Lie Algebras with Given Properties of Subalgebras and Elements

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Abstract We study finite-dimensional Lie algebras with given properties of subalgebras (like all proper subalgebras being abelian) and elements (like all elements being semisimple). We get results on both the structure of the whole class of algebras with the given property, and the structure of individual algebras in the class.

We study the following classes of Lie algebras: anisotropic (i.e., Lie algebras for which each adjoint operator $\mathsf{ad}\, x$ is semisimple), regular (i.e., Lie algebras in which each nonzero element is regular), minimal nonabelian (i.e., nonabelian Lie algebras all whose proper subalgebras are abelian), and algebras of depth 2 (i.e., Lie algebras all whose proper subalgebras are abelian or minimal nonabelian).

All algebras, Lie and associative, are assumed to be finite-dimensional and defined over a fixed field of characteristic zero (though some of the results, in a weaker form or under additional restrictions, will hold also in positive characteristic). We stress that the base field is not assumed to be algebraically closed (all the things considered here are collapsing to vacuous trivialities in the case of an algebraically closed base field).

Our notations are standard and largely follow Bourbaki [2]. The symbols $\dotplus$, $\oplus$, and $\oplus\!\!\!\!\subset$ denote direct sum of vector spaces, direct sum of Lie algebras, and semidirect sum of Lie algebras (the first summand acting on the second), respectively.

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1 Anisotropic Algebras

It is shown in [3, Propostion 1.2] that any anisotropic solvable Lie algebra is abelian. From this and the Levi–Malcev decomposition follows that any anisotropic Lie algebra is reductive.

Theorem 1.1 *For a reductive Lie algebra L the following are equivalent:*

  (i) *L is anisotropic;*
 (ii) *all proper subalgebras of L are anisotropic;*
(iii) *all proper subalgebras of L are reductive;*
(iv) *all 2-dimensional subalgebras of L are abelian;*
 (v) *L does not contain subalgebras isomorphic to* $\mathsf{sl}(2)$.

Proof (i) $\Rightarrow$ (ii). If S is a subalgebra of L, then for any $x \in S$, $\mathsf{ad}_S x$ is a restriction of $\mathsf{ad}_L x$, hence the semisimplicity of the latter implies the semisimplicity of the former.

(ii) $\Rightarrow$ (iii) follows from the observation above that any anisotropic Lie algebra is reductive.

(iii) $\Rightarrow$ (iv) follows from the obvious fact that a 2-dimensional reductive Lie algebra is abelian.

(iv) $\Rightarrow$ (v) follows from the obvious fact that $\mathsf{sl}(2)$ contains a 2-dimensional nonabelian subalgebra.

(v) $\Rightarrow$ (i). Write L as a direct sum $L = \mathfrak{g} \oplus A$, where $\mathfrak{g}$ is semisimple and A is abelian. Suppose $\mathfrak{g}$ is not anisotropic. As $\mathfrak{g}$ contains semisimple and nilpotent components of each of its elements ([2, Chap. I, Sect. 6, Theorem 3]), $\mathfrak{g}$ contains a nonzero nilpotent element, and by the Jacobson–Morozov theorem ([2, Chap. VIII, Sect. 11, Proposition 2]) $\mathfrak{g}$ contains $\mathsf{sl}(2)$ as a subalgebra, a contradiction. Hence $\mathfrak{g}$ is anisotropic and L is anisotropic.

