A Note on Jet and Geometric Approach to Higher Order Connections

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Abstract We compare two ways of interpreting higher order connections. The geometric approach lies in the decomposition of higher order tangent space into the horizontal and vertical structures while the jet-like approach considers a higher order connection as the section of a jet prolongation of a fibered manifold. Particularly, we use the Ehresmann prolongation of a general connection and study the result from the point of view of geometric theory. We pay attention to linear connections, too.

1 Introduction

Several models of real objects are given as a smooth manifold and one or more linear connections, e.g. material elasticity, see [1]. To obtain a manifold with just one characterization, one has to consider a concept of a higher order connection. In this paper, we recall the basic concepts of higher order connections from both geometric and jet–like point of view, Sects. 2 and 4. Let us note that the original ideas are those of Ehresmann, i.e. the definition of a connection by means of a horizontal distribution in a tangent space, the double fibered manifolds and holonomic and nonholonomic jets of fibered mappings. The first idea can be found in [2], the second one in [3]. The second idea was used for the case of vector bundles by Pradines, [4]. Finally, the concept of holonomic and nonholonomic jets is widely studied in [5–9]. The first idea was extended in [10], where the main formulae of higher order objects in multiple tangent spaces are derived, see also [11]. In this paper we compare the jet–like and

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geometric approach. We also recall a product of general connections which leads to the so called Ehresmann prolongation and show the reason why this operation is outstanding, especially concerning semiholonomic connections, Sect. 6.1. We study Ehresmann prolongation of a connection from both points of view and show the analogues in both approaches.

2 Jet Prolongation of a Fibered Manifold

Classical theory reads that r-th holonomic prolongation J^rY of $Y \to M$ is the space of r–jets of local sections $M \to Y$. The nonholonomic prolongation $\widetilde{J}^rY$ of $Y \to M$ is defined by the following iteration:

1. $\widetilde{J}^1Y = J^1Y$, i.e. $\widetilde{J}^1Y$ is a space of 1-jets of sections $M \to Y$ over the target space Y.
2. $\widetilde{J}^rY = J^1(\widetilde{J}^{r-1}Y \to M)$.

Clearly, we have an inclusion $J^rY \subset \widetilde{J}^rY$ given by $j^r_x\gamma \mapsto j^1_x(j^{r-1}\gamma)$. Further, r-th semiholonomic prolongation $\overline{J}^rY \subset \widetilde{J}^rY$ is defined by the following induction. First, by $\beta_1 = \beta_Y$ we denote the projection $J^1Y \to Y$ and by $\beta_r = \beta_{\widetilde{J}^{r-1}Y}$ the projection $\widetilde{J}^rY = J^1\widetilde{J}^{r-1}Y \to \widetilde{J}^{r-1}Y,\ r = 2, 3, \ldots$. If we set $\overline{J}^1Y = J^1Y$ and assume we have $\overline{J}^{r-1}Y \subset \widetilde{J}^{r-1}Y$ such that the restriction of the projection $\beta_{r-1} : \widetilde{J}^{r-1}Y \to \widetilde{J}^{r-2}Y$ maps $\overline{J}^{r-1}Y$ into $\overline{J}^{r-2}Y$, we can construct $J^1\beta_{r-1} : J^1\overline{J}^{r-1}Y \to J^1\overline{J}^{r-2}Y$ and define

$$\overline{J}^rY = \{A \in J^1\overline{J}^{r-1}Y;\ \beta_r(A) = J^1\beta_{r-1}(A) \in \overline{J}^{r-1}Y\}.$$

If we denote by $\mathscr{FM}_{m,n}$ the category with objects composed of fibered manifolds with m-dimensional bases and n-dimensional fibres and morphisms formed by locally invertible fiber-preserving mappings, then, obviously, J^r, $\overline{J}^r$ and $\widetilde{J}^r$ are bundle functors on $\mathscr{FM}_{m,n}$.

