Self-Adjointness and Conservation Laws for a Generalized Dullin-Gottwald-Holm Equation

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Abstract We consider the problem of group classification and conservation laws of some Generalized Dullin-Gottwald-Holm equations. We obtain the subclasses of these general equations which are self-adjoint. By using the recent Ibragimov's Theorem on conservation laws, we establish some conservation laws of the self-adjoint equations.

1 Introduction

Dullin, Gottwald and Holm derived a new equation describing unidirectional propagation of surface waves on a shallow layer of water which is at rest at infinity [4].

$$m_t + 2\omega u_x + 2mu_x + um_x = -\gamma u_{xxx}, \quad t > 0,\ x \in \mathscr{R} \tag{1}$$

where $m = u - \alpha^2 u_{xx}$, $u(x, t)$ stands for the fluid velocity, $x \in \mathscr{R}$ and $t > 0$. The constants α^2 and $\frac{\gamma}{c_0}$ are squares of length scales, $c_0 = \sqrt{gh}$ is the linear wave speed for undisturbed water at rest at spatial infinity.

Equation (1) is completely integrable and its traveling wave solutions contains both the Korteweg-de Vries solitons and the Camassa-Holm peakons as limiting cases [4]. When $\alpha \to 0$, this equation becomes the Korteweg-de Vries equation

$$u_t + 2\omega u_x + 3uu_x = -\gamma u_{xxx}$$

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which for $\omega = 0$, has the famous smooth soliton solution $u(x,t) = u_0 \text{sech}^2(x - ct)\sqrt{u_0\gamma/2}$. Instead taking $\gamma -> 0$ in the Eq. (1), it turns out to be the Camassa-Holm equation

$$u_t - \alpha^2 u_{xxt} + 2\omega u_x + 3uu_x = \alpha^2(2u_x u_{xx} + uu_{xxx}).$$

Tian, Fang and Gui, applying Kato's semigroup approach, obtained the well-posedness of the equation and showed the existence of global smooth solutions. The authors proved that the equation has solutions that exist for indefinite times as well as solutions that blow up in finite time, [13].

Biswas and Kara [1] obtained the 1-soliton solution by the aid of solitary wave ansatz. The conserved quantities were obtained by utilising the interplay between the multipliers and underlying Lie point symmetry generators of the equation.

In [10] Liu and Yin established the local well-posedness by using Kato's theory for the generalized Dullin-Gottwald-Holm equation

$$u_t - u_{txx} + (h(u))_x + bu_{xxx} = a\left(\frac{g'(u)}{2}u_x^2 + g(u)u_{xx}\right)_x. \tag{2}$$

They proved the orbital stability of the peaked solitary waves.

Symmetry groups have several different applications in the context of nonlinear differential equations. For example, they are used to obtain exact solutions and conservation laws of partial differential equations (PDEs) [3, 5]. The classical method for finding symmetry groups of PDEs is the Lie group method [2, 6, 11, 12].

In this work, we study Eq. (2) with $a, b \neq 0$ from the point of view of the theory of symmetry group transformations in PDEs. We determine the subclasses of equations which are self-adjoint. We also determine, by using the notation and techniques of the work [8, 9], some nontrivial conservation laws for Eq. (2). The paper is organized as follows. In Sect. 2 we give the Lie symmetries of (2) equation. In Sect. 3 we determine the subclasses of equations of (2) which is self-adjoint. In Sect. 4 we obtain some nontrivial conservation laws for Eq. (2). Finally, in Sect. 5 we give conclusions.

2 Classical Symmetries

To apply the Lie classical method to Eq. (2) we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \tag{3}$$
$$t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2), \tag{4}$$
$$u^* = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2), \tag{5}$$

where ε is the group parameter. We require that this transformation leaves invariant the set of solutions of Eq. (2). This yields to an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{6}$$

Having determined the infinitesimals, the symmetry variables are found by solving the characteristic equation which is equivalent to solving the invariant surface condition

$$\eta(x,t,u) - \xi(x,t,u)\frac{\partial u}{\partial x} - \tau(x,t,u)\frac{\partial u}{\partial t} = 0. \tag{7}$$

