Differential Geometry of Microlinear Frölicher Spaces IV-1

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Abstract The fourth paper of our series of papers entitled "Differential Geometry of Microlinear Frölicher Spaces" is concerned with jet bundles. We present three distinct approaches together with transmogrifications of the first into the second and of the second to the third. The affine bundle theorem and the equivalence of the three approaches with coordinates are relegated to a subsequent paper.

1 Introduction

As the fourth of our series of papers entitled "Differential Geometry of Microlinear Frölicher Spaces" [14–16], this paper will discuss jet bundles. Since the paper has become somewhat too long as a single paper, we have decided to divide it into two parts. In this first part we will present three distinct approaches to jet bundles in the general context of Weil exponentiable and microlinear Frölicher spaces. In the subsequent part [17], we will establish the affine bundle theorem in the second and the third approaches, and we will show that the three approaches are equivalent, as far as coordinates are available (i.e., in the classical context).

This part consisits of 7 sections. The first section is this introduction, while the second section is devoted to some preliminaries. We will present three distinct approaches to jet bundles in Sects. 3, 4 and 5. In Sect. 6 we will show how to translate the first approach into the second, while Sect. 7 is devoted to the transmogrification of the second approach into the third.

We have already discussed these three approaches to jet bundles in the context of synthetic differential geometry, for which the reader is referred to our previous work [8–13]. Now we have emancipated them to the real world of Frölicher spaces.

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2 Preliminaries

2.1 *Frölicher Spaces*

Frölicher and his followers have vigorously and consistently developed a general theory of smooth spaces, often called *Frölicher spaces* for his celebrity, which were intended to be the *maximal class of* spaces where smooth structures can live. A Frölicher space is an underlying set endowed with a class of real-valued functions on it (simply called *structure functions*) and a class of mappings from the set $\mathbb{R}$ of real numbers to the underlying set (simply called *structure curves*) subject to the condition that structure curves and structure functions should compose so as to yield smooth mappings from $\mathbb{R}$ to itself. It is required that the class of structure functions and that of structure curves should determine each other so that each of the two classes is maximal with respect to the other as far as they abide by the above condition. What is most important among many nice properties about the category **FS** of Frölicher spaces and smooth mappings is that it is cartesian closed, while neither the category of finite-dimensional smooth manifolds nor that of infinite-dimensional smooth manifolds modelled after any infinite-dimensional vector spaces such as Hilbert spaces, Banach spaces, Fréchet spaces or the like is so at all. For a standard reference on Frölicher spaces, the reader is referred to [2].

2.2 *Weil Algebras and Infinitesimal Objects*

2.2.1 The Category of Weil Algebras and the Category of Infinitesimal Objects

The notion of a Weil algebra was introduced by Weil himself in [18]. We denote by **W** the category of Weil algebras, which is well known to be left exact. Roughly speaking, each Weil algebra corresponds to an infinitesimal object in the shade. By way of example, the Weil algebra $\mathbb{R}[X]/(X^2)$ (=the quotient ring of the polynomial ring $\mathbb{R}[X]$ of an indeterminate X over $\mathbb{R}$ modulo the ideal (X^2) generated by X^2) corresponds to the infinitesimal object of first-order nilpotent infinitesimals, while the Weil algebra $\mathbb{R}[X]/(X^3)$ corresponds to the infinitesimal object of second-order nilpotent infinitesimals. Although an infinitesimal object is undoubtedly imaginary in the real world, as has harassed both mathematicians and philosophers of the 17th and the 18th centuries such as philosopher Berkley (because mathematicians at that time preferred to talk infinitesimal objects as if they were real entities), each Weil algebra yields its corresponding *Weil functor* or *Weil prolongation* on the category of smooth manifolds of some kind to itself, which is no doubt a real entity. By way of example, the Weil algebra $\mathbb{R}[X]/(X^2)$ yields the tangent bundle functor as its corresponding Weil functor. Intuitively speaking, the Weil functor corresponding to a Weil algebra stands for the exponentiation by the infinitesimal object corresponding to the Weil algebra at issue. For Weil functors on the category of finite-dimensional smooth manifolds, the reader is referred to §35 of [5], while the reader can find a readable treatment of Weil functors on the category of smooth manifolds modeled on convenient vector spaces in §31 of [6]. In [14] we have discussed how to assign, to each pair (X, W) of a Frölicher space X and a Weil algebra W, another Frölicher space $X \otimes W$ called the Weil prolongation *Weil prolongation of* X *with respect to* W, which is naturally extended to a bifunctor $\mathbf{FS} \times \mathbf{W} \to \mathbf{FS}$. And we have shown that, given a Weil algebra W, the functor assigning $X \otimes W$ to each object X in $\mathbf{FS}$ and $f \otimes \mathrm{id}_W$ to each morphism f in $\mathbf{FS}$, namely, the Weil functor on $\mathbf{FS}$ corresponding to W is product-preserving. The proof can easily be strengthened to

Theorem 2.1 *The Weil functor on the category* ***FS*** *corresponding to any Weil algebra is left exact.*

There is a canonical projection $\pi\colon X \otimes W \to X$. Given $x \in X$, we write $(X \otimes W)_x$ for the inverse image of x under the mapping π. We denote by $\mathbf{S}_n$ the symmetric group of the set $\{1, ..., n\}$, which is well known to be generated by $n-1$ transpositions $\langle i,\ i+1\rangle$ exchanging i and $i+1 (1 \leq i \leq n-1)$ while keeping the other elements fixed. Given $\sigma \in \mathbf{S}_n$ and $\gamma \in X \otimes \mathscr{W}_{D^n}$, we define $\gamma^\sigma \in X \otimes \mathscr{W}_{D^n}$ to be

$$\gamma^\sigma = \left(\mathrm{id}_X \otimes \mathscr{W}_{(d_1,\ldots,d_n)\in D^n \mapsto (d_{\sigma(1)},\ldots,d_{\sigma(n)})\in D^n}\right)(\gamma)$$

Given $\alpha \in \mathbb{R}$ and $\gamma \in X \otimes \mathscr{W}_{D^n}$, we define $\alpha \underset{i}{\cdot} \gamma \in \gamma \in X \otimes \mathscr{W}_{D^n}$ $(1 \leq i \leq n)$ to be

$$\alpha \underset{i}{\cdot} \gamma = \left(\mathrm{id}_X \otimes \mathscr{W}_{(d_1,\ldots,d_n)\in D^n \mapsto (d_1,\ldots,d_{i-1},\alpha d_i,d_{i+1},\ldots,d_n)\in D^n}\right)(\gamma)$$

Given $\alpha \in \mathbb{R}$ and $\gamma \in X \otimes \mathscr{W}_{D_n}$, we define $\alpha\gamma \in X \otimes \mathscr{W}_{D_n}$ $(1 \leq i \leq n)$ to be

$$\alpha\gamma = \left(\mathrm{id}_X \otimes \mathscr{W}_{d\in D_n \mapsto \alpha d\in D_n}\right)(\gamma)$$

for any $d \in D_n$. The restriction mapping $\gamma \in \mathbf{T}_x^{D_{n+1}}(M) \mapsto \gamma|_{D_n} \in \mathbf{T}_x^{D_n}(M)$ is often denoted by $\pi_{n+1,n}$.

