Commutants and Centers in a 6-Parameter Family of Quadratically Linked Quantum Plane Algebras

Fredrik Ekström and Sergei D. Silvestrov

Abstract We consider a family of associative algebras, defined as the quotient of a free algebra with the ideal generated by a set of multi-parameter deformed commutation relations between four generators consisting of five quantum plane relations between pairs of generators and one sub-quadratic relation inter-linking all four generators. For generic parameter vectors, the center and the commutants of the two of the generators are described and conditions on the parameters for these commutants to be itself commutative or non-commutative are obtained.

1 Introduction

Commuting elements in non-commutative algebra are important for representation theory, classifications, interplay with harmonic analysis and spectral theory, topology and algebraic geometry, operator algebras and applications in Physics and Engineering. For example, commutants or centralisers, maximal commutative subalgebras of crossed product C^*-algebras and von Neumann algebras play a central role in investigation of representations, classifications and in structure of state space [1–6]. In particular, maximal commutative subalgebras are essential objects for the famous Kadison-Singer conjecture stated in a pioneering 1959—paper by Kadison and Singer [7], equivalent to the paving conjecture [8, 9] and several conjectures important for wavelets and frames analysis and applications in signal and image processing, one of them the well-known Feichtinger conjecture [10] in frame theory. For the

F. Ekström (✉)
Centre for Mathematical Sciences, Lund University, Box 118, 22100 Lund, Sweden
e-mail: Fredrik.Ekstrom@math.lth.se

S. D. Silvestrov
Division of Applied Mathematics, The School of Education, Culture and Communication, Mälardalen University, Box 883, 72123 Västerås, Sweden
e-mail: sergei.silvestrov@mdh.se

A. Makhlouf et al. (eds.), *Algebra, Geometry and Mathematical Physics*, 37
Springer Proceedings in Mathematics & Statistics 85, DOI: 10.1007/978-3-642-55361-5_3,

key role of maximal commutative subalgebras for establishing interplay between Kadison-Singer conjecture, properties of projections, topological dynamical systems and compactifications of topological spaces see for example [11]. Commutants and maximal commutative subalgebras in generalized crossed product algebras arising from non-invertible dynamics and actions are used in the important ways in the general operator and spectral theory approach to wavelets analysis and investigation of wavelets on fractals [12–16]. The description of commuting elements and of corresponding commuting operators in the representing operator algebra, or in other words the problem of explicit description of commutative subalgebras is important in description and classifications of operator representations and applications of non-commutative algebras [17–27]. The commuting operators and commuting elements in rings and algebras also are important in study of integrable systems and non-linear equations. Further discussions in connection to this topic and numerous references can be found for instance in the book [28] devoted to commuting elements in the algebra defined by the q-deformed Heisenberg relations (see also [29, 30]).

The centers and commutants of elements or subsets in non-commutative algebras are fundamentally important subsets of an algebra or a ring in this context (see for example [31–39] and references therein). The center consists of elements commuting with all elements in the algebra, is the intersection of the commutants of all elements in the algebra and so is always a commutative subalgebra. The commutants of elements or subsets of elements in an associative algebra are subalgebras which contain the center of the whole algebra as its subalgebra, but may be commutative or may be not depending on the structure of the algebra and the subset for which the commutant is considered.

In this article we consider the centers and commutants for an interesting multi-parameter family of associative algebras generated by four generators and six sub-quadratic relations involving six deformation parameters. The five of these relations are the famous quantum plane relations playing important role in quantum groups, q-calculus and quantum mechanics, operator algebras and non-commutative geometry (rotation algebras, non-commutative tori, etc.). The sixth relation is interconnecting the four generators by a special q-deformed quadratic relation expressing the sum of two generators as q-commutator of the other two of the generators:

$$\begin{cases} AB - q_0BA = S + T & (a) \\ AT - q_1TA = 0 & (b) \\ BS - q_2SB = 0 & (c) \\ AS - q_3SA = 0 & (d) \\ BT - q_4TB = 0 & (e) \\ ST - q_5TS = 0 & (f) \end{cases} \tag{1}$$

where $\mathbf{q} = (q_0, q_1, q_2, q_3, q_4, q_5) \in \mathbb{C}^6$.

All of (1b)–(1f) are of the type $XY - qYX = 0$, which is the so called quantum plane relation studied in non-commutative geometry. Equation (1a) resembles the Sylvester equation $AX - XB = C$ and the Lyapunov equation $AX + XA^* =$

$-Q$, both of which are encountered in control theory. Specializing $S = \lambda I$ and $T = (1 - \lambda)I$ (where I denotes the multiplicative identity) and $q_1, \ldots, q_5 = 1$, the relations (1b)–(1f) become trivial and (1a) becomes $AB - q_0BA = I$. This is a generalization of the Heisenberg canonical commutation ($q_0 = 1$) and anti-commutation ($q_0 = -1$) relations, which are used in quantum mechanics to describe systems with one degree of freedom. More on the algebras defined by q-deformed Heisenberg relations (called also q-Weil relations) and commuting elements in such algebras can be found in the monograph [28] and references there.

It is a well known interesting issue whether it is possible to realize a given family of commutation relations in one or another way using matrices or differential operators or other types of linear operators, or any objects for which (1) makes sense, for example elements of some associative algebra. When the realization by the operators of a specific type is possible, further description and classifications of the representations of the relations by the operators of such type arise and often becomes a problem of great interest. It often requires insights both in the algebraic structure of the commutation relations and in the properties of the involved classes of operators. In algebraic contexts it often leads to interesting combinatorial identities and problems, while in the context of $*$-representations (involutive representations) and operator algebras it involves also spectral theory of possibly unbounded operators in the finite-dimensional or infinite-dimensional spaces.

The relations (1) provide an interesting example in this respect. For a first taste of what can happen in (1a)–(1c) when A, B, S, T are complex $(n \times n)$-matrices, consider the case when A and B are hermitian and q_0 lies on the unit circle. Note that $\|X\|_F^2 = \mathrm{tr}(X^*X)$ defines a norm $\|X\|_F$ on $\mathbb{C}^{n\times n}$ (this is the so called Frobenius norm). Since A, B are hermitian and $q_0q_0^* = 1$, $(AB - q_0BA)^* = -q_0^*(AB - q_0BA)$, and thus

$$\begin{aligned}\|AB - q_0BA\|_F^2 &= -q_0^*\,\mathrm{tr}((AB - q_0BA)^2) = -q_0^*\,\mathrm{tr}((AB - q_0BA)(S + T))\\ &= -q_0^*\,(\mathrm{tr}(ABS) + \mathrm{tr}(ABT) - q_0\,\mathrm{tr}(BAS) - q_0\,\mathrm{tr}(BAT))\,.\end{aligned}$$

Using (1b,c) and the fact that $\mathrm{tr}(XYZ) = \mathrm{tr}(ZXY)$ for all $(n \times n)$-matrices X, Y, Z, this implies that

$$\|AB - q_0BA\|_F^2 = -q_0^*((q_2 - q_0)\,\mathrm{tr}(ASB) + (1 - q_0q_1)\,\mathrm{tr}(BTA)).$$

Thus, if $q_2 = q_0$ and $q_0q_1 = 1$, A, B must satisfy $AB - q_0BA = 0$, and S, T must satisfy $S = -T$. In particular, if $q_0 = q_1 = q_2 = 1$, A and B must commute. It is not difficult to see that for many other conditions on the parameters this argument breaks down. This could be interpreted as an indication that the algebraic structures defined by these relations and their representations might have rich dependence on the interplay between the values of the six deformation parameters.

In this article, we provide further indication of richness of the structure of this family of algebras depending on the values of the deformation parameters, by considering some important properties of the algebra with generators and relations (1), especially focussing on centers and commutants. As these algebras are defined as the quotient algebra $\mathscr{F}/\mathscr{J}(\mathbf{q})$ of a free algebra on six generators by the ideal associated to the commutation relations (1), in order to be able to compute in this algebra it is particularly important to be able to decide the equality of the elements in the algebra, since using the relations (1), the same element can be expressed in many ways, and it is not obvious whether or not two given expressions are equal. To handle this, one needs a normal form for elements in $\mathscr{F}/\mathscr{J}(\mathbf{q})$, which for the relations of the type (1) amounts to finding a basis for $\mathscr{F}/\mathscr{J}(\mathbf{q})$ for various choices of parameters. In Sect. 2, we indicate that relations are more subtle than it may seem on the first sight as there are more relations than generators, and for many values of parameters these relations imply some further much more special relations between generators bringing significant restrictions on the size or the structure of the bases and thus on various further properties and computations in the algebra. Finding in a systematic way bases for the algebras for various choices of parameters becomes an elaborate task requiring non-trivial use of the Bergman's diamond lemma and relations (1) as well as some symmetries of the relations and their consequences for case reductions of various subtle parameter subsets. It appears in the course of this analysis, that the basis takes a somewhat simpler form for a large subset of parameters given by a system of certain inequalities. This set of "generic" parameters, as we call it, and the bases yield useful grading structures, used in Sect. 3 to describe the commutants of the main generators A and B by describing the spanning sets. In Sect. 4, the results from the preceding sections combined with further computations are used to describe explicitly the center by providing its basis depending on the deformation parameters. While the center of an algebra is always a commutative subalgebra, as an intersection of commutants of all elements of the algebra, the commutants of elements or non-trivial subsets of a non-commutative algebra are subalgebras which are not necessarily themselves commutative. For some classes of algebras it is possible to prove that commutants of the elements are commutative. Investigating whether this is a case and finding examples and counterexamples for such commutativity property for a particular family of algebras defined by generators and specific relations is an important problem which is often highly non-trivial, especially so when the defining relations are dependent on parameters. In Sect. 5, we provide necessary and sufficient conditions on $\mathbf{q}$ within the set of "generic" parameters, for $\mathscr{C}(A)$ and $\mathscr{C}(B)$ to be commutative, thus also providing necessary and sufficient conditions on "generic" parameters for when these commutants are not-commutative. The results of these paper suggest that the description and further in-depth analysis of the structure of the commutants of these and other elements and subsets for the family of algebras considered in this paper both for "generic" as well as for non-generic parameters is an interesting and rich open problem.

