

Causality from Dynamical Symmetry: An Example from Local Scale-Invariance

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Abstract Physical ageing phenomena far from equilibrium naturally lead to dynamical scaling. It has been proposed to consider the consequences of an extension to a larger Lie algebra of local scale-transformation. The best-tested applications of this are explicitly computed co-variant two-point functions which have been compared to non-equilibrium response functions in a large variety of statistical mechanics models. It is shown that the extension of the Schrödinger Lie algebra $\mathfrak{sch}(d)$ to a maximal parabolic sub-algebra, when combined with a dualisation approach, is sufficient to derive the causality condition required for the interpretation of two-point functions as physical response functions. The proof is presented for the recent logarithmic extension of the differential operator representation of the Schrödinger algebra.

1 Motivation and Background

Physicists have valued since a long time the important rôle of symmetries, be it for their usefulness in simplifying practical calculations, be it for making progress in issues of conceptual understanding. Arguably the most famous instance of this is *relativistic covariance* in mechanics and electrodynamics,¹ formally described by the Lie group of Lorentz transformations which has been introduced almost exactly a century ago [10, 40]. Almost three quarters of a century later, it has been realised that by the inclusion of scale-invariance and the subsequent extension

¹ Physicists carefully distinguish between *co*-variance and *in*variance: for example, a scalar is invariant under rotations, while a vector or a tensor transforms covariantly. Since the equations of mechanics or electrodynamics are in general vector or tensor equations, it is appropriate to speak of relativistic co-variance.

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of the Lorentz group to the *conformal group* considerable advances can be made, simultaneously in cooperative phenomena in statistical mechanics as well as in string theory. A special rôle is herein played by the case of two dimensions, where the infinite-dimensional Lie algebra of conformal transformations is centrally extended to the Virasoro algebra, in order to be able to take the physical effects of either thermal or quantum fluctuations into account [3].

Here, we shall consider a different example of covariance under a certain class of space-time transformations. Historically, these were found by considering the dynamical symmetries of what in physics is called by an abuse of language the ‘non-relativistic limit’ of mechanics where the speed of light $c \rightarrow \infty$. Specifically, we shall be interested in the transformations of the *Schrödinger group* $Sch(d)$ which is defined by the following transformation on space-time coordinates $(t, r) \in \mathbb{R} \times \mathbb{R}^d$:

$$t \mapsto t' := \frac{\alpha t + \beta}{\gamma t + \delta}, \quad r \mapsto r' := \frac{\mathcal{R}r + vt + a}{\gamma t + \delta}; \quad \alpha\delta - \beta\gamma = 1 \quad (1)$$

with $\mathcal{R} \in SO(d)$, $a, v \in \mathbb{R}^d$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Indeed, it has been known to mathematicians since a long time that free-particle motion (be it classical, quantum mechanical or probabilistic) is invariant under the Schrödinger group in the sense that a solution of the equation of motion is mapped onto a different solution of the same equation of motion in the transformed coordinates [36, 39]. During the past century, this has been re-discovered a couple of times, both in mathematics and physics, see e.g. [29] and references therein. It is often convenient to study instead the Lie algebra $\mathfrak{sch}(d) = \text{Lie}(Sch(d)) = \left\langle X_{0,\pm 1}, Y_{\pm 1/2}^{(j)}, M_0, R_0^{(jk)} \right\rangle_{j,k=1,\dots,d}$ with the explicit generators (where $\partial_j := \partial/\partial r_j$ and $\nabla_r = (\partial_1, \dots, \partial_d)^T$)

$$\begin{aligned} X_n &= -t^{n+1}\partial_t - \frac{n+1}{2}t^n r \cdot \nabla_r - \frac{\mathcal{M}}{2}(n+1)nt^{n-1}r^2 - \frac{n+1}{2}xt^n \\ Y_m^{(j)} &= -t^{m+1/2}\partial_j - \left(m + \frac{1}{2}\right)t^{m-1/2}\mathcal{M}r_j \\ M_n &= -t^n \mathcal{M} \\ R_n^{(jk)} &= -t^n (r_j \partial_k - r_k \partial_j) = -R_n^{(kj)} \end{aligned} \quad (2)$$

Herein, the non-derivative terms (characterised by a dimensionful constant \mathcal{M} (‘mass’) and a scaling dimension x) describe how the solution of a Schrödinger/diffusion equation will transform under the action of $\mathfrak{sch}(d)$. One has the non-vanishing commutation relations

$$\begin{aligned} [X_n, X_{n'}] &= (n - n')X_{n+n'} & , & \quad [X_n, Y_m^{(j)}] = \left(\frac{n}{2} - m\right)Y_{n+m}^{(j)} \\ [X_n, M_{n'}] &= -n'M_{n+n'} & , & \quad [X_n, R_{n'}^{(jk)}] = -n'R_{n+n'}^{(jk)} \\ [Y_m^{(j)}, Y_{m'}^{(k)}] &= \delta^{j,k} (m - m')M_{m+m'} & , & \quad [R_n^{(jk)}, Y_m^{(\ell)}] = \delta^{j,\ell} Y_{n+m}^{(k)} - \delta^{k,\ell} Y_{n+m}^{(j)} \end{aligned} \quad (3)$$

up to the commutators of $\mathfrak{so}(d)$, involving the $R_0^{(jk)}$, which are not spelled out. The Schrödinger algebra is also the Lie symmetry algebra of non-linear (systems of) equations. Probably one of the best-known examples of this kind are the Euler equations of a compressible fluid of velocity $u = u(t, r)$ and density $\rho = \rho(t, r)$

$$\partial_t \rho + \nabla \cdot (\rho u) = 0 \quad , \quad \rho(\partial_t + (u \cdot \nabla))u + \nabla P = 0 \tag{4}$$

together with the polytropic equation of state $P = \rho^{1+2/d}$. This has been known to russian and ukrainian mathematicians at least since the 1960s [15, 49] and was re-discovered by european physicists around the turn of the century [20, 48]. Many more Schrödinger-invariant non-linear equations and systems exist, see [14–16, 55]. Analogously to conformal invariance in $2D$, an infinite-dimensional extension of $\mathfrak{sch}(d)$ is the *Schrödinger-Virasoro algebra* $\mathfrak{sv}(d) = \langle X_n, Y_m^{(j)}, M_n, R_n^{(jk)} \rangle_{n \in \mathbb{Z}, m \in \mathbb{Z} + \frac{1}{2}, j, k \in \{1, \dots, d\}}$, with an explicit representation in (2) and an immediate extension of the commutators (3) [22]. The mathematical properties of \mathfrak{sv} are studied in detail in [57, 61], the geometry in [9] and physical applications are reviewed in [29].

