On Sinyukov's Equations in Their Relation to a Curvature Operator of Second Kind

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Abstract Many authors have studied Riemannian manifolds admitting a geodesic mapping. Fundamental results of the theory of geodesic mapping were settled by Sinyukov. In the present paper we analyze the Sinykov equations of the geodesic mappings of Riemannian manifolds by using the curvature operator of the second kind. This approach to the study of geodesic mapping is essentially new.

1 Introduction

In a Riemannian manifold, the Riemannian curvature tensor R defines two kinds of curvature operators: the operator $\mathring{R}$ of first kind, acting on 2-forms, and the operator $\mathring{R}$ of second kind, acting on symmetric 2-tensors. In our paper we analyze the Sinyukov equations of geodesic mappings of Riemannian manifolds by using the curvature operator of the second kind.

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A. Makhlouf et al. (eds.), *Algebra, Geometry and Mathematical Physics*, 489
Springer Proceedings in Mathematics & Statistics 85, DOI: 10.1007/978-3-642-55361-5_28,

2 Equation Systems of Geodesic Mappings and Einstein Manifolds

The condition, for which an n-dimensional ($n \geq 2$) Riemannian manifold (M, g) admits a geodesic mapping onto another n-dimensional Riemannian manifold $(\bar{M}, \bar{g})$, has the following form of differential equations of Cauchy type in covariant derivatives

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik}, \tag{1}$$

$$n\nabla_j \lambda_i = \mu g_{ij} - a_{ik} R^k_j + a^{kl} R_{ikjl}, \tag{2}$$

$$(n-1)\nabla_i \mu = -2(n+1)\lambda_k R^k_i + a_{kl}\left(\nabla_i R^{kl} - 2g^{kj}\nabla_j R^l_i\right). \tag{3}$$

These equations were obtained by Sinyukov more than fifty years ago (see [1–4]).

A geodesic mapping is *non trivial* (or *non affine*) if $\lambda \not\equiv \text{const}$.

Here $a = (a_{ij})$ is a regular symmetric 2-tensor, $Ric = (R_{ij})$ is the Ricci tensor whose components are given by $R_{jl} = g^{ik} R_{ijkl}$ for local components of the Riemannian curvature tensor $R = (R_{ijkl})$; λ_i and μ are defined in the following way

$$\lambda_i = \frac{1}{2}\nabla_i (g^{kl} a_{kl}); \tag{4}$$

$$\mu = g^{kl}\nabla_k \lambda_l. \tag{5}$$

With respect to the above Eqs. (4) and (5) Eq. (2) can be rewritten as

$$n\, \nabla_i \nabla_j \lambda = \Delta\lambda \cdot g_{ij} - a_{ik} R^k_j + a^{kl} R_{ikjl}, \tag{6}$$

where

$$\Delta\lambda = g^{ij}\nabla_i \nabla_j \lambda$$

is the Laplace operator acting on the scalar function $\lambda = \frac{1}{2}\, g^{kl} a_{kl}$.

From (6) follows (see [4, p. 138]):

$$a_{ik} R^k_j = R^k_i a_{kj}. \tag{7}$$

If we suppose that the manifold (M, g) is an Einstein manifold, then from Eq. (3) we conclude the following

$$(n-1)\nabla_i \Delta\lambda = -2\,\frac{n+1}{n}\, S\, \nabla_i \lambda \tag{8}$$

for the scalar curvature S (= const). After multiplying the left and right sides of (8) by $\nabla^i \lambda$ and after integration over the compact manifold we have

$$(n-1)\int_M (\Delta\lambda)^2 dv = 2\,\frac{n+1}{n}\,S\int_M \left(\nabla_i\lambda\cdot\nabla^i\lambda\right)dv, \tag{9}$$

because $\nabla_i\left(\Delta\lambda\cdot\nabla^i\lambda\right) = \nabla_i\Delta\lambda\cdot\nabla^i\lambda + (\Delta\lambda)^2$. From (9) we conclude that $S > 0$. This is in accordance with the paper Couty [5].

