quadratic such that there exists a morphism of Lie algebras $\varphi : \mathfrak{g}_2 \to \mathrm{Der}_a(\mathfrak{g}_1, B_1)$ where $\mathrm{Der}_a(\mathfrak{g}_1, B_1)$ is the set of the skew-symmetric derivations with respect to B_1, this set is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g}_1)$. Since $\varphi(\mathfrak{g}_2) \subseteq \mathrm{Der}_a(\mathfrak{g}_1, B_1)$, then the bilinear map $\psi : \mathfrak{g}_1 \times \mathfrak{g}_1 \to (\mathfrak{g}_2)^*$ is a 2−cocycle where $(\mathfrak{g}_2)^*$ is considered as a trivial $\mathfrak{g}_1$−module. Consequently, the vector space $\mathfrak{g}_1 \oplus (\mathfrak{g}_2)^*$ endowed with the multiplication:

$$[X_1 + f, Y_1 + h]_c := [X_1, Y_1]_1 + \psi(X_1, Y_1),\ \forall X_1, Y_1 \in \mathfrak{g}_1,\ f, h \in (\mathfrak{g}_2)^*,$$

is a Lie algebra. This Lie algebra is the central extension of $\mathfrak{g}_1$ by means of ψ.

Let π be the co-adjoint representation of $\mathfrak{g}_2$. If $X_2 \in \mathfrak{g}_2$, an easy computation prove that the endomorphism $\bar{\varphi}(X_2)$ of $\mathfrak{g}_1 \oplus (\mathfrak{g}_2)^*$ defined by: $\bar{\varphi}(X_2)(X_1 + f) := \varphi(X_2)(X_1) + \pi(X_2)(f), \forall X_1 \in \mathfrak{g}_1, f \in (\mathfrak{g}_2)^*$, is a derivation of Lie algebra $(\mathfrak{g}_1 \oplus (\mathfrak{g}_2)^*, [\ ,\ ]_c)$. Next, it is easy to see that the linear map $\bar{\varphi} : \mathfrak{g}_2 \to \mathrm{Der}(\mathfrak{g}_1 \oplus (\mathfrak{g}_2)^*)$ is a morphism of Lie algebras. Therefore, one can consider $\mathfrak{g} := \mathfrak{g}_2 \ltimes_{\bar{\varphi}} (\mathfrak{g}_1 \oplus (\mathfrak{g}_2)^*)$ the semi-direct product of $\mathfrak{g}_1 \oplus (\mathfrak{g}_2)^*$ by $\mathfrak{g}_2$ by means of $\bar{\varphi}$. As vector space $\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus (\mathfrak{g}_2)^*$ and the bracket of the Lie algebra $\mathfrak{g}$ is gien by:

$$\begin{aligned}[X_2 + X_1 + f, Y_2 + Y_1 + h] = [X_2, Y_2]_2 &+ \Big([X_1, Y_1]_1 + \varphi(X_2)(Y_1) - \varphi(Y_2)(X_1)\Big)\\ &+ \Big(\pi(X_2)(h) - \pi(Y_2)(f) + \psi(X_1, Y_1)\Big),\end{aligned}$$

$\forall (X_2, X_1, f), (Y_2, Y_1, h) \in \mathfrak{g}_2 \times \mathfrak{g}_1 \times (\mathfrak{g}_2)^*$. Moreover, if $\gamma : \mathfrak{g}_2 \times \mathfrak{g}_2 \to \mathbb{K}$ is an invariant, symmetric bilinear form on $\mathfrak{g}_2$, it is easy to see that the bilinear form $B_\gamma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ defined by:

$$B_\gamma(X_2 + X_1 + f, Y_2 + Y_1 + h) := \gamma(X_2, Y_2) + B(X_1, Y_1) + f(Y_2) + h(X_2),$$

$\forall (X_2, X_1, f), (Y_2, Y_1, h) \in \mathfrak{g}_2 \times \mathfrak{g}_1 \times (\mathfrak{g}_2)^*$, is an invariant scalar product on $\mathfrak{g}$. $\mathfrak{g}$ (or $(\mathfrak{g}, B_0)$) is called the double extension of $(\mathfrak{g}_1, [\ ,\ ]_1, B_1)$ by $\mathfrak{g}_2$ by means of φ.