2 First Steps: Reordering and Basis

Let $\mathscr{F}$ be the free unital associative algebra over $\mathbb{C}$ generated by the set $\{A, B, S, T\}$. For $\mathbf{q} = (q_0, q_1, q_2, q_3, q_4, q_5) \in \mathbb{C}^6$ let $\mathscr{J}(\mathbf{q})$ be the ideal generated by the set

$$G(\mathbf{q}) = \{AB - q_0BA - S - T,\ AT - q_1TA,\ BS - q_2SB,\\ AS - q_3SA,\ BT - q_4TB,\ ST - q_5TS\}, \tag{2}$$

or equivalently the ideal in $\mathscr{F}$ generated by the relations

$$\begin{aligned} AB - q_0BA &\equiv S + T \\ AT - q_1TA &\equiv 0 \\ BS - q_2SB &\equiv 0 \\ AS - q_3SA &\equiv 0 \\ BT - q_4TB &\equiv 0 \\ ST - q_5TS &\equiv 0, \end{aligned} \tag{3}$$

where $\equiv$ denotes equivalence modulo $\mathscr{J}(\mathbf{q})$.

From (3) it is not too hard to derive the additional relations

$$(1 - q_2q_3)S^2 \equiv (q_2q_3q_5 - 1)TS \tag{4}$$

$$(1 - q_1q_4)T^2 \equiv (q_1q_4 - q_5)TS. \tag{5}$$

The implications of these relations depend on which of the involved scalar expressions are zero and which are non-zero. There are also a few more expressions in the parameters that change the situation if they are zero. Only the generic case will be considered here.

Definition 2.1 A parameter vector $\mathbf{q} = (q_0, q_1, q_2, q_3, q_4, q_5) \in \mathbb{C}^6$ is *generic* if

$$\begin{cases} q_0, q_1, q_2, q_3, q_4, q_5 \neq 0, \\ 1 - q_5, 1 - q_1q_4, 1 - q_2q_3, q_1 - q_3, q_2 - q_4 \neq 0, \text{ and} \\ q_1q_4 - q_5 \neq 0 \text{ or } q_2q_3q_5 - 1 \neq 0. \end{cases}$$

For generic $\mathbf{q}$ it follows from (3), (4) and (5) that

$$XY \equiv 0 \text{ for all } X \in \{S^2,\ ST,\ TS,\ T^2\}, Y \in \{A,\ B,\ S,\ T\}. \tag{6}$$

Since (3) can be used to put the symbols in the monomials in the order T, S, B, A, this means that any monomial that has two symbols from $\{S, T\}$ and additionally one symbol from $\{A, B, S, T\}$ is $\equiv 0$. Thus it seems plausible that the set

$$\mathscr{B} = \{B^bA^a,\ SB^bA^a,\ TB^bA^a,\ TS;\ a, b \in \mathbb{N}\}$$

is a basis for $\mathscr{F}/\mathscr{J}(\mathbf{q})$. This can be shown using the Diamond Lemma for ring theory [40].

Sums of the form $\sum_{i=0}^{n-1} q^i$ will often appear in what follows, so it is convenient to have a more compact notation for them.

Definition 2.2 For $q \in \mathbb{C}$ and $n \in \mathbb{N}$, let

$$\{n\}_q = \sum_{i=0}^{n-1} q^i.$$

$\{n\}_q$ is called the *n:th q-natural number*.

To express the product of two general elements in the basis $\mathscr{B}$, it is necessary to be able to rewrite monomials of the form $A^m B^n$ so that the B:s are moved to the left of the A:s.

Lemma 2.1 *Let* $\mathbf{q} = (q_0, \ldots, q_5) \in \mathbb{C}^6$ *be generic. Then the following formula holds in* $\mathscr{F}/\mathscr{J}(\mathbf{q})$ *for* $m, n \geq 1$, $(m, n) \neq (2, 2)$.

$$\begin{aligned} A^m B^n \equiv\ & q_0^{mn} B^n A^m \\ & + q_0^{(m-1)n} \{m\}_{\frac{q_3}{q_0}} \{n\}_{q_0 q_2} S B^{n-1} A^{m-1} \\ & + q_0^{(m-1)n} \{m\}_{\frac{q_1}{q_0}} \{n\}_{q_0 q_4} T B^{n-1} A^{m-1}. \end{aligned} \tag{7}$$

The formula can be proved for most (m, n) by induction first on m and then on n, or by induction first on n and then on m. The exceptional point $(m, n) = (2, 2)$ makes it necessary to use both orders of induction to cover all $(m, n) \neq (2, 2)$. We omit the elaborate details of the proof.

Equation (7) does not hold for $(m, n) = (2, 2)$. The reordering formula for $(m, n) = (2, 2)$ is instead

$$\begin{aligned} A^2 B^2 \equiv\ & q_0^4 B^2 A^2 + q_0(q_0 + q_3)(1 + q_0 q_2) S B A + q_0(q_0 + q_1)(1 + q_0 q_4) T B A \\ & + (1 - q_5) \frac{q_1 - q_3 + q_0 q_1 q_4 - q_0 q_2 q_3 + q_1 q_3 q_4 - q_1 q_2 q_3}{(1 - q_1 q_4)(1 - q_2 q_3)} T S. \end{aligned} \tag{8}$$

This formula agrees with (7) except for the extra TS-term on the right side.

Let $\mathscr{M}$ be the set of monomials in $\mathscr{F}$ and define for $Y \in \mathscr{M}$

$$\begin{aligned} \deg_{A,S,T}(Y) &= \#A : s + \#S : s + \#T : s \text{ that occur in } Y \\ \deg_{B,S,T}(Y) &= \#B : s + \#S : s + \#T : s \text{ that occur in } Y. \end{aligned}$$

Then $\mathscr{F}$ has an $\mathbb{N}^2$-gradation $\{A_{(m,n)}\}_{(m,n)\in\mathbb{N}^2}$ given by

$$A_{(m,n)} = \mathrm{Span}\{Y \in \mathscr{M};\ \deg_{A,S,T}(Y) = m,\ \deg_{B,S,T}(Y) = n\}.$$

All elements in the generating set of $\mathscr{J}(\mathbf{q})$ are homogeneous in this gradation and thus the induced gradation of $\mathscr{F}/\mathscr{J}(\mathbf{q})$ is well defined. If $m, n \geq 1$ and $(m, n) \neq (2, 2)$ then a basis for the homogeneous component of degree (m, n) is given by

$$\{B^n A^m,\ S B^{n-1} A^{m-1},\ T B^{n-1} A^{m-1}\}.$$

3 Commutants of *A* and *B*

The *commutant* of an element $X \in \mathscr{F}/\mathscr{J}(\mathbf{q})$ is the set of all elements that commute with X. It will be denoted by $\mathscr{C}(X)$. In this section, spanning sets for $\mathscr{C}(A)$ and $\mathscr{C}(B)$ are described for generic $\mathbf{q}$.

In general, a commutant of a homogeneous element is spanned by the homogeneous elements of the commutant. This means that it is enough to find all *homogeneous* elements that commute with A or B. Let $X_{m,n}$ denote a general homogeneous element of degree (m, n). For $m, n \geq 1$, $(m, n) \neq (2, 2)$, such an element can be uniquely written as

$$X_{m,n} \equiv c_1 B^n A^m + c_2 S B^{n-1} A^{m-1} + c_3 T B^{n-1} A^{m-1}, \tag{9}$$

with $c_1, c_2, c_3 \in \mathbb{C}$. Since TS commutes with every $Y \in \{A, B, S, T\}$ (in fact $YTS \equiv TSY \equiv 0$ by (6)), it is general enough to consider $X_{m,n}$ of the form (9) when $(m, n) = (2, 2)$ as well. When $m = 0$ or $n = 0$, the homogeneous elements of degree (m, n) are of the form $X_{0,n} = c_1 B^n$ and $X_{m,0} = c_1 A^m$ respectively.

Using the defining relations (3) and the reordering formula (7), the commutators of $X_{m,n}$ with A and B can be computed. For $m, n \geq 1$, $(m, n) \neq (1, 2)$, the commutator of $X_{m,n}$ with A is

$$\begin{aligned}[X_{m,n}, A] \equiv\ & c_1(1 - q_0^n) B^n A^{m+1} \\ & + \left(-c_1 \{n\}_{q_0 q_2} + c_2(1 - q_0^{n-1} q_3)\right) S B^{n-1} A^m \\ & + \left(-c_1 \{n\}_{q_0 q_4} + c_3(1 - q_0^{n-1} q_1)\right) T B^{n-1} A^m .\end{aligned}$$

If $(m, n) = (1, 2)$ then there is an additional term

$$\frac{(1 - q_5)}{(1 - q_2 q_3)(1 - q_1 q_4)} (c_2 q_3 (1 - q_1 q_4) - c_3 q_1 (1 - q_2 q_3)) T S$$

on the right side. For $m = 0$, the commutator is

$$[X_{0,n}, A] \equiv c_1(1 - q_0^n) B^n A - c_1 \{n\}_{q_0 q_2} S B^{n-1} - c_1 \{n\}_{q_0 q_4} T B^{n-1}$$

and $[X_{m,0}, A] \equiv 0$ for all m.

For $m, n \geq 1$, $(m, n) \neq (2, 1)$, the commutator of $X_{m,n}$ with B is

$$\begin{aligned}[X_{m,n}, B] \equiv\ & c_1(q_0^m - 1) B^{n+1} A^m \\ & + \left(c_1 q_0^{m-1} q_2^n \{m\}_{q_3/q_0} + c_2(q_0^{m-1} - q_2)\right) S B^n A^{m-1} \\ & + \left(c_1 q_0^{m-1} q_4^n \{m\}_{q_1/q_0} + c_3(q_0^{m-1} - q_4)\right) T B^n A^{m-1} .\end{aligned}$$

If $(m, n) = (2, 1)$ then there is an additional term

$$-\frac{(1-q_5)}{(1-q_2q_3)(1-q_1q_4)}(c_2(1-q_1q_4)-c_3(1-q_2q_3))TS$$

on the right side. For $n = 0$, the commutator is

$$[X_{m,0}, B] \equiv c_1(q_0^m - 1)BA^m + c_1q_0^{m-1}\{m\}_{\frac{q_3}{q_0}} SA^{m-1} + c_1q_0^{m-1}\{m\}_{\frac{q_1}{q_0}} TA^{m-1}.$$

and $[X_{0,n}, B] \equiv 0$ for all n.

