On Dimensions of Vector Spaces of Conformal Killing Forms

Sergey E. Stepanov, Marek Jukl and Josef Mikeš

Abstract In this article there are found precise upper bounds of dimension of vector spaces of conformal Killing forms, closed and coclosed conformal Killing r-forms $(1 \leq r \leq n-1)$ on an n-dimensional manifold. It is proved that, in the case of n-dimensional closed Riemannian manifold, the vector space of conformal Killing r-forms is an orthogonal sum of the subspace of Killing forms and of the subspace of exact conformal Killing r-forms. This is a generalization of related local result of Tachibana and Kashiwada on pointwise decomposition of conformal Killing r-forms on a Riemannian manifold with constant curvature. It is shown that the following well known proposition may be derived as a consequence of our result: any closed Riemannian manifold having zero Betti number and admitting group of conformal mappings, which is non equal to the group of motions, is conformal equivalent to a hypersphere of Euclidean space.

1 Introduction

1.1 The history of conformal Killing forms has started almost a half of century ago by works of Tachibana and Kashiwada [7, 26]. During this long time, this topic has given rise to an active interest (see for example [8, 16, 17, 22]) because of great number of its applications (see e.g. [4, 10, 20]). This paper is devoted to the study of dimensions of vector spaces of Killing forms (see [17]).

S. E. Stepanov (✉)
Finance University Under the Government of the Russian Federation,
49-55, Leningradsky Prospect, Moscow, Russian Federation 125468
e-mail: s.e.stepanov@mail.ru

M. Jukl · J. Mikeš
Palacky University, 17. Listopadu 12, 77146 Olomouc, Czech Republic
e-mail: marek.jukl@upol.cz

J. Mikeš
e-mail: josef.mikes@upol.cz

A. Makhlouf et al. (eds.), *Algebra, Geometry and Mathematical Physics*,
Springer Proceedings in Mathematics & Statistics 85, DOI: 10.1007/978-3-642-55361-5_29,

1.2 In the second section, we investigate *conformal Killing differential r-forms* $(1 \le r \le n-1)$ on local coordinates of an arbitrary neighbourhood U of n-dimensional Riemannian manifold (M, g). We consider a vector space T^r of these forms and deal with two subspaces—the subspace P^r of *planar r-forms* (i.e. *closed conformal Killing forms)* and subspace K^r of *Killing r-forms (coclosed conformal Killing r-form)*. For $r = 1$ we obtain the following three vector spaces of 1-forms: dual to conformal Killing vector field, concircular vector field and Killing vector field. In this section there are found dimensions t_r, k_r and p_r of these free "local" vector spaces on the manifold (M, g) with constant curvature.

1.3 In the third section there are studied "complete" conformal Killing r-forms $(1 \le r \le n-1)$ on n-dimensional closed Riemannian manifold (M, g), vector space $T^r(M, \mathbb{R})$ of these forms and two subspaces of it $K^r(M, \mathbb{R})$ of Killing forms and $P^r(M, \mathbb{R})$ of planar r-forms; dimensions of these spaces are denoted by $t_r(M)$, $k_r(M)$ and $p_r(M)$, respectively. In the case of closed manifold (M, g) with zero Betti numbers $b_r(M) = 0$, we show the orthogonal decomposition of T^r in the form $T^r(M, \mathbb{R}) = K^r(M, \mathbb{R}) + P^r(M, \mathbb{R})$, which implies the relation $t_r(M) = k_r(M) + p_r(M)$. In the case $b_1(M) = 0$ and $t_1(M) \neq k_1(M)$, we will prove that (M, g) is globally conformal to the n-dimensional sphere S^n of the Euclidean space R^{n+1}.

2 Definitions and Notations

2.1 Let us consider n-dimensional Riemannian manifold (M, g) with Levi-Civita connection. Denote by $C^\infty(M)$ a vector space of C^∞-function on M and by $\Omega^r(M)$ a vector space of differentiable r-forms on M. Taking local orientation of M we introduce the Hodge operator $*$ defining an isomorphism $* : \Omega^r \to \Omega^{n-r}$ such that $g(\omega, *\Theta) = (-1)^{r(n-r)} g(*\omega, \Theta)$, for any $\omega \in \Omega^r(M)$, $\Theta \in \Omega^{n-r}(M)$, and $*^2 = (-1)^{r(n-r)} Id_{\Omega^r(M)}$ (see [2, definition 1.51], [13, p. 203]).

For the exterior differential operator $\mathrm{d} : C^\infty \Lambda^r(M) \to C^\infty \Lambda^{r+1}(M)$ there exists a formal adjoint operator $\delta : \Omega^{r+1}(M) \to \Omega^r(M)$ which is called *codifferential operator* (see [2, definition 1.56], [13, p. 203, 204], [15, § 25]) and it is defined by

$$\delta = (-1)^{(n-r)(r+1)} * \mathrm{d} * \tag{1}$$

2.2 Let us remind well known definitions of three types of Killing vector fields in Riemannian geometry.

A vector field Z on a Riemannian manifold (M, g) is called *infinitesimal conformal transformations* or *conformal Killing vector field* (see [5, § 69], [11, p. 120]) if $L_Z g = 2\sigma g$ for Lie derivative L_Z with respect to the vector field Z and some $\sigma \in C^\infty M$.