The set of solutions of Eq. (2) is invariant under the transformation (3)–(5) provided that

$$\mathrm{pr}^{(3)}\mathbf{v}(\Delta) = 0 \quad \text{when} \quad \Delta = 0,$$

where $\mathrm{pr}^{(3)}\mathbf{v}$ is the third prolongation of the vector field (6) given by

$$\mathrm{pr}^{(3)}\mathbf{v} = \mathbf{v} + \sum_J \eta^J(x,t,u^{(3)})\frac{\partial}{\partial u_J}$$

where

$$\eta^J(x,t,u^{(3)}) = D_J(\eta - \xi u_x - \tau u_t) + \xi u_{Jx} + \eta u_{Jt},$$

with $J = (j_1, \ldots, j_k)$, $1 \le j_k \le 2$ y $1 \le k \le 3$. Hence we obtain the following 13 determining equations for the infinitesimals:

$$\begin{aligned}
&\tau_u = 0,\\
&\tau_x = 0,\\
&\xi_u = 0,\\
&\eta_{uu} = 0,\\
&2\eta_{ux} - \xi_{xx} = 0,\\
&\eta_{uxx} - 2\xi_x = 0,\\
&3g_{uu}\eta_x + 8g_u\eta_{ux} - 4\xi_{xx}g_u = 0,\\
&g_u\eta_u + g_{uu}\eta + \tau_t g_u - \xi_x g_u = 0,\\
&2g_{uu}\eta_u + g_{uuu}\eta + \tau_t g_{uu} - \xi_x g_{uu} = 0,\\
&-ag\eta_{xxx} + b\eta_{xxx} + h_u\eta_x - \eta_{txx} + \eta_t = 0,\\
&ag_u\eta + a\tau_t g - a\xi_x g - b\tau_t + b\xi_x - \xi_t = 0,\\
&2ag_u\eta_x + 3ag\eta_{ux} - 3b\eta_{ux} + \eta_{tu} - 3a\xi_{xx}g + 3b\xi_{xx} - 2\xi_{tx} = 0,\\
&2ag_u\eta_{xx} + 3ag\eta_{uxx} - 3b\eta_{uxx} + 2\eta_{tux} - h_{uu}\eta - \tau_t h_u - \xi_x h_u - a\xi_{xxx}g\\
&+b\xi_{xxx} - \xi_{txx} + \xi_t = 0.
\end{aligned} \tag{8}$$

From (8) we obtain $g, h, \xi = \xi(x,t), \tau = \tau(t), \phi = \frac{(\delta+\xi_x)\,u}{2} + \nu$ with $\delta = \delta(t)$ and $\nu = \nu(x,t)$ where ξ, τ, δ and ν are related by the following conditions:

$$
\begin{array}{l}
\xi_{xxx} - 4\xi_x = 0,\\
g_{uu}\,(\xi_{xx}u + 2\nu_x) = 0,\\
2a\xi_{xx}g_u u + \delta_t + 4ag_u\nu_x - 3a\xi_{xx}g + 3b\xi_{xx} - 3\xi_{tx} = 0,\\
(ag_u\delta + a\xi_x g_u)\,u + 2ag_u\nu + (2a\tau_t - 2a\xi_x)\,g - 2b\tau_t + 2b\xi_x - 2\xi_t = 0,\\
(g_{uu}\delta + \xi_x g_{uu})\,u + g_u\delta + 2g_{uu}\nu + (2\tau_t - \xi_x)\,g_u = 0,\\
(g_{uuu}\delta + \xi_x g_{uuu})\,u + 2g_{uu}\delta + 2g_{uuu}\nu + 2\tau_t g_{uu} = 0,\\
(h_{uu}\delta + \xi_x h_{uu} - 2a\xi_{xxx}g_u)\,u - 4ag_u\nu_{xx}\\
+2h_{uu}\nu + (2\tau_t + 2\xi_x)\,h_u - a\xi_{xxx}g + b\xi_{xxx} - 2\xi_t = 0,\\
(\delta_t + \xi_{xx}h_u - a\xi_{xxxx}g + b\xi_{xxxx} - \xi_{txxx} + \xi_{tx})\,u + (2b - 2ag)\,\nu_{xxx}\\
+2h\nu_x - 2\nu_{txx} + 2\nu_t = 0.
\end{array}
\tag{9}
$$

Solving system (9) we obtain that if g and h are arbitrary functions the only symmetries admitted by (2) are

$$\xi = k_1, \quad \tau = k_2, \quad \eta = 0. \tag{10}$$

The generators are $\mathbf{X}_1 = \dfrac{\partial}{\partial x}$ (corresponding to space translational invariance) and $\mathbf{X}_2 = \dfrac{\partial}{\partial t}$ (time translational invariance). In the following cases Eq. (2) have extra symmetries:

Case 1: If $g = (a_1u + a_2)^n + a_3$ and $h = \left(\frac{b_1}{a_1}(a_1u + a_2)\right)^{n+1} + (aa_3 - b)u, n \neq 1$, $a_1 \neq 0$,

$$\xi = (b - aa_3)k_1 n\, t + k_2, \quad \tau = -k_1 n\, t + k_3, \quad \eta = \frac{k_1}{a_1}(a_1u + a_2).$$

The generators are: $\mathbf{X_1}, \mathbf{X_2}$ and $\mathbf{X_3^1} = \dfrac{(b - aa_4)n}{2}\, t\dfrac{\partial}{\partial x} - \dfrac{n}{2}t\dfrac{\partial}{\partial t} + \dfrac{1}{2a_1}(a_1u + a_2)\dfrac{\partial}{\partial u}$.

Case 2: If $g = a_1u + a_2$ and $h = \frac{b_1}{2}u^2 + b_2u, a_1 \neq 0, b_1 \neq aa_1$,

$$\xi = c_1k_1t + k_3, \quad \tau = k_1t + k_2, \quad \eta = -k_1(u + c_2).$$

We have $\mathbf{X_1}, \mathbf{X_2}$ and $\mathbf{X_3^2} = c_1t\dfrac{\partial}{\partial x} + t\dfrac{\partial}{\partial t} - (c_2 + u)\dfrac{\partial}{\partial u}$, where $c_1 = -\frac{aa_1b_2 + bb_1 - aa_2b_1}{b_1 - aa_1}$ and $c_2 = \frac{b_2 + b - aa_2}{b_1 - aa_1}$.

Case 3: If $g = a_1u + a_2$ and $h = \frac{aa_1}{2}u^2 + (aa_2 - b)u, a_1 \neq 0$

$$\xi = k_1t + k_2, \quad \tau = k_3t + k_4, \quad \eta = -k_3u + \frac{(b - aa_2)k_3 + k_1}{aa_1}.$$

The generators are $\mathbf{X_1}$, $\mathbf{X_2}$ and $\mathbf{X_3^3} = t\dfrac{\partial}{\partial x} + \dfrac{1}{aa_1}\dfrac{\partial}{\partial u}$, $\mathbf{X_4^3} = t\dfrac{\partial}{\partial t} - (u + c_1)\dfrac{\partial}{\partial u}$, where $c_1 = \frac{b-aa_2}{aa_1}$.

Case 4: If $g = c$=constant and $h = ku$ with $k \neq ac - b$,

$$\xi = k_1, \quad \tau = k_2, \quad \eta = k_3 u + \alpha(x,t),$$

where

$$(ac - b)\alpha_{xxx} - k\alpha_x + \alpha_{txx} - \alpha_t = 0. \tag{11}$$

In this case besides $\mathbf{X_1}$ and $\mathbf{X_2}$ we obtain the generators $\mathbf{X_3^4} = u\dfrac{\partial}{\partial u}$ and $\mathbf{X_\infty} = \alpha(x,t)\dfrac{\partial}{\partial u}$.

Case 5: If $g = c$ and $h = (ac - b)u$ with $c \neq \frac{b}{a}$,

$$\xi = k_1 e^{2x+2(b-ac)t} + k_3 e^{2(ac-b)t-2x} + (ac - b)\beta(t) + k_2, \quad \tau = \beta(t),$$

$$\eta = u\left(k_1 e^{2x+2(b-ac)t} - k_3 e^{2(ac-b)t-2x} + k_5\right) + \alpha(x,t),$$

where α satisfies Eq. (11) with $k = ac - b$.

We obtain the generators: $\mathbf{X_1}$, $\mathbf{X_2}$, $\mathbf{X_\infty}$, $\mathbf{X_3^5} = \left(\dfrac{\partial}{\partial x} + u\dfrac{\partial}{\partial u}\right)e^{2[x+(b-ac)t]}$, $\mathbf{X_4^5} = \left(\dfrac{\partial}{\partial x} - u\dfrac{\partial}{\partial u}\right)e^{-2[x+(b-ac)t]}$.