The computations are summarised in the following lemma.

Lemma 3.1 *Let* $\mathbf{q} = (q_0, \dots, q_5) \in \mathbb{C}^6$ *be generic.*

$\mathscr{C}(A)$ is the linear subspace of $\mathscr{F}/\mathscr{J}(\mathbf{q})$ spanned by the elements listed in the following table.

Element (m, n range over $\mathbf{N}^+$)	Conditions
I	—
TS	—
A^m	—
B^n	$1 - q_0^n = \{n\}_{q_0q_2} = \{n\}_{q_0q_4} = 0$
$c_1B^nA^m + (c_2S + c_3T)B^{n-1}A^{m-1}$	$K_{(m,n)} \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}^T = 0$

Here,

$$K_{(m,n)} = \begin{bmatrix} 1-q_0^n & 0 & 0 \\ -\{n\}_{q_0q_2} & 1-q_0^{n-1}q_3 & 0 \\ -\{n\}_{q_0q_4} & 0 & 1-q_0^{n-1}q_1 \end{bmatrix}$$

for $(m, n) \neq (1, 2)$ *and*

$$K_{(1,2)} = \begin{bmatrix} 1-q_0^2 & 0 & 0 \\ -(1+q_0q_2) & 1-q_0q_3 & 0 \\ -(1+q_0q_4) & 0 & 1-q_0q_1 \\ 0 & q_3(1-q_1q_4) & -q_1(1-q_2q_3) \end{bmatrix}.$$

$\mathscr{C}(B)$ is the linear subspace of $\mathscr{F}/\mathscr{J}(\mathbf{q})$ spanned by the elements listed in the following table.

Element (m, n range over $\mathbf{N}^+$)	Conditions
I	—
TS	—
A^m	$q_0^m - 1 = \{m\}_{q_1/q_0} = \{m\}_{q_3/q_0} = 0$
B^n	—
$c_1B^nA^m + (c_2S + c_3T)B^{n-1}A^{m-1}$	$L_{(m,n)} \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}^T = 0$

Here,

$$L_{(m,n)} = \begin{bmatrix} q_0^m - 1 & 0 & 0 \\ q_0^{m-1} q_2^n \{m\}_{q_3/q_0} & q_0^{m-1} - q_2 & 0 \\ q_0^{m-1} q_4^n \{m\}_{q_1/q_0} & 0 & q_0^{m-1} - q_4 \end{bmatrix}$$

for $(m, n) \neq (2, 1)$ *and*

$$L_{(2,1)} = \begin{bmatrix} q_0^2 - 1 & 0 & 0 \\ q_2(q_0 + q_3) & q_0 - q_2 & 0 \\ q_4(q_0 + q_1) & 0 & q_0 - q_4 \\ 0 & 1 - q_1 q_4 & -(1 - q_2 q_3) \end{bmatrix}.$$

4 The Center of $\mathscr{F}/\mathscr{J}(\mathbf{q})$

The *center* of $\mathscr{F}/\mathscr{J}(\mathbf{q})$ is the set of elements that commute with every element of $\mathscr{F}/\mathscr{J}(\mathbf{q})$. It will be denoted by $\mathscr{Z}$. In this section, $\mathscr{Z}$ is described for generic $\mathbf{q}$.

Lemma 4.1 *Let* $\mathbf{q} = (q_0, \ldots, q_5) \in \mathbb{C}^6$ *be generic. Then* $\mathscr{Z} = \mathscr{C}(A) \cap \mathscr{C}(B)$.

Proof Let X be a general homogeneous element that commutes with both A and B. Then X commutes with $S + T$ by (3a). It will be shown that X commutes with S and T as well. There are four cases depending on the degree of X.

If X has degree $(0, n)$, then $X \equiv c_1 B^n$ for some $c_1 \in \mathbb{C}$. Then

$$[X, S + T] \equiv 0 \implies c_1(q_2^n - 1)SB^n + c_1(q_4^n - 1)TB^n \equiv 0 \implies$$
$$c_1(q_2^n - 1) = c_1(q_4^n - 1) = 0 \implies [X, S] \equiv [X, T] \equiv 0.$$

If X has degree $(m, 0)$, then $X \equiv c_1 A^m$ for some $c_1 \in \mathbb{C}$. Then

$$[X, S + T] \equiv 0 \implies c_1(q_3^m - 1)SA^m + c_1(q_1^m - 1)TA^m \equiv 0 \implies$$
$$c_1(q_3^m - 1) = c_1(q_1^m - 1) = 0 \implies [X, S] \equiv [X, T] \equiv 0.$$

If X has degree $(1, 1)$, then $X \equiv c_1 BA + c_2 S + c_3 T$ for some $c_1, c_2, c_3 \in \mathbb{C}$. Then

$$[X, S + T] \equiv c_1(q_2 q_3 - 1)SBA + c_1(q_1 q_4 - 1)TBA + (c_3 - c_2)(1 - q_5)TS.$$

The right side can be zero only if $c_1 = c_3 - c_2 = 0$ since $\mathbf{q}$ is generic. But then $X \equiv c_3(S + T)$, so

$$[X, BA] \equiv c_3(1 - q_2 q_3)SBA + c_3(1 - q_1 q_4)TBA.$$

Since X is assumed to commute with A and B, the right side must be zero, which implies that $c_3 = 0$ since $\mathbf{q}$ is generic. Thus $X \equiv 0$, and so X commutes with S and T.

Finally, if X has degree (m, n) with $m, n \neq 0$ and $(m, n) \neq (1, 1)$ then

$$X \equiv c_1 B^n A^m + c_2 S B^{n-1} A^{m-1} + c_3 T B^{n-1} A^{m-1} + c_4 T S,$$

for some $c_1, c_2, c_3, c_4 \in \mathbb{C}$ (with $c_4 = 0$ unless $(m, n) = (2, 2)$). Then

$$[X, S + T] \equiv 0 \implies c_1(q_2^n q_3^m - 1) S B^n A^m + c_1(q_1^m q_4^n - 1) T B^n A^m \equiv 0 \implies c_1(q_2^n q_3^m - 1) = c_1(q_1^m q_4^n - 1) = 0 \implies [X, S] \equiv [X, T] \equiv 0.$$

Theorem 4.1 *Let* $\mathbf{q} = (q_0, \ldots, q_5) \in \mathbb{C}^6$ *be generic and suppose that* q_0 *is not a root of unity. Then a basis for* $\mathscr{Z}$ *is given by the elements listed in the following table.*

Element (m, n range over $\mathbb{N}^+$)	Conditions
I	—
TS	—
$SB^{n-1}A^{m-1}$	$1 - q_0^{n-1} q_3 = q_0^{m-1} - q_2 = 0$
$TB^{n-1}A^{m-1}$	$1 - q_0^{n-1} q_1 = q_0^{m-1} - q_4 = 0$

Moreover, this basis contains at most one element of the form $SB^{n-1}A^{m-1}$ *and at most one element of the form* $TB^{n-1}A^{m-1}$. *Thus,* $\mathscr{Z}$ *has dimension at most four.*

Proof By Lemma 4.1, it is enough to show that the listed elements form a basis for $\mathscr{C}(A) \cap \mathscr{C}(B)$. Spanning sets for $\mathscr{C}(A)$ and $\mathscr{C}(B)$ are given by Lemma 3.1; denote them by $\mathscr{B}(A)$ and $\mathscr{B}(B)$ respectively. Then $\mathscr{C}(A) \cap \mathscr{C}(B)$ is the linear space spanned by $\mathscr{B}(A) \cap \mathscr{B}(A)$. Now, $I, TS \in \mathscr{B}(A) \cap \mathscr{B}(B)$ always. For $m, n \geq 1$, $A^m \notin \mathscr{B}(B)$ and $B^n \notin \mathscr{B}(A)$ since q_0 is not a root of unity, and thus $A^m, B^n \notin \mathscr{B}(A) \cap \mathscr{B}(B)$. An element in $\mathscr{B}(A) \cap \mathscr{B}(B)$ of the form $c_1 B^n A^m + c_2 S B^{n-1} A^{m-1} + c_3 T B^{n-1} A^{m-1}$ with $m, n \geq 1$ must have $c_1 = 0$ since q_0 is not a root of unity. The coefficient c_2 may be non-zero if and only if $1 - q_0^{n-1} q_3 = q_0^{m-1} - q_2 = 0$. Since q_0 is not a root of unity, this can happen for at most one value of (m, n). Similarly, c_3 may be non-zero if and only if $1 - q_0^{n-1} q_1 = q_0^{m-1} - q_4 = 0$, and this can happen for at most one value of (m, n). Thus the listed elements span $\mathscr{C}(A) \cap \mathscr{C}(B)$, and since they are linearly independent they form a basis.