Defining for a vector field Z a dual 1-form ω by the relation $\omega = g(Z, \cdot)$ we introduce a denotation $\omega^\# = Z$ (see [2, denotation 1.38]). Now, the identity $L_Z g =$

$2\sigma \cdot g$, by which an infinitesimal conformal transformation is defined, may be written by

$$(L_Z g)(X, Y) \overset{\text{def}}{=} (\nabla_X)Y + (\nabla_Y \omega)X = -\frac{2}{n}(\delta\omega)g(X, Y) \tag{2}$$

or, equivalently, by

$$\nabla\omega = -\frac{1}{2}\mathrm{d}\omega - \frac{1}{n}g \cdot \delta\omega = 0. \tag{3}$$

Any vector field Z with $L_Z g = 0$ on a Riemannian manifold (M, g) is called *infinitesimal isometry* or Killing vector field.

Clearly, if $\sigma = 0$ then every infinitesimal conformal transformation Z is an infinitesimal isometry. Because $\sigma = n^{-1}(-\delta\omega)$, for 1-form ω dual to the vector field $Z = \omega^{\#}$, we see that any infinitesimal isometry may be considered as coclosed infinitesimal conformal transformation.

Let us remind that a vector field Z is called *concircular* (see [30]) if $\nabla Z = \rho\, Id_M$ for $\rho \in C^\infty M$. In this case, for every 1-form ω with $\omega^{\#} = Z$ we have $\nabla\omega = n^{-1}(-\delta\omega)g$. Therefore, a concircular vector field may be defined as closed infinitesimal conformal transformation.

2.3 Let us deal with a generalization of three types of Killing vector field defined above.

Let an n-dimensional Riemannian manifold (M, g) be given. A form $\omega \in \Omega^r(M)$ is called *conformal Killing r-form* (see [7]) if there exists a form $\Theta \in \Omega^{r-1}(M)$ with

$$\begin{gathered}(\nabla_Y\omega)(X, X_2, \ldots, X_r) + (\nabla_X\omega)(Y, X_2, \ldots, X_r) = 2g(X, Y)\Theta(X_2, \ldots, X_r) \\ - \sum_{a=2}^{r}(-1)^a \Big(g(Y, X_a)\Theta(X, X_2, \ldots, \hat{X}_a, \ldots, X_r + g(X, X_a)\Theta(Y, Y_2, \ldots, \hat{X}_a, \ldots, X_r)\Big)\end{gathered} \tag{4}$$

for any vector fields $Y, X, X_2, \ldots, X_r \in C^\infty(TM)$, where $\hat{X}_a$ means that X_a is omitted. The form $\Theta \in \Omega^{r-1}(M)$ is called an *associated form* of the conformal Killing form $\omega \in \Omega^r(M)$. Moreover, the identity (see [7])

$$\delta\omega = -(n - r + 1)\Theta \tag{5}$$

holds as the corollary of (3).

Equation (4) are a natural generalization of Eq. (2). Equation (4) may be written in the form (see [20, 22])

$$\nabla\omega = (r + 1)^{-1}\mathrm{d}\omega + (n - r + 1)^{-1}g \wedge \delta\omega. \tag{6}$$

These relations constitute necessary and sufficient conditions that an r-form ω is a conformal Killing form $(1 \le r \le n - 1)$.

The set of conformal Killing forms on a Riemannian manifold (T, g) forms a vector space T^r (with real coefficients) (see [19]).

A form $\omega \in \Omega^r(M)$ on an n-dimensional Riemannian manifold (M, g) is called a *Killing r-form* if it is a *coclosed conformal Killing r-form*. Such form $\omega \in \Omega^r(M)$ fulfils its definition equations (see [14] and [10], Definition 31.3.1)

$$(\nabla_Y \omega)(X, X_2, \ldots, X_r) + (\nabla_X \omega)(Y, X_2, \ldots, X_r) = 0 \tag{7}$$

or equivalent equations

$$\nabla\omega = (r+1)^{-1} \mathrm{d}\omega. \tag{8}$$

This form ω is a generalization of an 1-form $\omega \in \Omega^{-1}(M)$, which is dual to the Killing vector field $Z = \omega^{\#}$. The set of all Killing r-forms constitutes a vector space $K^r \subset T^r$ (see [19]).

A form $\omega \in C^\infty \Lambda^r(M)$ on an n-dimensional Riemannian manifold (M, g) is called a *planar r-form* if it is a *closed conformal Killing r-form* (see [19]). Such form $\omega \in C^\infty \Lambda^r(M)$ fulfils its definition equations

$$\nabla\omega = (n - r + 1)^{-1} g \wedge \delta\omega, \tag{9}$$

for $1 \le r \le n-1$.

This form ω is a generalization of an 1-form $\omega \in \Omega^1(M)$, which is dual to the concircular vector field $Z = \omega^{\#}$. The set of all such r-forms constitutes a vector space $P^r \subset T^r$ (see [19]).

3 Dimensions of Vector Spaces of Killing Forms and Vector Fields on Non-compact Riemannian Manifold

3.1 Let (M, g) be an n-dimensional connected Riemannian manifold. Let us remind some well known facts on dimensions of three types spaces of Killing vector fields on (M, g). It is known that the dimension of a Lie algebra of a group $C(M, g)$ of infinitesimal conformal transformations of connected Riemannian n-dimensional manifold (M, g) is not greater than $\frac{1}{2}(n+1)(n+2)$ and this algebra is a vector space of conformal Killing vector fields (see [11, p. 120]). The equality is obtained in the case of conformally flat Riemannian manifold (M, g).