Contrary to a widespread belief, when taking the non-relativistic limit of the conformal algebra, one does *not* obtain the Schrödinger algebra, but a different Lie algebra, which by now is usually called the *conformal Galilean algebra* $\text{CGA}(d) = \langle X_{\pm 1, 0}, Y_{\pm 1, 0}^{(j)}, R_0^{(jk)} \rangle_{j, k=1, \dots, d}$ [1, 21, 23, 26, 42, 47, 64]. Its most general known differential operator representation is [7]

$$\begin{aligned} X_n &= -t^{n+1} \partial_t - (n+1)t^n r \cdot \nabla_r - n(n+1)t^{n-1} \gamma \cdot r - x(n+1)t^n \\ Y_n^{(j)} &= -t^{n+1} \partial_j - (n+1)t^n \gamma_j \\ R_n^{(jk)} &= -t^n (r_j \partial_k - r_k \partial_j) - t^n (\gamma_j \partial_{\gamma_k} - \gamma_k \partial_{\gamma_j}) = -R_n^{(kj)} \end{aligned} \tag{5}$$

where $\gamma = (\gamma_1, \dots, \gamma_d)$ is a vector of dimensionful constants and x is again a scaling dimension. Its non-vanishing commutators read, again up to those of $\mathfrak{so}(d)$

$$\begin{aligned} [X_n, X_{n'}] &= (n - n')X_{n+n'} \quad , \quad [X_n, Y_m^{(j)}] = (n - m)Y_{n+m}^{(j)} \\ [X_n, R_{n'}^{(jk)}] &= -n'R_{n+n'}^{(jk)} \quad , \quad [R_n^{(jk)}, Y_m^{(\ell)}] = \delta^{j, \ell} Y_{n+m}^{(k)} - \delta^{k, \ell} Y_{n+m}^{(j)} \end{aligned} \tag{6}$$

The non-linear systems for which $\text{CGA}(d)$ arises as a (conditional) dynamical symmetry are distinct from (4) [7, 62]. As before, the systematic organisation of the generators allows for an immediate infinite-dimensional extension $\mathfrak{av}(d) := \langle X_n, Y_n^{(j)}, R_n^{(jk)} \rangle_{n \in \mathbb{Z}, j, k=1, \dots, d}$ [23, 50] (*‘altern-Virasoro algebra’*).

In $d = 2$ spatial dimensions, it was recently shown [41] that the conformal Galilean algebra admits a so-called ‘exotic’ central extension. This is achieved by adding to the commutator relations (6) the following commutator

$$[Y_n^{(1)}, Y_m^{(2)}] = \delta_{n+m, 0} (3\delta_{n, 0} - 2) \Theta, \quad n, m \in \{\pm 1, 0\}, \tag{7}$$

where the new central generator Θ is needed for this central extension. Physicists usually call this central extension of CGA(2) the *exotic Galilean conformal algebra*, and we shall denote it by ECGA = CGA(2) + $\mathbb{C}\Theta$.² A differential operator representation of ECGA reads [7, 42] (ε_{jk} is the totally antisymmetric tensor)

$$\begin{aligned} X_n &= -t^{n+1}\partial_t - (n+1)t^n r \cdot \nabla_r - \lambda(n+1)t^n - (n+1)nt^{n-1}\gamma \cdot r - (n+1)nh \cdot r \\ Y_n^{(j)} &= -t^{n+1}\partial_j - (n+1)t^n \gamma_j - (n+1)t^n h_j - (n+1)n(\varepsilon_{jk}r_k)\theta \\ R_0^{(12)} &= -(r_1\partial_2 - r_2\partial_1) - (\gamma_1\partial_{\gamma_2} - \gamma_2\partial_{\gamma_1}) - \frac{1}{2\theta}h \cdot h \end{aligned} \tag{8}$$

where $n \in \{\pm 1, 0\}$ and $j, k \in \{1, 2\}$.³ Because of Schur’s lemma, the central generator Θ can be replaced by its eigenvalue $\theta \neq 0$. The components of the vector-operator $h = (h_1, h_2)$ are connected by the commutator $[h_1, h_2] = \Theta$. For illustration, we quote the following non-linear system which has ECGA as a Lie symmetry [7]

$$\nabla \wedge u = 0 \quad , \quad \partial_t u + (u \cdot \nabla)u + \frac{1}{2}(u \wedge \nabla) \wedge u = q \nabla \wedge \omega \tag{9}$$

where q is a constant, $u = u(t, r) = (u_1, u_2, 0)^T$ is a planar vector embedded into \mathbb{R}^3 (and similarly for ∇) and $\omega = (0, 0, w)^T$ is constructed from the coordinate dual to the central charge according to $\Theta = \partial_w$. Clearly, (9) is very different from (4).

Remark In analogy to the Virasoro algebra of 2D conformal invariance, it is natural to ask if the full definition of algebras such as $\mathfrak{sv}(d)$ or $\mathfrak{av}(d)$ may include central extensions. For the Schrödinger-Virasoro algebra $\mathfrak{sv}(1)$, one merely has the central Virasoro-like extension of $[X_n, X_m]$ [22, 57, 61]. On the other hand, if in $\mathfrak{sv}(1)$ one considers the generators Y_n with *integer* indices $n \in \mathbb{Z}$, then three distinct central extensions are possible [57], [61, Theorem 7.4]. For the ‘altern-Virasoro algebra’ or the infinite-dimensional extension of CGA(1) one has the central extensions [28, 50]

$$\begin{aligned} [X_n, X_{n'}] &= (n - n')X_{n+n'} + \frac{c_X}{12}\delta_{n+n',0}(n^3 - n) \\ [X_n, Y_{n'}] &= (n - n')X_{n+n'} + \frac{c_Y}{12}\delta_{n+n',0}(n^3 - n) \end{aligned} \tag{10}$$

with two independent central charges. The independence of the two central charges $c_{X,Y}$ can be illustrated by the following example: let L_n and L'_n with $n \in \mathbb{Z}$ stand for the generators of two commuting Virasoro algebras with central charges c and c' . Then the generators

$$X_n := \begin{pmatrix} L_n + L'_n & 0 \\ 0 & L_n + L'_n \end{pmatrix}, \quad Y_n := \begin{pmatrix} 0 & L_n \\ 0 & 0 \end{pmatrix}, \quad K_X := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_Y := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{11}$$

² All Lie algebras are complex, unless explicitly stated otherwise.

³ An infinite-dimensional extension of ECGA does not appear to be possible.

obey the commutators (10), with $c_X = (c + c')K_X$ and $c_Y = cK_Y$ [28] [29, Exercise 5.5].

In statistical physics, many situations are known and well-understood where the usual space-time symmetries of temporal and spatial translation-invariance and rotation-invariance are supplemented by dilatation (or scale-) invariance.⁴ The paradigmatic examples are provided by various phase transitions—often-mentioned examples include the liquid-gas transition, the ferromagnetic-paramagnetic transition, the transition between normal conductivity and superconductivity, the electroweak phase transition in the early universe and so on. Here, we shall be interested in instances of *dynamical* scaling, which involves the space-time rescaling $t \mapsto b^z t$, $r \mapsto br$ and is characterised by a constant, the *dynamical exponent* z . It arises naturally in various many-body systems far from equilibrium, often without having to fine-tune external parameters. Paradigmatic examples are *ageing phenomena*, which may arise in systems quenched, from some initial state, either (i) into a coexistence phase with more than one stable equilibrium state or else (ii) onto a critical point of the stationary state, see [4, 8, 29] for reviews. Phenomenologically, ageing can be characterised through three (symmetry) properties: namely [29]

1. slow, non-exponential relaxation,
2. breaking of time-translation-invariance
3. dynamical scaling.