Alternatively, one can define the r-th order semiholonomic prolongation $\overline{J}^rY$ by means of natural target projections of nonholonomic jets, see [9]. For $r \geq q \geq 0$ let us denote by π^r_q the target surjection $\pi^r_q : \widetilde{J}^rY \to \widetilde{J}^qY$ with π^r_r being the identity on $\widetilde{J}^rY$. We note that the restriction of these projections to the subspace of semiholonomic jet prolongations will be denoted by the same symbol. By applying the functor J^k we have also the surjections $J^k\pi^{r-k}_{q-k} : \widetilde{J}^rY \to \widetilde{J}^qY$ and, consequently, the element $X \in \widetilde{J}^rY$ is semiholonomic if and only if

$$(J^k\pi^{r-k}_{q-k})(X) = \pi^r_q(X) \text{ for any integers } 1 \leq k \leq q \leq r. \tag{1}$$

In [9], the proof of this property can be found and the author finds it useful when handling semiholonomic connections and their prolongations.

Now let us recall local coordinates on higher order jet prolongations of a fibered manifold $Y \to M$. Let us denote by $x^i,\ i = 1, \dots, m$ the local coordinates on M and $y^p,\ p = 1, \dots, n$ the fiber coordinates of $Y \to M$. We recall that the induced coordinates on the holonomic prolongation $J^r Y$ are given by (x^i, y^p_α), where α is a multiindex of range m satisfying $|\alpha| \le r$. Clearly, the coordinates y^p_α on $J^r Y$ are characterized by the complete symmetry in the indices of α. Having the nonholonomic prolongation $\widetilde{J}^r Y$ constructed by the iteration, we define the local coordinates inductively as follows:

(1) Suppose that the induced coordinates on $\widetilde{J}^{r-1} Y$ are of the form

$$(x^i, y^p_{k_1 \dots k_{r-1}}),\ k_1, \dots, k_{r-1} = 0, 1, \dots, m.$$

(2) We define the induced coordinates on $\widetilde{J}^r Y$ by

$$(x^i, y^p_{k_1 \dots k_{r-1} 0} = y^p_{k_1 \dots k_{r-1}}, y^p_{k_1 \dots k_{r-1} i} = \frac{\partial}{\partial x^i} y^p_{k_1 \dots k_{r-1}}),$$

i.e. induced coordinates are partial derivatives are obtained as partial derivatives of fiber coordinates with respect to the base coordinates.

It remains to describe coordinates on the semiholonomic prolongation $\overline{J}^r Y$. Let $(k_1, \dots, k_r),\ k_1, \dots, k_r = 0, 1, \dots, m$ be a sequence of indices and denote by $\langle k_1, \dots, k_s \rangle, s \le r$ the sequence of non-zero indices in $(k_1, \dots, k_r)$ respecting the order. Then the definition of $\overline{J}^r Y$ reads that the point $(x^i, y^p_{k_1 \dots k_r}) \in \widetilde{J}^r Y$ belongs to $\overline{J}^r Y$ if and only if $y^p_{k_1 \dots k_r} = y^p_{l_1 \dots l_r}$ whenever $\langle k_1, \dots, k_r \rangle = \langle l_1, \dots, l_r \rangle$

3 Iterated Tangents

Another concept, in this paper called geometric, of a connection rises from the theory of iterated tangent spaces. Let us recall that the bundle $T^k M \to T^{k-1} M$ is equipped with the structure of a k-fold vector bundle. Particularly, $T^k M$ admits k different projections to $T^{k-1} M$,

$$\rho_s := T^{k-s} \pi_s : T^k M \to T^{k-1} M,$$

where π_s is the natural projection $T^s M \to T^{s-1} M$, $s = 1, 2, \dots, k$. Each projection defines a vector bundle with basis $T^{k-1} M$ and the total space is composed of $2^{k-1} n$-dimensional vector spaces as fibers. The local coordinates on the neighborhoods

$$T^s U \subset T^s M, \quad \text{where} \quad T^{s-1} U = \pi_s(T^s U),\ s = 1, 2, \dots, k,$$

are derived from coordinates, or coordinate mappings, (u^i), which are given on the neighborhood $U \subset M$:

U: (u^i), $i = 1, 2, \dots, n$,
TU: (u^i, u^i_1), where $u^i := u^i \circ \pi_1$, $u^i_1 := du^i$,
T^2U: $(u^i, u^i_1, u^i_2, u^i_{12})$,
where $u^i := u^i \circ \pi_1\pi_2$, $u^i_1 := du^i \circ \pi_2$, $u^i_2 := d(u^i \circ \pi_1)$, $u^i_{12} := d^2u^i$,
etc.