Though the proof is elementary, and all the necessary ingredients are contained in [3] anyway (in particular, the implication (i) $\Rightarrow$ (iv) is noted in [3, Sect. 1], and the equivalence (i) $\Leftrightarrow$ (v) in the case of semisimple L is proved, with a slightly different argument, in [3, Theorem 2.1]), we find this explicit formulation of Theorem 1.1 interesting enough. There are many works in the literature devoted to study of *minimal non-$\mathscr{P}$* Lie algebras, i.e. Lie algebras not satisfying $\mathscr{P}$ and such that all their proper subalgebras satisfy $\mathscr{P}$, where $\mathscr{P}$ is a certain "natural" property of Lie algebras (abelianity, nilpotency, solvability, simplicity, modularity of the lattice of subalgebras, …). In all the cases studied so far, the class of minimal non-$\mathscr{P}$ algebras turns out to be highly nontrivial (without further assumptions about the base field, such as algebraic or quadratic closedness, triviality of the Brauer group, etc.), with lot of simple objects. To the contrary, from the Levi–Malcev decomposition and Theorem 1.1 it follows that the class of minimal nonanisotropic Lie algebras is relatively trivial: those are exactly solvable minimal nonabelian Lie algebras. One may ask a "philosophical" question: what makes the condition of being anisotropic different in that regard from other conditions? Where is a borderline for a property $\mathscr{P}$ which makes the class of minimal non-$\mathscr{P}$ Lie algebras small and "simple" (or even empty)?

Corollary 1.1 *A simple Lie algebra all whose proper subalgebras are not simple, is either minimal nonabelian, or isomorphic to* sl(2).

Proof Let L be a reductive Lie algebra all whose proper subalgebras are not simple. By implication (v) $\Rightarrow$ (iii) of Theorem 1.1, either L is isomorphic to sl(2), or all proper subalgebras of L are reductive. As any nonabelian reductive Lie algebra contains a simple subalgebra, in the latter case all proper subalgebras of L are abelian.

In [16, Theorem 2.2] a statement similar to the corollary is proved about simple Lie algebras, all whose proper subalgebras are supersolvable.

Theorem 1.2 *Let $\mathscr{L}$ be a nonempty class of Lie algebras satisfying the following properties:*

(i) *$\mathscr{L}$ is closed with respect to subalgebras;*
(ii) *if each proper subalgebra of a reductive Lie algebra L belongs to $\mathscr{L}$, then L belongs to $\mathscr{L}$;*
(iii) *solvable Lie algebras belonging to $\mathscr{L}$ are abelian.*

Then $\mathscr{L}$ is the class of all anisotropic Lie algebras.

Proof Any class of Lie algebras satisfying conditions (i) and (iii) consists of anisotropic algebras. Indeed, from the Levi–Malcev decomposition and condition (iii) it follows that any algebra in $\mathscr{L}$ is reductive. Then from implication (iii) $\Rightarrow$ (i) of Theorem 1.1 and condition (i), it follows that any algebra in $\mathscr{L}$ is anisotropic.

Now, suppose that there is an anisotropic Lie algebra not belonging to $\mathscr{L}$, and consider such algebra L of the minimal possible dimension. Then all proper subalgebras of L belong to $\mathscr{L}$, and by condition (ii) L itself belongs to $\mathscr{L}$, a contradiction.

2 Regular Algebras

If N is a nilpotent subalgebra of a Lie algebra L, by $L^0(N)$ is denoted the Fitting 0-component with respect to the N-action on L (i.e., the set of all elements of L on which N acts nilpotently).

Recall ([2, Chap. VII, Sect. 2.2]) that *rank* rk L of a Lie algebra L is the minimal possible non-vanishing power of the characteristic polynomial of ad x, $x \in L$, and elements of L for which this minimal number is attained are called *regular*. Another characterization of $x \in L$ to be a regular element is the equality dim $L^0(x) =$ rk L.

If each nonzero element of L is regular, then L itself is called *regular*.

It is clear that any nilpotent Lie algebra is regular, with rank equal the dimension of the algebra. If a regular Lie algebra L is not semisimple, i.e., contains a nonzero abelian ideal I, then for any $x \in I$, $(\mathrm{ad}\, x)^2 = 0$, hence each element in L is nilpotent, and by the Engel theorem L is nilpotent. It is clear also that a regular semisimple Lie algebra is simple (see [2, Chap. VII, Sect. 2.2, Proposition 7]), and that a regular simple Lie algebra is anisotropic (see [2, Chap. VII, Sect. 2.4, Corollary 2]).