Now, we are going to construct quadratic Lie algebras from quadratic Poisson-admissible algebras by using this concept of double extension.

Let $(\mathscr{A}, ., B)$ be a quadratic Poisson-admissible algebra. Then, $(\mathscr{A}^-, [.,.], B)$ is a quadratic Lie algebra and $(\mathscr{A}^+, \circ, B)$ is a symmetric commutative algebra. Let us consider the three-dimensional Lie algebra $\mathfrak{sl}(2)$. Recall that there exists a basis

$\{H, E, F\}$ of $\mathfrak{sl}(2)$ such that $[H, E] = E, [H, F] = -F, [E, F] = 2H$. The vector space $\mathfrak{sl}(2) \otimes \mathscr{A}^+$ with the bracket $[.,.]_1$ defined by:

$$[x \otimes a, y \otimes b]_1 := [x, y] \otimes a \circ b, \ \forall (x, a), (y, b) \in \mathfrak{sl}(2) \times \mathscr{A},$$

is a Lie algebra. Moreover, the bilinear form $B_1 : (\mathfrak{sl}(2)\otimes\mathscr{A}^+)\times(\mathfrak{sl}(2)\otimes\mathscr{A}^+) \to \mathbb{K}$ defined by:

$$B_1(x \otimes a, y \otimes b) := \mathscr{K}(x, y)B(a, b), \ \forall (x, a), (y, b) \in \mathfrak{sl}(2) \times \mathscr{A},$$

is an invariant scalar product on the Lie algebra $(\mathfrak{sl}(2) \otimes \mathscr{A}^+, [.,.]_1)$ (ie. $(\mathfrak{sl}(2) \oplus \mathscr{A}^+, [.,.]_1, B_1)$ is a quadratic Lie algebra) where $\mathscr{K}$ is the Killing form of $\mathfrak{sl}(2)$.

It is cleat that if D is a derivation of $(\mathscr{A}^+, \circ)$, then the endomorphism $\bar{D} := \mathrm{id}_{\mathfrak{sl}(2)} \otimes D$ of $\mathfrak{sl}(2) \otimes \mathscr{A}^+$ defined by:

$$\bar{D}(x \otimes a) := x \otimes D(a), \ \forall (x, a) \in \mathfrak{sl}(2) \times \mathscr{A},$$

is a derivation of the Lie algebra $(\mathfrak{sl}(2) \otimes \mathscr{A}^+, [.,.]_1)$. In addition, if D is skew-symmetric with respect to B, then $\bar{D}$ is skew-symmetric with respect to B_1. In fact, let $(x, a), (y, b)$ be two elements of $\mathfrak{sl}(2) \times \mathscr{A}$,

$$\begin{aligned} B_1(\bar{D}(x \otimes a), y \otimes b) &= \mathscr{K}(x, y)B(D(a), b) \\ &= -\mathscr{K}(x, y)B(a, D(b)) = -B_1(x \otimes a, \bar{D}(y \otimes b)). \end{aligned}$$

Claim $\bar{D}$ is an inner derivation of the Lie algebra $(\mathfrak{sl}(2) \otimes \mathscr{A}^+, [.,.]_1)$ if and only if $D = 0$.

Proof of claim Let us suppose that the derivation $\bar{D}$ is inner, then

$$\bar{D} = \mathrm{ad}(H \otimes a_1) + \mathrm{ad}(E \otimes a_2) + \mathrm{ad}(F \otimes a_3),$$

where $a_1, a_2, a_3 \in \mathscr{A}$. Let $a \in \mathscr{A}$, then $H \otimes D(a) = -E \otimes a \circ a_2 + F \otimes a \circ a_3$, so $D(a) = 0$. We conclude that $D = 0$.