Proposition 3.1 *Coordinate mappings given on the neighborhood $T^{s-1}U$ induce coordinate mappings on the neighborhood T^sU with respect to the projection π_s by adding the differentials of these mappings.*

Local coordinates are obtained by the following principle:
to the coordinates of a point of a manifold we attach the coordinates of the vector tangent to the manifold at that point. We use the following notation: the coordinates of a neighborhood T^kU consist of two copies of local coordinates on $T^{k-1}U$ where the second copy is equipped with an additional subscript k. This principle is suitable in the sense that the coordinates with index s are recognized as the fiber coordinates for projections ρ_s, $s = 1, 2, \dots, k$, i.e. the coordinates with index s disappear after the application of projection ρ_s.

The coordinate form of the three projections $\rho_s : T^3U \to T^2U$, $s = 1, 2, 3$, is given by the following diagram:

$$\begin{array}{ccc} & (u^i, u^i_1, u^i_2, u^i_{12}, u^i_3, u^i_{13}, u^i_{23}, u^i_{123}) & \\ \rho_1 \swarrow & \rho_2 \downarrow & \searrow \rho_3 \\ (u^i, u^i_2, u^i_3, u^i_{23}) & (u^i, u^i_1, u^i_3, u^i_{13}) & (u^i, u^i_1, u^i_2, u^i_{12}). \end{array}$$

Remark 3.1 Let us note that the semiholonomity condition is connected to the notion of the osculating bundle, see [11], and can be defined as the equalizer of all possible projections, which corresponds to (1).

4 Connections

We start with the jet–like approach to connections. This rather structural description is quite suitable for determining natural operators on connections, for details see [5].

Definition 4.1 A general connection on the fibered manifold $Y \to M$ is a section $\Gamma : Y \to J^1Y$ of the first jet prolongation $J^1Y \to Y$.

Further generalization of this idea leads us to the definition of r-th order connection, which is a section of r-th order jet prolongation of a fibered manifold. According to the character of the target space we distinguish holonomic, semiholonomic and nonholonomic general connections. The coordinate form of a second order nonholonomic connection $\Delta : Y \to \widetilde{J}^2 Y$ is given by

$$y_i^p = F_i^p(x, y), \quad y_{0i}^p = G_i^p(x, y), \quad y_{ij}^p = H_{ij}^p(x, y),$$

where F, G, H are arbitrary smooth functions. In case of linear connections all functions are linear in fiber coordinates.

Let us now recall the geometric concept of a connection and its extension to higher order connections. The following section is based on the paper [11].

Definition 4.2 A connection on bundle $\pi : M_1 \to M$ is defined by the structure $\triangle_h \oplus \triangle_v$ on a manifold M_1 where $\triangle_v = \ker T\pi$ is *vertical distribution* tangent to the fibers and $\triangle_h$ is *horizontal distribution* complementary to the distribution $\triangle_v$. The transport of the fibers along the path $\gamma \subset M$ is realized by the horizontal lifts given by the distribution $\triangle_h$ on the surface $\pi^{-1}(\gamma)$. If the bundle is a vector one and the transport of fibers along an arbitrary path is linear, then the connection is called linear.

We will assume that the base manifold M is of dimension n and the fibers are of dimension r. Then

$$\dim \triangle_h = n\,, \quad \dim \triangle_v = r\,.$$

On the neighborhood $U \subset M_1$, let us consider local base and fiber coordinates:

$$(u^i, u^\alpha)\,,\ i = 1, 2,\ \ldots,\ n\,;\ \alpha = n+1,\ \ldots,\ n+r.$$

Base coordinates (u^i) are determined by the projection π and the coordinates $(\bar{u}^i)$ on a neighborhood $\bar{U} = \pi(U)$, $u^i = \bar{u}^i \circ \pi$.