Between $X \otimes \mathscr{W}_{D^n}$ and $X \otimes \mathscr{W}_{D^{n+1}}$ there are $2n+2$ canonical mappings:

$$X \otimes \mathscr{W}_{D^{n+1}} \underset{\mathbf{s}_i}{\overset{\mathbf{d}_i}{\rightleftarrows}} X \otimes \mathscr{W}_{D^n} \quad (1 \leq i \leq n+1)$$

For any $\gamma \in X \otimes \mathscr{W}_{D^n}$, we define $\mathbf{s}_i(\gamma) \in X \otimes \mathscr{W}_{D^{n+1}}$ to be

$$\mathbf{s}_i(\gamma) = \left(\mathrm{id}_X \otimes \mathscr{W}_{(d_1,\ldots,d_{n+1})\in D^{n+1} \mapsto (d_1,\ldots,d_{i-1},d_{i+1},\ldots,d_{n+1})\in D^n}\right)(\gamma)$$

For any $\gamma \in X \otimes \mathscr{W}_{D^{n+1}}$, we define $\mathbf{d}_i(\gamma) \in X \otimes \mathscr{W}_{D^n}$ to be

$$\mathbf{d}_i(\gamma) = \left(\mathrm{id}_X \otimes \mathscr{W}_{(d_1,\ldots,d_n)\in D^n \mapsto (d_1,\ldots,d_{i-1},0,d_i,\ldots,d_n)\in D^{n+1}}\right)(\gamma)$$

These operations satisfy the so-called simplicial identities (cf. Goerss and Jardine [3]), so that the family of $X \otimes \mathscr{W}_{D^n}$'s together with mappings $\mathbf{s}_i$'s and $\mathbf{d}_i$'s form a so-called simplicial set.

Synthetic differential geometry (usually abbreviated to SDG), which is a kind of differential geometry with a cornucopia of nilpotent infinitesimals, was forced to invent its models, in which nilpotent infinitesimals were visible. For a standard textbook on SDG, the reader is referred to [7], while he or she is referred to [4] for the model theory of SDG constructed vigorously by Dubuc [1] and others. Although we do not get involved in SDG herein, we will exploit locutions in terms of infinitesimal objects so as to make the paper highly readable. Thus we prefer to write $\mathscr{W}_D$ and $\mathscr{W}_{D_2}$ in place of $\mathbb{R}[X]/(X^2)$ and $\mathbb{R}[X]/(X^3)$ respectively, where D stands for the infinitesimal object of first-order nilpotent infinitesimals, and D_2 stands for the infinitesimal object of second-order nilpotent infinitesimals. To Newton and Leibniz, D stood for

$$\{d \in \mathbb{R} \mid d^2 = 0\}$$

while D_2 stood for

$$\{d \in \mathbb{R} \mid d^3 = 0\}$$

More generally, given a natural number n, we denote by D_n the set

$$\{d \in \mathbb{R} | d^{n+1} = 0\},$$

which stands for the infinitesimal object corresponding to the Weil algebra $\mathbb{R}[X]/(X^{n+1})$. Even more generally, given natural numbers m, n, we denote by $D(m)_n$ the infinitesimal object

$$\{(d_1, ..., d_m) \in \mathbb{R}^m | d_{i_1}...d_{i_{n+1}} = 0\},$$

where $i_1, ..., i_{n+1}$ shall range over natural numbers between 1 and m including both ends. It corresponds to the Weil algebra $\mathbb{R}[X_1, ..., X_m]/I$, where I is the ideal generated by $X_{i_1}...X_{i_{n+1}}$'s. Therefore we have

$$D(1)_n = D_n$$
$$D(m)_1 = D(m)$$

Trivially we have

$$D(m)_n \subseteq D(m)_{n+1}$$

It is easy to see that

$$D(m_1)_n \times D(m_2)_1 \subseteq D(m_1 + m_2)_{n+1}$$
$$D(m_1 + m_2)_n \subseteq D(m_1)_n \times D(m_2)_n$$

By convention, we have

$$D^0 = D_0 = \{0\} = 1$$

A polynomial ρ of $d \in D_n$ is called a *simple* polynomial of $d \in D_n$ if every coefficient of ρ is either 1 or 0, and if the constant term is 0. A simple polynomial ρ of $d \in D_n$ is said to be of dimension m, in notation $\dim(\rho) = m$, provided that m is the least integer with $\rho^{m+1} = 0$. By way of example, letting $d \in D_3$, we have

$$\dim(d) = \dim(d + d^2) = \dim(d + d^3) = 3$$
$$\dim(d^2) = \dim(d^3) = \dim(d^2 + d^3) = 1$$

We will write $\mathscr{W}_{d \in D_2 \mapsto d^2 \in D}$ for the homomorphism of Weil algebras $\mathbb{R}[X]/(X^2) \to \mathbb{R}[X]/(X^3)$ induced by the homomorphism $X \to X^2$ of the polynomial ring $\mathbb{R}[X]$ to itself. Such locutions are justifiable, because the category **W** of Weil algebras in the real world and the category **D** of infinitesimal objects in the shade are dual to each other in a sense. Thus we have a contravariant functor $\mathscr{W}$ from the category of infinitesimal objects in the shade to the category of Weil algebras in the real world. Its inverse contravariant functor from the category of Weil algebras in the real world to the category of infinitesimal objects in the shade is denoted by $\mathscr{D}$. By way of example, $\mathscr{D}_{\mathbb{R}[X]/(X^2)}$ and $\mathscr{D}_{\mathbb{R}[X]/(X^3)}$ stand for D and D_2, respectively. Since the category **W** is left exact, the category **D** is right exact, in which we write $\mathbb{D} \oplus \mathbb{D}'$ for the coproduct of infinitesimal objects $\mathbb{D}$ and $\mathbb{D}'$. For any two infinitesimal objects $\mathbb{D}, \mathbb{D}'$ with $\mathbb{D} \subseteq \mathbb{D}'$, we write i or $i_{\mathbb{D} \to \mathbb{D}'}$ for its natural injection of $\mathbb{D}$ into $\mathbb{D}'$. We write $\mathbf{m}$ or $\mathbf{m}_{D_n \times D_m \to D_n}$ for the mapping $(d, d') \in D_n \times D_m \mapsto dd' \in D_n$. Given $\alpha \in \mathbb{R}$, we write $\left(\underset{i}{\alpha \cdot}\right)_{D^n}$ for the mapping

$$(d_1, ..., d_n) \in D^n \mapsto (d_1, ...d_{i-1}, \alpha d_i, d_{i+1}, ..., d_n) \in D^n$$