For equilibrium critical phenomena, it was believed for a long time that under relatively weak conditions scale-invariance could be extended to conformal invariance. Recent work has considerably clarified that this conclusion cannot always be drawn so readily [56], although there exist many theoretical models which are indeed both scale- and conformally invariant, with many important consequences [3, 53]. Drawing on this analogy, we look for situations when dynamical scaling can be extended to a larger group, such as the Schrödinger group when $z = 2$. Quite analogously with respect to conformal invariance, one is looking for co-variant two-point functions, such that the co-variance under Schrödinger transformations leads to a set of differential equations for the said two-point function. However, in contrast to conformal invariance, it has turned out that this kind of co-variance condition is *not* satisfied by correlation functions but rather by the so-called *response functions*. As an example, we quote the basic prediction of Schrödinger-invariance for the linear two-time auto-response function [24–27]

$$R(t, s) = \left. \frac{\delta \langle \phi(t, r) \rangle}{\delta h(s, r)} \right|_{h=0} = \langle \phi(t, r) \tilde{\phi}(s, r) \rangle = s^{-1-a} f_R \left(\frac{t}{s} \right),$$

$$f_R(y) = f_0 y^{1+a'-\lambda_R/z} (y-1)^{-1-a'} \Theta(y-1) \tag{12}$$

which measures the linear response of the order-parameter $\phi(t, r)$ with respect to its canonically conjugated external field $h(s, r)$. In stochastic field-theory using the

⁴ In the physicists terminology: at an equilibrium critical point, the partition function is *invariant* under dilatations, whereas correlators of physical observables transform *co-variantly*.

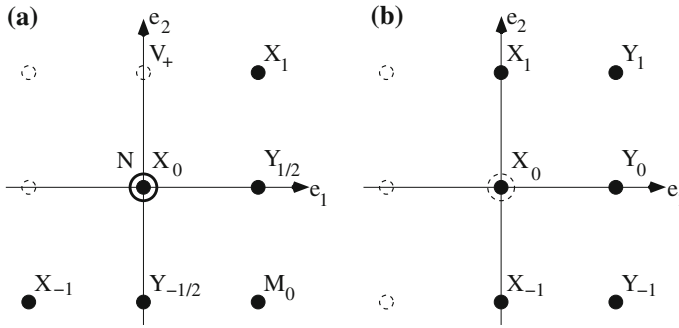


Fig. 1 Root diagrammes of some sub-algebras of the complex Lie algebra B_2 . The roots of B_2 are indicated by the full and broken dots, those of the sub-algebras by the full dots only. **a** Schrödinger algebra $\mathfrak{sch}(1) = \langle X_{\pm 1,0}, Y_{\pm 1/2}, M_0 \rangle$ and the maximal parabolic sub-algebra $\widetilde{\mathfrak{sch}}(1) = \mathfrak{sch}(1) + \mathbb{C}N$. **b** Conformal Galilean algebra $\text{CGA}(1) = \langle X_{\pm 1,0}, Y_{\pm 1,0} \rangle$

Janssen-de Dominicis formalism, see e.g. [8, 29], it can be shown that response functions can be written as a correlator between the order-parameter ϕ and an associated ‘response field’ $\tilde{\phi}$.⁵ The auto-response exponent λ_R and the ageing exponents a, a' are universal non-equilibrium exponents.⁶ This prediction has been tested extensively, and the computation of correlators can be understood along different lines, as reviewed in [29].

The main distinction of response functions with respect to correlation functions is the *causality condition* $t > s$, which is spelt out in (12) through the Heaviside Θ -function. Here, we shall show *how the origin of this causality condition can be understood from an algebraic symmetry hypothesis*. The central observation is that there exists a natural way to imbed the Schrödinger algebra $\mathfrak{sch}(d)$ into a (semi-simple) conformal Lie algebra in $d + 2$ dimensions [5, 26]. This opens the route to introduce a powerful mathematical concept, namely the parabolic sub-algebras of that conformal Lie algebra. By definition, a (standard) *parabolic sub-algebra* is made up by the Cartan sub-algebra and a selected set of positive roots [37]. It turns out that *a sufficient condition for deriving a causality condition for the co-variant two-point functions as in (12)* is the co-variance under a maximal parabolic sub-algebra dualised in such a way that translation-invariance in the dual variable becomes part of the algebra. For example, rather than requiring Schrödinger-covariance under the algebra $\mathfrak{sch}(d)$, one considers an extended co-variance under the *maximal parabolic sub-algebra* $\widetilde{\mathfrak{sch}}(d) = \mathfrak{sch}(d) + \mathbb{C}N$, with a single extra generator N , to be specified below [26]. In Fig. 1a, we illustrate the inclusion, for the $d = 1$ case, $\widetilde{\mathfrak{sch}}(1) = \mathfrak{sch}(1) + \mathbb{C}N \subset B_2$ to the simple complex Lie algebra B_2 , isomorphic to the conformal algebra

⁵ The example of the free field equations of motion already shows that while the order-parameter ϕ has a positive ‘mass’ $\mathcal{M} > 0$, the ‘mass’ associated to the response field is negative $\tilde{\mathcal{M}} = -\mathcal{M} < 0$.

⁶ In magnets, the temperature is rapidly lowered (‘quenched’) from a very high initial value to a finite value T . Mean-field theory suggests that usually $a = a'$ for low final temperatures $T < T_c$ and $a \neq a'$ for critical quenches at $T = T_c$, where T_c is the equilibrium critical temperature [29].

$B_2 \cong (\text{conf}(3))_{\mathbb{C}}$ in three dimensions. Similarly, Fig. 1b illustrates the inclusion $\text{CGA}(1) \subset B_2$ and an extension by the second independent generator in the Cartan sub-algebra would give an inclusion $\widetilde{\text{CGA}}(1) \subset B_2$. Maximal parabolic sub-algebras of B_2 are distinguished in that the addition of any further generator produces the entire conformal algebra. Furthermore, in view of many important physical applications (some of them to be mentioned briefly below), we shall see that the same kind of causality condition is also obtained for the novel logarithmic extensions of the Schrödinger and/or conformal Galilean algebras [30, 32–34, 58].

This paper is organised as follows. The first sections recall basic facts on the ingredients required. In Sect. 2, we recall briefly those elements of *logarithmic* conformal invariance as required here and quote the corresponding logarithmic extensions of $\mathfrak{sch}(d)$ - and $\text{CGA}(d)$ -invariance. In Sect. 3, specialising to $d = 1$ for brevity, we describe the inclusion of the Schrödinger algebra into B_2 by a canonical dualisation procedure and its extension to the logarithmic case. In Sect. 4, the shapes of the dual logarithmic Schrödinger-covariant two-point functions will be derived and we shall see that Schrödinger-covariance alone is *not* enough to derive a causality condition. In Sect. 5 we finally derive our main result, namely that $\mathfrak{sch}(1)$ -covariant two-point functions automatically must obey causality. In this way, a combination of dualisation with an extended dynamical co-variance requirement allows to derive the causality condition algebraically.

2 Logarithmic Conformal Invariance

In various physical situations presenting an equilibrium phase transition, for example disordered systems [6], percolation [12, 43] or sand-pile models [52], it has been useful to consider degenerate vacuum states. Formally, this can be implemented [19, 54] by replacing the order parameter ϕ by a vector $\begin{pmatrix} \psi \\ \phi \end{pmatrix}$ and the scaling dimension x by a Jordan matrix $\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$. For reviews, see [11, 17].

Here, we consider an analogous extension of the representations of the Schrödinger and conformal Galilean algebras. Consider the two-point functions⁷

$$\begin{aligned} F &:= \langle \phi_1(t_1, r_1) \phi_2(t_2, r_2) \rangle \quad , \quad G := \langle \phi_1(t_1, r_1) \psi_2(t_2, r_2) \rangle \quad , \\ H &:= \langle \psi_1(t_1, r_1) \psi_2(t_2, r_2) \rangle \end{aligned} \tag{13}$$

Temporal and spatial translation-invariance imply that they are functions $F = F(t, r)$, $G = G(t, r)$ and $H = H(t, r)$ with $t = t_1 - t_2$ and $r = r_1 - r_2$. Since we shall explain the method in more detail below, we now simply quote the results and generalise them immediately to an arbitrary space dimension d . Co-variance under the logarithmic extension of either $\mathfrak{sch}(d)$ or $\text{CGA}(d)$ implies $x_1 = x_2 =: x$ and

⁷ Here and later, $\langle \cdot \rangle$ refers to an average over the thermal noise.