The dimension of a Lie algebra of a subgroup $I(M, g)$ of infinitesimal transformations is not greater than $\frac{1}{2}(n+1)n$ and this algebra is a vector space of Killing vector fields (see [11, p. 101]). The equality is obtained in the case of Riemannian manifold (M, g) with constant curvature.

The dimension of a vector space of concircular vector fields on connected n-dimensional manifold (M, g) is not greater than $n+1$ and the equality is obtained for manifold with constant curvature (see [6]).

3.2 To generalize facts presented above we will find precise upper bounds of dimensions of vector spaces of three types of Killing r-forms ($1 \le r \le n-1$). Let us investigate a connected n-dimensional Riemannian manifold (M, g). To form $\omega \in C^\infty \Lambda^r(M)$ the condition of integrability of arbitrary equation of a systems of Eqs. (5), (7) or (8) is the *Ricci identity* ([5, § 11]). This identity has in any local coordinate system $x^1, \ldots, x^n$ of a manifold (M, g) the following expression:

$$\nabla_j \nabla_k \omega_{i_1 i_2 \ldots i_r} - \nabla_k \nabla_j \omega_{i_1 i_2 \ldots i_r} = -\sum_{\alpha=1}^{r} \omega_{i_1 i_2 \ldots i_{\alpha-1} l i_{\alpha+1} \ldots i_r} R^l_{i_\alpha jk}\,, \tag{10}$$

where $\omega_{i_1, i_2 \ldots i_p} = \omega(X_{i_1}, X_{i_2}, \ldots, X_{i_p})$ and $R^i_{jkl} X_i = R(X_k, X_l) X_j$ are local components of a conformal Killing r-form and curvature tensor R for $X_k = \frac{\partial}{\partial x^k}$ and $\nabla_k = \nabla_{X_k}$.

Ricci identity (10) establishes restrictions not only the choice of components of r-form ω but also on the curvature tensor R of the manifold (M, g). We have the following theorem.

Theorem 3.1 *On an n-dimensional connected Riemannian manifold (M, g), the dimensions t_r, k_r and p_r of vector spaces of conformal Killing r-form T^r, co-closed confomal Killing (Killing) r-form K^r and closed conformal Killing r-form P^r, $(1 \leq r \leq n-1)$, respectively have the following upper bounds*

$$t_r \leq \frac{(n+2)!}{(r+1)!(n-r+1)!}, \quad k_r \leq \frac{(n+1)!}{(r+1)!(n-r)!}, \quad p_r \leq \frac{(n+1)!}{r!(n-r+1)!}.$$

The equalities are obtained in the case of Riemannian manifold (M, g) with constant nonzero curvature.

Proof The case when (M, g) is locally flat manifold is trivial; in [19, 20] on the basis of (10) there are defined components $\omega_{i_1 \ldots i_r} = A_{k i_1 \ldots i_r} x^k + B_{i_1 \ldots i_r}$ of Killing r-form ω, for an arbitrary local Cartesian coordinate system $x^1, \ldots, x^n$. Here, $A_{k i_1 \ldots i_r}$, $B_{i_1 \ldots i_r}$ are local components of constant skew-symmetric $(r+1)$-forms and r-forms, respectively.

With respect to this result, in [18] there for some special coordinate system $x^1, \ldots, x^n$ of manifold (M, g) with constant sectional curvature $C \neq 0$ were found components $\omega_{i_1 \ldots i_r} = \mathrm{e}^{(r+1)\varphi}(A_{k i_1 \ldots i_r} x^k + B_{i_1 \ldots i_r})$, $\varphi = \frac{1}{2(n+1)} \ln(\det g)$, of Killing r-form ω. Therefore the dimension k_r of a space K^r of coclosed conformal Killing r-forms on Riemannian manifold with constant curvature is equal to

$$k_r = \binom{n}{r+1} + \binom{n}{r} = \binom{n+1}{r+1} = \frac{(n+1)!}{(r+1)!(n-r)!}.$$

In the case of an arbitrary connected manifold (M, g), it is evident that the dimension k_r of a space K^r is not greater then this number, i. e. $k_r \leq \frac{(n+1)!}{(r+1)!(n-r)!}$.

It is known (see [8, 19, 20]) that there exists an isomorphism $* : K^{n-r} \to P^r$. It gives the possibility to count the dimension p_r of a space P^r of closed conformal Killing r-forms on a manifold (M, g) with nonzero constant sectional curvature

$C \neq 0$; this dimension is equal to $p_r = \frac{(n+1)!}{r!(n-r+1)!}$. For arbitrary connected manifold (M, g), it is evident that the dimension p_r of a space P^r is not greater then this number, i. e. $p_r \leq \frac{(n+1)!}{r!(n-r+1)!}$.

For manifold (M, g) with constant sectional curvature $C \neq 0$, in [7, 26] there by direct calculation was obtained decomposition of any conformal Killing r-form ω into direct sum $\omega = \omega_1 + \omega_2$ of a coclosed conformal Killing (Killing) r-form ω_1 and of closed conformal Killing (planar) r-form $\omega_2 = \nabla\Theta$ for Killing $(r-1)$-form Θ.