Theorem 2.1 *For a simple Lie algebra L the following are equivalent:*

  (i) *L is regular;*
 (ii) *all proper subalgebras of L are regular;*
(iii) *all proper subalgebras of L are either simple, or abelian.*

Proof (i) $\Rightarrow$ (ii) follows from the fact that if S is a subalgebra of L, and $x \in S$ is a regular element in L, then x is a regular element in S ([2, Chap. VII, Sect. 2.2, Proposition 9]).

   (ii) $\Rightarrow$ (iii). By the observation above, any proper subalgebra of L is either simple, or nilpotent. Hence L does not contain a 2-dimensional nonabelian Lie algebra, and by implication (iv) $\Rightarrow$ (iii) of Theorem 1.1, all proper subalgebras of L are reductive, and all its nilpotent subalgebras are abelian.

   (iii) $\Rightarrow$ (i). By implication (iii) $\Rightarrow$ (i) of Theorem 1.1, L is anisotropic. In any Lie algebra, Cartan subalgebras are exactly nilpotent subalgebras N such that $L^0(N) = N$ ([2, Chap. VII, Sect 2.1, Proposition 4]). But nilpotent subalgebras of L are abelian, and $L^0(N)$ coincides with the centralizer of N, so Cartan subalgebras of L are exactly abelian subalgebras coinciding with their own centralizer. For an arbitrary nonzero element $x \in L$, its centralizer $Z_L(x)$ cannot be simple, hence it is abelian. But, obviously, $Z_L(x)$ coincides with its own centralizer, hence $Z_L(x)$ is a Cartan subalgebra of L, $\dim Z_L(x) = \dim L^0(x) = \mathrm{rk}\, L$, and x is regular.

   Note that similar to the anisotropic case, minimal nonregular Lie algebras are exactly solvable minimal nonnilpotent Lie algebras.

Theorem 2.2 *Let $\mathscr{L}$ be a nonempty class of Lie algebras satisfying the following properties:*

  (i) *$\mathscr{L}$ is closed with respect to subalgebras;*
 (ii) *if each proper subalgebra of a Lie algebra L belongs to $\mathscr{L}$, then L belongs to $\mathscr{L}$;*
(iii) *non-semisimple Lie algebras belonging to $\mathscr{L}$ are nilpotent.*

Then $\mathscr{L}$ is the class of all regular Lie algebras.

Proof Any class of Lie algebras satisfying conditions (i) and (iii) consists of regular algebras. Indeed, from the Levi–Malcev decomposition and condition (iii) it follows that any algebra L in $\mathscr{L}$ is either semisimple, or nilpotent. In the former case, write $L = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ as the direct sum of simple components. If $n > 1$, by condition (i) the subalgebra of L of the form $\mathfrak{g}_1 \oplus Kx$, where x is an arbitrary nonzero element of $\mathfrak{g}_2$, belongs to $\mathscr{L}$, and by condition (iii) it is nilpotent, a contradiction. Hence $n = 1$, that is, L is simple. By conditions (i) and (iii) L does not contain 2-dimensional nonabelian subalgebra, and by implication (iv) $\Rightarrow$ (iii) of Theorem 1.1, all subalgebras of L are reductive. This, together with conditions (i) and (iii) again, implies that all subalgebras of L are either simple, or abelian, and by implication (iii) $\Rightarrow$ (i) of Theorem 2.1, L is regular.

   Now, the same elementary reasoning utilizing condition (ii) as at the end of the proof of Theorem 1.2, shows that any regular Lie algebra belongs to $\mathscr{L}$.

3 Minimal Nonabelian Algebras

It follows from the Levi–Malcev decomposition that any minimal nonabelian Lie algebra is either simple, or solvable. Solvable minimal nonabelian Lie algebras (even in a slightly more general minimal nonnilpotent setting) were described in [7], [14], and [15]. A simple minimal nonabelian Lie algebra is regular. Simple minimal nonabelian Lie algebras were studied in [4] and [6], but their full description remains an open problem.