To familiarize himself or herself with such locutions, the reader is strongly encouraged to read the first two chapters of [7], even if he or she is not interested in SDG at all.

2.2.2 Simplicial Infinitesimal Objects

Definition 2.1 1. Simplicial infinitesimal spaces are objects of the form

$$D\{m; \mathscr{S}\} = \{(d_1, ..., d_m) \in D^m | d_{i_1} ... d_{i_k} = 0 \text{ for any } (i_1, ..., i_k) \in \mathscr{S}\},$$

where $\mathscr{S}$ is a finite set of sequences $(i_1, ..., i_k)$ of natural numbers with $1 \leq i_1 < \cdots < i_k \leq m$.

2. A simplicial infinitesimal object $D\{m; \mathscr{S}\}$ is said to be *symmetric* if $(d_1, ..., d_m) \in D\{m; \mathscr{S}\}$ and $\sigma \in \mathbf{S}_m$ always imply $(d_{\sigma(1)}, ..., d_{\sigma(m)}) \in D\{m; \mathscr{S}\}$.

To give examples of simplicial infinitesimal spaces, we have

$$D(2) = D\{2; (1,2)\}$$
$$D(3) = D\{3; (1,2), (1,3), (2,3)\},$$

which are all symmetric.

Definition 2.2 1. The number m is called the *degree* of $D\{m; \mathcal{S}\}$, in notation: $m = \deg D\{m; \mathcal{S}\}$.
2. The maximum number n such that there exists a sequence $(i_1, ..., i_n)$ of natural numbers of length n with $1 \leq i_1 < \cdots < i_n \leq m$ containing no subsequence in $\mathcal{S}$ is called the *dimension* of $D\{m; \mathcal{S}\}$, in notation: $n = \dim D\{m; \mathcal{S}\}$.

By way of example, we have

$$\deg D(3) = \deg D\{3; (1,2)\} = \deg D\{3; (1,2), (1,3)\} = \deg D^3 = 3$$
$$\dim D(3) = 1$$
$$\dim D\{3; (1,2)\} = \dim D\{3; (1,2), (1,3)\} = 2$$
$$\dim D^3 = 3$$

It is easy to see that

Proposition 2.1 *if* $n = \dim D\{m; \mathcal{S}\}$, *then*

$$d_1 + \cdots + d_m \in D_n$$

for any $(d_1, ..., d_m) \in D\{m; \mathcal{S}\}$, *so that we have the mapping*

$$+_{D\{m;\mathcal{S}\} \to D_n} : D\{m; \mathcal{S}\} \to D_n$$

Definition 2.3 Infinitesimal objects of the form D^m are called basic infinitesimal objects.

Definition 2.4 Given two simplicial infinitesimal objects $D\{m; \mathcal{S}\}$ and $D\left\{m'; \mathcal{S}'\right\}$, a mapping

$$\varphi = (\varphi_1, ..., \varphi_{m'}) : D\{m; \mathcal{S}\} \to D\left\{m'; \mathcal{S}'\right\}$$

is called a *monomial mapping* if every φ_j is a monomial in $d_1, ..., d_m$ with coefficient 1.

Notation 2.2 *We denote by* $D\{m\}_n$ *the infinitesimal object*

$$\{(d_1, ..., d_m) \in D^m | d_{i_1} ... d_{i_{n+1}} = 0\},$$

where $i_1, ..., i_{n+1}$ *shall range over natural numbers between* 1 *and* m *including both ends.*

2.2.3 Quasi-Colimit Diagrams

Definition 2.5 A diagram in the category **D** is called a quasi-colimit diagram if its dually corresponding diagram in the category **W** is a limit diagram.

Theorem 2.3 (The Fundamental Theorem on Simplicial Infinitesimal Objects) *Any simplicial infinitesimal object* $\mathbb{D}$ *of dimension* n *is the quasi-colimit of a finite diagram whose objects are of the form* D^k*'s* ($0 \leq k \leq n$) *and whose arrows are natural injections.*

Proof Let $\mathbb{D} = \mathscr{D}(m; \mathscr{S})$. For any maximal sequence $1 \leq i_1 < \cdots < i_k \leq m$ of natural numbers containing no subsequence in $\mathscr{S}$ (maximal in the sense that it is not a proper subsequence of such a sequence), we have a natural injection of D^k into $\mathbb{D}$. By collecting all such D^k's together with their natural injections into $\mathbb{D}$, we have an overlapping representation of $\mathbb{D}$ in terms of basic infinitesimal spaces. This representation is completed into a quasi-colimit representation of $\mathbb{D}$ by taking D^l together with its natural injections into D^{k_1} and D^{k_2} for any two basic infinitesimal spaces D^{k_1} and D^{k_2} in the overlapping representation of $\mathbb{D}$, where if D^{k_1} and D^{k_2} come from the sequences $1 \leq i_1 < \cdots < i_{k_1} \leq m$ and $1 \leq \overline{i}_1 < \cdots < \overline{i}_{k_2} \leq m$ in the above manner, then D^l together with its natural injections into D^{k_1} and D^{k_2} comes from the maximal common subsequence $1 \leq \widetilde{i}_1 < \cdots < \widetilde{i}_l \leq m$ of both the preceding sequences of natural numbers in the above manner. By way of example, the above method leads to the following quasi-colimit representation of $\mathbb{D}{=}D\{3\}_2$:

$$\begin{array}{ccccccc}
 & & & D^2 & & & \\
 & i_1 \nearrow & & & & \nwarrow i_2 & \\
D & & & \downarrow i_{12} & & & D \\
i_1 \downarrow & & & D(3)_2 & & & \downarrow i_1 \\
D^2 & i_{13} \nearrow & & & & \nwarrow i_{23} & D^2 \\
 & i_2 \nwarrow & & & & \nearrow i_2 & \\
 & & & D & & &
\end{array}$$

In the above representation i_{jk}'s and i_j's are as follows:

1. the j-th and k-th components of $i_{jk}(d_1, d_2) \in D(3)_2$ are d_1 and d_2, respectively, while the remaining component is 0;
2. the j-th component of $i_j(d) \in D^2$ is d, while the other component is 0.