Theorem 4.2 *Let* $\mathbf{q} = (q_0, \ldots, q_5) \in \mathbb{C}^6$ *be generic and suppose that* q_0 *is a root of unity. Let* d *be the smallest positive integer such that* $q_0^d = 1$. *Then a basis for* $\mathscr{Z}$ *is given by the elements listed in the following table.*

Element (m, n range over $\mathbb{N}^+$)	Conditions
I	—
TS	—
A^m	$d\|m, q_1^m = q_3^m = 1, q_1/q_0, q_3/q_0 \neq 1$
B^n	$d\|n, q_2^n = q_4^n = 1, q_0q_2, q_0q_4 \neq 1$
$SB^{n-1}A^{m-1}$	$d\|m-1-r_2, d\|n-1+r_3, q_2 = q_0^{r_2}, q_3 = q_0^{r_3}$
$TB^{n-1}A^{m-1}$	$d\|n-1+r_1, d\|m-1-r_4, q_1 = q_0^{r_1}, q_4 = q_0^{r_4}$
$B^nA^m + (c_SS + c_TT)B^{n-1}A^{m-1}$	$d\|m, d\|n, q_1^m q_4^n = q_2^n q_3^m = 1$
	and in addition
	$q_2 + 1/q_2 = q_4 + 1/q_4$ if $(m, n) = (1, 2)$
	$q_1 + 1/q_1 = q_3 + 1/q_3$ if $(m, n) = (2, 1)$

Here,

$$c_S = \begin{cases} \frac{1-q_2^n}{(1-q_0q_2)(1-q_3/q_0)} = \frac{1-1/q_3^m}{(1-q_0q_2)(1-q_3/q_0)} & \text{if } 1-q_0q_2, 1-q_3/q_0 \neq 0 \\ \frac{-m}{1-q_0q_2} & \text{if } 1-q_0q_2 \neq 0, 1-q_3/q_0 = 0 \\ \frac{n}{1-q_3/q_0} & \text{if } 1-q_0q_2 = 0, 1-q_3/q_0 \neq 0 \end{cases}$$

and

$$c_T = \begin{cases} \frac{1-q_4^n}{(1-q_0q_4)(1-q_1/q_0)} = \frac{1-1/q_1^m}{(1-q_0q_4)(1-q_1/q_0)} & \text{if } 1-q_0q_4, 1-q_1/q_0 \neq 0 \\ \frac{-m}{1-q_0q_4} & \text{if } 1-q_0q_4 \neq 0, 1-q_1/q_0 = 0 \\ \frac{n}{1-q_1/q_0} & \text{if } 1-q_0q_4 = 0, 1-q_1/q_0 \neq 0. \end{cases}$$

Proof By Lemma 4.1, it is enough to show that the listed elements form a basis for $\mathscr{C}(A) \cap \mathscr{C}(B)$. Spanning sets for $\mathscr{C}(A)$ and $\mathscr{C}(B)$ are given by Lemma 3.1; denote them by $\mathscr{B}(A)$ and $\mathscr{B}(B)$ respectively. Then $\mathscr{C}(A) \cap \mathscr{C}(B)$ is the linear space spanned by $\mathscr{B}(A) \cap \mathscr{B}(B)$. Now, $I, TS \in \mathscr{B}(A) \cap \mathscr{B}(B)$ always, and

$$\begin{aligned} A^m \in \mathscr{B}(A) \cap \mathscr{B}(B) &\iff q_0^m - 1 = \{m\}_{q_1/q_0} = \{m\}_{q_3/q_0} = 0 \\ &\iff q_0^m = q_1^m = q_3^m = 1, q_1/q_0, q_3/q_0 \neq 1 \end{aligned}$$

and

$$\begin{aligned} B^n \in \mathscr{B}(A) \cap \mathscr{B}(B) &\iff 1 - q_0^n = \{n\}_{q_0q_2} = \{n\}_{q_0q_4} = 0 \\ &\iff q_0^n = q_2^n = q_4^n = 1, q_0q_2, q_0q_4 \neq 1. \end{aligned}$$

Assume now that $m, n \geq 1$, $(m, n) \neq (1, 2)$ and $(m, n) \neq (2, 1)$. An element of the form $c_1B^nA^m + c_2SB^{n-1}A^{m-1} + c_3TB^{n-1}A^{m-1}$ lies in $\mathscr{B}(A) \cap \mathscr{B}(B)$ if and only if $K_{(m,n)}[c_1\, c_2\, c_3]^T = 0$ and $L_{(m,n)}[c_1\, c_2\, c_3]^T = 0$, where $K_{(m,n)}$ and $L_{(m,n)}$ are defined as in Lemma 3.1. Regrouping these equations gives the equivalent equation system

$$\begin{cases} (1-q_0^n)c_1 = (q_0^m - 1)c_1 = 0 & \text{(first rows of } K_{(m,n)} \text{ and } L_{(m,n)}) \\ M_S \begin{bmatrix} c_1 & c_2 \end{bmatrix}^T = 0 & \text{(second rows of } K_{(m,n)} \text{ and } L_{(m,n)}) \\ M_T \begin{bmatrix} c_1 & c_3 \end{bmatrix}^T = 0 & \text{(third rows of } K_{(m,n)} \text{ and } L_{(m,n)}), \end{cases} \tag{10}$$

where

$$M_S = \begin{bmatrix} -\{n\}_{q_0q_2} & 1-q_0^{n-1}q_3 \\ q_0^{m-1}q_2^n\{m\}_{q_3/q_0} & q_0^{m-1}-q_2 \end{bmatrix} \quad M_T = \begin{bmatrix} -\{n\}_{q_0q_4} & 1-q_0^{n-1}q_1 \\ q_0^{m-1}q_4^n\{m\}_{q_1/q_0} & q_0^{m-1}-q_4 \end{bmatrix}.$$

There are two types of possible solutions to (10): Those with $c_1 = 0$ and those with $c_1 \neq 0$. Note that $[0\, c_2\, c_3]^T$ satisfies (10) if and only if $[0\, c_2\, 0]^T$ and $[0\, 0\, c_3]^T$ do. Thus for the case $c_1 = 0$, it is enough to consider elements of the forms $SB^{n-1}A^{m-1}$ and $TB^{n-1}A^{m-1}$ separately, rather that a general linear combination $c_2SB^{n-1}A^{m-1} + c_3TB^{n-1}A^{m-1}$.

Now, $SB^{n-1}A^{m-1} \in \mathscr{B}(A) \cap \mathscr{B}(B)$ if and only if $M_S[0\, 1]^T = 0$, that is, iff

$$q_0^{n-1}q_3 = 1 \text{ and } q_0^{m-1} = q_2. \tag{11}$$

This implies (by raising both sides of the equations to the power of d) that $q_2^d = q_3^d = 1$, so q_2 and q_3 are d:th roots of unity and thus $q_2 = q_0^{r_2}$ and $q_3 = q_0^{r_3}$ for some $r_2, r_3 \in \{0, \ldots, d-1\}$ (since q_0 generates the group of d:th roots of unity). Then (11) holds if and only if $q_0^{n-1+r_3} = q_0^{m-1-r_2} = 1$, that is, d divides both $n-1+r_3$ and $m-1-r_2$.

Similarly, $TB^{n-1}A^{m-1} \in \mathscr{B}(A) \cap \mathscr{B}(B)$ if and only if $M_T[0\, 1]^T = 0$, that is, iff

$$q_0^{n-1}q_1 = 1 \text{ and } q_0^{m-1} = q_4. \tag{12}$$

This implies that $q_1^d = q_4^d = 1$ and thus that $q_1 = q_0^{r_1}$ and $q_4 = q_0^{r_4}$ for some $r_1, r_4 \in \{0, \ldots, d-1\}$. Then (12) holds if and only if $q_0^{n-1+r_1} = q_0^{m-1-r_4} = 1$, that is, d divides both $n-1+r_1$ and $m-1-r_4$.

If there is a solution of (10) with $c_1 \neq 0$ then $q_0^m = q_0^n = 1$ so $d|m$ and $d|n$. In addition, $\det(M_S) = \det(M_T) = 0$, which is equivalent (using $q_0^m = q_0^n = 1$ and $\{k\}_q(1-q) = 1-q^k$) to $q_2^nq_3^m = q_1^mq_4^n = 1$. When M_S is singular, either of the equations of the system $M_S[c_1\, c_2]^T = 0$ can be used to solve for c_2. One gets

$$c_2 = \begin{cases} \frac{1-q_2^n}{(1-q_0q_2)(1-q_3/q_0)}c_1 = \frac{1-1/q_3^m}{(1-q_0q_2)(1-q_3/q_0)}c_1 & \text{if } 1-q_0q_2,\, 1-q_3/q_0 \neq 0 \\ \frac{-m}{1-q_0q_2}c_1 & \text{if } 1-q_0q_2 \neq 0,\, 1-q_3/q_0 = 0 \\ \frac{n}{1-q_3/q_0}c_1 & \text{if } 1-q_0q_2 = 0,\, 1-q_3/q_0 \neq 0 \end{cases} \tag{13}$$

(the case $1-q_0q_2 = 1-q_3/q_0 = 0$ is excluded since $q_2q_3 \neq 1$ when $\mathbf{q}$ is generic). Similarly, when M_T is singular, one can use either of the equations of the system $M_T[c_1\, c_3]^T = 0$ to solve for c_3. One gets

$$c_3 = \begin{cases} \frac{1-q_4^n}{(1-q_0q_4)(1-q_1/q_0)}c_1 = \frac{1-1/q_1^m}{(1-q_0q_4)(1-q_1/q_0)}c_1 & \text{if } 1-q_0q_4, 1-q_1/q_0 \neq 0 \\ \frac{-m}{1-q_0q_4}c_1 & \text{if } 1-q_0q_4 \neq 0, 1-q_1/q_0 = 0 \\ \frac{n}{1-q_1/q_0}c_1 & \text{if } 1-q_0q_4 = 0, 1-q_1/q_0 \neq 0 \end{cases} \tag{14}$$

(the case $1-q_0q_4 = 1-q_1/q_0 = 0$ is excluded since $q_1q_4 \neq 1$ when $\mathbf{q}$ is generic).

When $(m,n) = (1,2)$ or $(m,n) = (2,1)$ then (10) is still a necessary condition for $c_1B^nA^m + c_2SB^{n-1}A^{m-1} + c_3TB^{n-1}A^{m-1}$ to lie in $\mathscr{C}(A,B)$, but c_2, c_3 must also satisfy

$$\begin{cases} c_2q_3(1-q_1q_4) = c_3q_1(1-q_2q_3) & \text{if } (m,n) = (1,2) \\ c_2(1-q_1q_4) = c_3(1-q_2q_3) & \text{if } (m,n) = (2,1). \end{cases} \tag{15}$$

If $c_1 = 0$ in any of these cases then either $c_2 = 0$ or $c_3 = 0$, since if $c_2, c_3 \neq 0$ then (10) would imply $q_1 = q_3\ (= 1/q_0^{n-1})$ and $q_2 = q_4\ (= q_0^{m-1})$. But if one of c_2, c_3 is $= 0$, then so is the other by (15). Thus, there are no non-trivial solutions with $c_0 = 0$ when $(m,n) = (1,2)$ or $(m,n) = (2,1)$. The element $B^nA^m + c_2SB^{n-1}A^{m-1} + c_3TB^{n-1}A^{m-1}$ lies in $\mathscr{C}(A,B)$ if and only if $[1\, c_2\, c_3]^T$ satisfies both (10) and (15). Using the expressions (13) and (14) for c_2 and c_3 together with $q_0 = q_1^mq_4^n = q_2^nq_3^m = 1$ (that is, using the conditions that have just been shown to be equivalent to (10); note that when $m = 1$ or $n = 1$, $d|m$ and $d|n$ iff $q_0 = 1$. Also note that $q_1^mq_4^n = q_2^nq_3^m = 1$ implies $q_2, q_4 \neq 1$ when $(m,n) = (1,2)$ and $q_1, q_3 \neq 1$ when $(m,n) = (2,1)$ since $\mathbf{q}$ is generic), (15) can be simplified to

$$\begin{cases} q_2 + 1/q_2 = q_4 + 1/q_4 & \text{if } (m,n) = (1,2) \\ q_1 + 1/q_1 = q_3 + 1/q_3 & \text{if } (m,n) = (2,1). \end{cases}$$