Definition 4.3 On a neighborhood $U \subset M_1$ we define a local (adapted) basis of the structure $\triangle_h \oplus \triangle_v$,

$$(X_i\ X_\alpha) = \begin{pmatrix} \dfrac{\partial}{\partial u^j} & \dfrac{\partial}{\partial u^\beta} \end{pmatrix} \cdot \begin{pmatrix} \delta_i^j & 0 \\ \Gamma_i^\beta & \delta_\alpha^\beta \end{pmatrix}, \quad \begin{pmatrix} \omega^i \\ \omega^\alpha \end{pmatrix} = \begin{pmatrix} \delta_j^i & 0 \\ -\Gamma_j^\alpha & \delta_\beta^\alpha \end{pmatrix} \cdot \begin{pmatrix} du^j \\ du^\beta \end{pmatrix}.$$

The horizontal distribution $\triangle_h$ is the linear span of the vector fields (X_i) and the annihilator of the forms (ω^α),

$$X_i = \partial_i + \Gamma_i^\beta \partial_\beta, \quad \omega^\alpha = du^\alpha - \Gamma_i^\alpha du^i.$$

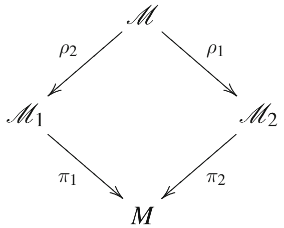
Definition 4.4 A classical affine connection on manifold M is seen as a linear connection on the bundle $\pi_1 : TM \to M$. On the tangent bundle $TM \to M$ one can define the structure $\triangle_h \oplus \triangle_v$. The indices in the formulas are denoted by Latin letters all of them ranging from 1 to n. The functions Γ_i^α, X_i, ω^α are of the form (in Γ_i^α the sign is changed to comply with the classical theory):

$$\begin{aligned} \Gamma_i^\alpha \quad &\rightsquigarrow \quad -\Gamma_{jk}^i u_1^k \,, \\ X_i = \partial_i + \Gamma_i^\alpha \partial_\alpha \quad &\rightsquigarrow \quad X_i = \partial_i - \Gamma_{ij}^k u_1^i \partial_k^1 \,, \\ \omega^\alpha = du^\alpha - \Gamma_i^\alpha du^i \quad &\rightsquigarrow \quad U_{12}^i = u_{12}^i + \Gamma_{jk}^i u_1^k u_2^j \,. \end{aligned}$$

Definition 4.5 Higher order connections are defined as follows: on tangent bundle *TM* the structure $\triangle \oplus \triangle_1$ is defined where $\ker T\rho_1 = \triangle_1$, on $T(TM)$ the structure $\Delta \oplus \Delta_1 \oplus \Delta_2 \oplus \Delta_{12}$ is defined where $\ker T\rho_s = \Delta_s \oplus \Delta_{12}$, $s = 1, 2$, etc.

5 Connections on Two-Fold Fibered Manifolds

More generally, one can define a second order connection by means of a two-fold fibered manifold. Note that the Definition 4.5 is a special case of the following. A two-fold fibered manifold is a commutative diagram



where ρ_1, ρ_2 and π_1, π_2—four fibered manifolds
$\dim M = n, \quad \dim \mathscr{M}_1 = n + r_1, \quad \dim \mathscr{M}_2 = n + r_2, \quad \dim \mathscr{M} = n + r_1 + r_2 + r_{12}$.
The double projection

$$\pi = \pi_1 \circ \rho_2 = \pi_2 \circ \rho_1 : \mathscr{M} \to M$$

divides a manifold $\mathscr{M}$ to n-parameter family of fibers of dimensions $(r_1 + r_2 + r_{12})$. Each fiber carries structure of another two fibers of dimensions $r_1 + r_{12}$ and $r_2 + r_{12}$ and these two fibers have the common intersection of dimension r_{12}.