Clearly, the arbitrariness of choice of an r-form $\omega_2 = \nabla\Theta$ is given by the number of parameters on which a Killing $(r-1)$-form depends. This number is equal to $\binom{n}{r} + \binom{n}{n-1} = \binom{n+1}{r}$. In the case of an arbitrary connected manifold (M, g), it is obvious that the arbitrariness of determination of an exact conformal Killing r-form $\omega_2 = \nabla\Theta$ does not be greater then this number.

Based on a pointwise decomposition $\omega = \omega_1 + \omega_2$ the expression of an arbitrary conformal Killing r-form ω in some special local coordinate system $x^1, \ldots, x^n$ on a manifold (M, g) with nonzero constant curvature $C \neq 0$

$$\omega_{i_1\ldots i_r} = \mathrm{e}^{(r+1)\varphi}\left(A_{ki_1\ldots i_r}x^k + B_{i_1\ldots i_r} - \frac{1}{C}\left(\varphi_{[i_1}C_{ki_2\ldots i_r]}x^k + \varphi_{[i_1}D_{i_2\ldots i_r]} + \frac{1}{r}C_{i_1\ldots i_r}\right)\right),$$

where $A_{k,i_1\ldots i_r}$, $B_{i_1\ldots i_r}$, $C_{i_1 i_2\ldots i_r}$ and $D_{i_1\ldots i_{r-1}}$ are local components of constant skew-symmetric forms. Now, we may compute the dimension t_r of a space T^r of conformal Killing r-forms on a manifold with constant curvature $C \neq 0$ which is equal to the following summ

$$t_r = \binom{n+1}{r+1} + \binom{n+1}{r} = \binom{n+2}{r+1} = \frac{(n+2)!}{(r+1)!(n-r+1)!}.$$

In the case of an arbitrary connected manifold (M, g), it is obvious that the dimension of a space T^r is not greater then this number, i. e.

$$t_r \leq \frac{(n+2)!}{(r+1)!(n-r+1)!}.$$

We have proved the theorem.

Considering the upper bounds of dimension of vector space of conformal Killing forms T^r and of space of Killing r-forms K^r as found in Theorem 3.1 we for $r = 1$ obtain the following well known proposition.

Corollary 3.1 *The dimensions of vector spaces of conformal Killing vector field T^1, Killing vector fields K^1 and concircular vector fields P^1 on connected Riemannian manifold (M, g) do not be greater then $\frac{1}{2}(n+1)(n+2)$, $\frac{1}{2}(n+1)n$ and $n+1$, respectively. The equalities are obtained in the case of manifold with constant nonzero section curvature.*

3.3 Formulated in the theorem and its corollary results on dimensions of vector spaces of Killing and conformal Killing forms and of vector fields are *substantially local*.

As an example, let us consider an n-dimensional ($n \geq 2$) hyperbolic space which is a Riemannian manifold with constant negative curvature. As we have proved above, in this space the dimension k_r of the space of coclosed conformal Killing (Killing) r-forms ($1 \leq r \leq n-1$) and, especially, the dimension k_1 of Killing vector spaces is equal to $\frac{(n+1)!}{(r+1)!(n-r)!}$ and $\frac{1}{2}(n+1)n$, respectively. Factorizing hyperbolic space according to a suitable discrete group of motions (see [29, § 2.4]) we obtain a compact manifold with constant curvature. On this manifold there exists "generally" no nonzero Killing r-form (see [32, § 1 of Chap. VI]). Therefore, our result on the dimension of a space of Killing r-forms as well as well known result on the dimension of a space of Killing vector field deals with "local dimensions" k_r and k_1, especially.

An analogous conclusion may be obtained for dimensions of spaces of conformal Killing forms and vector fields and also for closed conformal Killing forms and concircular vector field.

4 Dimensions of Spaces of Conformal Killing Forms on a Closed Riemannian Manifold

4.1 Let (M, g) be a closed (i.e. compact without the border ∂M) oriented Riemannian manifold. Let us denote by $\langle \cdot, \cdot \rangle$ the global inner product

$$\langle \omega, \omega' \rangle = \int_M \frac{1}{r!} g(\omega, \omega') dv \tag{11}$$

for arbitrary compact carriers $\omega, \omega' \in \Omega^r(M)$ of r-form and volume element dv. Then the exterior differential operator d: $\Omega^r(M) \rightarrow \Omega^{r+1}(M)$ and the adjoin codifferential operator δ: $\Omega^{r+1}(M) \rightarrow \Omega^r(M)$ are connected by the following equality (see [13, p. 204])

$$\langle \mathrm{d}\omega, \Theta \rangle = \langle \omega, \delta\Theta \rangle \tag{12}$$

for any $\omega \in \Omega^r(M)$ and $\Theta \in \Omega^{r+1}(M)$.