Definition 2.6 The quasi-colimit representation of $\mathbb{D}$ depicted in the proof of the above theorem is called *standard*.

Remark 2.1 Generally speaking, there are multiple ways of quasi-colimit representation of a given simplicial infinitesimal space. By way of example, two quasi-colimit representations of $D\{3; (1, 3), (2, 3)\}$ $(= (D \times D) \oplus D)$ were given in Lavendhomme [7, pp. 92–93] (§3.4, pp. 92–93), only the second one being standard.

2.3 Weil-Exponentiability and Microlinearity

2.3.1 Weil-Exponentiability

We have no reason to hold that all Frölicher spaces credit Weil prolongations as exponentiations by infinitesimal objects in the shade. Therefore we need a notion which distinguishes Frölicher spaces that do so from those that do not.

Definition 2.7 A Frölicher space X is called *Weil exponentiable* if

$$(X \otimes (W_1 \otimes_\infty W_2))^Y = (X \otimes W_1)^Y \otimes W_2 \qquad (1)$$

holds naturally for any Frölicher space Y and any Weil algebras W_1 and W_2.

If $Y = 1$, then (1) degenerates into

$$X \otimes (W_1 \otimes_\infty W_2) = (X \otimes W_1) \otimes W_2$$

If $W_1 = \mathbb{R}$, then (1) degenerates into

$$(X \otimes W_2)^Y = X^Y \otimes W_2$$

The following three propositions have been established in our previous paper [14].

Proposition 2.2 *Convenient vector spaces are Weil exponentiable.*

Corollary 2.1 *C^∞-manifolds in the sense of* [6] *(cf. Section 27) are Weil exponentiable.*

Proposition 2.3 *If X is a Weil exponentiable Frölicher space, then so is $X \otimes W$ for any Weil algebra W.*

Proposition 2.4 *If X and Y are Weil exponentiable Frölicher spaces, then so is $X \times Y$.*

The last proposition can be strengthened to

Proposition 2.5 *The limit of a diagram in* **FS** *whose objects are all Weil-exponentiable is also Weil-exponentiable.*

Proof Let Γ be a diagram in **FS**. Given a Weil algebra W, we write $\Gamma \otimes W$ for the diagram obtained from Γ by putting $\otimes W$ to the right of every object in Γ and $\otimes \mathrm{id}_W$ to the right of every morphism in Γ.We have

$$
\begin{aligned}
&((\mathrm{Lim}\ \Gamma) \otimes (W_1 \otimes_\infty W_2))^Y \\
&= (\mathrm{Lim}\ (\Gamma \otimes (W_1 \otimes_\infty W_2)))^Y \\
&= \mathrm{Lim}\ (\Gamma \otimes (W_1 \otimes_\infty W_2))^Y \\
&= \mathrm{Lim}\ \left((\Gamma \otimes W_1)^Y \otimes W_2\right) \\
&= \left(\mathrm{Lim}\ (\Gamma \otimes W_1)^Y\right) \otimes W_2 \\
&= (\mathrm{Lim}\ (\Gamma \otimes W_1))^Y \otimes W_2 \\
&= ((\mathrm{Lim}\ \Gamma) \otimes W_1)^Y \otimes W_2
\end{aligned}
$$

so that we have the coveted result.

We have already established the following proposition and theorem in in our previous paper [14].

Proposition 2.6 *If X is a Weil exponentiable Frölicher space, then so is X^Y for any Frölicher space Y.*

Theorem 2.4 *Weil exponentiable Frölicher spaces, together with smooth mappings among them, form a Cartesian closed subcategory* $\mathbf{FS_{WE}}$ *of the category* **FS**.

2.3.2 Microlinearity

The central object of study in SDG is *microlinear* spaces. Although the notion of a manifold (=a pasting of copies of a certain linear space) is defined on the local level, the notion of microlinearity is defined on the genuinely infinitesimal level. For the historical account of microlinearity, the reader is referred to §§2.4 of [7] or Appendix D of [4]. To get an adequately restricted cartesian closed subcategory of Frölicher spaces, we have emancipated microlinearity from within a well-adapted model of SDG to Frölicher spaces in the real world in [15]. Recall that

Definition 2.8 A Frölicher space X is called *microlinear* providing that any finite limit diagram Γ in **W** yields a limit diagram $X \otimes \Gamma$ in **FS**, where $X \otimes \Gamma$ is obtained from Γ by putting $X\otimes$ to the left of every object in Γ and $\mathrm{id}_X\otimes$ to the left of every morphism in Γ.

Generally speaking, limits in the category **FS** are bamboozling. The notion of limit in **FS** should be elaborated geometrically.

Definition 2.9 A finite cone Γ in **FS** is called a *transversal limit diagram* providing that $\Gamma \otimes W$ is a limit diagram in **FS** for any Weil algebra W, where the diagram $\Gamma \otimes W$ is obtained from Γ by putting $\otimes W$ to the right of every object in Γ and $\otimes \mathrm{id}_W$ to the right of every morphism in Γ. The limit of a finite diagram of Frölicher spaces is said to be *transversal* providing that its limit diagram is a transversal limit diagram.

Remark 2.2 By taking $W = \mathbb{R}$, we see that a transversal limit diagram in **FS** is always a limit diagram in **FS**.

We have already established the following two propositions in [15].

Proposition 2.7 *If Γ is a transversal limit diagram in* **FS** *whose objects are all Weil exponentiable, then Γ^X is also a transversal limit diagram for any Frölicher space X, where Γ^X is obtained from Γ by putting X as the exponential over every object in Γ and over every morphism in Γ.*

Proposition 2.8 *If Γ is a transversal limit diagram in* **FS** *whose objects are all Weil exponentiable, then $\Gamma \otimes W$ is also a transversal limit diagram for any Weil algebra W.*

The following results have been established in [15].