$F = 0$. For logarithmic Schrödinger invariance [32]

$$G = G_0 |t|^{-x} \exp \left[-\frac{\mathcal{M}}{2} \frac{r^2}{t} \right], \quad H = (H_0 - G_0 \ln |t|) |t|^{-x} \exp \left[-\frac{\mathcal{M}}{2} \frac{r^2}{t} \right] \quad (14)$$

subject to the constraint [2] $\mathcal{M} := \mathcal{M}_1 = -\mathcal{M}_2$.⁸ For the case of logarithmic conformal Galilean invariance [30]

$$G = G_0 |t|^{-2x} \exp \left[-2 \frac{\gamma \cdot r}{t} \right], \quad H = (H_0 - 2G_0 \ln |t|) |t|^{-2x} \exp \left[-2 \frac{\gamma \cdot r}{t} \right] \quad (15)$$

together with the constraint $\gamma := \gamma_1 = \gamma_2$. Here, G_0, H_0 are normalisation constants. The presence of the logarithmic terms explain the name of ‘logarithmic extension’.

3 Extension to Maximal Parabolic Sub-Algebras

Clearly, the results (14, 15) do not contain any information on causality. In order to write down the required extension of the symmetry algebras, we first *consider the ‘mass’ parameter \mathcal{M} as a further variable* (for the moment for the scalar case) and write [18]

$$\widehat{\phi}(\zeta, t, r) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\mathcal{M} e^{i\mathcal{M}\zeta} \phi_{\mathcal{M}}(t, r) \quad (16)$$

which defines the coordinate ζ dual to \mathcal{M} which we shall consider as a ‘ $(-1)^{\text{st}}$ ’ coordinate.⁹ From now on, we concentrate on the case $d = 1$ for simplicity. The generators of $\mathfrak{sch}(1)$ become

⁸ In order to keep the physical convention of non-negative masses $\mathcal{M} \geq 0$, one may introduce a ‘complex conjugate’ ϕ^* to the scaling field ϕ , with $\mathcal{M}^* = -\mathcal{M}$. In dynamics, co-variant two-point functions are interpreted as response functions, written as $R(t, s) = \langle \phi(t) \tilde{\phi}(s) \rangle$ in the context of Janssen-de Dominicis theory, where the response field $\tilde{\phi}$ has a mass $\mathcal{M} = -\mathcal{M}$, see e.g. [8, 29] for details.

Furthermore, the physical relevant equations are *stochastic* Langevin equations, whose noise terms do break any interesting extended dynamical scale-invariance. However, one may identify a ‘deterministic part’ which may be Schrödinger-invariant, such that the predictions (14) remain valid even in the presence of noise [51]. This was rediscovered recently under name of ‘time-dependent deformation of Schrödinger geometry’ [46].

⁹ In string theory and non-relativistic versions of the celebrated AdS/CFT correspondence [44], an analogous construction is used [38, 45, 59], with interesting applications to cold atoms [13].

$$\begin{aligned}
 X_n &= \frac{i}{2}(n+1)nt^{n-1}r^2\partial_\zeta - t^{n+1}\partial_t - \frac{n+1}{2}t^n r\partial_r - \frac{n+1}{2}xt^n \\
 Y_m &= i\left(m + \frac{1}{2}\right)t^{m-1/2}r\partial_\zeta - t^{m+1/2}\partial_j \\
 M_n &= it^n\partial_\zeta
 \end{aligned}
 \tag{17}$$

The extension to the maximal parabolic sub-algebra $\widetilde{\mathfrak{sch}}(1) = \mathfrak{sch}(1) + \mathbb{C}N$ is achieved by including the generator

$$N := \zeta\partial_\zeta - t\partial_t + \xi. \tag{18}$$

In order to understand the origin of the constant term ξ , which in what follows will turn out to play the rôle of a second scaling dimension, consider another representation of the conformal Galilean algebra $\text{CGA}(1) = \langle X_1, Y_{\pm 1/2}, M_0, V_+, 2X_0 - N \rangle$, see Fig. 1a. Herein, the generator X_1 takes a slightly generalised form¹⁰

$$X_1 = ir^2\partial_\zeta - t^2\partial_t - tr\partial_r - (x + \xi)t \tag{19}$$

along with the new generator

$$V_+ = -\zeta r\partial_\zeta - tr\partial_t - \left(i\zeta t + \frac{r^2}{2}\right)\partial_r - (x + \xi)r \tag{20}$$

All other generators are as in (17). One readily verifies that $[V_+, Y_{-1/2}] = 2X_0 - N$, with the explicitly given forms and this explains the presence of the constant ξ in (18).

The chosen normalisation of the generators is clarified by the commutator $[V_+, Y_{1/2}] = X_1$ and the remaining commutators of $\text{CGA}(1)$ are promptly verified. *These generators act as a dynamical symmetry of the Schrödinger equation*

$$\mathcal{S}\widehat{\phi} = 0, \quad \mathcal{S} = -2i\partial_\zeta\partial_t - \partial_r^2 - 2i\left(x + \xi - \frac{1}{2}\right)t^{-1}\partial_\zeta \tag{21}$$

in the sense that the generators of $\text{CGA}(1)$ map solutions of $\mathcal{S}\widehat{\phi} = 0$ onto another solution.

Proof To check this, it suffices to verify the commutators

$$\begin{aligned}
 [\mathcal{S}, V_+] &= -2r\mathcal{S}, \quad [\mathcal{S}, X_1] = -2t\mathcal{S}, \quad [\mathcal{S}, X_0] = -\mathcal{S} \\
 [\mathcal{S}, N] &= [\mathcal{S}, Y_{-1/2}] = [\mathcal{S}, M_0] = 0
 \end{aligned}$$

¹⁰ The same form of X_1 also arises in the ageing sub-algebra $\text{age}(1) = \langle X_{1,0}, Y_{\pm 1/2}, M_0 \rangle \subset \mathfrak{sch}(1)$. Physically, the presence of ξ , together with the absence of the time-translations $X_{-1} = -\partial_t$, leads to distinct exponents a and a' in (12).

and to recall that $X\widehat{\phi}$ with $X \in \text{CGA}(1)$ generates an infinitesimal transformation on the solution $\widehat{\phi}$. □

In general, a *standard parabolic sub-algebra* of a simple complex Lie algebra is spanned by the Cartan sub-algebra \mathfrak{h} and a set of ‘positive’ generators [37]. We illustrate this for the example B_2 , using Fig. 1a. The separation between positive and non-positive generators can be introduced by drawing a straight line through the Cartan sub-algebra \mathfrak{h} , indicated by the double point in the centre and then defining all generators who are represented by a dot to the right of this line as ‘positive’. It is well-known that the Weyl group (which acts on the root diagramme) maps isomorphic sub-algebras onto each other. Hence, it is enough to consider the cases when the straight line mentioned above has a slope between unity and infinity. Then one finds the following classification of the non-isomorphic maximal standard parabolic sub-algebras of B_2 [26]: (i) if the slope is unity, one has $\widetilde{\text{sch}}(1)$, (ii) for a finite slope larger than unity, one has $\widetilde{\text{age}}(1) = \langle X_{0,1}, Y_{\pm 1/2}, M_0, N \rangle$ and (iii) if the slope is infinite, one has $\widetilde{\text{CGA}}(1)$.