On a closed manifold (M, g), it holds the following Hodge-de Ram orthogonal decomposition with respect to the global inner product (11) (see [12]):

$$\Omega^r(M) = \mathrm{Im}\,\mathrm{d} \oplus \mathrm{Im}\,\delta \oplus \mathrm{Ker}\,\Delta, \tag{13}$$

where $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$ is Laplace operator with $\mathrm{Ker}\,\Delta = \mathrm{Im}\,\mathrm{d} \cap \mathrm{Ker}\,\delta$. In addition, there are following orthogonal decompositions (see [12])

$$\mathrm{Ker}\,\delta = \mathrm{Im}\,\delta \oplus \mathrm{Ker}\,\Delta, \tag{14}$$

$$\operatorname{Ker} \mathrm{d} = \operatorname{Im} \mathrm{d} \oplus \operatorname{Ker} \Delta . \tag{15}$$

The kernel of Laplace operator Δ on (M, g) is a finite dimension vector space $H^r(M, \mathbb{R}) = \{\omega \in \Omega^r(M) | \Delta\omega = 0\}$ of harmonic r-forms, for $r = 1, \ldots, n-1$ (see [15, § 25], [12]). The dimension of $H^r(M, \mathbb{R})$ is equal to the Betti number $b_r(M)$ of a manifold (M, g), i.e. $b_r(M) = \dim_{\mathbb{R}} \operatorname{Ker} \Delta$. It is known (see [12]), that Betti numbers of a manifold (M, g) are dual in the sense of $b_r(M) = b_{n-r}(M)$ and for $n = 2r$ Betti numbers are invariant with respect to the conformal transformation of metric $\overline{g} = e^{2F} g$, because in this case $\overline{\delta} = \delta$ (see [2, Corollary 1.162]).

4.2 Let an n-dimensional closed manifold (M, g) be given and let us consider a natural with respect to isometric dipheomorphisms differential operator of the first order D: $\Omega^r(M) \to \Omega^1(M) \otimes \Omega^1(M)$ being define by the following (see [3, pp. 312–313], [16, 20])

$$\mathrm{D} = \nabla - (r+1)^{-1}\mathrm{d} - (n-r+1)^{-1} g \wedge \delta, \tag{16}$$

where $\wedge$ denotes multiplication of an $(r-1)$-form $\delta\omega$ by a metric tensor which is defined by the following rule

$$(g \wedge \delta\omega)(X_0, X_1, \ldots, X_r) = \sum_{a=1}^{r} (-1)^a g(X_0, X_a)(\delta\omega)(X_1, \ldots, \widehat{X_a}, \ldots, x_r)$$

for arbitrary $(X_0, \ldots, X_r) \in C^\infty(TM)$.

Then the condition $\omega \in \operatorname{Ker} \mathrm{D}$, which is equal to the identity $\nabla\omega = (r+1)^{-1}\mathrm{d}\omega + (\mathrm{n} - \mathrm{r} + 1)^{-1} g \wedge \delta\omega$, is a necessary and sufficient condition that an r-form ω is a conformal Killing form, $r = 1, \ldots, n-1$ (see [16, 19, 20]).

Especially, it follows from this that for a conformal Killing vector field $Z = \omega^{\#}$ any 1-form ω belongs to the kernel of differential operator of the first order

$$\mathrm{D} := \nabla\omega - \frac{1}{2}\mathrm{d}\omega + \frac{1}{\mathrm{n}} g \cdot \delta\omega,$$

which is called Ahlfors operator (see [14]).

In the [23] there is for operator D, defined by (16), found an adjoint operator D^*. Moreover, there is also the rough Laplacian constructed by (see [2, Definition 1.135], [3, pp. 316–317])

$$\mathrm{D}^*\mathrm{D} = \frac{1}{r(r+1)} \left(\nabla^*\nabla - \frac{1}{r+1}\delta \mathrm{d} - \frac{1}{n-r+1}\mathrm{d}\delta \right), \tag{17}$$

where $\nabla^*\nabla$ is the *rough Bochner Laplacian* ([21]).

It follows from general theory that rough Laplacian on a closed Riemannian manifold (M, g) is positive and elliptic (see [3, pp. 316–317]). Because it is an elliptic operator, its kernel is a finite dimensional vector subspace. Therefore, the kernel of a rough Laplacian $\operatorname{Ker} \mathrm{D}^*\mathrm{D}$ is a finite dimensional vector space $T^r(M, \mathbb{R}) = \{\omega \in$

$\Omega^r(M) \mid D^*D\omega = 0\}$. Using the identity $\langle D^*D\omega, \omega\rangle = \langle D\omega, D\omega\rangle$, we have that this kernel consists of conformal Killing r-forms, for all $r = 1, \ldots, n-1$ (see [16]).

In [17] and [24], the dimension $t_r(M) = \dim_{\mathbb{R}} \operatorname{Ker} D^*D$, for all $r=1, \ldots, n-1$ is called *Tachibana number* of closed Riemannian manifold (M, g) as an analogy to the Betti number $b_r(M) = \dim_{\mathbb{R}} \operatorname{Ker} \Delta$. Tachibana numbers as well as Betti numbers are dual in the sense of $t_r(M) = t_{n-r}(M)$ (see [17, 22, 24].) Evidently,

$$t_r(M) \leq \frac{(n+2)!}{(r+1)!(n-r+1)!}.$$

One of the most important properties of conformal Killing forms is their conformal invariance (see [4]), i.e. for an arbitrary conformal Killing r-form ω the form $\bar{\omega} = e^{(r+1)f}\omega$ is a conformal Killing form with respect to the conformally equivalent metric $\bar{g} = e^{2f}g$. This implies that Tachibana numbers $t_r(M)$ for $r = 1, \ldots, n-1$ are conformal scalar invariants of a closed Riemannian manifold (see [17, 24]).