Case 6: If $g = a_1\, e^{a_2\, u} + a_3$ and $h_u = k\, e^{a_2\, u} - b + a\, a_3$

$$\xi = a_2\,(b - a\, a_3)\; k_2\, t + k_1, \quad \tau = k_3 - a_2\, k_2\, t, \quad \eta = k_2.$$

The generators are: $\mathbf{X_1}$, $\mathbf{X_2}$ and $\mathbf{X_3^6} = a_2\,(b - a\, a_3)\; t\dfrac{\partial}{\partial x} - a_2 t\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial u}$.

Case 7: If $g = a_2\ln(a_1\, u + b_1) + b_2$ and $h_u = aa_2\, \ln(a_1 u + b_1) + b_3$,

$$\xi = k_3 t + k_1, \quad \tau = k_2, \quad \eta = \frac{k_3}{aa_1 a_2}(a_1 u + b_1).$$

The generators are $\mathbf{X_1}$, $\mathbf{X_2}$ and $\mathbf{X_3^7} = t\dfrac{\partial}{\partial x} + \dfrac{a_1 u + b_1}{aa_1 a_2}\dfrac{\partial}{\partial u}$.

3 Determination of Self-Adjoint Equations

In [8] Ibragimov introduced a new theorem on conservation laws. The theorem is valid for any system of differential equations where the number of equations is equal to the number of dependent variables. The new theorem does not require existence of a Lagrangian and this theorem is based on a concept of an adjoint equation for nonlinear equations.

Consider an sth-order partial differential equation

$$F(x, u, u_{(1)}, \dots, u_{(s)}) = 0 \tag{12}$$

with independent variables $x = (x^1, \dots, x^n)$ and a dependent variable u, where $u_{(1)} = \{u_i\}$, $u_{(2)} = \{u_{ij}\}, \dots$ denote the sets of the partial derivatives of the first, second, etc. orders, $u_i = \partial u/\partial x^i$, $u_{ij} = \partial^2 u/\partial x^i \partial x^j$. The adjoint equation to (12) is

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \dots, u_{(s)}, v_{(s)}) = 0, \tag{13}$$

with

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \dots, u_{(s)}, v_{(s)}) = \frac{\delta(vF)}{\delta u}, \tag{14}$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}} \tag{15}$$

denotes the variational derivative (the Euler-Lagrange operator), and v is a new dependent variable. Here

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots$$

are the total differentiations.

Equation (12) is said to be *self-adjoint* if the equation obtained from the adjoint Eq. (13) by the substitution $v = u$:

$$F^*(x, u, u, u_{(1)}, u_{(1)}, \dots, u_{(s)}, u_{(s)}) = 0,$$

is identical to the original Eq. (12). In other words, if

$$F^*(x, u, u_{(1)}, u_{(1)}, \dots, u_{(s)}, u_{(s)}) = \lambda(x, u, u_{(1)}, \dots) F(x, u, u_{(1)}, \dots, u_{(s)}). \tag{16}$$

Let us single out self-adjoint equations from the equation of the form (2). Equation (14) yields

$$\begin{aligned} F^* \equiv\ & agv_{xxx} - bv_{xxx} + ag_u u_x v_{xx} + ag_u u_{xx} v_x - ag_{uu} (u_x)^2 v_x \\ & -h_u v_x + v_{txx} - v_t - 3ag_{uu} u_x u_{xx} v - \tfrac{3}{2} ag_{uuu} (u_x)^3 v. \end{aligned} \tag{17}$$

By substituting $v = u$ into (17) we obtain

$$\begin{aligned} F^* \equiv\ & agu_{xxx} - bu_{xxx} - 3ag_{uu} u u_x u_{xx} + 2ag_u u_x u_{xx} - \tfrac{3}{2} ag_{uuu} u (u_x)^3 \\ & -ag_{uu} (u_x)^3 - h_u u_x + u_{txx} - u_t. \end{aligned} \tag{18}$$

Comparing F^* with F we obtain the following result:

Proposition Equation $F \equiv u_t - u_{txx} + (h(u))_x + bu_{xxx} - a\left(\frac{g'(u)}{2} u_x^2 + g(u) u_{xx}\right)_x = 0$ is self-adjoint if g and h are arbitrary functions.