A two-fold fibered manifold is called a vector bundle if both fibrations π_1, π_2, ρ_1 and ρ_2—form vector bundles.

An example of a two-fold fibered manifold is the second order tangent bundle T^2M of a manifold M. In this case $n = r_1 = r_2 = r_{12}$.

Definition 5.1 A connection on a two-fold fibered manifold is defined by a structure on a manifold $\mathscr{M}$:

$$\Delta \otimes \Delta_1 \otimes \Delta_2 \otimes \Delta_{12} \,, \tag{2}$$

$$\dim \Delta = n, \quad \dim \Delta_1 = r_1 \,, \quad \dim \Delta_2 = r_2 \,, \quad \dim \Delta_{12} = r_{12} \,,$$

$$\mathrm{Ker} T\rho_2 = \Delta_2 \oplus \Delta_{12}\,, \qquad \mathrm{Ker} T\rho_1 = \Delta_1 \oplus \Delta_{12}$$

$$T\rho_2(\Delta \oplus \Delta_1) = T\mathscr{M}_1, \quad T\rho_1(\Delta \oplus \Delta_2) = T\mathscr{M}_2\,,$$

$$T\pi\Delta = TM.$$

Remark 5.1 A connection on a two-fold vector fibered manifold is called linear if the structure (2) induces on the manifolds π_1, π_2, ρ_1 and ρ_2 linear connections.

Remark 5.2 Similarly, one can define a connection on a k–fold fibered manifold. In such case the commutative diagram would be represented by a k–dimensional cube. These manifolds would correspond to the k–th tangent bundle $T^k M$ of a manifold M.

On the neighborhoods

$$\mathscr{U} \subset \mathscr{M},\;\; \mathscr{U}_1 = \rho_2(\mathscr{U}) \subset \mathscr{M}_1,\;\; \mathscr{U}_2 = \rho_1(\mathscr{U}) \subset \mathscr{M}_2,\;\; U = \pi(\mathscr{U}) \subset M$$

we have the coordinate systems
$(u^i,\; u^{\alpha_1},\; u^{\alpha_2},\; u^{\alpha_{12}}),\;\; (u^i,\; u^{\alpha_1}),\;\; (u^i,\; u^{\alpha_1}),\;\; (u^i)$.

The transformation of coordinates on the neighborhoods $\mathscr{U}$,

$$(u^i, u^{\alpha_1}, u^{\alpha_2}, u^{\alpha_{12}}) \rightsquigarrow (\tilde{u}^i, \tilde{u}^{\alpha_1}, \tilde{u}^{\alpha_2}, \tilde{u}^{\alpha_{12}}) = (a^i, a^{\alpha_1}, a^{\alpha_2}, a^{\alpha_{12}}),$$

gives a Jacobi matrix:

$$\begin{pmatrix} a^i_j & 0 & 0 & 0 \\ a^{\alpha_1}_j & a^{\alpha_1}_{\beta_1} & 0 & 0 \\ a^{\alpha_2}_j & 0 & a^{\alpha_2}_{\beta_2} & 0 \\ a^{\alpha_{12}}_j & a^{\alpha_{12}}_{\beta_1} & a^{\alpha_{12}}_{\beta_2} & a^{\alpha_{12}}_{\beta_{12}} \end{pmatrix}.$$

See [10, 12]. Let us mention that the local (adapted) basis of such decomposition is represented by a matrix of the form

$$\begin{pmatrix} \delta^i_j & 0 & 0 & 0 \\ \Gamma^{\alpha_1}_j & \delta^{\alpha_1}_{\beta_1} & 0 & 0 \\ \Gamma^{\alpha_2}_j & 0 & \delta^{\alpha_2}_{\beta_2} & 0 \\ \Gamma^{\alpha_{12}}_j & \Gamma^{\alpha_{12}}_{\beta_1} & \Gamma^{\alpha_{12}}_{\beta_2} & \delta^{\alpha_{12}}_{\beta_{12}} \end{pmatrix}. \tag{3}$$