Thus, the elements listed in the theorem form a spanning set for $\mathscr{C}(A) \cap \mathscr{C}(B)$. To see that they are linearly independent, note that for a fixed (m,n) it is impossible that both $B^nA^m + c_2SB^{n-1}A^{m-1} + c_3TB^{n-1}A^{m-1}$ and $SB^{n-1}A^{m-1}$ lie in $\mathscr{C}(A) \cap \mathscr{C}(B)$, for that would imply

$$\left.\begin{array}{l} m \equiv n \pmod d \\ m-1-r_2 \equiv n-1+r_3 \pmod d \end{array}\right\} \implies -r_2 \equiv r_3 \pmod d \implies q_2q_3 = 1.$$

Similarly, it is impossible that both $B^nA^m + c_2SB^{n-1}A^{m-1} + c_3TB^{n-1}A^{m-1}$ and $TB^{n-1}A^{m-1}$ lie in $\mathscr{C}(A) \cap \mathscr{C}(B)$, for that would imply $q_1q_4 = 1$. Thus, the listed elements are linearly independent, and so they form a basis for $\mathscr{C}(A) \cap \mathscr{C}(B)$.

5 Commutativity of $\mathscr{C}(A)$ and $\mathscr{C}(B)$

This section gives, for generic $\mathbf{q}$, necessary and sufficient conditions on $\mathbf{q}$ for $\mathscr{C}(A)$ and $\mathscr{C}(B)$ to be commutative. As before, it is enough to consider homogeneous elements of $\mathscr{C}(A)$ and $\mathscr{C}(B)$, and the TS-components of the homogeneous elements of degree $(2, 2)$ can be disregarded since $TS \in \mathscr{Z}$. Thus, $\mathscr{C}(A)$ is commutative if and only if the set

$$\mathscr{H}_A = \{ H \in \mathscr{C}(A);\ H \text{ is homogeneous and has no } TS\text{-component}\}$$

is, and $\mathscr{C}(B)$ is commutative if and only if the set

$$\mathscr{H}_B = \{ H \in \mathscr{C}(B);\ H \text{ is homogeneous and has no } TS\text{-component}\}$$

is. Define further the sets

$$\begin{aligned}
\mathscr{X}_A &= \left\{X \in \mathscr{H}_A;\ X \text{ is homogeneous of degree } (m, n) \text{ and } q_0^n = 1\right\} \\
\mathscr{Y}_A &= \left\{Y \in \mathscr{H}_A;\ Y \text{ is homogeneous of degree } (m, n) \text{ and } q_0^n \neq 1\right\} \cup \{0\} \\
\mathscr{X}_B &= \left\{X \in \mathscr{H}_B;\ X \text{ is homogeneous of degree } (m, n) \text{ and } q_0^m = 1\right\} \\
\mathscr{Y}_B &= \left\{Y \in \mathscr{H}_B;\ Y \text{ is homogeneous of degree } (m, n) \text{ and } q_0^m \neq 1\right\} \cup \{0\}.
\end{aligned}$$

Then $\mathscr{H}_A = \mathscr{X}_A \cup \mathscr{Y}_A$ and $\mathscr{H}_B = \mathscr{X}_B \cup \mathscr{Y}_B$, and $\mathscr{X}_A \cap \mathscr{Y}_A = \mathscr{X}_B \cap \mathscr{Y}_B = \{0\}$, since 0 is homogeneous of all degrees. (The reason for explicitly including 0 in $\mathscr{Y}_A$ and $\mathscr{Y}_B$ is to make sure that they always contain 0: If 0 were not explicitly included then the case $q_0 = 1$ would be exceptional.) These sets do, of course, depend on $\mathbf{q}$; when this dependence needs to be emphasised the notation will be $\mathscr{X}_A(\mathbf{q})$, $\mathscr{Y}_A(\mathbf{q})$ and so on.

Consider two general homogeneous elements

$$\begin{aligned}
X_{k,l} &\equiv b_1 B^l A^k + b_2 S B^{l-1} A^{k-1} + b_3 T B^{l-1} A^{k-1} \\
X_{m,n} &\equiv c_1 B^n A^m + c_2 S B^{n-1} A^{m-1} + c_3 T B^{n-1} A^{m-1}
\end{aligned}$$

of degrees (k, l) and (m, n) respectively. A somewhat lengthy calculation using the reordering formula (7) and Eq. (6) shows that

$$\begin{aligned}
& X_{k,l} X_{m,n} \equiv \dots \\
& \equiv b_1 c_1 q_0^{kn} B^{l+n} A^{k+m} \\
& \quad + q_0^{(k-1)(n-1)} \left(b_1 c_1 q_0^{k-1} q_2^l \{k\}_{\frac{q_3}{q_0}} \{n\}_{q_0 q_2} + b_1 c_2 q_0^{n-1} q_2^l q_3^k + b_2 c_1 q_0^{k-1} \right) \\
& \qquad S B^{l+n-1} A^{k+m-1} \\
& \quad + q_0^{(k-1)(n-1)} \left(b_1 c_1 q_0^{k-1} q_4^l \{k\}_{\frac{q_1}{q_0}} \{n\}_{q_0 q_4} + b_1 c_3 q_0^{n-1} q_1^k q_4^l + b_3 c_1 q_0^{k-1} \right) \\
& \qquad T B^{l+n-1} A^{k+m-1}.
\end{aligned}$$

This holds for all $k, l, m, n \geq 1$ except for $k = l = m = n = 1$ (if $(k, l) = (2, 2)$ or $(m, n) = (2, 2)$ then there is an additional TS-term in the expression for $X_{k,l}$ or

$X_{m,n}$, but that makes no difference for the product because of (6)). The commutator of $X_{k,l}$ and $X_{m,n}$ then is

$$
\begin{aligned}
&[X_{k,l}, X_{m,n}] \equiv b_1c_1(q_0^{kn} - q_0^{ml})B^{l+n}A^{k+m} \\
&+ \Bigg(q_0^{(k-1)(n-1)}\Bigg(b_1c_1q_0^{k-1}q_2^l\{k\}_{\frac{q_3}{q_0}}\{n\}_{q_0q_2} + b_1c_2q_0^{n-1}q_2^lq_3^k + b_2c_1q_0^{k-1}\Bigg) \\
&- q_0^{(m-1)(l-1)}\Bigg(b_1c_1q_0^{m-1}q_2^n\{m\}_{\frac{q_3}{q_0}}\{l\}_{q_0q_2} + b_2c_1q_0^{l-1}q_2^nq_3^m + b_1c_2q_0^{m-1}\Bigg)\Bigg) \\
&\quad SB^{l+n-1}A^{k+m-1} \\
&+ \Bigg(q_0^{(k-1)(n-1)}\Bigg(b_1c_1q_0^{k-1}q_4^l\{k\}_{\frac{q_1}{q_0}}\{n\}_{q_0q_4} + b_1c_3q_0^{n-1}q_1^kq_4^l + b_3c_1q_0^{k-1}\Bigg) \\
&- q_0^{(m-1)(l-1)}\Bigg(b_1c_1q_0^{m-1}q_4^n\{m\}_{\frac{q_1}{q_0}}\{l\}_{q_0q_4} + b_3c_1q_0^{l-1}q_1^mq_4^n + b_1c_3q_0^{m-1}\Bigg)\Bigg) \\
&\quad TB^{l+n-1}A^{k+m-1}.
\end{aligned}
\tag{16}
$$

The switch $A \leftrightarrow B$ will be used in the proofs below. Assume that $q_0 \neq 0$ (as is the case when $\mathbf{q}$ is generic) and let $f_{AB} : \mathscr{F} \to \mathscr{F}$ be the isomorphism defined by

$$f_{AB}(A) = B, \quad f_{AB}(B) = A, \quad f_{AB}(S) = -q_0S, \quad f_{AB}(T) = -q_0T.$$

The image under f_{AB} of $G(\mathbf{q})$, defined in (2), is

$$
\begin{aligned}
&\{-q_0(AB - \frac{1}{q_0}BA - S - T),\ -q_0(BT - q_1TB),\ -q_0(AS - q_2SA), \\
&-q_0(BS - q_3SB),\ -q_0(AT - q_4TA),\ q_0^2(ST - q_5TS)\}.
\end{aligned}
$$

Thus $f_{AB}(G(\mathbf{q}))$ generates the ideal $\mathscr{J}(\hat{\mathbf{q}})$ where $\hat{\mathbf{q}} = (\frac{1}{q_0}, q_4, q_3, q_2, q_1, q_5)$, and consequently $f_{AB}(\mathscr{J}(\mathbf{q})) = \mathscr{J}(\hat{\mathbf{q}})$. This makes it possible to define an isomorphism $h_{AB} : \mathscr{F}/\mathscr{J}(\mathbf{q}) \to \mathscr{F}/\mathscr{J}(\hat{\mathbf{q}})$ by

$$h_{AB}(X + \mathscr{J}(\mathbf{q})) = f_{AB}(X) + \mathscr{J}(\hat{\mathbf{q}}). \tag{17}$$

It is easily checked that $\hat{\mathbf{q}}$ is generic whenever $\mathbf{q}$ is.