Coclosed conformal Killing (Killing) r-forms $(1 \leq r \leq n-1)$ form the vector space $K^r(M, \mathbb{R}) = \{\omega \in \Omega^r(M) \mid D^*D\omega = \delta\omega = 0\}$. The dimension $k_r(M) = \dim_{\mathbb{R}}(\operatorname{Ker} D^*D \cap \operatorname{Ker} \delta)$ was in [17] and [24] called *Killing number* of closed Riemannian manifold (M, g). Evidently,

$$k_r(M) \leq \frac{(n+1)!}{(r+1)!(n-r)!}.$$

Closed conformal Killing (planar) r-forms $(1 \leq r \leq n-1)$ form the vector space $P^r(M, \mathbb{R}) = \{\omega \in \Omega^r(M) \mid D^*D\omega = d\omega = 0\}$. Its dimension $p_r(M) = \dim_{\mathbb{R}}(\operatorname{Ker} D^*D \cap \operatorname{Ker} d)$ was in [17] and [24] called *planar number* of closed Riemannian manifold (M, g). Evidently,

$$p_r(M) \leq \frac{(n+1)!}{r!(n-r+1)!}.$$

One of the most important properties of Killing and planar r-forms is their conformal invariance (see [4]), i.e. for an arbitrary conformal Killing (or planar) r-form ω the form

$$\bar{\omega} = e^{-(r+1)f}\omega \ \text{ for } \ f = (n+1)^{-1} \ln \sqrt{\frac{\det g}{\det \bar{g}}}$$

is a Killing (respectively, planar) r-form with respect to the projectively equivalent metric $\bar{g}$ (see [24]).This implies that the Killing numbers $k_r(M)$ and planar numbers $p_r(M)$ for $r = 1, \ldots, n-1$ are projective scalar invariants of the Riemannian manifold (M, g).

To summarize, we may formulate

Proposition 4.1 *On an n-dimensional closed Riemannian manifold (M, g) the following hold for all r, $r=1, \ldots, n-1$*

1. *Tachibana numbers $t_r(M)$ are conformal scalar invariants, they are dual in the sense $t_r(M) = t_{n-r}(M)$ and they fulfill the relation*

$$t_r(M) \leq \frac{(n+2)!}{(r+1)!(n-r+1)!};$$

2. *Killing numbers $t_r(M)$ and planar numbers $p_r(M)$ are projective scalar invariants, they are dual in the sense $k_r(M) = p_{n-r}(M)$ and they fulfill the relations*

$$k_r(M) \leq \frac{(n+1)!}{(r+1)!(n-r)!} \quad \textit{and} \quad p_r(M) \leq \frac{(n+1)!}{r!(n-r+1)!}.$$

4.3 We establish a connection between Betti numbers and Tachibana numbers. Two following theorems hold.

Theorem 4.1 *If Ricci tensor Ric of an n-dimensional compact and oriented conformal planar Riemannian manifold (M, g), $n \geq 2$, is definite, then Tachibana $t_k(M)$ and Betti $b_l(M)$ numbers cannot be different from zero for arbitrary pair of indices $k, l = 1, \ldots, n-1$.*

Proof Let Ricci tensor Ric be definite on a compact manifold (M, g), i.e. quadratic form $Ric(X, X)$ is definite for arbitrary nonzero vector field $X \in C^\infty TM$. Let us suppose that quadratic form $Ric(X, X)$ is positive definite and manifold (M, g) be compact and oriented conformal flat manifold. Then $b_1(M) = \cdots = b_{n-1}(M) = 0$, in accordance with [31].

Further, to get a new expression of the rough Laplacian (17), let us use the classical Bochner-Weitzenböck formula [2], which gives $\Delta = \nabla^*\nabla + F_r$, where F_r may be a algebraically (even linearly) expressed in terms of the curvature tensor R of manifold (M, g). Now, the rough Laplacian may be written in the form

$$\mathrm{D}^*\mathrm{D} = \frac{1}{r(r+1)}\left(\mathrm{d}^*\mathrm{d} - \frac{n-r}{n-r+1}\mathrm{d}\mathrm{d}^* - F_r\right).$$

Then any conformal Killing r-form ω must fulfill the following equation

$$\int_M g(F_r(\omega), \omega)dv = \frac{r}{r+1}\langle \mathrm{d}\omega, \mathrm{d}\omega\rangle + \frac{n-r}{n-r+1}\langle \mathrm{d}^*\omega, \mathrm{d}^*\omega\rangle.$$

If we suppose that quadratic form $Ric(X, X)$ is negative definite, then on a compact conformal flat manifold we have the following inequality [31]

$$g(F_r(\Theta), \Theta) \leq -\frac{n-r}{n-1}\lambda \cdot g(\Theta, \Theta),$$

for the greatest (negative) eigenvalue $-\lambda$ of the matrix $\|Ric\|$, for all $1 \leq r \leq n-1$ and any nonzero form $\Theta \in \Omega^r(M)$. Consequently, any conformal Killing r-form ω must fulfill the inequality

$$\frac{r}{(n-r)(r+1)}\langle \mathrm{d}\omega, \mathrm{d}\omega\rangle + \frac{1}{n-r+1}\langle \mathrm{d}^*\omega, \mathrm{d}^*\omega\rangle \le -\frac{n-r}{n-1}\lambda\langle \omega, \omega\rangle.$$

This is possible only if a conformal Killing r-form vanishes at any point of manifold (M, g) and then $t_1(M) = \cdots = t_{n-1}(M) = 0$. The theorem is proved.