Proposition 2.9 *Convenient vector spaces are microlinear.*

Corollary 2.2 *C^∞-manifolds in the sense of* [6] *(cf. Section 27) are microlinear.*

Proposition 2.10 *If X is a Weil exponentiable and microlinear Frölicher space, then so is $X \otimes W$ for any Weil algebra W.*

Proposition 2.11 *The class of microlinear Frölicher spaces is closed under transversal limits.*

Corollary 2.3 *Direct products are transversal limits, so that if X and Y are microlinear Frölicher spaces, then so is $X \times Y$.*

Proposition 2.12 *If X is a Weil exponentiable and microlinear Frölicher space, then so is X^Y for any Frölicher space Y.*

Proposition 2.13 *If a Weil exponentiable Frölicher space X is microlinear, then any finite limit diagram Γ in* **W** *yields a transversal limit diagram $X \otimes \Gamma$ in* **FS**.

Theorem 2.5 *Weil exponentiable and microlinear Frölicher spaces, together with smooth mappings among them, form a cartesian closed subcategory $\mathbf{FS}_{\mathbf{WE},\mathbf{ML}}$ of the category* **FS**.

2.4 Convention

Unless stated to the contrary, every Frölicher space occurring in the sequel is assumed to be microlinear and Weil exponentiable. We will fix a smooth mapping $\pi : E \to M$ arbitrarily. In this paper we will naively speak of *bundles* simply as smooth mappings of microlinear and Weil exponentiable Frölicher spaces, for which we will develop three theories of jet bundles. We say that $t \in M \otimes \mathcal{W}_D$ is *degenerate* providing that

$$t = \left(i_{\{x\}\to M} \otimes \mathrm{id}_{\mathscr{W}_D}\right)\left(t'\right)$$

for some $x \in M$ and some $t' \in \{x\} \otimes \mathscr{W}_D$. We say that $t \in E \otimes \mathscr{W}_D$ is *vertical* provided that $\left(\pi \otimes \mathrm{id}_{\mathscr{W}_D}\right)(t)$ is degenerate. We write $(E \otimes \mathscr{W}_D)^{\perp}$ for the totality of vertical $t \in E \otimes \mathscr{W}_D$.

3 The First Approach to Jets

Definition 3.1 A 1-*tangential over* the bundle $\pi\colon E \to M$ *at* $x \in E$ is a mapping $\nabla_x\colon (M \otimes \mathscr{W}_D)_{\pi(x)} \to (E \otimes \mathscr{W}_D)_x$ subject to the following three conditions:

1. We have
$$\left(\pi \otimes \mathrm{id}_{\mathscr{W}_D}\right)(\nabla_x(t)) = t$$
for any $t \in (M \otimes \mathscr{W}_D)_{\pi(x)}$.
2. We have
$$\nabla_x(\alpha t) = \alpha \nabla_x(t)$$
for any $t \in (M \otimes \mathscr{W}_D)_{\pi(x)}$ and any $\alpha \in \mathbb{R}$.
3. The diagram
$$\begin{array}{ccc} (M \otimes \mathscr{W}_D)_{\pi(x)} & \xrightarrow{\mathrm{id}_M \otimes \mathscr{W}_{(d,e)\in D\times D_m \mapsto ed \in D}} & (M \otimes \mathscr{W}_D)_{\pi(x)} \otimes \mathscr{W}_{D_m} \\ \nabla_x \downarrow & & \downarrow \nabla_x \otimes \mathrm{id}_{\mathscr{W}_{D_m}} \\ (E \otimes \mathscr{W}_D)_x & \xrightarrow[\mathrm{id}_E \otimes \mathscr{W}_{(d,e)\in D\times D_m \mapsto ed \in D}]{} & (E \otimes \mathscr{W}_D)_x \otimes \mathscr{W}_{D_m} \end{array}$$
is commutative, where m is an arbitrary natural number.

We note in passing that condition (1.2) implies that ∇_x is linear by dint of Proposition 10 in §1.2 of [7].

Notation 3.1 *We denote by* $\mathbf{J}^1_x(\pi)$ *the totality of* 1*-tangentials* ∇_x *over the bundle* $\pi\colon E \to M$ *at* $x \in E$*. We denote by* $\mathbf{J}^1(\pi)$ *the set-theoretic union of* $\mathbf{J}^1_x(\pi)$*'s for all* $x \in E$*. The canonical projection* $\mathbf{J}^1(\pi) \to E$ *is denoted by* $\pi_{1,0}$ *with*

$$\pi_1 = \left(\pi \otimes \mathrm{id}_{\mathscr{W}_D}\right) \circ \pi_{1,0}.$$

Definition 3.2 Let F be a morphism of bundles over M from π to π' over the same base space M. We say that a 1 *-tangential* ∇_x over π at a point x of E is F-related *to* a 1-*tangential* $\nabla_{F(x)}$ over π' at $F(x)$ of E' provided that

$$\left(F \otimes \mathrm{id}_{\mathscr{W}_D}\right)(\nabla_x(t)) = \nabla_{F(x)}(t)$$

for any $t \in (M \otimes \mathscr{W}_D)_{\pi(x)}$.

Notation 3.2 *By convention, we let*

$$\tilde{\mathbf{J}}^0(\pi) = \hat{\mathbf{J}}^0(\pi) = \mathbf{J}^0(\pi) = E$$

with

$$\tilde{\pi}_{0,0} = \hat{\pi}_{0,0} = \pi_{0,0} = \mathrm{id}_E$$

and

$$\tilde{\pi}_0 = \hat{\pi}_0 = \pi_0 = \pi$$

We let

$$\tilde{\mathbf{J}}^1(\pi) = \hat{\mathbf{J}}^1(\pi) = \mathbf{J}^1(\pi)$$

with

$$\tilde{\pi}_{1,0} = \hat{\pi}_{1,0} = \pi_{1,0}$$

and

$$\tilde{\pi}_1 = \hat{\pi}_1 = \pi_1$$

Notation 3.3 *Now we are going to define $\tilde{\mathbf{J}}^{k+1}(\pi)$, $\hat{\mathbf{J}}^{k+1}(\pi)$ and $\mathbf{J}^{k+1}(\pi)$ together with mappings $\tilde{\pi}_{k+1,k}$: $\tilde{\mathbf{J}}^{k+1}(\pi) \to \tilde{\mathbf{J}}^k(\pi)$, $\hat{\pi}_{k+1,k}$: $\hat{\mathbf{J}}^{k+1}(\pi) \to \hat{\mathbf{J}}^k(\pi)$ and $\pi_{k+1,k}$: $\mathbf{J}^{k+1}(\pi) \to \mathbf{J}^k(\pi)$ by induction on $k \geq 1$. Intuitively speaking, these are intended for non-holonomic, semi-holonomic and holonomic jet bundles in order. We let $\tilde{\pi}_{k+1} = \tilde{\pi}_k \circ \tilde{\pi}_{k+1,k}$, $\hat{\pi}_{k+1} = \hat{\pi}_k \circ \hat{\pi}_{k+1,k}$ and $\pi_{k+1} = \pi_k \circ \pi_{k+1,k}$.*