Since $(\mathscr{A}, .)$ is a Poisson-amissible algebra, then for all $X \in \mathscr{A}$ we have $\delta_X := \mathrm{ad}_{\mathscr{A}^-} X$ is a derivation of $(\mathscr{A}^+, \circ)$ and in addition this derivation is skew-symmetric with respect to B because B is associative. Therefore for all $X \in \mathscr{A}$, $\bar{\delta_X}$ is a skew-symmetric derivation of $(\mathfrak{sl}(2) \otimes \mathscr{A}^+, [.,.]_1, B_1)$ and $\bar{\delta_X}$ is not inner if $\mathrm{ad}_{\mathscr{A}^-} X \neq 0$ (ie. X is not in the center of $\mathscr{A}^-$). Then we can consider $\mathfrak{g}(\mathscr{A}) := \mathscr{A}^- \oplus (\mathfrak{sl}(2) \otimes \mathscr{A}^+) \oplus (\mathscr{A}^-)^*$ the double extension of $(\mathfrak{sl}(2) \otimes \mathscr{A}^+, [.,.]_1, B_1)$ by the Lie algebra $\mathscr{A}^-$ by means the morphism of Lie algebra $\varphi : \mathscr{A}^- \to \mathrm{Der}_a(\mathfrak{sl}(2) \otimes \mathscr{A}^+, B_1)$ defined by: $\varphi(X) := \delta_X, \ \forall X \in \mathscr{A}$. Let us remark that the dimension of this quadratic Lie algebra obtained by double extension is $5n$ where n is the dimension of $\mathscr{A}$. Recall that the bilinear form $T_0 : \mathfrak{g}(\mathscr{A}) \times \mathfrak{g}(\mathscr{A}) \to \mathbb{K}$ defined by:

$$T_0(X + s \otimes a + f, Y + s' \otimes b + h) := \mathscr{K}(s, s')B(a, b) + f(Y) + h(X),$$

for all $X, Y, a, b \in \mathscr{A}$, $f, h \in \mathscr{A}^*$, is an invariant scalar product on $\mathfrak{g}(\mathscr{A})$.

Remark 3.1 In the construction above, one can replace $\mathfrak{sl}(2)$ by an arbitrary simple Lie algebra.

In [2], symplectic quadratic Lie algebras are studied. Now, we are going to show how we can construct symplectic quadratic Lie algebras from symplectic quadratic Poisson-admissible algebras.

By easy computation, we prove the following lemma.

Lemma 3.1 *If D is a derivation of a quadratic Poisson-admissible algebra $(\mathscr{A}, ., B)$, then the endomorphism $\tilde{D}$ of $\mathfrak{g}(\mathscr{A})$ defined by:*

$$\tilde{D}(x) := D(x),\ \tilde{D}(f) := -f \circ D;\ \tilde{D}(s \otimes a) := s \otimes D(a),$$
$$\forall a, x \in \mathscr{A}, f \in \mathscr{A}^*, s \in \mathfrak{sl}(2),$$

is a derivation of Lie algebra $\mathfrak{g}(\mathscr{A})$. Moreover, if D is invertible (resp. skew-symmetric with respect to B), then $\tilde{D}$ is invertible (resp. skew-symmetric with respect to T_0).

Consequently, if $(\mathscr{A}, B, \omega)$ is a symplectic quadratic Poisson-admissible algebras and D the skew-symmetric (with respect to B) invertible derivation of $\mathscr{A}$ such that $\omega(x, y) = B(D(x), y)$, $\forall x, y \in \mathscr{A}$, then $(\mathfrak{g}(\mathscr{A}), T, \Omega)$ is a symplectic quadratic Lie algebra where Ω is the symplectic structure on Lie algebra $\mathfrak{g}(\mathscr{A})$ defined by:

$$\Omega(X, Y) := T(\tilde{D}(X), Y),\ \forall X, Y \in \mathfrak{g}(\mathscr{A}).$$

3.1 Structures of Symplectic Quadratic Poisson-Admissible Algebras

Recall that if $(\mathscr{A}, \circ, B)$ is a commutative associative algebra, we denote by $\mathrm{End}_s(\mathscr{A}, B)$ the set of symmetric endomorphisms of the vector space $\mathscr{A}$ with respect to B. It is clear that $\mathrm{End}_s(\mathscr{A}, B)$ is a subalgebra of $\mathrm{End}(\mathscr{A})$ (the associative algebra of the endomorphisms of $\mathscr{A}$).