The dual frame is given by the system of 1–forms:

$$\begin{aligned} \omega^i &= du^i, \\ \omega^{\alpha_1} &= du^{\alpha_1} - \Gamma^{\alpha_1}_i du^i, \\ \omega^{\alpha_2} &= du^{\alpha_2} - \Gamma^{\alpha_2}_i du^i, \\ \omega^{\alpha_{12}} &= du^{\alpha_{12}} - \Gamma^{\alpha_{12}}_{\alpha_1} du^{\alpha_1} - \Gamma^{\alpha_{12}}_{\alpha_2} du^{\alpha_2} - \bar{\Gamma}^{\alpha_{12}}_i du^i, \end{aligned}$$

$$\text{where} \quad \Gamma_i^{\alpha_{12}} - \bar{\Gamma}_i^{\alpha_{12}} = \Gamma_{\beta_1}^{\alpha_{12}} \Gamma_i^{\beta_1} + \Gamma_{\beta_2}^{\alpha_{12}} \Gamma_i^{\beta_2}.$$

In case of linear connection the elements of the matrix (3) are of the form

$$\Gamma_j^{\alpha_1} = \Gamma_{j\beta_1}^{\alpha_1} u^{\beta_1}, \quad \Gamma_j^{\alpha_2} = \Gamma_{j\beta_2}^{\alpha_2} u^{\beta_2},$$
$$\Gamma_{\beta_1}^{\alpha_{12}} = \Gamma_{\beta_1\beta_2}^{\alpha_{12}} u^{\beta_2}, \quad \Gamma_{\beta_2}^{\alpha_{12}} = \Gamma_{\beta_2\beta_1}^{\alpha_{12}} u^{\beta_1},$$
$$\Gamma_j^{\alpha_{12}} = \Gamma_{j\beta_1\beta_2}^{\alpha_{12}} u^{\beta_1} u^{\beta_2} + \Gamma_{j\beta_{12}}^{\alpha_{12}} u^{\beta_{12}}, \quad \bar{\Gamma}_j^{\alpha_{12}} = \bar{\Gamma}_{j\beta_1\beta_2}^{\alpha_{12}} u^{\beta_1} u^{\beta_2} + \bar{\Gamma}_{j\beta_{12}}^{\alpha_{12}} u^{\beta_{12}},$$

$$\Gamma_{j\beta_1\beta_2}^{\alpha_{12}} - \bar{\Gamma}_{j\beta_1\beta_2}^{\alpha_{12}} = \Gamma_{\gamma_2\beta_1}^{\alpha_{12}} \Gamma_{j\beta_2}^{\gamma_2},$$

where the coefficients depend on the base coordinates u^i only.

6 Ehresmann Prolongation

First, let us now recall a concept of a product of two connections.

Given two higher order connections $\Gamma : Y \to \widetilde{J}^r Y$ and $\overline{\Gamma} : Y \to \widetilde{J}^s Y$, the product of Γ and $\overline{\Gamma}$ is the $(r+s)$-th order connection $\Gamma * \overline{\Gamma} : Y \to \widetilde{J}^{r+s} Y$ defined by

$$\Gamma * \overline{\Gamma} = \widetilde{J}^s \Gamma \circ \overline{\Gamma}.$$

Particularly, if both Γ and $\overline{\Gamma}$ are of the first order, then $\Gamma * \overline{\Gamma} : Y \to \widetilde{J}^2 Y$ is semiholonomic if and only if $\Gamma = \overline{\Gamma}$ and $\Gamma * \overline{\Gamma}$ is holonomic if and only if Γ is curvature-free, [9, 13].