1. *First we deal with $\tilde{\mathbf{J}}^{k+1}(\pi)$, which is defined to be $\mathbf{J}^1(\tilde{\pi}_k)$ with $\tilde{\pi}_{k+1,k} = (\tilde{\pi}_k)_{1,0}$.*
2. *Next we deal with $\hat{\mathbf{J}}^{k+1}(\pi)$, which is defined to be the subspace of $\mathbf{J}^1(\hat{\pi}_k)$ consisting of ∇_x's with $x = \nabla_y \in \hat{\mathbf{J}}^k(\pi)$ abiding by the condition that ∇_x is $\hat{\pi}_{k,k-1}$-related to ∇_y.*
3. *Finally we deal with $\mathbf{J}^{k+1}(\pi)$, which is defined to be the subspace of $\mathbf{J}^1(\pi_k)$ consisting of ∇_x's with $x = \nabla_y \in \mathbf{J}^k(\pi)$ abiding by the conditions that ∇_x is $\pi_{k,k-1}$-related to ∇_y and that the composition of mappings*

$$\begin{aligned}
&\left(M \otimes \mathcal{W}_{D^2}\right)_{\pi_k(x)} \\
&\xrightarrow{\left\langle \mathrm{id}_M \otimes \mathcal{W}_{d \in D \mapsto (d,0) \in D^2},\ \mathrm{id}_M \otimes \mathcal{W}_{(d_1.d_2) \in D^2 \mapsto (d_2.d_1) \in D^2} \right\rangle} \\
&\left((M \otimes \mathcal{W}_D) \times_{M \otimes \mathcal{W}_D} \left(M \otimes \mathcal{W}_{D^2}\right)\right)_{\pi_k(x)} \\
&\xrightarrow{\nabla_x \times \mathrm{id}_{M \otimes \mathcal{W}_{D^2}}} \\
&\left(\left(\mathbf{J}^k(\pi) \otimes \mathcal{W}_D\right) \times_{M \otimes \mathcal{W}_D} \left(M \otimes \mathcal{W}_{D^2}\right)\right)_{\pi_k(x)} \\
&= \left(\left(\mathbf{J}^k(\pi) \otimes \mathcal{W}_D\right) \times_{M \otimes \mathcal{W}_D} ((M \otimes \mathcal{W}_D) \otimes \mathcal{W}_D)\right)_{\pi_k(x)} \\
&= \left(\left(\mathbf{J}^k(\pi) \times_M (M \otimes \mathcal{W}_D)\right) \otimes \mathcal{W}_D\right)_{\pi_k(x)}
\end{aligned}$$

$$\xrightarrow{\left((\nabla, t) \in \mathbf{J}^k(\pi) \times_M (M \otimes \mathscr{W}_D) \mapsto \nabla(t) \in \left(\mathbf{J}^{k-1}(\pi) \otimes \mathscr{W}_D\right)\right) \otimes \mathrm{id}_{\mathscr{W}_D}}$$

$$\left(\mathbf{J}^{k-1}(\pi) \otimes \mathscr{W}_D\right) \otimes \mathscr{W}_D$$

$$= \mathbf{J}^{k-1}(\pi) \otimes \mathscr{W}_{D^2}$$

is equal to the composition of mappings

$$\left(M \otimes \mathscr{W}_{D^2}\right)_{\pi_k(x)}$$

$$\xrightarrow{\left\langle \mathrm{id}_M \otimes \mathscr{W}_{d \in D \mapsto (0,d) \in D^2}, \mathrm{id}_{M \otimes \mathscr{W}_{D^2}} \right\rangle}$$

$$\left((M \otimes \mathscr{W}_D) \times_{M \otimes \mathscr{W}_D} \left(M \otimes \mathscr{W}_{D^2}\right)\right)_{\pi_k(x)}$$

$$\xrightarrow{\nabla_x \times \mathrm{id}_{M \otimes \mathscr{W}_{D^2}}}$$

$$\left(\left(\mathbf{J}^k(\pi) \otimes \mathscr{W}_D\right) \times_{M \otimes \mathscr{W}_D} \left(M \otimes \mathscr{W}_{D^2}\right)\right)_{\pi_k(x)}$$

$$= \left(\left(\mathbf{J}^k(\pi) \otimes \mathscr{W}_D\right) \times_{M \otimes \mathscr{W}_D} ((M \otimes \mathscr{W}_D) \otimes \mathscr{W}_D)\right)_{\pi_k(x)}$$

$$= \left(\left(\mathbf{J}^k(\pi) \times_M (M \otimes \mathscr{W}_D)\right) \otimes \mathscr{W}_D\right)_{\pi_k(x)}$$

$$\xrightarrow{\left((\nabla, t) \in \mathbf{J}^k(\pi) \times_M (M \otimes \mathscr{W}_D) \mapsto \nabla(t) \in \left(\mathbf{J}^{k-1}(\pi) \otimes \mathscr{W}_D\right)\right) \otimes \mathrm{id}_{\mathscr{W}_D}}$$

$$\left(\mathbf{J}^{k-1}(\pi) \otimes \mathscr{W}_D\right) \otimes \mathscr{W}_D$$

$$= \mathbf{J}^{k-1}(\pi) \otimes \mathscr{W}_{D^2}$$

$$\xrightarrow{\mathrm{id}_{\mathbf{J}^{k-1}(\pi)} \otimes \mathscr{W}_{(d_1,d_2) \in D^2 \mapsto (d_2,d_1) \in D^2}}$$

$$\mathbf{J}^{k-1}(\pi) \otimes \mathscr{W}_{D^2}$$

Definition 3.3 Elements of $\tilde{\mathbf{J}}^n(\pi)$ are called n-subtangentials, while elements of $\hat{\mathbf{J}}^n(\pi)$ are called n-quasitangentials. Elements of $\mathbf{J}^n(\pi)$ are called n-tangentials.