Let $(\delta, a_0) \in \mathrm{End}_s(\mathscr{A}, B) \times \mathscr{A}$. In [3], (δ, a_0) is called a pre-admissible element of $\mathrm{End}_s(\mathscr{A}, B) \times \mathscr{A}$ if

$$\delta \circ L_x = L_x \circ \delta \text{ and } \delta^2 = L_{a_0} \ \text{(ie. } \delta(x \circ y) = x \circ \delta(y),\ \delta^2(x) = a_0 \circ x,\ \forall x, y \in \mathscr{A}).$$

Now, Let $(\mathscr{W}, ., B)$ be a quadratic Poisson-admissible algebra. Let $\Delta \in \mathrm{Der}_a(\mathscr{W}^-, B)$ and (δ, a_0) be a pre-admissible element of $\mathrm{End}_s(\mathscr{W}^+, B) \times \mathscr{W}^+$. Then,

1. The vector space $\mathscr{A} := \mathbb{K}e \oplus \mathscr{W} \oplus \mathbb{K}e^*$ with the skew-symmetric bilinear map $[\ ,\ ] : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ defined by:

$$[x, y] := [x, y]_{\mathscr{W}^-} + B(\Delta(x), y)e^*;\quad [e, x] := \Delta(x);\quad [e^*, \mathscr{A}] = \{0\},$$

$\forall x, y \in \mathscr{W}$, is a Lie algebra.
2. The vector space $\mathscr{A} := \mathbb{K}e \oplus \mathscr{W} \oplus \mathbb{K}e^*$ with the symmetric bilinear map $\star : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ defined by:

$$x \star y := x \circ y + B(\delta(x), y)e^*;\quad e \star x := \delta(x) + B(a_0, x)e^*;$$
$$e \star e := a_0 + ke^*;\quad e^* \star \mathscr{A} = \{0\},$$

$\forall x, y \in \mathscr{W}$, is an associative commutative algebra.

Moreover, if we consider the symmetric bilinear form $T : \mathscr{A} \times \mathscr{A} \to \mathbb{K}$ defined by:

$$T_{|\mathscr{W}\times\mathscr{W}} := B;\ T(e, e^*) = 1;\ T(e, \mathbb{K}e \oplus \mathscr{W}) = T(e^*, \mathbb{K}e^* \oplus \mathscr{W}) = \{0\},$$

then $(\mathscr{A}, [\ ,\ ], T)$ is a quadratic Lie algebra (called the double extension of $(\mathscr{W}^-, [\ ,\ ]_{\mathscr{W}^-}, B)$ by the one-dimensional Lie algebra by means of D (see [13])) and $(\mathscr{A}, \star, T)$ is a symmetric commutative associative algebra (called generalized double extension of $(\mathscr{W}^+, \circ, B)$ by the one dimensional algebra with null product by means of (δ, a_0) (see [3])).

In addition, if we suppose that

$$\Delta \circ \delta = \delta \circ \Delta = \frac{1}{2}\mathrm{ad}_{\mathscr{W}^-}(a_0)\ \text{(ie.} = \frac{1}{2}[a_0, .]_{\mathscr{W}^-});\ \ \Delta(a_0) = 0;$$

$$\Delta \in \mathrm{Der}(\mathscr{W}^+);\quad \delta([x, y]_{\mathscr{W}^-}) = \Delta(x) \circ y + [x, \delta(y)]_{\mathscr{W}^-},\ \forall x, y \in \mathscr{W},$$

then $(\mathscr{A}, [\ ,\ ], \star)$ is a Poisson algebra, so $(\mathscr{A}, [\ ,\ ], \star, T)$ is a quadratic Poisson algebra. We call this quadratic Poisson algebra the double extension of the quadratic Poisson algebra $(\mathscr{W}, [\ ,\ ]_{\mathscr{W}^-}, \circ, B)$ by means of (Δ, δ, a_0, k).