As an example we show the coordinate expression of an arbitrary nonholonomic second order connection and of the product of two first order connections. The coordinate form of $\Delta : Y \to \widetilde{J}^2 Y$ is

$$y_i^p = F_i^p(x, y), \quad y_{0i}^p = G_i^p(x, y), \quad y_{ij}^p = H_{ij}^p(x, y),$$

where F, G, H are arbitrary smooth functions. Further, if the coordinate expressions of two first order connections $\Gamma, \overline{\Gamma} : Y \to J^1 Y$ are

$$\Gamma : \quad y_i^p = F_i^p(x, y), \qquad \overline{\Gamma} : \quad y_i^p = G_i^p(x, y), \tag{4}$$

then the second order connection $\Gamma * \overline{\Gamma} : Y \to \widetilde{J}^2 Y$ has equations

$$y_i^p = F_i^p, \quad y_{0i}^p = G_i^p, \quad y_{ij}^p = \frac{\partial F_i^p}{\partial x^j} + \frac{\partial F_i^p}{\partial y^q} G_j^q.$$

For linear connections, the coordinate form would be obtained by substitution

$$F_i^p = F_{iq}^p y^q,$$
$$G_i^p = G_{iq}^p y^q$$

in the Eq. (4), where F_{iq}^p and G_{iq}^p are functions of the base manifold coordinates x_i. For order three see [8].

In the above process, if $\Gamma = \overline{\Gamma}$, the connection $\Gamma * \Gamma$ is called the Ehresmann prolongation of Γ, iteratively we obtain the r–th Ehresmann prolongation of Γ. We show that Ehresmann prolongation plays an important role in determining all natural operators transforming first order connections into higher order connections. Let us note that also natural transformations of semiholonomic jet prolongation functor $\overline{J}^r$ are involved. To find the details about this topic we refer to [5–7]. For our purposes, it is enough to consider $r = 2$. We use the notation of [5], where the map $e : \overline{J}^2 Y \to \overline{J}^2 Y$ is obtained from the natural exchange map $e_\Lambda : J^1 J^1 Y \to J^1 J^1 Y$ as a restriction to the subbundle $\overline{J}^2 Y \subset J^1 J^1 Y$. Note that while e_Λ depends on the linear connection Λ on M, its restriction e is independent of any auxiliary connections. We remark, that originally the map e_Λ was introduced by M. Modugno. We also recall that J. Pradines introduced a natural map $\overline{J}^2 Y \to \overline{J}^2 Y$ with the same coordinate expression.

Now we are ready to recall the following assertion, see [7] for the proof.

Proposition 6.1 *All natural operators transforming first order connection $\Gamma : Y \to J^1 Y$ into second order semiholonomic connection $Y \to \overline{J}^2 Y$ form a one parameter family*

$$\Gamma \mapsto k \cdot (\Gamma * \Gamma) + (1 - k) \cdot e(\Gamma * \Gamma), \qquad k \in \mathbb{R}.$$

This shows the importance of Ehresmann prolongation in the theory of prolongations of connections.

7 Tangent Functor and Ehresmann Prolongation

If we apply the tangent functor T two times on a projection $\pi : E \to M$ and a section $\sigma : M \to E$ we obtain

$$T\pi : TE \to TM\,,\ \ T^2\pi : T^2 E \to T^2 M,$$

$$T\sigma : TM \to TE,\ T^2\sigma : T^2 M \to T^2 E,$$

respectively. The mappings $\sigma,\ T\sigma$ and $T^2\sigma$ define the sections of fibered manifolds $\pi,\ T\pi$ and $T^2\pi$.

Let us consider local coordinates on the following manifolds in the form

$$\text{on} \quad M,\ TM,\ T^2M\ :\ \ (x^i),\ (x^i, x_1^i),\ (x^i, x_1^i, x_2^i, x_{12}^i),$$

$$\text{and on} \quad E,\ TE,\ T^2E\ :\ \ (y^p),\ (y^p, y_1^p),\ (y^p, y_1^p, y_2^p, y_{12}^p).$$

Let us also consider for a function f defined on a manifold M, its following differentials on T^2M in local coordinate form:

$$f_1 \doteq f_i x_1^i, \quad f_2 \doteq f_i x_2^i, \quad f_{12} \doteq f_{ij} x_1^i x_2^j + f_i x_{12}^i, \quad \text{where } f_i = \frac{\partial f}{\partial x^i}\,, \; f_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}.$$

Furthermore, $f_1 = df \circ \rho_1\,,\; f_2 = df \circ \rho_2\,,\; f_{12} = d^2 f$. We use these notations in the formulae bellow.