Recall that an algebra is called *central* if its centroid coincides with the base field. For simple algebras this is equivalent to the condition that the algebra remains simple under extension of the base field.

Theorem 3.1 *There are no central simple minimal nonabelian Lie algebras of types* B_l *(l $\geq$ 2),* C_l *(l $\geq$ 3, l $\neq$ 2^k),* D_l *(l $\geq$ 5, l $\neq$ 2^k),* G_2*, and* F_4*.*

Proof The proof follows from the known classification of central simple Lie algebras of these types (see, for example, [13, Chap. IV]).

Types B–D. Each central simple Lie algebra of this type (with the exception of D_4) is isomorphic to a Lie algebra of J-skew-symmetric elements $S^-(A, J) = \{x \in A \mid J(x) = -x\}$, where A is a central simple associative algebra of dimension $n^2 > 16$ with involution J of the first kind (smaller dimensions of A are covered by "occasional" isomorphisms between "small" algebras of different types, including type A). By a known description of such algebras (see, for example, [10, Theorem 5.1.23]), A is isomorphic to $M_m(D)$, a matrix algebra of size $m \times m$ over a central division algebra D with involution j, and J has the form

$$(d_{k\ell})_{k,\ell=1}^m \mapsto \mathrm{diag}(g_1, \ldots, g_m)(j(d_{k\ell}))^\top \mathrm{diag}(g_1^{-1}, \ldots, g_m^{-1})$$

for some $g_1, \ldots, g_m \in D$ such that $j(g_k) = g_k$, $k = 1, \ldots, m$.

If D coincides with the base field, i.e. A is a full matrix algebra, than the Lie algebra $S^-(A, J)$ is split and, obviously, contains a lot of proper nonabelian subalgebras. Hence we may assume $\dim D \geq 4$. From the description above it is clear that, provided $m > 1$, the subalgebra B of A of all matrices with vanishing last row and column, is isomorphic to $M_{m-1}(D)$ and is stable under J, hence $S^-(B, J)$ is a Lie subalgebra of $S^-(A, J)$. Since $\dim A = m^2 \dim D \geq 25$, we have $\dim B = (m-1)^2 \dim D = s^2 \geq 9$, and this subalgebra is a central simple Lie algebra of dimension $\frac{s(s-1)}{2}$ or $\frac{s(s+1)}{2}$. Therefore, if $m > 1$, $S^-(A, J)$ contains proper nonabelian subalgebras, and it remains to consider the case where $A = D$ is a division algebra.

Since D has an involution, its exponent is equal to 2, and its dimension n^2 is equal to some power of 4. This excludes all the types mentioned in the statement of the theorem.

Type G_2. Each central simple Lie algebra of this type is a derivation algebra of a 8-dimensional Cayley algebra $\mathbb{O}$. The latter is obtained by the doubling (Cayley–Dickson) process from the 4-dimensional associative quaternion algebra $\mathbb{H}$, and it is known that each derivation of $\mathbb{H}$ can be extended to a derivation of $\mathbb{O}$ (see, for

example [12, Theorem 2]). Thus, $\mathrm{Der}(\mathbb{O})$ always contains a 3-dimensional central simple Lie algebra $\mathrm{Der}(\mathbb{H})$ as a subalgebra, and hence cannot be minimal nonabelian.

Type F_4. Each central simple Lie algebra of this type is a derivation algebra of a 27-dimensional exceptional simple Jordan algebra $\mathbb{J}$. It is known that derivations of $\mathbb{J}$ mapping a cubic subfield of $\mathbb{J}$ to zero form a central simple Lie algebra of type D_4 (see, for example, [9, Chap. IX, Sect. 11, Exercise 5]).

We conjecture that the remaining types not covered by Theorem 3.1—C_{2^k} and D_{2^k}—cannot occur as well.