Also the switch $S \leftrightarrow T$ will be used below. Let $f_{ST} : \mathscr{F} \to \mathscr{F}$ be the isomorphism defined by

$$f_{ST}(A) = A, \quad f_{ST}(B) = B, \quad f_{ST}(S) = T, \quad f_{ST}(T) = S.$$

The image under f_{ST} of $G(\mathbf{q})$ is (assuming that $q_5 \neq 0$)

$$\{AB - q_0BA - S - T,\ AS - q_1SA,\ BT - q_2TB,$$

$$AT - q_3TA,\ BS - q_4SB,\ -q_5(ST - \frac{1}{q_5}TS)\}.$$

Thus $f_{ST}(G(\mathbf{q}))$ generates the ideal $\mathscr{J}(\tilde{\mathbf{q}})$, where $\tilde{\mathbf{q}} = (q_0, q_3, q_4, q_1, q_2, \frac{1}{q_5})$, so $f_{ST}(\mathscr{J}(\mathbf{q})) = \mathscr{J}(\tilde{\mathbf{q}})$ and an isomorphism $h_{ST} : \mathscr{F}/\mathscr{J}(\mathbf{q}) \to \mathscr{F}/\mathscr{J}(\tilde{\mathbf{q}})$ can be defined by

$$h_{ST}(X + \mathscr{J}(\mathbf{q})) = f_{ST}(X) + \mathscr{J}(\tilde{\mathbf{q}}). \tag{18}$$

Again, it is easily checked that $\tilde{\mathbf{q}}$ is generic whenever $\mathbf{q}$ is.

Lemma 5.1 *Let $\mathbf{q} = (q_0, \ldots, q_5) \in \mathbb{C}^6$ be generic. Then the sets $\mathscr{X}_A$ and $\mathscr{X}_B$ are commutative.*

Proof Pick any $X_1, X_2 \in \mathscr{X}_A$. Then $X_1, X_2 \in \mathscr{C}(A)$ and they are homogeneous, say of degrees (k, l) and (m, n) respectively with $q_0^l = q_0^n = 1$. If $k = 0$ then $X_1 \equiv b_1B^l$ and Lemma 3.1 implies that $X_1 \equiv 0$ or $q_2^l = q_4^l = 1$. In either case, $X_1 \in \mathscr{Z}$, so in particular X_1 commutes with X_2. Similarly, if $m = 0$ then $X_2 \in \mathscr{Z}$ and thus commutes with X_1. If $l = 0$ then $X_1 = b_1A^k$ and if $n = 0$ then $X_2 = c_1A^m$; in both cases X_1 and X_2 commute. Thus it may be assumed that $k, l, m, n \geq 1$, so that X_1 and X_2 can be written as

$$\begin{aligned} X_1 &\equiv b_1B^lA^k + b_2SB^{l-1}A^{k-1} + b_3TB^{l-1}A^{k-1} \qquad k, l \geq 1 \\ X_2 &\equiv c_1B^nA^m + c_2SB^{n-1}A^{m-1} + c_3TB^{n-1}A^{m-1} \qquad m, n \geq 1 \end{aligned}$$

with coefficients that satisfy

$$K_{(k,l)}[b_1\, b_2\, b_3]^T = 0, \quad K_{(m,n)}[c_1\, c_2\, c_3]^T = 0, \tag{19}$$

where $K_{(k,l)}, K_{(m,n)}$ are defined as in Lemma 3.1. If $k = l = m = n = 1$ then X_1 and X_2 are parallel, because $K_{(1,1)}$ has rank at least two (using that $q_1 \neq q_3$ since $\mathbf{q}$ is generic). Hence it may be assumed that at least one of k, l, m, n is ≥ 2, so that the commutator $[X_1, X_2]$ is given by (16). Since $q_0^l = q_0^n = 1$, the coefficient of $B^{l+n}A^{k+m}$ in (16) is 0—it has to be shown that the coefficients of $SB^{l+n-1}A^{k+m-1}$ and $TB^{l+n-1}A^{k+m-1}$ are 0 as well.

If $q_3 \neq q_0$ then (19) implies that

$$b_2 = \frac{\{l\}_{q_0q_2}}{1 - q_3/q_0}b_1, \qquad c_2 = \frac{\{n\}_{q_0q_2}}{1 - q_3/q_0}c_1$$

and the $SB^{l+n-1}A^{k+m-1}$-coefficient in (16) can be simplified to

$$\begin{aligned} &b_1c_1q_2^l\frac{1 - (q_3/q_0)^k}{1 - q_3/q_0}\{n\}_{q_0q_2} + b_1c_1q_0^{-k}q_2^lq_3^k\frac{\{n\}_{q_0q_2}}{1 - q_3/q_0} + b_1c_1\frac{\{l\}_{q_0q_2}}{1 - q_3/q_0} \\ &\quad - b_1c_1q_2^n\frac{1 - (q_3/q_0)^m}{1 - q_3/q_0}\{l\}_{q_0q_2} - b_1c_1q_0^{-m}q_2^nq_3^m\frac{\{l\}_{q_0q_2}}{1 - q_3/q_0} - b_1c_1\frac{\{n\}_{q_0q_2}}{1 - q_3/q_0} \end{aligned}$$

$$= \frac{b_1c_1}{1-q_3/q_0}\Big(q_2^l\,\{n\}_{q_0q_2} + \{l\}_{q_0q_2} - q_2^n\,\{l\}_{q_0q_2} - \{n\}_{q_0q_2}\Big) = 0. \tag{20}$$

The last equality holds by the identity $\{a\}_q + q^a\,\{b\}_q = \{a+b\}_q$ for q-natural numbers, since $q_2^l = (q_0q_2)^l$, $q_2^n = (q_0q_2)^n$. Similarly if $q_1 \neq q_0$ then

$$b_3 = \frac{\{l\}_{q_0q_4}}{1-q_1/q_0}b_1, \qquad c_3 = \frac{\{n\}_{q_0q_4}}{1-q_1/q_0}c_1$$

and the $TB^{l+n-1}A^{k+m-1}$-coefficient in (16) can be simplified to

$$\begin{aligned} &b_1c_1q_4^l\frac{1-(q_1/q_0)^k}{1-q_1/q_0}\,\{n\}_{q_0q_4} + b_1c_1q_0^{-k}q_1^kq_4^l\frac{\{n\}_{q_0q_4}}{1-q_1/q_0} + b_1c_1\frac{\{l\}_{q_0q_4}}{1-q_1/q_0} \\ &\quad - b_1c_1q_4^n\frac{1-(q_1/q_0)^m}{1-q_1/q_0}\,\{l\}_{q_0q_4} - b_1c_1q_0^{-m}q_1^mq_4^n\frac{\{l\}_{q_0q_4}}{1-q_1/q_0} - b_1c_1\frac{\{n\}_{q_0q_4}}{1-q_1/q_0} \\ &= \frac{b_1c_1}{1-q_1/q_0}\Big(q_4^l\,\{n\}_{q_0q_4} + \{l\}_{q_0q_4} - q_4^n\,\{l\}_{q_0q_4} - \{n\}_{q_0q_4}\Big) = 0. \end{aligned} \tag{21}$$

Now there are three cases to consider ($q_1 = q_3 = q_0$ is impossible since $\mathbf{q}$ is generic).

1. If $q_1, q_3 \neq q_0$ then the coefficients of $SB^{l+n-1}A^{k+m-1}$ and $TB^{l+n-1}A^{k+m-1}$ in (16) are 0 by (20) and (21).

2. If $q_1 = q_0$, $q_3 \neq q_0$, then the computation (20) is still valid, that is, the coefficient of $SB^{l+n-1}A^{k+m-1}$ in (16) is 0. Further, (19) implies that

$$\{l\}_{q_0q_4}\,b_1 = \{n\}_{q_0q_4}\,c_1 = 0$$

and thus also

$$(1-q_4^l)b_1 = (1-q_4^n)c_1 = 0.$$

Then the $TB^{l+n-1}A^{k+m-1}$-coefficient in (16) can be simplified to

$$b_1c_3q_4^l + b_3c_1 - b_3c_1q_4^n - b_1c_3 = -b_1c_3(1-q_4^l) + b_3c_1(1-q_4^n) = 0.$$

3. If $q_1 \neq q_0, q_3 = q_0$ then the computation (21) is still valid, that is, the coefficient of $TB^{l+n-1}A^{k+m-1}$ in (16) is 0. The condition (19) implies that

$$\{l\}_{q_0q_2}\,b_1 = \{n\}_{q_0q_2}\,c_1 = (1-q_2^l)b_1 = (1-q_2^n)c_1 = 0,$$

and the $SB^{l+n-1}A^{k+m-1}$-coefficient becomes

$$b_1c_2q_2^l + b_2c_1 - b_2c_1q_2^n - b_1c_2 = -b_1c_2(1-q_2^l) + b_2c_1(1-q_2^n) = 0.$$

Thus it has been shown that $\mathscr{X}_A$ is commutative.

In order to see that $\mathscr{X}_B = \mathscr{X}_B(\mathbf{q})$ is commutative, let $h_{AB} : \mathscr{F}/\mathscr{J}(\mathbf{q}) \to \mathscr{F}/\mathscr{J}(\hat{\mathbf{q}})$ be the isomorphism defined as in (17). Then $\mathscr{X}_A(\hat{\mathbf{q}})$ is commutative by the above proof, so $\mathscr{X}_B(\mathbf{q}) = h_{AB}^{-1}(\mathscr{X}_A(\hat{\mathbf{q}}))$ is commutative as well.

Lemma 5.2 *Let $\mathbf{q} = (q_0, \ldots, q_5) \in \mathbb{C}^6$ be generic. Then the sets $\mathscr{Y}_A$ and $\mathscr{Y}_B$ are commutative.*

Proof Pick any $Y_1, Y_2 \in \mathscr{Y}_A$. Then $Y_1, Y_2 \in \mathscr{C}(A)$ and they are homogeneous, say of degrees (k, l) and (m, n) respectively with $q_0^l, q_0^n \neq 1$. It is then impossible that $l = 0$ or $n = 0$, and by Lemma 3.1 it cannot be that $Y_1 = b_1 B^l$ or $Y_2 = c_1 B^n$ with $b_1, c_1 \neq 0$. Thus it may be assumed that $k, l, m, n \geq 1$, and Y_1, Y_2 can be written as

$$Y_1 \equiv b_1 B^l A^k + b_2 S B^{l-1} A^{k-1} + b_3 T B^{l-1} A^{k-1} \qquad k, l \geq 1$$
$$Y_2 \equiv c_1 B^n A^m + c_2 S B^{n-1} A^{m-1} + c_3 T B^{n-1} A^{m-1} \qquad m, n \geq 1$$

with coefficients that satisfy

$$K_{(k,l)}[b_1\, b_2\, b_3]^T = 0, \quad K_{(m,n)}[c_1\, c_2\, c_3]^T = 0, \tag{22}$$

where $K_{(k,l)}, K_{(m,n)}$ are defined as in Lemma 3.1. If $k = l = m = n = 1$ then Y_1 and Y_2 are parallel, because $K_{(1,1)}$ has rank at least two (using that $q_1 \neq q_3$ since $\mathbf{q}$ is generic). Hence it may be assumed that at least one of k, l, m, n is ≥ 2. Since $q_0^l, q_0^n \neq 1$, (22) implies that $b_1 = c_1 = 0$. But then $Y_1 Y_2 \equiv Y_2 Y_1 \equiv 0$ by (6), so Y_1, Y_2 commute.