4 Dual Logarithmic Schrödinger-Invariance

We now describe the consequences of logarithmic Schrödinger-invariance for the ‘dual’ formulation introduced in the previous section. This representation is constructed from (17) by the formal substitution $x \rightarrow \begin{pmatrix} x & x' \\ 0 & x \end{pmatrix}$, where we explicitly keep the two possibilities $x' = 0$ and $x' = 1$. Only the generators $X_{0,1}$ are modified and now read

$$\begin{aligned} X_0 &= -t\partial_t - \frac{1}{2}r\partial_r - \frac{1}{2} \begin{pmatrix} x & x' \\ 0 & x \end{pmatrix} \\ X_1 &= \frac{i}{2}r^2\partial_\zeta - t^2\partial_t - tr\partial_r - t \begin{pmatrix} x & x' \\ 0 & x \end{pmatrix} \end{aligned} \tag{22}$$

The co-variant two-point functions, built from quasi-primary scaling operators $\begin{pmatrix} \phi_i \\ \psi_i \end{pmatrix}$ which are characterised by the values of x_i and x'_i , to be studied are

$$\begin{aligned} \widehat{F}(\zeta, t, r) &:= \langle \widehat{\phi}_1(\zeta_1, t_1, r_1) \widehat{\phi}_2(\zeta_2, t_2, r_2) \rangle \\ \widehat{G}_{12}(\zeta, t, r) &:= \langle \widehat{\phi}_1(\zeta_1, t_1, r_1) \widehat{\psi}_2(\zeta_2, t_2, r_2) \rangle \\ \widehat{G}_{21}(\zeta, t, r) &:= \langle \widehat{\psi}_1(\zeta_1, t_1, r_1) \widehat{\phi}_2(\zeta_2, t_2, r_2) \rangle \\ \widehat{H}(\zeta, t, r) &:= \langle \widehat{\psi}_1(\zeta_1, t_1, r_1) \widehat{\psi}_2(\zeta_2, t_2, r_2) \rangle \end{aligned} \tag{23}$$

where $\zeta = \zeta_1 - \zeta_2, t = t_1 - t_2$ and $r = r_1 - r_2$. This form already takes translation-invariance in the three variables ζ, t, r into account which in turn follow from the co-

variance under $M_0, Y_{-1/2}, X_{-1}$, respectively.¹¹ Next, we consider the consequences of co-variance under the Galilei-transformations generated by $Y_{1/2}$. For the first of the two-point functions (23) this implies the differential equation (called ‘projective Ward identity’ in physics¹²)

$$(i(r_1 - r_2)\partial_\zeta - (t_1 - t_2)\partial_r) \widehat{F} = 0 \tag{24}$$

whose general solution (and similarly for the other two-point functions) is

$$\widehat{F} = \widehat{F}(t, u) \ , \ \widehat{G}_{12} = \widehat{G}_{12}(t, u) \ , \ \widehat{G}_{21} = \widehat{G}_{21}(t, u) \ , \ \widehat{H} = \widehat{H}(t, u) \ ; \\ u := 2\zeta t + ir^2 \tag{25}$$

The new specific information of the logarithmic representations becomes first evident from dilatation-covariance, generated by X_0 . When taking the previous results (25) into account, the projective Ward identities become, for the four distinct functions in (23)

$$\begin{aligned} \left(-t\partial_t - u\partial_u - \frac{1}{2}(x_1 + x_2)\right) \widehat{F}(t, u) &= 0 \\ \left(-t\partial_t - u\partial_u - \frac{1}{2}(x_1 + x_2)\right) \widehat{G}_{12}(t, u) &= \frac{x'_2}{2} \widehat{F}(t, u) \\ \left(-t\partial_t - u\partial_u - \frac{1}{2}(x_1 + x_2)\right) \widehat{G}_{21}(t, u) &= \frac{x'_1}{2} \widehat{F}(t, u) \\ \left(-t\partial_t - u\partial_u - \frac{1}{2}(x_1 + x_2)\right) \widehat{H}(t, u) &= \frac{x'_1}{2} \widehat{G}_{12}(t, u) + \frac{x'_2}{2} \widehat{G}_{21}(t, u) \end{aligned} \tag{26}$$

Rather than solving this directly, it is more efficient to use first the information coming from the special Schrödinger transformations generated by X_1 . Applied to the first two-point function \widehat{F} , the use of (24, 26) gives

$$\left(\frac{i}{2}r^2\partial_\zeta - t^2\partial_t - tr\partial_r - tx_1\right) \widehat{F}(t, u) = 0 \tag{27}$$

Applying again (26), we have the system

$$\left. \begin{aligned} (-t\partial_t - u\partial_u - x_1) \widehat{F} &= 0 \\ (-t\partial_t - u\partial_u - (x_1 + x_2)/2) \widehat{F} &= 0 \end{aligned} \right\} \implies (x_1 - x_2) \widehat{F} = 0 \tag{28}$$

¹¹ Since the kinetic term of the invariant Schrödinger Eq.(21) reduces to a Laplace operator in a convenient basis, the calculations are analogous to those of logarithmic conformal invariance.

¹² We prefer to include the terms describing the transformation of the physical scaling operators right into the generators, while many authors only include them into the projective Ward identities. The end result is the same, the difference corresponds to the distinction between active and passive transformations.

and we have proven the following

Proposition 1 *If $\begin{pmatrix} \phi \\ \psi \end{pmatrix}$ is a quasi-primary scaling operator of logarithmic Schrödinger-invariance with generators (17, 22), it the two-point function $\widehat{F} = \langle \widehat{\phi}_1 \widehat{\phi}_2 \rangle$ satisfies one of the following conditions: (i) $x_1 = x_2$, (ii) $\widehat{F} = 0$.*

Now, we consider the mixed two-point functions \widehat{G}_{12} and \widehat{G}_{21} . In complete analogy with the above calculations, we find

$$\left. \begin{aligned} (-t\partial_t - u\partial_u - x_1)\widehat{G}_{12} &= 0 \\ (-t\partial_t - u\partial_u - (x_1 + x_2)/2)\widehat{G}_{12} - \frac{1}{2}x'_2\widehat{F} &= 0 \end{aligned} \right\} \implies (x_1 - x_2)\widehat{G}_{12} = x'_2\widehat{F} \tag{29}$$

$$\left. \begin{aligned} (-t\partial_t - u\partial_u - x_1)\widehat{G}_{21} &= 0 \\ (-t\partial_t - u\partial_u - (x_1 + x_2)/2)\widehat{G}_{21} - \frac{1}{2}x'_1\widehat{F} &= 0 \end{aligned} \right\} \implies (x_1 - x_2)\widehat{G}_{21} = x'_1\widehat{F} \tag{30}$$

Proposition 2 *If either $x'_2 \neq 0$ and $\widehat{G}_{12} \neq 0$ or else $x'_1 \neq 0$ and $\widehat{G}_{21} \neq 0$, then both (i) $x := x_1 = x_2$ and (ii) $\widehat{F} = 0$ hold true.*