3 An Algebraic Operator Associated with the Curvature Tensor

We will consider the space of symmetric 2-forms S^2M over the Riemannian manifold (M, g). In particular, the tensor $a = (a_{ij})$ is a smooth cross-section of S^2M. The space S^2M (see [6]) has the pointwise orthogonal decomposition

$$S^2M = C^\infty M\cdot g\oplus S_0^2M,$$

where $C^\infty M$ is a space C^∞-functions on M and S_0^2M is a subspace of the space S^2M, which contains symmetric 2-forms with zero traces.

We introduce (see [7, 8]) a curvature operator of second kind $\mathring{R}\colon S^2M\to S^2M$ with components

$$R^{ij}{}_{kl} = \frac{1}{2}\left(g^{im}R^j_{kml} + g^{jm}R^i_{kml}\right)$$

for the curvature tensor $R = (R^i_{jkl})$, whose actions are defined by the formulas $\mathring{R}(b_{ij}) = R_{ikjl}b^{kl}$ for any smooth cross-section $b = (b_{ij})$ of S^2M. On the basis of the curvature operator of second kind (see [9]) we can define a linear symmetric operator $B_2\colon S^2M\to S_0^2M$ with components

$$B^{ij}_{kl} = \frac{1}{2}\left(g^{im}R^j_{kml} + g^{jm}R^i_{kml}\right) + \frac{1}{4}\left(\delta^i_kR^j_l + \delta^j_kR^i_l + \delta^i_lR^j_k + \delta^j_lR^i_k\right) \tag{10}$$

for the Ricci operator $Ric^* = (g^{im}R_{mj})$. From (10) we have

$$B_2(b_{ij}) = R_{ikjl}b^{kl} - \frac{1}{2}\left(R^m_ib_{mj} + R^m_jb_{mi}\right) \tag{11}$$

for any smooth section $b = (b_{ij})$ on S^2M. The operator $B_2\colon S^2M\to S_0^2M$ is a linear and symmetric operator. Then there exists a pointwise orthogonal decomposition $S^2M = ImB_2\oplus KerB_2$ of the space S^2M of symmetric 2-tensors on M. It is obvious that $C^\infty M\cdot g\subset KerB_2$ and $ImB_2\subset S_0^2M$. The following theorem holds.

Theorem 3.1 *If a complete Riemannian manifold (M, g) of dimension $n\geq 2$ admits geodesic mapping is a non affine and the tensor $a = (a_{ij})$ belongs to the kernel of the symmetric linear operator $B_2\colon S^2M\to S_0^2M$ then (M, g) is conformal to a sphere S^n in $(n+1)$ dimensional Euclidean space.*

Proof According to (11) we can rewrite Eq. (6) in the following form

$$\nabla_i \nabla_j \lambda - \frac{1}{n} \Delta\lambda g_{ij} = -B_2(a_{ij}). \tag{12}$$

If we assume that $B_2(a_{ij}) = 0$ in (12), then we have the following

$$\nabla_i \nabla_j \lambda = \frac{1}{n} \Delta\lambda g_{ij}. \tag{13}$$

We note that (see [10]) the complete Riemannian manifold (M, g) of dimension $n \geq 2$ is conformal to a sphere S^n in $(n+1)$-dimensional Euclidean space if on (M, g) exists a non-constant function $\lambda \in C^\infty M$ satisfying Eq. (13). Therefore a complete Riemannian manifold (M, g) of dimension $n \geq 2$ admitting geodesic mappings onto another n-dimensional Riemannian manifold $(\bar{M}, \bar{g})$ is conformal to a sphere S^n in $(n+1)$-dimensional Euclidean space if the tensor $a = (a_{ij})$ from the equations of geodesic mappings (1–3) belongs to the kernel $Ker B_2$ of the linear operator B_2: $S^2M \to S_0^2 M$.

As a corollary to the Theorem 3.1, we can deduce the following theorem.