4 The Second Approach to Jets

Definition 4.1 Let n be a natural number. A D^n-pseudotangential *over* the bundle $\pi: E \to M$ *at* $x \in E$ is a mapping $\nabla_x: (M \otimes \mathscr{W}_{D^n})_{\pi(x)} \to (E \otimes \mathscr{W}_{D^n})_x$ abiding by the following conditions:

1. We have

$$\left(\pi \otimes \mathrm{id}_{\mathscr{W}_{D^n}}\right)(\nabla_x(\gamma)) = \gamma$$

for any $\gamma \in (M \otimes \mathscr{W}_{D^n})_{\pi(x)}$.

2. We have
$$\nabla_x(\alpha \underset{i}{\cdot} \gamma) = \alpha \underset{i}{\cdot} \nabla_x(\gamma) \qquad (1 \leq i \leq n)$$
for any $\gamma \in (M \otimes \mathcal{W}_{D^n})_{\pi(x)}$ and any $\alpha \in \mathbb{R}$.
3. The diagram
$$\begin{array}{ccc} (M \otimes \mathcal{W}_{D^n})_{\pi(x)} & \rightarrow & (M \otimes \mathcal{W}_{D^n})_{\pi(x)} \otimes \mathcal{W}_{D_m} \\ \nabla_x \downarrow & & \downarrow \nabla_x \otimes \mathrm{id}_{\mathcal{W}_{D_m}} \\ (E \otimes \mathcal{W}_{D^n})_x & \rightarrow & (E \otimes \mathcal{W}_{D^n})_x \otimes \mathcal{W}_{D_m} \end{array}$$
is commutative, where m is an arbitrary natural number, the upper horizontal arrow is
$$\mathrm{id}_M \otimes \mathcal{W}_{(d_1,\ldots,d_n,e) \in D^n \times D_m \mapsto (d_1,\ldots,d_{i-1},ed_i,d_{i+1},\ldots d_n) \in D^n},$$
and the lower horizontal arrow is
$$\mathrm{id}_E \otimes \mathcal{W}_{(d_1,\ldots,d_n,e) \in D^n \times D_m \mapsto (d_1,\ldots,d_{i-1},ed_i,d_{i+1},\ldots d_n) \in D^n}.$$
4. We have
$$\nabla_x(\gamma^\sigma) = (\nabla_x(\gamma))^\sigma$$
for any $\gamma \in (M \otimes \mathcal{W}_{D^n})_{\pi(x)}$ and for any $\sigma \in \mathbf{S}_n$.

Remark 4.1 The third condition in the above definition claims what is called infinitesimal multilinearity, while the second claims what is authentic multilinearity.

Notation 4.1 *We denote by $\hat{\mathbb{J}}_x^{D^n}(\pi)$ the totality of D^n-pseudotangentials ∇_x over the bundle $\pi : E \to M$ at $x \in E$. We denote by $\hat{\mathbb{J}}^{D^n}(\pi)$ the set-theoretic union of $\hat{\mathbb{J}}_x^{D^n}(\pi)$'s for all $x \in E$. In particular, $\hat{\mathbb{J}}^{D^0}(\pi) = E$ by convention.*

Lemma 4.1 *The diagram*
$$E \otimes \mathcal{W}_{D^n} \xrightarrow{\mathrm{id}_E \otimes \mathcal{W}_{(d_1,\ldots,d_n,d_{n+1}) \in D^{n+1} \mapsto (d_1,\ldots,d_n) \in D^n}} E \otimes \mathcal{W}_{D^{n+1}}$$
$$\underset{\mathrm{id}_E \otimes \mathcal{W}_{(d_1,\ldots,d_n,d_{n+1}) \in D^{n+1} \mapsto (d_1,\ldots,d_n,0) \in D^{n+1}}}{\overset{\mathrm{id}_{E \otimes \mathcal{W}_{D^{n+1}}}}{\rightrightarrows}} E \otimes \mathcal{W}_{D^{n+1}}$$
is an equalizer.

Proof It is well known that the diagram
$$\mathcal{W}_{D^n} \xrightarrow{\mathcal{W}_{(d_1,\ldots,d_n,d_{n+1}) \in D^{n+1} \mapsto (d_1,\ldots,d_n) \in D^n}} \mathcal{W}_{D^{n+1}} \underset{\mathcal{W}_{(d_1,\ldots,d_n,d_{n+1}) \in D^{n+1} \mapsto (d_1,\ldots,d_n,0) \in D^{n+1}}}{\overset{\mathrm{id}_{\mathcal{W}_{D^{n+1}}}}{\rightrightarrows}} \mathcal{W}_{D^{n+1}}$$
is an equalizer in the category of Weil algebras, so that the desired result follows from the microlinearity of E.

Corollary 4.1 *$\gamma \in E \otimes \mathcal{W}_{D^{n+1}}$ is in the equalizer of*

$$E \otimes \mathcal{W}_{D^{n+1}} \underset{\mathrm{id}_E \otimes \mathcal{W}_{(d_1,\ldots,d_n,d_{n+1}) \in D^{n+1} \mapsto (d_1,\ldots,d_n,0) \in D^{n+1}}}{\overset{\mathrm{id}_{E \otimes \mathcal{W}_{D^{n+1}}}}{\rightrightarrows}} E \otimes \mathcal{W}_{D^{n+1}}$$

iff

$$\gamma = (\mathbf{s}_{n+1} \circ \mathbf{d}_{n+1})(\gamma)$$

Proof This follows simply from

$$\mathbf{s}_{n+1} \circ \mathbf{d}_{n+1} = \mathrm{id}_E \otimes \mathcal{W}_{(d_1,\ldots,d_n,d_{n+1}) \in D^{n+1} \mapsto (d_1,\ldots,d_n,0) \in D^{n+1}}$$

Proposition 4.1 *Let ∇_x be a D^{n+1}-pseudotangential over the bundle $\pi \colon E \to M$ at $x \in E$. Let $\gamma \in (M \otimes \mathcal{W}_{D^n})_{\pi(x)}$. Then we have*

$$\nabla_x(\mathbf{s}_{n+1}(\gamma)) = \left(\mathrm{id}_E \otimes \mathcal{W}_{(d_1,\ldots,d_n,d_{n+1}) \in D^{n+1} \mapsto (d_1,\ldots,d_n,0) \in D^{n+1}}\right)(\nabla_x(\mathbf{s}_{n+1}(\gamma)))$$

so that

$$\nabla_x(\mathbf{s}_{n+1}(\gamma)) = (\mathbf{s}_{n+1} \circ \mathbf{d}_{n+1})(\nabla_x(\mathbf{s}_{n+1}(\gamma)))$$

Proof For any $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} &\alpha \underset{n+1}{\cdot} (\nabla_x(\mathbf{s}_{n+1}(\gamma))) \\ &= \nabla_x(\alpha \underset{n+1}{\cdot} (\mathbf{s}_{n+1}(\gamma))) \\ &= \nabla_x(\mathbf{s}_{n+1}(\gamma)) \end{aligned}$$

Therefore we have the desired result by letting $\alpha = 0$ in the above calculation.