Obviously, at least one of \widehat{G}_{12} or \widehat{G}_{21} must be non-zero in order to have a non-trivial answer. More information is obtained from the last two-point function \widehat{H} , for which covariance under the generators $X_{0,1}$ implies, using also that $x_1 = x_2$

$$\left. \begin{aligned} (-t\partial_t - u\partial_u - x_1)\widehat{H} - x'_1\widehat{G}_{12} &= 0 \\ (-t\partial_t - u\partial_u - (x_1 + x_2)/2)\widehat{H} - \frac{1}{2}x'_1\widehat{G}_{12} - \frac{1}{2}x'_2\widehat{G}_{21} &= 0 \end{aligned} \right\} \implies x'_1\widehat{G}_{12} = x'_2\widehat{G}_{21} \tag{31}$$

Consequently, one must distinguish two essentially distinct cases:

$x'_1 = x'_2 = 1$. We shall refer to this situation as the **symmetric case**. The scaling operators $\begin{pmatrix} \widehat{\phi}_1 \\ \widehat{\psi}_1 \end{pmatrix}$ and $\begin{pmatrix} \widehat{\phi}_2 \\ \widehat{\psi}_2 \end{pmatrix}$ are identical. Since under the exchange of the two operators, one has $t \mapsto -t$ and $u \mapsto u$, it follows that $\widehat{G}_{12} = \widehat{G}(t, u)$ and $\widehat{G}_{21} = \widehat{G}(-t, u)$. Because of (31), the function $\widehat{G}(t, u) = \widehat{G}(-t, u)$ is symmetric. Solving the differential Eq. (29), we have

$$\widehat{G}(t, u) = |t|^{-x} \widehat{g}(u|t|^{-1}) \tag{32}$$

where \widehat{g} is a differentiable scaling function. Inserting into (31) and integrating

$$\widehat{H}(t, u) = |t|^{-x} \left(\widehat{h}(u|t|^{-1}) - \ln |t| \widehat{g}(u|t|^{-1}) \right) \tag{33}$$

Finally, we return to the formulation with fixed masses $\mathcal{M}_{1,2}$, which gives

Proposition 3 *The co-variant two-point functions of the logarithmic representation (17, 22) of $\text{sch}(1)$ are, with $x := x_1 = x_2$*

$$\begin{aligned}
 F(t, r) &= 0 \\
 G(t, r) &= \delta(\mathcal{M}_1 + \mathcal{M}_2) |t|^{-x} \exp \left[-\frac{\mathcal{M}_1 r^2}{2 t} \right] g_0(\text{sign}(t), \mathcal{M}_1) \quad (34) \\
 H(t, r) &= \delta(\mathcal{M}_1 + \mathcal{M}_2) |t|^{-x} \exp \left[-\frac{\mathcal{M}_1 r^2}{2 t} \right] \\
 &\quad \times (h_0(\text{sign}(t), \mathcal{M}_1) - \ln |t| g_0(\text{sign}(t), \mathcal{M}_1))
 \end{aligned}$$

where g_0 and h_0 are unspecified functions and $\delta(\mathcal{M})$ is the Dirac distribution.

Comparing with the prediction (14), we can identify $G_0 = g_0$ and $H_0 = h_0$. Notice: logarithmic Schrödinger-invariance did *not* produce the causality constraint $t > 0$!

Proof We illustrate the proof of (34) for $G(t, r)$. Using $\zeta = \zeta_1 - \zeta_2$, $\eta = \zeta_1 + \zeta_2$, we have

$$\begin{aligned}
 G(t, r) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} d\zeta_1 d\zeta_2 e^{-i\mathcal{M}_1\zeta_1 - i\mathcal{M}_2\zeta_2} |t|^{-x} \widehat{g} \left(\frac{2(\zeta_1 - \zeta_2)t + ir^2}{|t|} \right) \\
 &= \frac{1}{4\pi} |t|^{-x} \int_{\mathbb{R}} d\eta e^{-i(\mathcal{M}_1 + \mathcal{M}_2)\eta/2} \int_{\mathbb{R}} d\zeta e^{-i(\mathcal{M}_1 - \mathcal{M}_2)\zeta/2} \widehat{g} \left(2\text{sign}(t) \left(\zeta + \frac{i}{2} \frac{r^2}{\text{sign}(t) |t|} \right) \right) \\
 &= \delta(\mathcal{M}_1 + \mathcal{M}_2) |t|^{-x} \int_{\mathbb{R}} d\zeta e^{-i\mathcal{M}_1\zeta} \widehat{g} \left(2\text{sign}(t) \left(\zeta + \frac{i}{2} \frac{r^2}{t} \right) \right) \\
 &= \delta(\mathcal{M}_1 + \mathcal{M}_2) |t|^{-x} \exp \left[-\frac{\mathcal{M}_1 r^2}{2 t} \right] \underbrace{\int_{\mathbb{R}} d\zeta e^{-i\mathcal{M}_1\zeta} \widehat{g}(2\text{sign}(t)\zeta)}_{= g_0(\text{sign}(t), \mathcal{M}_1)}
 \end{aligned}$$

with a change of variables in the last line and we have also assumed that \widehat{g} has no singularity ‘near to’ the real axis which could prevent shifting the contour. H is derived similarly. □

$\boxed{\begin{matrix} x'_1 = 0 \\ x'_2 = 1 \end{matrix}}$. This is called the **asymmetric case**. The mirror situation $x'_1 = 1, x'_2 = 0$ is analogous. Now, from (31) we have $G_{21} = 0$. Inserting into and solving (29, 31), we have

$$\widehat{G}_{12}(t, u) = t^{-x} \widehat{g}(ut^{-1}) \quad , \quad \widehat{H}(t, u) = t^{-x} \widehat{h}(ut^{-1}) \quad (35)$$

without any logarithmic term ! Again, no causality condition is produced.

5 Causality in Maximal Parabolic Sub-Algebras

In the previous section we had seen that $\mathfrak{sch}(1)$ -covariance alone is not strong enough to derive the causality condition $t > 0$ for the two-point function. We now show that indeed *causality is implied if covariance under the maximal parabolic*

sub-algebra $\widetilde{\mathfrak{sch}}(1)$ is required. In what follows, it will be essential that $M_0 = i\partial_\zeta$ generates translations in the dual coordinate. In consequence, the M_0 -covariant two-point functions merely depend on $\zeta = \zeta_1 - \zeta_2$.

We begin by extending N to a logarithmic representation by replacing the second scaling dimension ξ by a matrix $\mathcal{E} = \begin{pmatrix} \xi & \xi' \\ \xi'' & \xi \end{pmatrix}$ and write

$$N = \zeta \partial_\zeta - t \partial_t + \begin{pmatrix} \xi & \xi' \\ \xi'' & \xi \end{pmatrix}. \tag{36}$$

Proposition 4 *One can always arrange in (36) for $\xi'' = 0$.*

Proof Since both X_0 and N are in the Cartan sub-algebra of B_2 , see Fig. 1a, we must have $[X_0, N] = \frac{1}{2}x'\xi'' \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0$, hence $x'\xi'' = 0$. If $x' = 0$, one asks whether \mathcal{E} can be diagonalised. If that is so, one has the non-interesting case of a pair of non-logarithmic quasi-primary operators. If \mathcal{E} cannot be diagonalised, it can be brought to a Jordan form and one can always arrange for $\xi'' = 0$. Therefore, we can set $\xi'' = 0$ in (36) without restriction of the generality. One can check the commutators of $\widetilde{\mathfrak{sch}}(1)$, notably $[X_1, N] = X_1$. \square

Using the results of Sect. 4, co-variance under N yields

$$N\widehat{G}_{12}(t, u) = (-t\partial_t + \xi_1 + \xi_2)\widehat{G}_{12}(t, u) = 0 \tag{37}$$

Solving this first for $t > 0$, this implies $\widehat{G}_{12}(t, u) = t^{\xi_1+\xi_2}\widehat{\mathcal{V}}(u)$. Comparison with the scaling form (32) leads to $\widehat{G}_{12} = \widehat{g}_0 t^{\xi_1+\xi_2} u^{-x-\xi_1-\xi_2}$. Together with the results of Sect. 4, and setting $v = u/t$, we have the scaling function

$$\widehat{g}(v) = \widehat{g}_0 v^{-x-\xi_1-\xi_2} \tag{38}$$

where \widehat{g}_0 is a normalisation constant. The last two-point function \widehat{H} can be found from

$$N\widehat{H}(t, u) = (-t\partial_t + \xi_1 + \xi_2)\widehat{H}(t, u) + \xi'_1\widehat{G}_{12}(t, u) + \xi'_2\widehat{G}_{21}(t, u) = 0. \tag{39}$$

We now look at the two cases defined in Sect. 4.