Conjecture 3.1 *There are no central simple minimal nonabelian Lie algebras of types* B–D *(except of* D_4*).*

Let us provide some evidence in support of this conjecture.

Lemma 3.1 *Let D be a central division algebra of dimension* n^2 *over a field K with involution J of the first kind, such that* $\mathrm{S}^-(D, J)$ *is a minimal nonabelian Lie algebra. Then for any J-symmetric or J-skew-symmetric element x in D, not lying in K, one of the following holds:*

   (i) *x is J-symmetric and of degree* 2*;*
   (ii) *K[x] is of degree* $\frac{n}{2}$*, and* $\dim_{K[x]} \mathrm{C}_D(x) = 4$*;*
   (iii) *K[x] is a maximal subfield of D.*

Proof The associative centralizer of x in D, $\mathrm{C}_D(x)$, is a proper simple associative subalgebra of D. By the Double Centralizer Theorem (see, for example, [11, Sect. 12.7]),

$$\dim K[x] \cdot \dim \mathrm{C}_D(x) = n^2, \tag{1}$$

and the associative center of $\mathrm{C}_D(x)$ coincides with $K[x]$.

As $\mathrm{C}_D(x)$ is stable under J, $\mathrm{S}^-(\mathrm{C}_D(x), J)$ is a Lie subalgebra of $\mathrm{S}^-(D, J)$. If it coincides with the whole $\mathrm{S}^-(D, J)$, then $\mathrm{S}^-(D, J) \subseteq \mathrm{C}_D(x)$, and by (1), $\dim K[x] \leq \frac{n^2}{\frac{n(n-1)}{2}} < 3$, hence $\dim K[x] = 2$, i.e. $K[x]$ is a quadratic extension of K, the case (i). Note that in this case x cannot be J-skew-symmetric, as otherwise it lies in the Lie center of $\mathrm{S}^-(D, J)$, a contradiction.

If $\mathrm{S}^-(\mathrm{C}_D(x), J)$ is a proper subalgebra of $\mathrm{S}^-(D, J)$, then it is abelian, and by [8, Theorem 2.2], $\mathrm{C}_D(x)$ is either commutative (i.e., a subfield of D), or is 4-dimensional over its center $K[x]$. In the former case, since the degree (= dimension) over K of each intermediate field between K and D is $\leq n$ (actually, a divisor of n), and since $K[x] \subseteq \mathrm{C}_D(x)$, we have $\dim K[x] = \dim \mathrm{C}_D(x) = n$, and $\mathrm{C}_D(x) = K[x]$, the case (iii). In the latter case, from (1) we have $\dim K[x] = \frac{n}{2}$ and $\dim \mathrm{C}_D(x) = 2n$, the case (ii).

For example, if the division algebra D is cyclic (what always happens over number fields), then, considering the conditions of the lemma simultaneously for a J-skew-symmetric element x generating a cyclic extension of the base field, and even powers of x (which are J-symmetric), one quickly arrives to a contradiction.

For the remaining exceptional types, the question seems to be much more difficult, and it is treated in [5] using the language and technique of algebraic groups and Galois cohomology. There are central simple minimal nonabelian Lie algebras of types D_4 and E_8. For types E_6 and E_7 partial answers are available.

Central simple minimal nonabelian Lie algebras of type A of the form $D^{(-)}/K1$ (i.e., quotient of D, considered as a Lie algebra subject to commutator $[a, b] = ab - ba$, by the 1-dimensional center spanned by the unit 1 of D), where D is a central division associative algebra, were studied in [6]. A necessary, but not sufficient condition for such Lie algebra to be minimal nonabelian is D to be minimal noncommutative (i.e., all proper subalgebras of D are commutative). In this connection the following observation is of interest:

Theorem 3.2 *Let D be a central division associative algebra. Then the Lie algebra $D^{(-)}/K1$ is regular if and only if D is a minimal noncommutative algebra.*

Proof Let the dimension of D over the base field K is equal to n^2, so dim $D^{(-)}/K1 = n^2 - 1$. The Lie algebra $D^{(-)}/K1$ is regular if and only if the Lie centralizer of any nonzero element $\bar{x} \in D^{(-)}/K1$ is a Cartan subalgebra of dimension $n - 1$, what, in associative terms, is equivalent to the condition that the associative centralizer $C_D(x)$ of any element $x \in D\backslash K$, is a maximal subfield of D of dimension n over K. Taking this into account, the proof is an easy application of the Double Centralizer Theorem, with reasonings similar to those used in the proof of the lemma above.