If the section σ is defined by local functions Γ^p, then the sections $T\sigma$ and $T^2\sigma$ are defined by its differentials Γ_1^p , Γ_2^p and Γ_{12}^p,

$$\begin{aligned}
&\sigma : \; x^i \rightsquigarrow y^p = \Gamma^p,\\
&T\sigma : \; (x^i, x_1^i) \rightsquigarrow (y^p, y_1^p) = (\Gamma^p, \Gamma_1^p),\\
&T^2\sigma : \; (x^i, x_1^i, x_2^i, x_{12}^i) \rightsquigarrow (y^p, y_1^p, y_2^p, y_{12}^p) = (\Gamma^p, \Gamma_1^p, \Gamma_2^p, \Gamma_{12}^p),\\
&\text{where} \quad \Gamma_1^p = \Gamma_i^p x_1^i, \quad \Gamma_2^p = \Gamma_i^p x_2^i, \quad \Gamma_{12}^p = \Gamma_{ij}^p x_1^i x_2^j + \Gamma_i^p x_{12}^i.
\end{aligned} \tag{5}$$

The case when the coefficients Γ_i^p, Γ_{ij}^p in (5) are arbitrary functions, corresponds to a nonholonomic connection on the fibered manifold π.

The case when $\Gamma_{ij}^p = \dfrac{\partial \Gamma_i^p}{\partial x^j}$, where Γ_i^p are arbitrary functions corresponds to a semiholonomic connection on the fibered manifold π.

The case when $\Gamma_1^p = d\Gamma^p \circ \rho_1\,,\; \Gamma_2^p = d\Gamma^p \circ \rho_2\,,\; \Gamma_{12}^p = d^2\Gamma^p\,,$ corresponds to a holonomic connection on the fibered manifold π.

The functions Γ_i^p, Γ_{ij}^p define nonholonomic, semiholonomic or holonomic Ehresmann prolongation of a connection, respectively.

Remark 7.1 Nonholonomic prolongation induces a connection on a double fibered manifold

$$J \to E \to M : \; y_i^p \rightsquigarrow y^p \rightsquigarrow x^i.$$

On the fibered manifold $E \to M$ the fiber transformations are given by the Pfaff system

$$\omega^p \equiv dy^p - \Gamma_i^p dx^i = 0,$$

more precisely, along a curve $x^i(t)$ – by the system of first order ODEs

$$\dot{y}^p = \Gamma_i^p \dot{x}^i. \tag{6}$$

In case $(\Gamma_{12}^p, x_1^i, x_2^j, x_{12}^i) \rightsquigarrow (\ddot{y}^p, \dot{x}^i, \dot{x}^j, \ddot{x}^i)$ we obtain the system of second order ODEs:

$$\Gamma_{12}^p = \Gamma_{ij}^p x_1^i x_2^j + \Gamma_i^p x_{12}^i \rightsquigarrow \ddot{y}^p = \Gamma_{ij}^p \dot{x}^i \dot{x}^j + \Gamma_i^p \ddot{x}^i.$$

Considering the system (6), we obtain for fiber coordinates y^α, y_i^α system of first order ODEs

$$\begin{cases} \dot{y}^p = \Gamma_i^p \, \dot{x}^i, \\ \dot{y}_i^p = \Gamma_{ij}^p \, \dot{x}^j. \end{cases}$$

The sections of fibers along a curve $x^i(t)$ are given.
The horizontal distribution $\triangle_h$ is n-dimensional and described by the vector field

$$X_i = \partial_i + \Gamma_i^p \, \partial_p + \Gamma_{ij}^p \, \partial_p^j, \quad \text{where} \quad \partial_i = \frac{\partial}{\partial x^i}, \ \partial_p = \frac{\partial}{\partial y^p}, \ \partial_p^j = \frac{\partial}{\partial y_j^p}.$$

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