4 General Theorem on Conservation Laws

We use the following theorem on conservation laws proved in [8]. Any Lie point, Lie-Bäcklund or non-local symmetry

$$X = \xi^i(x, u, u_{(1)}, \ldots) \frac{\partial}{\partial x^i} + \eta(x, u, u_{(1)}, \ldots) \frac{\partial}{\partial u} \tag{19}$$

of Eq. (12) provides a conservation law $D_i(C^i) = 0$ for the simultaneous system (12), (13). The conserved vector is given by

$$\begin{aligned} C^i = \xi^i \mathscr{L} + W & \left[\frac{\partial \mathscr{L}}{\partial u_i} - D_j \left(\frac{\partial \mathscr{L}}{\partial u_{ij}} \right) + D_j D_k \left(\frac{\partial \mathscr{L}}{\partial u_{ijk}} \right) - \cdots \right] \\ & + D_j(W) \left[\frac{\partial \mathscr{L}}{\partial u_{ij}} - D_k \left(\frac{\partial \mathscr{L}}{\partial u_{ijk}} \right) + \cdots \right] + D_j D_k(W) \left[\frac{\partial \mathscr{L}}{\partial u_{ijk}} - \cdots \right] + \cdots, \end{aligned} \tag{20}$$

where W and $\mathscr{L}$ are defined as follows:

$$W = \eta - \xi^j u_j, \quad \mathscr{L} = v\, F\left(x, u, u_{(1)}, \ldots, u_{(s)}\right). \tag{21}$$

The proof is based on the following operator identity (N.H. Ibragimov 1979):

$$X + D_i(\xi^i) = W \frac{\delta}{\delta u} + D_i \mathscr{N}^i, \tag{22}$$

where X is operator (19) taken in the prolonged form:

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \zeta_i \frac{\partial}{\partial u_i} + \zeta_{i_1 i_2} \frac{\partial}{\partial u_{i_1 i_2}} + \cdots,$$

$$\zeta_i = D_i(\eta) - u_j D_i(\xi^j), \quad \zeta_{i_1 i_2} = D_{i_2}(\zeta_{i_1}) - u_{ji_1} D_{i_2}(\xi^j), \ \ldots .$$

For the expression of operator $\mathscr{N}^i$ and a discussion of the identity (22) in the general case of several dependent variables to see [7] (Sect. 8.4.4).

We will write the generators of a point transformation group admitted by Eq. (2) in the form

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u}$$

by setting $t = x^1,\ x = x^2$. The conservation law will be written

$$D_t(C^1) + D_x(C^2) = 0. \tag{23}$$

Now we use the Ibragimov's Theorem on conservation laws to establish some conservation laws of Eq. (2). We obtain trivial conservation laws for $g = a_1 u + a_2$, from generators $\mathbf{X}_1$ and $\mathbf{X}_2$.
For **Case 2**, $g = a_1 u + a_2$, $h = |\frac{b_1}{2} + b_2 u$, from generator $\mathbf{X}_2^3$ we obtain the conserved vector associated,

$$C^1 = -(u_x)^2 - u^2 - \tfrac{b_2 + b - a a_2}{b_1 - a a_1}\, u,$$

$$\begin{aligned} C^2 = \tfrac{1}{6(b_1 - a a_1)} \Big[& \Big(\big(12 a a_1 b_1 - 12 a^2 {a_1}^2\big) u^2 \\ & + \big(6 a a_1 b_2 + (12 a a_2 - 12 b)\, b_1 + 18 a a_1 b - 18 a^2 a_1 a_2\big) u \\ & + (6 a a_2 - 6b)\, b_2 - 6 b^2 + 12 a a_2 b - 6 a^2 {a_2}^2 \Big) u_{xx} \\ & + \big(3 a a_1 b_2 + (6b - 6 a a_2)\, b_1 - 3 a a_1 b + 3 a^2 a_1 a_2\big) (u_x)^2 \\ & + ((12 b_1 - 12 a a_1)\, u + 6 b_2 + 6b - 6 a a_2)\, u_{tx} + \big(4 a a_1 b_1 - 4 {b_1}^2\big) u^3 \\ & + ((6 a a_1 - 9 b_1)\, b_2 + (3 a a_2 - 3b)\, b_1)\, u^2 + \big((6 a a_2 - 6b)\, b_2 - 6 {b_2}^2\big) u \Big]. \end{aligned}$$

We use the symmetry of the **Case 3** of Eq. (2) for $g = a_1 u + a_2$ and $h = \frac{a a_1}{2} u^2 + (a a_2 - b) u$ with $a_1 \neq 1$. Proceeding as before we obtain the conserved vector associated with the following symmetries.
For $\mathbf{X_3^3}$:

$$C^1 = \frac{1}{a a_1} u,$$

$$C^2 = -u u_{xx} + \frac{b}{a a_1} u_{xx} - \frac{a_2}{a_1} u_{xx} - \frac{1}{2} (u_x)^2 - \frac{1}{a a_1} u_{tx} + \frac{1}{2} u^2 - \frac{b}{a a_1} u + \frac{a_2}{a_1} u. \tag{24}$$