Now, the vector space $\mathscr{A}$ with the product:

$$X \boxtimes Y := \frac{1}{2}[X, Y] + X \star Y,\ \forall\ X, Y \in \mathscr{A},$$

is Poisson-admissible algebra. Then $(\mathscr{A}, \boxtimes, T)$ is a quadratic Poisson-admissible algebra called the double extension of the quadratic Poisson-admissible algebra $(\mathscr{W}, ., B)$ by means of (Δ, δ, a_0, k).

More precisely, the product $\boxtimes$ is given by:

$$x \boxtimes y = x.y + B(\Omega(x), y)e^*,$$

$$e \boxtimes x = \Omega(x) + B(a_0, x)e^*,\ \ x \boxtimes e = \Omega^*(x) + B(a_0, x)e^*,$$

$$e \boxtimes e = e \star e := a_0 + ke^*;\ e^* \star \mathscr{A} = \mathscr{A} \star e^* = \{0\}, \forall x, y \in \mathscr{W},$$

where $\Omega := \frac{1}{2}\Delta + \delta$ and $\Omega^* := -\frac{1}{2}\Delta + \delta$.

Let us consider a symplectic quadratic Poisson-admissible algebra $(\mathscr{W}, ., B, \omega)$. Then there exists a unique invertible skew-symmetric derivation of $\mathscr{W}$ such that $\omega(x, y) = B(D(x), y),\ \forall x, y \in \mathscr{W}$. Next, we consider a double extension $(\mathscr{A}, \boxtimes, T)$ of $(\mathscr{W}, ., B)$ by means (Δ, δ, a_0, k).

By [2, 3], if there exist $ł \in \mathbb{K}$ and $c_0 \in \mathscr{W}$ such that:

$$[D, \Delta] + ł\Delta = \mathrm{ad}_{\mathscr{W}^-}(c_0),$$

$$[D, \delta] + ł\delta = L_{c_0} \ \ and \ \ \delta(c_0) = ła_0 + \frac{1}{2}D(a_0),$$

Then the endomorphism Γ of $\mathscr{A}$ defined by:

$$\Gamma(x) := D(x) - B(c_0, x)e^*; \quad \Gamma(e^*) := łe^*; \quad \Gamma(e) := c_0 - łe^*,$$

is an invertible derivation of $(\mathscr{A}, \boxtimes)$ and Γ is skew-symmetric with respect to T. It follows that the skew-symmetric bilinear form $\diamondsuit : \mathscr{A} \times \mathscr{A} \to \mathbb{K}$ defined by:

$$\diamondsuit(X, Y) := T(\Gamma(X), Y),\ \ \forall X, Y \in \mathscr{A},$$

is a symplectic structure on $(\mathscr{A}, \boxtimes)$. Therefore, $(\mathscr{A}, \boxtimes, T, \diamondsuit)$ is a symplectic quadratic Poisson-admissible algebra called the double extension of $(\mathscr{W}, ., B, \omega)$ (by means of $(\Delta, \delta, a_0, c_0, k, ł)$).

Proposition 3.1 *Let $(\mathscr{A}, \boxtimes, T)$ be a quadratic Poisson-admissible algebra. Suppose that there exists $e^* \in Ann(\mathscr{A})\backslash\{0\}$ such that $B(e^*, e^*) = 0$. Then, $(\mathscr{A}, \boxtimes, T)$ is a double extension of a quadratic Poisson-admissible algebra $(\mathscr{W}, ., B)$. More precisely, $\mathscr{W} := (\mathbb{K}e^*)^\perp/\mathbb{K}e^*$ and*

$$(x + \mathbb{K}e^*).(y + \mathbb{K}e^*) := (x \boxtimes y) + \mathbb{K}e^*,$$

$$B(x + \mathbb{K}e^*, (y + \mathbb{K}e^*) := T(x, y),\ \ \forall x, y \in (\mathbb{K}e^*)^\perp.$$