To see that $\mathscr{Y}_B = \mathscr{Y}_B(\mathbf{q})$ is commutative, let $h_{AB} : \mathscr{F}/\mathscr{J}(\mathbf{q}) \to \mathscr{F}/\mathscr{J}(\hat{\mathbf{q}})$ be the isomorphism defined as in (17). Then $\mathscr{Y}_A(\hat{\mathbf{q}})$ is commutative by the above proof, so $\mathscr{Y}_B(\mathbf{q}) = h_{AB}^{-1}(\mathscr{Y}_A(\hat{\mathbf{q}}))$ is commutative as well.

Because of Lemma 5.1 and Lemma 5.2, it is enough to check whether the elements of $\mathscr{X}_A$ commute with the elements of $\mathscr{Y}_A$ to see if $\mathscr{C}(A)$ is commutative, and to check whether the elements of $\mathscr{X}_B$ commute with the elements of $\mathscr{Y}_B$ to see if $\mathscr{C}(B)$ is commutative.

Theorem 5.1 *Let $\mathbf{q} = (q_0, \ldots, q_5) \in \mathbb{C}^6$ be generic and suppose that q_0 is not a root of unity. Then $\mathscr{C}(A)$ and $\mathscr{C}(B)$ are commutative.*

Proof Since q_0 is not a root of unity,

$$\mathscr{X}_A = \{cA^m;\ c \in \mathbb{C},\ m \in \mathbf{N}\}.$$

Thus, every element of $\mathscr{X}_A$ commutes with every element of $\mathscr{Y}_A$, and it follows from Lemma 5.1 and Lemma 5.2 that $\mathscr{C}(A)$ is commutative. Similarly,

$$\mathscr{X}_B = \{cB^n;\ c \in \mathbb{C},\ n \in \mathbf{N}\};$$

thus every element of $\mathscr{X}_B$ commutes with every element of $\mathscr{Y}_B$ and thus $\mathscr{C}(B)$ is commutative.

Theorem 5.2 *Let* $\mathbf{q} = (q_0, \ldots, q_5) \in \mathbb{C}^6$ *be generic, and suppose that* q_0 *is a root of unity with* d *being the smallest positive integer such that* $q_0^d = 1$. *Then* $\mathscr{C}(A)$ *is commutative if and only if*

$$\Big(\{d\}_{q_3/q_0} \neq 0 \text{ or } q_2^d = 1 \text{ or } (q_1 = q_0 \text{ and } q_4^d \neq 1)\Big) \tag{23}$$

and

$$\Big(\{d\}_{q_1/q_0} \neq 0 \text{ or } q_4^d = 1 \text{ or } (q_3 = q_0 \text{ and } q_2^d \neq 1)\Big), \tag{24}$$

and $\mathscr{C}(B)$ *is commutative if and only if*

$$\Big(\{d\}_{q_0q_2} \neq 0 \text{ or } q_3^d = 1 \text{ or } (q_4 = q_0^{-1} \text{ and } q_1^d \neq 1)\Big) \tag{25}$$

and

$$\Big(\{d\}_{q_0q_4} \neq 0 \text{ or } q_1^d = 1 \text{ or } (q_2 = q_0^{-1} \text{ and } q_3^d \neq 1)\Big). \tag{26}$$

Proof It follows from Lemma 5.1 and Lemma 5.2 that $\mathscr{C}(A)$ is commutative if and only if every element of $\mathscr{X}_A$ commutes with every element of $\mathscr{Y}_A$. A non-zero element of $\mathscr{Y}_A$ cannot have degree $(m, 0)$ since $q_0^0 = 1$, and it cannot have degree $(0, n)$ by Lemma 3.1. Thus any non-zero $Y \in \mathscr{Y}_A$ is of the form

$$Y \equiv c_1 B^n A^m + c_2 S B^{n-1} A^{m-1} + c_3 B^{n-1} A^{m-1}$$

for some n with $q_0^n \neq 1$ and with coefficients that satisfy

$$\begin{bmatrix} 1 - q_0^n & 0 & 0 \\ -\{n\}_{q_0q_2} & 1 - q_0^{n-1} q_3 & 0 \\ -\{n\}_{q_0q_4} & 0 & 1 - q_0^{n-1} q_1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0. \tag{27}$$

Since $q_1 \neq q_3$ (because $\mathbf{q}$ is generic), the matrix in (27) has one of the forms

$$\begin{bmatrix} * & 0 & 0 \\ ? & * & 0 \\ ? & 0 & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 \\ ? & 0 & 0 \\ ? & 0 & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 \\ ? & * & 0 \\ ? & 0 & 0 \end{bmatrix},$$

where $*$ indicates a non-zero element and ? indicates an element that may or may not be zero. Thus the solutions to (27) are

$$\begin{array}{ll} c_1 = c_2 = c_3 = 0 & \text{if } q_0^{n-1} q_3, q_0^{n-1} q_1 \neq 1 \\ c_1 = c_3 = 0 & \text{if } q_0^{n-1} q_3 = 1, q_0^{n-1} q_1 \neq 1 \\ c_1 = c_2 = 0 & \text{if } q_0^{n-1} q_3 \neq 1, q_0^{n-1} q_1 = 1. \end{array}$$

Since it is impossible that both c_2 and c_3 are non-zero, Y must actually have the form $c_2 S B^{n-1} A^{m-1}$ or $c_3 T B^{n-1} A^{m-1}$. Thus $\mathscr{Y}_A$ can be decomposed as $\mathscr{Y}_A^S \cup \mathscr{Y}_A^T \cup \{0\}$, where

$$\mathscr{Y}_A^S = \left\{ c S B^{n-1} A^{m-1} \in \mathscr{Y}_A;\ q_0^n \neq 1,\ q_0^{n-1} q_3 = 1 \right\}$$
$$\mathscr{Y}_A^T = \left\{ c T B^{n-1} A^{m-1} \in \mathscr{Y}_A;\ q_0^n \neq 1,\ q_0^{n-1} q_1 = 1 \right\}$$

(here, c ranges over $\mathbb{C}$ and m, n range over $\mathbf{N}^+$), and $\mathscr{C}(A)$ is commutative if and only if every element of $\mathscr{X}_A$ commutes with every element of $\mathscr{Y}_A^S$ and $\mathscr{Y}_A^T$.

Consider first $\mathscr{Y}_A^S$ and the conditions (23). If $\{d\}_{q_3/q_0} \neq 0$ then $q_3^d \neq 1$ or $q_3 = q_0$ and it cannot be that $q_0^n \neq 1$ and $q_0^{n-1} q_3 = 1$; thus $\mathscr{Y}_A^S = \emptyset$. Otherwise, there is an $r_3 \in \{2, \ldots, d\}$ such that $q_3 = q_0^{r_3}$, and

$$\mathscr{Y}_A^S = \left\{ c S B^{n-1} A^{m-1};\ q_0^{n-1+r_3} = 1 \right\}$$

is non-empty. Now pick any $X \in \mathscr{X}_A$. If $X = b_1 A^k$ then X obviously commutes with every element in $\mathscr{Y}_A^S$. Otherwise, X has one of the forms

$$b_1 B^l, \quad b_1 B^l A^k + b_2 S B^{l-1} A^{k-1} + b_3 T B^{l-1} A^{k-1}$$

with $d | l$. Then the commutator of X with an element of $\mathscr{Y}_A^S$ is (note that $l = n = 1$ is impossible and use (6))

$$[X, c S B^{n-1} A^{m-1}] \equiv b_1 c (q_2^l - 1) S B^{l+n-1} A^{k+m-1}. \tag{28}$$

Thus if $\{d\}_{q_3/q_0} \neq 0$, $q_2^d = 1$ then X commutes with everything in $\mathscr{Y}_A^S$. Finally, if $q_1 = q_0$ and $q_4^d \neq 1$ then Lemma 3.1 implies that $b_1 = 0$, so that the commutator (28) is 0, and again X commutes with everything in $\mathscr{Y}_A^S$.

On the other hand, if none of the conditions

$$q_3^d \neq 1,\ q_3 = q_0,\ q_2^d = 1,\ (q_1 = q_0 \text{ and } q_4^d \neq 1)$$

is satisfied, then

$$B^d A^2 + \frac{\{d\}_{q_0 q_2}}{1 - q_0^{d-1+r_3}} S B^{d-1} A + b_T T B^{d-1} A \in \mathscr{X}_A,$$

where $r_3 \in \{2, \ldots, d\}$ is such that $q_3 = q_0^{r_3}$, and

$$b_T = \begin{cases} \{d\}_{q_0 q_4} / (1 - q_0^{d-1} q_1) & \text{if } q_1 \neq q_0 \\ \text{arbitrary} & \text{if } q_1 = q_0 \end{cases}$$

(b_T arbitrary if $q_1 = q_0$ works because then $q_4^d = 1$ and thus $\{d\}_{q_0 q_4} = 0$ since $\mathbf{q}$ generic implies $q_4 \neq q_0^{-1}$). This element of $\mathscr{X}_A$ does not commute with SB^{2d-r_3} in $\mathscr{Y}_A^S$; their commutator is $(q_2^d - 1)SB^{3d-r_3}A^2$. Thus it has been shown that every element of $\mathscr{X}_A$ commutes with every element of $\mathscr{Y}_A^S$ if and only if (23) is satisfied.