Theorem 4.2 *If on an n-dimensional closed Riemannian manifold* (M, g) *Betti number* $b_r(M) = 0$*, for* $1 \le r \le n-1$*, and Tachibana number* $t_r(M) > k_r(M) \ne 0$*, then for Killing numbers* $k_r(M)$ *and planar numbers* $p_r(M)$ *it holds* $t_r(M) = k_r(M) + p_r(M)$*.*

Proof Let on an n-dimensional closed Riemannian manifold (M, g) Betti number $b_r(M) = 0$, for $1 \le r \le n-1$. Then in decompositions (13–15) there is $\mathrm{Ker}\,\Delta = 0$ and therefore we have the following orthogonal decomposition

$$\Omega^r(M) = \mathrm{Im}\,\mathrm{d} \oplus \mathrm{Im}\,\delta \tag{18}$$

and besides $\mathrm{Ker}\,\mathrm{d} = \mathrm{Im}\,\mathrm{d}$ and $\mathrm{Ker}\,\delta = \mathrm{Im}\,\delta$. Since $t_r(M) > k_r(M) \ne 0$ we obtain for Killing number $k_r(M) = \dim_{\mathbb{R}}(\mathrm{Ker}\,\mathrm{D}^*\mathrm{D} \cap \mathrm{Im}\,\mathrm{d}^*)$ and for planar number $p_r(M) = \dim_{\mathbb{R}}(\mathrm{Ker}\,\mathrm{D}^*\mathrm{D} \cap \mathrm{Im}\,\mathrm{d}) \ne 0$. Consequently, it is clear that decomposition (18) implies the following orthogonal decomposition

$$\mathrm{Ker}\,\mathrm{D}^*\mathrm{D} = (\mathrm{Ker}\,\mathrm{D}^*\mathrm{D} \cap \mathrm{Im}\,\delta) \oplus (\mathrm{Ker}\,\mathrm{D}^*\mathrm{D} \cap \mathrm{Im}\,\mathrm{d}). \tag{19}$$

It may be rewritten in the form $T^r(M, \mathbb{R}) = K^r(\mathbb{R}, M) \oplus P^r(M, \mathbb{R})$, $1 \le r \le n-1$, where in accordance with (14) a space $K^r(M, \mathbb{R})$ consists of co-exact conformal Killing r-forms and in accordance with (15) a space $P^r(M, \mathbb{R})$ consists of exact ones. It follows from this the equality $t_r(M) = k_r(M) + p_r(M)$. The summands on the right side represent dimensions of namely these vector spaces. Let us remark that in the case $t_r(M) = 0$ the equality which may be proved turns into an identity.

The proof is finished.

Remark 4.1 In the article [25], an analogical orthogonal decomposition $T^r(M, \mathbb{R}) = K^r(M, \mathbb{R}) + P^r(M, \mathbb{R})$ was established on an $2r$-dimensional closed conformal flat Riemannian manifold (M, g) with constant positive scalar curvature. Let us remark, that $b_r(M) = 0$ is fulfilled in such manifold in accordance with [31].

In [17] this decomposition is established for a closed manifold with positive curvature operator, where $b_1(M) = \cdots = b_{n-1}(M) = 0$ in accordance with [13]. It is important to say that in the both decompositions the space $P^r(M, \mathbb{R})$ consists of exact conformal Killing r-forms, $1 \le r \le n-1$, and the space $K^r(M, \mathbb{R})$ consists of coexact ones. These facts does not contained in the cited article.

Especially, for $r = 1$ we have the following corollary.

Corollary 4.1 *If for an n-dimensional closed Riemannian the first Betti number* $b_1(M) = 0$ *and at the same time* $t_1(M) \ne 0$*,* $t_1(M) \ne k_1(M)$*, then* (M, g) *is globally conformal to an n-dimensional sphere* S^n *of Euclidean space* $\mathbb{R}^{n+1}$*. If* $s = const$*,*

$s > 0$, additionally, then a manifold (M, g) is globally isometric to the sphere S^n. If for an n-dimensional closed Riemannian manifold $s = const$ and $s = 0$ or $s = const$ and $s < 0$, then $t_1(M) = k_1(M)$.

Proof For an n-dimensional closed Riemannian manifold (M, g) with $b_1(M) = 0$, $t_1(M) \neq 0,\ t_1(M) \neq k_1(M)$ we have the following orthogonal decomposition with respect to the global inner product

$$T^1(M, \mathbb{R}) = K^1(M, \mathbb{R}) + P^1(M, \mathbb{R}), \tag{20}$$

where $P^1(M, \mathbb{R})$ contains at least one exact conformal Killing 1-form, i.e. a form which may be expressed as $\omega = \text{gradf}$ with $\nabla\nabla f = \rho g$, where $\rho = -\frac{1}{n}\Delta f$. In this case, manifold (M, g) is globally conformal to an Euclidean sphere S^n with the standard metric $\bar{g}_{can}$ (see [27]). It is known (see [32]), that the presence of orthogonal decomposition (20) on closed Riemannian manifold (M, g) with an additional condition $s = const$ implies that (M, g) is globally isometric to the Euclidean sphere S^n. Let us remark, that for a compact Riemannian manifold (M, g) with constant negative or zero curvature it holds $t_1(M) = k_1(M)$, because in this case any conformal Killing vector field is a Killing field (see [9]).