Proof Since B is non-degenerate, then there exists $e \in \mathscr{A}$ such that $B(e^*, e) = 1$. Consequently, if $\mathscr{W} := (\mathbb{K}e^* \oplus \mathbb{K}e)^\perp$ denotes the orthogonal of $\mathbb{K}e^* \oplus \mathbb{K}e$ with respect to T, with the Poisson-admissible structure . induced by the one of $(\mathbb{K}e^*)^\perp/\mathbb{K}e^*$, one easily verifies that $B := T_{|\mathscr{W} \times \mathscr{W}}$ is an invariant scalar product on Poisson-admissible algebra $(\mathscr{W}, .)$

Let us remark that there exit $a_0 \in \mathscr{W}$ and $k \in \mathbb{K}$ such that $e \boxtimes e = a_0 + ke^*$ because $B(e \boxtimes e, e^*) = B(e, e \boxtimes e^*) = 0$,

Let us consider Ω the endomorphism of $\mathscr{W}$ defined by:

$$\Omega(x) := P_{\mathscr{W}}(e \boxtimes x),\ \forall x \in \mathscr{W},$$

where $P_{\mathscr{W}}:\mathscr{A} \to \mathscr{W}$ is the natural projection.

Next, we consider $\Delta := \Omega - \Omega^*$, $\delta := \frac{1}{2}(\Omega + \Omega^*)$, where Ω^* the endomorphism of $\mathscr{W}$ defined by $B(\Omega(x), y) = B(x, \Omega^*(y))$, for all $x, y \in \mathscr{W}$ (ie. Ω^* is the adjoint of Ω with respect to B). It easy to verify that $(\mathscr{A}, \boxtimes, T)$ is the double extension of the quadratic Poisson-admissible algebra $(\mathscr{W}, ., B)$ by means of (Δ, δ, a_0, k).

Lemma 3.2 *Let $(\mathscr{A}, .)$ be a Poisson-admissible algebra. If $\mathscr{A}$ admits an invertible derivation, then Ann$(\mathscr{A}) \neq \{0\}$.*

Proof If $(\mathscr{A}, .)$ admits an invertible derivation Γ then Γ is either an invertible derivation of $(\mathscr{A}^-, [\ ,\ ])$ and an invertible derivation of $(\mathscr{A}^+, \circ)$. Consequently, by [11] (rep. by [3]), $(\mathscr{A}^-, [\ ,\ ])$ is a nilpotent Lie algebra (resp. $(\mathscr{A}^+, \circ)$ is a nilpotent associative algebra). It follows that $\mathfrak{z}(\mathscr{A}^-) \neq \{0\}$ and Ann$(\mathscr{A}^+) \neq \{0\}$. Since $\mathrm{ad}_{\mathscr{A}^-} X$ is a derivation of $(\mathscr{A}^+, \circ)$, for all $X \in \mathscr{A}$, then Ann$(\mathscr{A}^+)$ is an ideal of $(\mathscr{A}^-, [\ ,\ ])$. Therefore Ann$(\mathscr{A}^+) \cap \mathfrak{z}(\mathscr{A}^-) \neq \{0\}$ because $(\mathscr{A}^-, [\ ,\ ])$ is a nilpotent Lie algebra. The fact that Ann$(\mathscr{A}^+) \cap \mathfrak{z}(\mathscr{A}^-) \subseteq$ Ann$(\mathscr{A})$ implies that Ann$(\mathscr{A}) \neq \{0\}$.

Theorem 3.1 *If $\mathbb{K}$ is algebraically closed, then every non-zero symplectic quadratic Poisson-admissible algebra $(\mathscr{A}, \boxtimes, T, \diamondsuit)$ is a double extension of a symplectic quadratic Poisson-admissible algebra $(\mathscr{W}, ., B, \omega)$*