The "only if" part. Suppose that for any $x \in D\backslash K$, $C_D(x)$ is a maximal subfield of D. Consider a subfield $K[x] \subseteq C_D(x)$ of D. We have $C_D(x) = C_D(K[x])$, and by the Double Centralizer Theorem, dim $K[x] \cdot$ dim $C_D(x) = n^2$. But the degree (= dimension) over K of each intermediate field between K and D is $\leq n$ (actually, a divisor of n), hence dim $K[x] = $ dim $C_D(x) = n$, and $C_D(x) = K[x]$. That means that there are no intermediate fields between K and the maximal subfields of D.

If A is a noncommutative proper subalgebra of D, then, obviously, A is a division algebra. Its center $Z(A)$, being a field extension of K, either coincides with K, or is a maximal subfield of D. In the former case A is central of dimension m^2, where $1 < m < n$, and its maximal subfield has degree m over K, a contradiction. In the latter case, we have dim $A >$ dim $Z(A) = n$. Applying again the Double Centralizer Theorem, we have dim $A \cdot$ dim $C_D(A) = n^2$. Since $Z(A) \subseteq C_D(A)$, we have dim $C_D(A) \geq$ dim $Z(A) = n$, a contradiction.

The "if" part. Suppose D is minimal noncommutative. For an arbitrary $x \in D$ not lying in the base field K, its centralizer $C_D(x)$ is a subfield of D. By the Double Centralizer Theorem, $C_D(C_D(x))$ is a simple subalgebra of D (and, hence, is also a subfield), and dim $C_D(x) \cdot$ dim $C_D(C_D(x)) = n^2$. By the same argument as above about degrees of intermediate fields between K and D, dim $C_D(x) = $ dim $C_D(C_D(x)) = n$. Since $C_D(x) \subseteq C_D(C_D(x))$, this implies $C_D(x) = C_D(C_D(x))$, and $C_D(x)$ is a maximal subfield of D.

4 Algebras of Depth 2

Define the *depth* of a Lie algebra in the following inductive way: a Lie algebra has depth 0 if and only if it is abelian, and has depth $n > 0$ if and only if it does not have depth $< n$ and all its proper subalgebras have depth $< n$. Thus, minimal nonabelian Lie algebras are exactly algebras of depth 1.

   Many of the algebras considered below arise as semidirect sums $L \oplus V$ of a Lie algebra L and an L-module module V (in such a situation, we will always assume that V is an abelian ideal: $[V, V] = 0$). It is clear that the depth of such semidirect sums is related to depth of L and the maximal length of chains of subspaces of V invariant under action of subalgebras of L, though the exact formulation in the general case seems to be out of reach. In the particular case where L is 1-dimensional, the depth of such semidirect sum is equal to the maximal length of chains in V of invariant subspaces with nontrivial L-action.

   The following can be considered as an extension of the corresponding results from [7], [14], and [15].