5.1 Symmetric Case

A straightforward calculation gives, using (32, 33, 38, 39)

$$\widehat{G}(\zeta, t, r) = \widehat{g}_0 |t|^{-x} \left(\frac{2\zeta t + ir^2}{|t|} \right)^{-x-\xi_1-\xi_2}$$

$$\widehat{H}(\zeta, t, r) = |t|^{-x} \left(\frac{2\zeta t + ir^2}{|t|} \right)^{-x-\xi_1-\xi_2} \times \left(\widehat{h}_0 + \widehat{g}_0(1 + \xi'_1 + \xi'_2) \ln \left(\frac{2\zeta t + ir^2}{|t|} \right) - \widehat{g}_0 \ln |t| \right) \tag{40}$$

where \widehat{g}_0 and \widehat{h}_0 are normalisation constants. We can now state the main result.

Theorem *Quasi-primary scaling operators $\begin{pmatrix} \phi_i \\ \psi_i \end{pmatrix}$, which are scalars under spatial rotations and transform co-variantly under a logarithmic representation of the parabolic sub-algebra $\mathfrak{sch}(d)$, are characterised by the simultaneous Jordan matrices $\begin{pmatrix} x_i & x'_i \\ 0 & x_i \end{pmatrix}$ and $\begin{pmatrix} \xi_i & \xi'_i \\ 0 & \xi_i \end{pmatrix}$ and the masses \mathcal{M}_i . Assume that $\mathcal{M}_1 > 0$ and furthermore that $\frac{1}{2}(x_1 + x_2) + \xi_1 + \xi_2 > 0$. If $x'_1 = x'_2 = 1$, the co-variant two-point functions (13) have the following causal forms*

$$\begin{aligned} F(t, r) &= 0 \\ G(t, r) &= \delta(\mathcal{M}_1 + \mathcal{M}_2) \delta_{x_1, x_2} \Theta(t) t^{-x_1} \exp \left[-\frac{\mathcal{M}_1 r^2}{2 t} \right] G_0 \\ H(t, r) &= \delta(\mathcal{M}_1 + \mathcal{M}_2) \delta_{x_1, x_2} \Theta(t) t^{-x_1} \exp \left[-\frac{\mathcal{M}_1 r^2}{2 t} \right] (H_0 - G_0 \ln t) \end{aligned} \tag{41}$$

where G_0 and H_0 are normalisation constants, $\Theta(t)$ is the Heaviside function and $\delta_{a,b} = 1$ if $a = b$ and zero otherwise.

Here, we are mainly interested in the causality statement which is essentially contained in the following

Proposition 5 *Let $x > 0$, n be a non-negative integer and consider the integrals, in the limit $\varepsilon \rightarrow 0+$*

$$I_{\pm}^{(n)}(x) := \int_{\mathbb{R} \pm i\varepsilon} d\zeta e^{-i\zeta} \zeta^{-x} \ln^n \zeta \tag{42}$$

Then $I_{-}^{(n)}(x) = 0$. There is no simple known expression for $I_{+}^{(n)}(x)$.

Proof To prove the proposition, consider the contour integrals

$$J_{\pm} := \oint_{C_{\pm}} d\zeta e^{-i\zeta} \zeta^{-x} \ln^n \zeta$$

where the contours C_{\pm} correspond to $t > 0$ and $t < 0$, respectively, as we shall see below and are indicated in Fig. 2. For $x > 0$, the only singularity is the cut along the negative real axis, hence $J_{\pm} = 0$. We now estimate the contribution of the lower half-circle, $J_{-, \text{inf}}$. Setting $\zeta = Re^{i\theta}$ such that $\ln R > 1$, one has

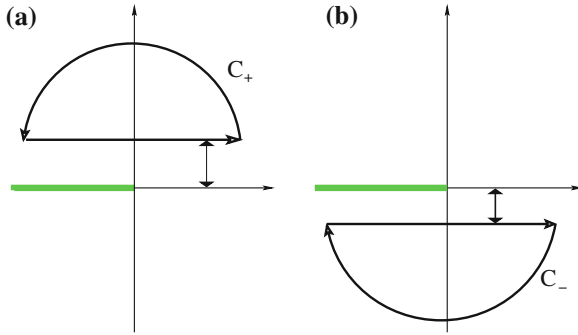


Fig. 2 Integration contours **a** C_+ for $t > 0$ and **b** C_- for $t < 0$. The cut is indicated by the *thick line*

$$J_{-, \text{inf}} = \frac{1}{i} \int_0^\pi d\theta R^{1-x} e^{i\theta(x-1) - iR \cos \theta} e^{-R \sin \theta} (\ln R e^{-i\theta})^n$$

Computing the complex logarithm via the binomial theorem, one has the estimate

$$\begin{aligned} |J_{-, \text{inf}}| &\leq \int_0^\pi d\theta R^{1-x} e^{-R \sin \theta} \sum_{k=0}^n \binom{n}{k} \ln^{n-k} R \theta^k \\ &\leq \sum_{k=0}^n \binom{n}{k} R^{1-x} (\ln R)^{n-k} \underbrace{\pi^k \int_0^\pi d\theta e^{-R \sin \theta}}_{\leq \pi R^{-1}} \\ &\leq \pi R^{-x} (\pi + \ln R)^n \rightarrow 0 \end{aligned}$$

as $R \rightarrow \infty$. Hence, since $J_- = I_-^{(n)}(x) + J_{-, \text{inf}} = 0$, the assertion follows. \square

Proof (of the Theorem) In order to prove the theorem, recall first that for quasi-primary operators which are scalars under rotations, one can always reduce to the case $d = 1$. Hence the spatial dependence in (41) is a direct consequence of (34). Writing $\xi := \xi_1 + \xi_2$, we use the physical convention of positive masses $\mathcal{M}_1 > 0$ and have along the lines of the proof of proposition 3

$$\begin{aligned} G &= \delta(\mathcal{M}_1 + \mathcal{M}_2) |t|^{-x} \widehat{g}_0 \int_{\mathbb{R}} d\zeta e^{-i\mathcal{M}_1 \zeta} (2\text{sign}(t))^{-x-\xi} \left(\zeta + \frac{ir^2}{2\text{sign}(t)|t|} \right)^{-x-\xi} \\ &= \delta(\mathcal{M}_1 + \mathcal{M}_2) (2\text{sign}(t))^{-x-\xi} \mathcal{M}_1^{x+\xi-1} |t|^{-x} \widehat{g}_0 \underbrace{\int_{\mathbb{R} + \frac{i\mathcal{M}_1 r^2}{2t}} d\zeta e^{-i\zeta} \zeta^{-x-\xi}}_{I_{\pm}^{(0)}(x+\xi)} e^{-\frac{\mathcal{M}_1 r^2}{2t}} \\ &= \delta(\mathcal{M}_1 + \mathcal{M}_2) |t|^{-x} \underbrace{2^{-x-\xi} \mathcal{M}_1^{x+\xi-1} \widehat{g}_0}_{=: G_0} I_+^{(0)}(x+\xi) e^{-\frac{\mathcal{M}_1 r^2}{2t}} \Theta(t) \end{aligned}$$