Proof Let $(\mathscr{A}, \boxtimes, T, \diamondsuit)$ is a non-zero symplectic quadratic Poisson-admissible. There exists a unique skew-symmetric (with respect to T) invertible derivation Γ of $(\mathscr{A}, \boxtimes)$ such that $\diamondsuit(X, Y) = T(\Gamma(X), Y)$, for all $X, Y \in \mathscr{A}$. Then, By Lemma 3.2, Ann$(\mathscr{A}) \neq \{0\}$. Since Γ is a derivation of $(\mathscr{A}, \boxtimes)$, then $\Gamma(\mathrm{Ann}(\mathscr{A})) \subseteq \mathrm{Ann}(\mathscr{A})$. It follows that there exist $e^* \in \mathrm{Ann}(\mathscr{A})\backslash\{0\}$ and $\l \in \mathbb{K}\backslash\{0\}$ such that $\Gamma(e^*) = \l e^*$. The fact that Γ is skew-symmetric with respect to T implies that $T(e^*, e^*) = 0$. By, Proposition 3.1, $(\mathscr{A}, \boxtimes, T)$ is a double extension of a quadratic Poisson-admissible algebra $(\mathscr{W} = \mathbb{K}e^*)^{\perp}/\mathbb{K}e^*, ., B)$ by means of (Δ, δ, a_0, k) (see the proof of Proposition 3.1 for definitions of Δ, δ, a_0 and k). Therefore, $\mathscr{A} = \mathbb{K}e^* \oplus \mathscr{W} \oplus \mathbb{K}e$ with $T(e, e^*) = 1$ and $\mathscr{W} = (\mathbb{K}e^* \oplus \mathbb{K}e)^{\perp}$.

Since the ideal $\mathbb{K}e^*$ of $(\mathscr{A}, \boxtimes)$ is invariant by the skew-symmetric derivation Γ, so is its orthogonal (with respect to T) $\mathbb{K}e^* \oplus \mathscr{W}$. Now, if $p : \mathbb{K}e^* \oplus \mathscr{W} \to \mathscr{W}$ denotes the projection $p(te^* + x) := x$, for $t \in \mathbb{K}, x \in \mathscr{W}$, then one can easily verify that $D := p \circ \Gamma_{|\mathscr{W}}$ is an invertible skew-symmetric derivation of $(\mathscr{W}, ., B)$. Since Γ is skew-symmetric with respect to T and $T(e^*, e) = 1$, one immediately obtains that there exists $c_0 \in \mathscr{W}$ such that $\Gamma(e) := c_0 - \l e^*$ and $\Gamma_{|\mathscr{W}} = D - B(c_0, .)e^*$. Since Γ is a derivation of $(\mathscr{A}, \boxtimes)$, then Γ is either a derivation of $\mathscr{A}^-$ and a derivation of $\mathscr{A}^+$, one easily deduces that

$$[D, \Delta] + \l\Delta = \mathrm{ad}_{\mathscr{W}^-}(c_0),$$

$$[D, \delta] + \l\delta = L_{c_0} \ \ and \ \ \delta(c_0) = \l a_0 + \frac{1}{2}D(a_0).$$

Therefore $(\mathscr{A}, \boxtimes, T, \diamond)$ is the double extension of $(\mathscr{W}, ., B, \omega := B(D(.), .))$ (by means of $(\Delta, \delta, a_0, c_0, k, \text{ł}))$.

Now, we denote by $\mathscr{M}$ the 2−dimensional Poisson-admissible algebra with zero product. If $\{e, f\}$ is a basis of the vector space $\mathscr{M}$, the symmetric (resp. skew-symmetric) bilinear form B_0 (resp. ω_0) of $\mathscr{M}$ defined by $B_0(e, e) = B_0(f, f) = 1$, $B_0(e, f) = 0$ (resp. $\omega_0(e, f) = 1$), is quadratic (resp. symplectic) structure on $\mathscr{M}$. By Theorem 3.1, The following result follows easily:

Corollary 3.1 *If $\mathbb{K}$ is algebraically closed, then every non-zero symplectic quadratic Poisson-admissible algebra $(\mathscr{A}, \boxtimes, T, \diamond)$ is obtained from the 2−dimensional symplectic quadratic Poisson-admissible algebra $(\mathscr{M}, B_0, \omega_0)$ by a sequence of double extensions.*

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