Remark 4.2 Conformal Killing and closed conformal Killing vector field is dual to conformal Killing and planar 1-form, respectively. At the end of the past and beginning of this century, these vector fields have been objects of an intensive interest in connection with the study of groups of infinitesimal conformal transformations of search criteria for conformity and isometry Riemannian manifold to the Euclidean sphere (see [1, 6, 23, 28] and others.) Therefore, there exists a great number of propositions, which are analogical to our proved corollary. Let us mention one of them (see [23]). It says that any compact Riemannian manifold having finite fundamental group $\pi_1(M)$ and admitting a closed conformal Killing vector field, which is not an infinitesimal isometry, is diffeomorphic to the Euclidean sphere. Adding the condition of constancy of the scalar curvature we obtain that such manifold must be isometric to the Euclidean sphere (see [28]). Le us add, the finiteness of a fundamental group $\pi_1(M)$ implies automatically that first Betti number $b_1(M)$ is equal to zero and closed conformal Killing vector field is the gradient at the same time.

Acknowledgments The paper was supported by grant P201/11/0356 of The Czech Science Foundation.

References

1. Alodan, H.: Conformal gradient vector fields. Differential Geometry—Dynamical System **12**, 1–3 (2010)
2. Besse, A.L.: Einstein Manifolds. Springer, Berlin (1987)
3. Besse, A.L.: Géométrie Riemannienne en dimension 4. Cedic-Fernand Nathan, Paris (1981)

4. Benn, I.M., Chalton, P.: Dirac symmetry operators from conformal Killing-Yano tensors. Class. Quantum Gravity **14**(5), 1037–1042 (1997)
5. Eisenhart, L.P.: Riemannian Geometry. Princeton University Press, Princeton (1926)
6. Kühnel, W., Rademacher, H.-B.: Conformal transformations of pseudo-Riemannian manifolds. Recent developments in pseudo-Riemannian geometry. Zürich Eur. Math. Soc. 261–298 (2008)
7. Kashiwada, T.: On conformal Killing tensor. Nat. Sci. Rep. Ochanomizu Univ **19**(2), 67–74 (1968)
8. Kora, M.: On conformal Killing forms and the proper space of for p-forms. Math. J. Okayama Univ. **22**, 195–204 (1980)
9. Lichnerowicz, A.: Géométrie des groupes de transformations. Dunod, Paris (1958)
10. Maccallum, M., Herlt, E., Schmutzer, E.E.: Exact Solutions of the Einstein Field Equations. Deutscher Verlag der Wissenschaften, Berlin (1980)
11. Mikeš, J., Vanžurová, A., Hinterleitner, I.: Geodesic Mappings and Some Generalizations. Palacký University Press, Olomouc (2009)
12. Novikov, S.P.: Topology I. Encycl. Math. Sci. **12**, 1–310 (1996) (translation from Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya **12**, 5–252 (1986))
13. Petersen, P.: Riemannian Geometry. Springer, New York (1997)
14. Pierzchalski, A.: Ricci curvature and quasiconformal deformations of a Riemannian manifold. Manuscripta Math. **66**, 113–127 (1989)
15. De Rham, G.: Variétés Différentiables. Hermann, Paris (1955)
16. Stepanov, S.E.: A new strong Laplacian on differential forms. Math. Notes **76**(3), 420–425 (2004)
17. Stepanov, S.E.: Curvature and Tachibana numbers. Sbornik Math. **202**(7), 135–146 (2011)
18. Stepanov, S.E.: The Killing-Yano tensor. Theoret. Math. Phys. **134**(3), 333–338 (2003)
19. Stepanov, S.E.: The vector space of conformal Killing forms on a Riemannian manifold. J. Math. Sci. **110**(4), 2892–2906 (2002)
20. Stepanov, S.E.: On conformal Killing 2-form of the electromagnetic field. J. Geom. Phys. **33**, 191–209 (2000)
21. Stepanov, S.E.: Vanishing theorems in affine, Riemannian and Lorenz geometries. J. Math. Sci. **141**(1), 929–964 (2007)
22. Semmelmann, U.: Conformal Killing forms on Riemannian manifolds. Math. Z. **245**(3), 503–627 (2003)
23. Suyama, Y., Tsukamoto, Y.: Riemannian manifolds admitting a certain conformal transformation group. J. Diff. Geom. **5**, 415–426 (1971)
24. Stepanov, S.E.: Some conformal and projective scalar invariants of Riemannian manifolds. Math. Notes **80**(6), 848–852 (2006)
25. Tachibana, S.: On the proper space of for m-forms im 2m dimensional conformal flat Riemannian manifolds. Nat. Sci. Rep. Ochanomizu Univ. **28**, 111–115 (1978)
26. Tachibana, S.: On conformal Killing tensor in a Riemannian space. Tohoku Math. J. **21**, 56–64 (1969)
27. Tashiro, Y.: Complete Riemannian manifolds and some vector fields. Trans. Am. Math. Soc. **117**, 251–275 (1965)
28. Tanno, S., Weber, W.C.: Closed conformal vector fields. J. Diff. Geom. **3**, 361–366 (1969)
29. Wolf, J.A.: Space of Constant Curvature. California University Press, Berkley (1972)
30. Yano, K.: Concircular geometry. Proc. Imp. Acad. Tokyo **16**, 195–200, 354–360, 442–448, 505–511 (1940)
31. Yano, K., Bochner, C.: Curvature and Betti Numbers. Princeton University Press, Princeton (1953)
32. Yano, K., Sawaki, S.: Riemannian manifolds admitting a conformal transformation group. J. Diff. Geom. **2**, 161–184 (1968)