where in the second line we see that for $t > 0$ ($t < 0$) the contour is slightly above (below) the real axis and we need $I_+^{(0)}$ ($I_-^{(0)}$). In the last line, we used the statement $I_-^{(0)}(x + \xi) = 0$ of proposition 5 and expressed this by the Heaviside function. Similarly, for H we use (41) and obtain along the same lines

$$H = \delta(\mathcal{M}_1 + \mathcal{M}_2) |t|^{-x} e^{-\frac{\mathcal{M}_1}{2} \frac{t^2}{r}} 2^{-x-\xi} \mathcal{M}_1^{x+\xi-1} \left[-\widehat{g}_0 \ln |t| I_{\pm}^{(0)}(x + \xi) + (\widehat{h}_0 + \widehat{g}_0(1 + \xi'_1 + \xi'_2) \ln(2 \text{sign}(t)/\mathcal{M}_1)) I_{\pm}^{(0)}(x + \xi) + \widehat{g}_0(1 + \xi'_1 + \xi'_2) I_{\pm}^{(1)}(x + \xi) \right]$$

and by proposition 5 and defining H_0 from the constants in the second line, the announced causal form follows. □

Remarks and Generalisations:

(a) Equation (41) reproduces the known form (14) [32] of logarithmic Schrödinger-covariance, but adds the causality condition $t > 0$ described by the extra factor $\Theta(t)$. Our derivation generalises earlier causality proofs for the non-logarithmic case and under the more strong condition $x > 0$ [26].

The extension to $d > 1$ dimensions is immediate.

(b) For physical applications, recall the form (12) of the response function $R = \langle \phi \widetilde{\phi} \rangle$ with a positive mass $\mathcal{M}_{\widetilde{\phi}} > 0$ and a negative mass $\mathcal{M}_{\phi} = -\mathcal{M}_{\widetilde{\phi}} < 0$ such that the ‘mass conservation’ following from galilean invariance is accounted for. The response field $\widetilde{\phi}$ is associated with the complex conjugate ϕ^* in (16).

(c) Since the generator of time-translations $X_{-1} \in \mathfrak{sch}(1)$, the proven scaling forms (41) correspond to $a = a'$ in (12). However, the specific form (18, 36) of the generator N is already compatible with the more general representations (or equivalently the Ward identities) required for the maximal parabolic extension of the ageing algebra, $\widetilde{\mathfrak{age}}(1)$ [27, 30]. Hence the causality arguments presented here explicitly for Schrödinger-invariance can be directly generalised to ageing-invariance, including the logarithmic extension. Hence our present results also provide a mathematical justification for the successful empirical comparison of numerical data of response functions from critical directed percolation [30] and the 1D KPZ Eq. [31] with the co-variant two-point function of logarithmic ageing-invariance.

(d) Galilei-covariance is an essential assumption. While it seems to be well confirmed in many numerical tests of specific models, see [29] and references therein, it is very difficult to prove formally. Finding such an argument remains an important open problem.¹³

(e) The second essential ingredient is the dualisation with respect to the mass \mathcal{M} , and that co-variance under the corresponding generator $M_0 = i\partial_{\zeta}$ takes the form of translation-invariance in the dual coordinate ζ . For illustration of this ingredient, consider the (non-logarithmic) representation $\widetilde{\text{CGA}}(1) = \langle X_1, Y_{\pm 1/2}, D, M_0, V_+, N \rangle$ from Sect. 3, with the dilatations $D = 2X_0 - N = -\zeta \partial_{\zeta} - t \partial_t - r \partial_r - (x + \xi)$.

¹³ At present, the nearest one might come to a formal proof is to consider the models in the dualised form as introduced in Sect. 4. Then, one trades the phase changes of the usual solution of the ‘Schrödinger equation’ $\mathcal{S}\phi = 0$ for a transformation in the dual coordinate ζ . This seriously modifies the equation under study, but galilean co-variance can be checked [60].

Hence, the only effective scaling dimension appearing is $x + \xi$, hence the dual CGA(1)-covariant two-point function can be read from the literature [26, 28]

$$\begin{aligned} \langle \widehat{\phi}_1(\xi_1, t_1, r_1) \widehat{\phi}_2(\xi_2, t_2, r_2) \rangle &= (t_1 - t_2)^{-\frac{1}{2}(x_1 + \xi_1 + x_2 + \xi_2)} \left(\frac{t_1}{t_2} \right)^{\frac{1}{2}(x_2 + \xi_2 - x_1 - \xi_1)} \\ &\times f \left(\xi_1 - \xi_2 + \frac{i}{2} \frac{(r_1 - r_2)^2}{t_1 - t_2} \right) \end{aligned} \tag{43}$$

Requiring the co-variance $N \langle \widehat{\phi}_1 \widehat{\phi}_2 \rangle = 0$, with N given by (18), leads as before to $f(u) = \widehat{f}_0 u^{-(x_1 + 3\xi_1 + x_2 + 3\xi_2)/2}$ and transforming back, we recover the causality condition $t_1 - t_2 > 0$, provided only that $x_1 + 3\xi_1 + x_2 + 3\xi_2 > 0$.

(f) M_0 plays the rôle of a central extension in the *Schrödinger* algebra. Such a central extension does not exist for CGA(d) with $d \neq 2$, but we expect that an argument similar to the one used here should apply to the exotic central generator Θ in the ECGA, after dualisation. This should allow, after the identification of the corresponding parabolic sub-algebra, to derive causality conditions in this case as well. We hope to return to this question in the future.

5.2 Asymmetric Case

Applying the conditions (37, 39) to the previously derived scaling forms (35), we promptly have

$$\widehat{g}(v) = \widehat{g}_0 v^{-x - \xi_1 - \xi_2}, \quad \widehat{h}(v) = v^{-x - \xi_1 - \xi_2} (\widehat{h}_0 - \xi_1' \widehat{g}_0 \ln v) \tag{44}$$

Transforming back as before to the situation with fixed masses, we obtain under the same conditions as for the main theorem, but now with $x'_1 = 0$ and $x'_2 = 1$, that $F(t, r) = G_{21}(t, r) = 0$ and the causal, but non-logarithmic forms

$$\begin{aligned} G_{12}(t, r) &= G_0 \delta(\mathcal{M}_1 + \mathcal{M}_2) \delta_{x_1, x_2} \Theta(t) t^{-x_1} \exp \left[-\frac{\mathcal{M}_1}{2} \frac{r^2}{t} \right] \\ H(t, r) &= H_0 \delta(\mathcal{M}_1 + \mathcal{M}_2) \delta_{x_1, x_2} \Theta(t) t^{-x_1} \exp \left[-\frac{\mathcal{M}_1}{2} \frac{r^2}{t} \right] \end{aligned} \tag{45}$$

Note added in proof: For the representation (5) of the non-exotic CGA, an analogous dualisation and parabolic extension rather shows that $\langle \phi_1(t) \phi_2(s) \rangle = \langle \phi_1(s) \phi_2(t) \rangle$ is fully symmetric [63].

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