Theorem 4.1 *A non-simple Lie algebra of depth* 2 *over a field K is isomorphic to one of the following algebras:*

 (i) *A 4-dimensional solvable Lie algebra having the basis $\{x, y, z, t\}$ and the following multiplication table:*

$$[x, y] = z, \quad [x, z] = 0, \quad [y, z] = 0, \quad [z, t] = 0,$$

   with $\mathsf{ad}\, t$ *acting on the space $Kx \dotplus Ky$ invariantly, without nonzero eigenvectors, and with trace zero.*
 (ii) *A 4-dimensional solvable Lie algebra having the basis $\{x, y, z, t\}$ and the following multiplication table:*

$$[x, y] = z, \quad [x, z] = 0, \quad [y, z] = 0, \quad [z, t] = z,$$

   with $\mathsf{ad}\, t$ *acting on the space $Kx \dotplus Ky$ invariantly, without nonzero eigenvectors, and with trace 1.*
 (iii) *A direct sum of a simple minimal nonabelian Lie algebra and 1-dimensional algebra.*
 (iv) *A semidirect sum $S \oplus V$, where S is either the 2-dimensional nonabelian Lie algebra, or a 3-dimensional simple minimal nonabelian Lie algebra, and V is an S-module such that each nonzero element of S acts on V irreducibly.*
 (v) *A semidirect sum $S \oplus V$, where S is an abelian 1- or 2-dimensional Lie algebra, and V is an S-module such that for each nonzero element $x \in S$, the maximal length of chains of x-invariant subspaces of V is equal to 2 (what is equivalent to saying that any proper x-invariant subspace does not contain proper x-invariant subspaces).*

Proof It is a straightforward verification that in each of these cases the corresponding Lie algebras have depth 2, so let us prove that each non-simple Lie algebra L of depth 2 has one of the indicated forms.

Note that L cannot be semisimple. For, in this case it is decomposed into the direct sum of simple components: $L = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, $n > 1$, and any subalgebra of the form $\mathfrak{g}_1 \oplus Kx$, $x \in \mathfrak{g}_2$, $x \neq 0$, is not minimal nonabelian.

Suppose that L is non-semisimple and non-solvable, and let $L = \mathfrak{g} \oplus \mathrm{Rad}(L)$ be its Levi–Malcev decomposition. Then $\mathfrak{g}$ is minimal nonabelian and hence is simple. Further, $\mathrm{Rad}(L)$ abelian, as otherwise $\mathfrak{g} \oplus [\mathrm{Rad}(L), \mathrm{Rad}(L)]$ is a proper subalgebra of L which is not minimal nonabelian. Suppose now that $\mathrm{rk}\,\mathfrak{g} > 1$, and $\mathfrak{g}$ acts on $\mathrm{Rad}(L)$ nontrivially. Then taking $x \in \mathfrak{g}$ with a nontrivial action on $\mathrm{Rad}(L)$, and the Cartan subalgebra H of $\mathfrak{g}$ of dimension > 1 containing x, we get a subalgebra $H \oplus \mathrm{Rad}(L)$ of L which is not minimal nonabelian. Hence in the case $\mathrm{rk}\,\mathfrak{g} > 1$, $\mathrm{Rad}(L)$ is a trivial (and then, obviously, 1-dimensional) $\mathfrak{g}$-module, and we arrive at case (iii). If $\mathrm{rk}\,\mathfrak{g} = 1$, then $\mathfrak{g}$ is 3-dimensional. If some nonzero $x \in \mathfrak{g}$ acts on $\mathrm{Rad}(L)$ trivially, then so is $[x, \mathfrak{g}]$, and, since $\mathfrak{g}$ is generated by the latter subspace, the whole $\mathfrak{g}$ acts on $\mathrm{Rad}(L)$ trivially, a case covered by (iii). Assume that any nonzero $x \in \mathfrak{g}$ acts on $\mathrm{Rad}(L)$ nontrivially. The Lie subalgebra $Kx \oplus \mathrm{Rad}(L)$ contains, in its turn, a subalgebra $Kx \oplus V$ for any proper $\mathrm{ad}\,x$-invariant subspace V of $\mathrm{Rad}(L)$, what shows that x acts trivially on V. Letting here V to be the Fitting 1-component with respect to the x-action on $\mathrm{Rad}(L)$, we see that $\mathrm{Rad}(L) = V$, what means that x acts on $\mathrm{Rad}(L)$ nondegenerately, and hence, irreducibly. We arrive at case (iv).