To see that every element of $\mathscr{X}_A$ commutes with every element of $\mathscr{Y}_A^T$ consider the isomorphism $h_{ST} : \mathscr{F}/\mathscr{J}(\mathbf{q}) \to \mathscr{F}/\mathscr{J}(\tilde{\mathbf{q}})$, where $\tilde{\mathbf{q}} = (q_0, q_3, q_4, q_1, q_2, \frac{1}{q_5})$, as defined in (18). Note that $\tilde{\mathbf{q}}$ is generic, $\tilde{q}_0$ is a root of unity with d being the smallest positive integer such that $(\tilde{q}_0)^d = 1$ and $\tilde{\mathbf{q}}$ satisfies (23) if and only if $\mathbf{q}$ satisfies (24). Furthermore, $h_{ST}(\mathscr{X}_A(\mathbf{q})) = \mathscr{X}_A(\tilde{\mathbf{q}})$ and $h_{ST}(\mathscr{Y}_A^T(\mathbf{q})) = \mathscr{Y}_A^S(\tilde{\mathbf{q}})$. Thus, using what has already been proved,

$$\begin{aligned} &\mathbf{q} \text{ satisfies (24)} \iff \tilde{\mathbf{q}} \text{ satisfies (23)} \iff \\ &\text{every } \tilde{X} \in \mathscr{X}_A(\tilde{\mathbf{q}}) \text{ commutes with every } \tilde{Y} \in \mathscr{Y}_A^S(\tilde{\mathbf{q}}) \iff \\ &\text{every } X \in \mathscr{X}_A(\mathbf{q}) \text{ commutes with every } Y \in \mathscr{Y}_A^T(\mathbf{q}). \end{aligned}$$

This concludes the proof of the first part of the theorem, namely that $\mathscr{C}(A)$ is commutative if and only if (23) and (24) are satisfied.

For the second part of the theorem, consider the isomorphism $h_{AB} : \mathscr{F}/\mathscr{J}(\mathbf{q}) \to \mathscr{F}/\mathscr{J}(\hat{\mathbf{q}})$, where $\hat{\mathbf{q}} = (\frac{1}{q_0}, q_4, q_3, q_2, q_1, q_5)$, as defined in (17). Note that $\hat{\mathbf{q}}$ is generic, $\hat{q}_0$ is a root of unity with d being the smallest positive integer such that $(\hat{q}_0)^d = 1$ and $\hat{\mathbf{q}}$ satisfies (23) and (24) if and only if $\mathbf{q}$ satisfies (25) and (26). Moreover, $h_{AB}(\mathscr{C}(B)(\mathbf{q})) = \mathscr{C}(A)(\hat{\mathbf{q}})$ and thus using what has already been proved,

$$\begin{aligned} &\mathbf{q} \text{ satisfies (25) and (26)} \iff \hat{\mathbf{q}} \text{ satisfies (23) and (24)} \iff \\ &\mathscr{C}(A)(\hat{\mathbf{q}}) \text{ is commutative} \iff \mathscr{C}(B)(\mathbf{q}) \text{ is commutative.} \end{aligned}$$

Acknowledgments This research was supported in part by the Swedish Research Council (621-2007-6338), Swedish Foundation for International Cooperation in Research and Higher Education (STINT), Swedish Royal Academy of Sciences, Crafoord Foundation and the Nordforsk network "Operator algebra and dynamics" (grant # 11580).

References

1. Bratteli, O., Robinson, D.W.: Operator Algebras and Statistical Mechanics. Springer, New York, I 1981, (1979)
2. Dixmier, J.: Les C^*-algèbres et Leurs Représentations, (French), Cahiers Scientifiques, Fasc. XXIX, Gauthier-Villars, Paris (1964). Russian translation: Nauka, Moscow (1974). English, translation: North-Holland, Amsterdam (1982)

3. Sakai, S.: Operator Algebras in Dynamical Systems. Cambridge University Press, Cambridge (1991)
4. Silvestrov, S.D., Tomiyama, Y.: Topological dynamical systems of type i. Expositiones Mathe. **20**, 117–142 (2002)
5. Tomiyama, J.: Invitation to the C^*-algebras and topological dynamics. World Scientific, Singapore (1987)
6. Tomiyama, J.: C^*-algebras and topological dynamical systems. Rev. Math. Phys. **8**, 741–760 (1996)
7. Kadison, R.V., Singer, I.M.: Extensions of pure states. Amer. J. Math. **81**, 383–400 (1959)
8. Anderson, J.: Extensions, restrictions and representations of states on c*-algebras. Trans. Am. Math. Soc. **249**, 303–329 (1979)
9. Anderson, J.: Extreme points in sets of linear maps on b(h). J. Func. Anal. **31**, 195–217 (1979)
10. Casazza, P.G., Fickus, M., Tremain, J.C., Weber, E.: The Kadison-Singer Problem in Mathematics and Engineering:A Detailed Account, Operator Theory, Operator Algebras, and Applications, vol. 414, pp. 299–355. Contemporary Mathematics—American Mathematical Society, Providence, RI (2006)
11. Paulsen, V.I.: A dynamical systems approach to the kadison-singer problem. J. Funct Anal. **255**(1), 120–132 (2008)
12. Bratteli, O., Jorgensen, P.E.T.: Wavelets Through a Looking Glass. Birkhäuser Verlag, Basel (2002)
13. Dutkay, D.E., Silvestrov, S.: Reducibility of the wavelet representation associated to the Cantor set. Proc. Am. Math. Soc. **139**(10), 3657–3664 (2011) MR2813395
14. Dutkay, D.E., Larson, D.R., Silvestrov, S.: Irreducible wavelet representations and ergodic automorphisms on solenoids. Oper. Matrices **5**(2), 201–219 (2011) MR2830593
15. Dutkay, D.E., Jorgensen, P.E.T., Silvestrov, S.: Decomposition of wavelet representations and martin boundaries. J. Func. Anal. **262**(3), 1043–1061 (2012)
16. Jorgensen, P.E.T.: Ruelle Operators: Functions which are Harmonic with Respect to a Transfer Operator. The University of Iowa, Iowa City (1998). (Preprint)
17. Davidson, K.R.: C*-algebras by example. Fields Institute Monographs, vol. 6. American Mathematical Society, Providence, RI (1996)
18. Jorgensen, P.E.T.: Operators and representation theory. Canonical models for algebras of operators arising in quantum mechanics. North-Holland Mathematical Studies, 147, Notas de Matematica 120, p. 337. North-Holland Publishing Company, Amsterdam (1988)
19. Li, B.R.: Introduction To Operator Algebras. World Scientific, Singapore (1992)
20. Mackey, G.W.: Induced Representations of Groups and Quantum Mechanics. W.A. Benjamin New York. Boringhieri Editore, Turin (1968)
21. Mackey, G.W.: The Theory of Unitary Group Representations. The University of Chicago Press, Chicago (1976)
22. Mackey, G.W.: Unitary Group Representations in Physics, Probability, and Number Theory. Addison-Wesley, Boston (1989)
23. Nazaikinskii, V.E., Shatalov, V.E., Yu, B.: Sternin, Methods of Noncommutative Analysis. Theory and Applications, De Gruyter Studies in Mathematics, vol .22. De Gruyter, Berlin (1976)
24. Ostrovskyi, V., Yu, S.: Introduction to the theory of representations of finitely presented *-algebras. I. Representations by bounded operators. Reviews in Mathematics and Mathematical Physics, 11, pt.1. Harwood Academic Publishers, Amsterdam (1999)
25. Pedersen, G.K.: C^*-algebras and their automorphism group. Academic Press, Salt Lake City (1979)
26. Samoĭlenko, Y.S.: Spectral Theory of Families of Selfadjoint Operators. Naukova Dumka, Kiev (1984) (Russian). Extended English transl.: Kluwer Academic Publication, Boston (1991)
27. Schmüdgen, K.: Unbounded Operator Algebras and Representation Theory. Birkhäuser Verlag, Basel (1990)
28. Hellström, L., Silvestrov, S.D.: Commuting Elements in q-deformed Heisenberg Algebras, p. 256. World Scientific, Singapore (2000). ISBN:981-02-4403-7

29. de Jeu, M., Svensson, C., Silvestrov, S.: Algebraic Curves for Commuting Elements in the q-deformed Heisenberg Algebra. J. Algebra **321**(4), 1239–1255 (2009). (Preprints in Mathematical Sciences 2007:23, LUTFMA-5089-2007, Centre for Mathematical Sciences, p. 17. Lund Institute of Technology, Lund University, Lund (2007). arXiv:0710.2748
30. Hellström, L., Silvestrov, S.D.: Ergodipotent maps and commutativity of elements in noncommutative rings and algebras with twisted intertwining. J. Algebra **314**(1), 17–41 (2007)
31. Carlsen, T.M., Silvestrov, S.: On the exel crossed product of topological covering maps. Acta Applicandae Mathe. **108**(3), 573–583 (2009)
32. Nystedt, P., Öinert, J.: Simple skew category algebras associated with minimal partially defined dynamical systems. Discrete Continuous Dyn. Syst. Ser. A (DCDS-A), **33**(9), 4157–4171 (2013)
33. Öinert, J., Silvestrov, S.D.: Commutativity and ideals in algebraic crossed products. J. Gen. Lie Theory. Appl. **2**(4), 287–302 (2008)
34. Öinert, J., Silvestrov, S., Theohari-Apostolidi, Th, Vavatsoulas, H.: Commutativity and ideals in strongly graded rings. Acta Applicandae Math. **108**(3), 585–602 (2009)
35. Öinert, J., Silvestrov, S.: Commutativity and ideals in pre-crystalline graded rings. Acta Applicandae Math. **108**(3), 603–615 (2009)
36. Öinert, J., Richter, J., Silvestrov, S.: Maximal ommutative subrings and simplicity of Ore extensions J. Algebra Appl. **12**(4), 1250192-1-16 (2013)
37. Persson, T., Silvestrov, S.: Commuting operators for representations of commutation relations defined by dynamical systems. Numer. Funct. Anal. Optim. **33**(7–9) (2012)
38. Svensson, C., Silvestrov, S., de Jeu, M.: Dynamical systems and commutants in crossed products. Int. J. Math. **18**(4), 455–471 (2007)
39. Svensson, C., Silvestrov, S., de Jeu, M.: Dynamical systems associated with crossed products. Acta Applicandae Mathe. **108**(3), 547–559 (2009)
40. Bergman, G.M.: The diamond lemma for ring theory. Avd. Math. **29**, 178–218 (1978)