Theorem 3.2 *If a compact Riemannian manifold (M, g) of dimension $n \geq 2$ admits geodesic mapping is a non affine and the tensor $a = (a_{ij})$ belongs to the kernel of the symmetric linear operator B_2: $S^2M \to S_o^2 M$ then (M, g) is isometric to a sphere S^n in $(n+1)$-dimensional Euclidean space.*

Proof It is know (see [11]) that if a compact Riemannian manifold (M, g) of n-dimension of $n \geq 2$ admits an infinitesimal conformal transformation which is not an isometry:

$$L_X g_{ij} = 2\rho g_{ij} \tag{14}$$

for $\rho \neq 0$, and if the vector field X is a gradient of a scalar function then (M, g) is isometric to a Euclidean n-sphere (see [11]). Here L_X is the operator of the Lie derivation with respect to X. If we assume that $X = \operatorname{grad} \lambda$ then (14) we can rewritten as $\nabla_i \nabla_j \lambda = \frac{1}{n} \Delta\lambda\, g_{ij}$. Then our Theorem 3.2 follows from Theorem 3.1 and the above result by Lichnerowicz.

Remark The Lichnerowicz Laplacian (see [6], p. 54) acting on symmetric covariant 2-tensor is $\Delta_L = \bar{\Delta} - 2B_2$ where we denote by $\bar{\Delta}$ the rough Bochner Laplacian (see [6], p. 52). Then $B_2(b) = 0$ if and only if $\Delta_L b = \bar{\Delta} b$ for a symmetric covariant 2-tensor b.

4 Principle Directions of the Ricci Tensor in the Case of Degenerate Geodesic Mappings

From Eq. (7) we conclude that the Ricci tensor $Ric = (R_{ij})$ can be diagonalised in any point $x \in M$ in the same orthonormal basis $\{e_1, \dots, e_n\}$ as the symmetric non-degenerate tensor $a = (a_{ij})$. Therefore in any point $x \in M$ vectors of the orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space $T_x M$ define principle directions not only of the tensor $a = (a_{ij})$, but also principle directions of the Ricci tensor (see [12, § 34]). In this case the basis $\{e_1, \dots, e_n\}$ gets an invariant meaning for the manifold (M, g), independent of the tensor $a = (a_{ij})$.

The following theorem holds.

Theorem 4.1 *Let the Riemannian manifold (M, g) of dimension $n \geq 2$ admit a geodesic mapping and the tensor $a = (a_{ij})$ belong to the kernel of the symmetric linear operator $B_2 : S^2 M \to S_0^2 M$. If in each point $x \in M$ the sectional curvature $K(e_i, e_j) > 0$ (or $K(e_i, e_j) < 0$) to the direction $e_i \wedge e_j$ for the ortonormal basis $\{e_1, \dots, e_n\}$ of the vectors of principle directions of the Ricci tensor then the geodesic mapping is an affine mapping.*

Proof The quadratic form $\Phi_2(b_{ij}) = g(B_2(b_{ij}), b_{ij})$ can be written in the following from (see [6, § 16.9]).

$$\Phi_2(b_{ij}) = -\sum_{i<j} K(e_i, e_j)(b_i - b_j)^2,$$

where $K(e_i, e_j)$ is the sectional curvature to the direction $e_i \wedge e_j$ for any vectors of the orthonormal basis $\{e_1, \dots, e_n\}$ of eigenvectors of the tensor $b = (b_{ij})$ of the space $T_x M$ in any point $x \in M$ i.e. $b(e_i, e_j) = b_i \delta_{ij}$, where δ_{ij} is the Kroneker symbol. Then from the condition $B_2(a_{ij}) = 0$ follows $\Phi_2(a_{ij}) = 0$ and under the condition $K(e_i, e_j) > 0$ (or $K(e_i, e_j) < 0$) from the equality $\Phi_2(g_{ij}) = 0$ follows that $a(e_i, e_j) = a\delta_{ij}$ for the corresponding ortonormal basis $\{e_1, \dots, e_n\}$ of $T_x M$ in any point $x \in M$. This means that the geodesic mappings is an affine mapping (see [3, p. 93]).

Acknowledgments The paper was supported by the project FAST-S-11-47 of the Brno University of Technology, by grant P201/11/0356 of The Czech Science Foundation, and by the project POST-UP CZ 1.07/2.3.00/30.0004.

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