It remains to consider the case of L solvable. Take any subspace A of L of codimension 1 containing $[L, L]$, and a complimentary 1-dimensional subspace:

$$L = Kt \dotplus A, \tag{2}$$

$\mathrm{ad}\,t$ acts on A. Since A is a proper ideal of L, it is either abelian or minimal nonabelian. In the former case, we arrive at the semidirect sum $Kt \oplus A$, and it is easy to see that any proper nonabelian subalgebra of L is isomorphic to the semidirect sum $Kt \oplus V$, where V is a proper $\mathrm{ad}\,t$-invariant subspace of A. Thus, for L to be of depth 2 is equivalent to the condition described in case (v) (with S 1-dimensional).

Suppose now that A is minimal nonabelian. According to [7, Theorem 4] (also implicit in [14] and [15]), each solvable minimal nonabelian Lie algebra is either isomorphic to the 3-dimensional nilpotent Lie algebra, or to the semidirect sum $Kx \oplus V$ such that $\mathrm{ad}\,x$ acts on V irreducibly (in particular, $\mathrm{ad}\,x|_V$ is nondegenerate). Further, $\mathrm{ad}\,t$ is a derivation of A, and subtracting from t an appropriate element of A, we may assume that either t is central, i.e. (2) is the direct sum of A and 1-dimensional algebra, or $\mathrm{ad}\,t$ is an outer derivation of A.

Suppose first that A is 3-dimensional nilpotent, i.e., has a basis $\{x, y, z\}$ with multiplication table $[x, y] = z$, $[x, z] = [y, z] = 0$. If t is central, we arrive at a particular case of (i). Straightforward computation shows that each outer derivation of A is equivalent to a derivation d which acts invariantly on the space $Kx \dotplus Ky$, and either $d|_{Kx \dotplus Ky}$ has trace zero, and $d(z) = 0$, or $d|_{Kx \dotplus Ky}$ has trace 1, and $d(z) = z$. These two cases correspond to the cases (i) and (ii) respectively, with the condition

of absence of nonzero eigenvectors to ensure the absence of subalgebras which are not minimal nonabelian.

Suppose now that $A = Kx \notin V$, $\mathsf{ad}\,x$ acts on V irreducibly. If t is central, $L \simeq Kx \in (V \dotplus Kt)$ (with $\mathsf{ad}\,x$ acting on t trivially), a case covered by (v) (with S 1-dimensional). Straightforward computation shows that each outer derivation of A is equivalent to a derivation d which acts on V invariantly, and either $[\mathsf{ad}\,x, d] = 0$ in the Lie algebra $\mathsf{gl}(V)$, and $d(x) = 0$, or $[\mathsf{ad}\,x, d] = \mathsf{ad}\,x$ and $d(x) = x$. These two cases correspond to the cases (v) and (iv) respectively (with S 2-dimensional), with the respective conditions to ensure the absence of subalgebras which are not minimal nonabelian.

Corollary 4.1 (to Theorems 1.1 and 2.1) *A simple Lie algebra of depth* 2 *is either isomorphic to* $\mathsf{sl}(2)$*, or regular.*

Proof It is clear that $\mathsf{sl}(2)$ has depth 2. Hence a simple Lie algebra L of depth 2 is either isomorphic to $\mathsf{sl}(2)$, or does not contain $\mathsf{sl}(2)$ as a proper subalgebra. In the latter case, by implication (v) $\Rightarrow$ (iii) of Theorem 1.1, all subalgebras of L are reductive. But as each minimal nonabelian Lie algebra is either simple, or solvable, all subalgebras of L are either simple, or abelian, and by implication (iii) $\Rightarrow$ (i) of Theorem 2.1, L is regular.

In group theory, a notion analogous to depth in the class of finite p-groups is called $\mathscr{A}_n$-*groups*, see [1, Sect. 65] for their discussion and for a partial description of $\mathscr{A}_2$-groups.

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