For $\mathbf{X_4^3}$:

$$C^1 = -(u_x)^2 - u^2 + \left(\frac{b}{a a_1} - \frac{a_2}{a_1} \right) u,$$

$$C^2 = 2aa_1u^2u_{xx} + 3(aa_2 - b)uu_{xx} + \frac{(b - aa_2)^2}{aa_1}u_{xx} + \frac{b - aa_2}{2}(u_x)^2 + 2uu_{tx} + \frac{a_2a - b}{a_1}u_{tx} - \frac{2}{3}aa_1u^3 + \frac{3}{2}(b - aa_2)u^2 - \frac{(b - aa_2)^2}{aa_1}u. \tag{25}$$

For **Case 4**, if $g = c$ and $h = ku$, from generator $\mathbf{X}_\infty = \alpha(x, t)$, where α satisfies Eq. (11), the normal form for this group is $W = \alpha(x, t)$. By applying (20) the vector components are

$$\begin{aligned} C^1 &= -\tfrac{1}{3}\alpha v_{xx} + \tfrac{1}{3}\alpha_x v_x - \tfrac{1}{3}\alpha_{xx}v + \alpha v = 0. \\ C^2 &= -a\alpha c v_{xx} + \alpha b v_{xx} + a\alpha_x c v_x - \alpha_x b v_x + \tfrac{1}{3}\alpha_t v_x - \tfrac{2}{3}\alpha v_{tx} + \tfrac{1}{3}\alpha_x v_t \\ &\quad + \alpha k v - a\alpha_{xx}cv + \alpha_{xx}bv - \tfrac{2}{3}\alpha_{tx}v. \end{aligned} \tag{26}$$

Setting $v = u$ in (26)

$$\begin{aligned} C^1 &= -\tfrac{\alpha u_{xx}}{3} + \tfrac{\alpha_x u_x}{3} - \tfrac{\alpha_{xx}u}{3} + \alpha u. \\ C^2 &= -a\alpha c u_{xx} + \alpha b u_{xx} + a\alpha_x c u_x - \alpha_x b u_x + \tfrac{1}{3}\alpha_t u_x - \tfrac{2}{3}\alpha u_{tx} + \tfrac{1}{3}\alpha_x u_t \\ &\quad + \alpha k u - a\alpha_{xx}cu + \alpha_{xx}bu - \tfrac{2}{3}\alpha_{tx}u. \end{aligned} \tag{27}$$

We simplify the conserved vector (27) by transferring the terms of the form $D_x(\ldots)$ from C^1 to C^2 and obtain

$$\begin{aligned} C^1 &= (\alpha - \alpha_{xx})u. \\ C^2 &= \alpha(b - ac)u_{xx} + \alpha_x(ac - b)u_x - \alpha u_{tx} + \alpha_x u_t + \alpha k u - \alpha_{xx}(ac - b)u. \end{aligned}$$

For **Case 5**, if $g = c$ and $h = (ac - b)u$, from generators $\mathbf{X_3^5} = \frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$ and $\mathbf{X_4^5} = \frac{\partial}{\partial x} - u\frac{\partial}{\partial u}$, proceeding as in the Case 1 we obtain the conserved vector associated,

$$\begin{aligned} C^1 &= n\left((u_x)^2 + u^2\right), \\ C^2 &= 2(b - ac)\,nuu_{xx} + (ac - b)\,n\,(u_x)^2 - 2nuu_{tx} + (ac - b)\,nu^2, \end{aligned}$$

where $n = \pm 1$.

5 Conclusions

In this work we have considered the generalized Dullin-Gottwald-Holm Eq. (2). We have derived the Lie classical symmetries. We have determined the subclasses of Eq. (2) which are self-adjoint. By using a general theorem on conservation laws proved by Nail Ibragimov we found conservation laws for some of these partial differential equations without classical Lagrangians. We point out that in physical systems, many conservation laws that arise can usually be identified with a physical quantity, like energy or linear momentum, being conserved. Finally, we remark that the search for conservation laws is also useful to determine potential symmetries.

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