Corollary 4.2 *The assignment*

$$\gamma \in (M \otimes \mathcal{W}_{D^n})_{\pi(x)} \longmapsto \mathbf{d}_{n+1}(\nabla_x(\mathbf{s}_{n+1}(\gamma))) \in (E \otimes \mathcal{W}_{D^n})_x$$

is an n-pseudotangential over the bundle $\pi : E \to M$ at x.

Notation 4.2 *By this Corollary, we have canonical projections $\widehat{\pi}_{n+1,n} \colon \hat{\mathbb{J}}^{D^{n+1}}(\pi) \to \hat{\mathbb{J}}^{D^n}(\pi)$. By assigning $\pi(x) \in M$ to each n-pseudotangential ∇_x over the bundle $\pi \colon E \to M$ at $x \in E$, we have the canonical projections $\widehat{\pi}_n : \hat{\mathbb{J}}^{D^n}(\pi) \to M$. Note that $\widehat{\pi}_n \circ \widehat{\pi}_{n+1,n} = \widehat{\pi}_{n+1}$ For any natural numbers n, m with $m \leq n$, we define $\widehat{\pi}_{n,m} \colon \hat{\mathbb{J}}^{D^n}(\pi) \to \hat{\mathbb{J}}^{D^m}(\pi)$ to be $\widehat{\pi}_{m+1,m} \circ \ldots \circ \widehat{\pi}_{n,n-1}$.*

Now we are going to show that

Proposition 4.2 *Let* $\nabla_x \in \hat{\mathbb{J}}^{D^{n+1}}(\pi)$. *Then the following diagrams are commutative:*

$$\begin{array}{ccc} \left(M \otimes \mathscr{W}_{D^{n+1}}\right)_{\pi(x)} & \xrightarrow{\nabla_x} & \left(E \otimes \mathscr{W}_{D^{n+1}}\right)_x \\ \mathbf{s}_i \uparrow & & \uparrow \mathbf{s}_i \\ (M \otimes \mathscr{W}_{D^n})_{\pi(x)} & \xrightarrow[\widehat{\pi}_{n+1,n}(\nabla_x)]{} & (E \otimes \mathscr{W}_{D^n})_x \end{array}$$

$$\begin{array}{ccc} \left(M \otimes \mathscr{W}_{D^{n+1}}\right)_{\pi(x)} & \xrightarrow{\nabla_x} & \left(E \otimes \mathscr{W}_{D^{n+1}}\right)_x \\ \mathbf{d}_i \downarrow & & \downarrow \mathbf{d}_i \\ (M \otimes \mathscr{W}_{D^n})_{\pi(x)} & \xrightarrow[\widehat{\pi}_{n+1,n}(\nabla_x)]{} & (E \otimes \mathscr{W}_{D^n})_x \end{array}$$

Proof By the very definition of $\widehat{\pi}_{n+1,n}$, we have

$$\mathbf{s}_{n+1}(\widehat{\pi}_{n+1}(\nabla_x)(\gamma)) = \nabla_x(\mathbf{s}_{n+1}(\gamma))$$

for any $\gamma \in (M \otimes \mathscr{W}_{D^n})_{\pi(x)}$. For $i \neq n+1$, we have

$$\begin{aligned} & \mathbf{s}_i(\widehat{\pi}_{n+1,n}(\nabla_x)(\gamma)) \\ & = \left(\left(\mathbf{s}_{n+1}(\widehat{\pi}_{n+1,n}(\nabla_x)(\gamma))\right)^{\langle i,n+1\rangle}\right)^{\langle i+1,i+2,\ldots,n,n+1\rangle} \\ & = \left((\nabla_x(\mathbf{s}_{n+1}(\gamma)))^{\langle i,n+1\rangle}\right)^{\langle i+1,i+2,\ldots,n,n+1\rangle} \\ & = \left(\nabla_x\left((\mathbf{s}_{n+1}(\gamma))^{\langle i,n+1\rangle}\right)\right)^{\langle i+1,i+2,\ldots,n,n+1\rangle} \\ & = \nabla_x\left(\left((\mathbf{s}_{n+1}(\gamma))^{\langle i,n+1\rangle}\right)^{\langle i+1,i+2,\ldots,n,n+1\rangle}\right) \\ & = \nabla_x(\mathbf{s}_i(\gamma)) \end{aligned}$$

Now we are going to show that

$$\mathbf{d}_i(\nabla_x(\gamma)) = (\widehat{\pi}_{n+1,n}(\nabla_x))(\mathbf{d}_i(\gamma))$$

for any $\gamma \in \left(M \otimes \mathscr{W}_{D^{n+1}}\right)_{\pi(x)}$. First we deal with the case of $i = n+1$. We have

$$\begin{aligned} & \mathbf{d}_{n+1}(\nabla_x(\gamma)) \\ & = \mathbf{d}_{n+1}(0 \underset{n+1}{\cdot} \nabla_x(\gamma)) \\ & = \mathbf{d}_{n+1}(\nabla_x(0 \underset{n+1}{\cdot} \gamma)) \\ & = \mathbf{d}_{n+1}(\nabla_x(\mathbf{s}_{n+1}(\mathbf{d}_{n+1}(\gamma)))) \\ & = (\widehat{\pi}_{n+1,n}(\nabla_x))(\mathbf{d}_{n+1}(\gamma)) \end{aligned}$$

For $i \neq n+1$, we have

$$\begin{aligned}
&\mathbf{d}_i(\nabla_x(\gamma)) \\
&= \left(\mathbf{d}_{n+1}\left((\nabla_x(\gamma))^{\langle i,n+1\rangle}\right)\right)^{\langle n,n-1,\ldots,i+1,i\rangle} \\
&= \left(\mathbf{d}_{n+1}(\nabla_x(\gamma^{\langle i,n+1\rangle}))\right)^{\langle n,n-1,\ldots,i+1,i\rangle} \\
&= (((\widehat{\pi}_{n+1,n}(\nabla_x))\left(\mathbf{d}_{n+1}(\gamma^{\langle i,n+1\rangle}))\right)^{\langle n,n-1,\ldots,i+1,i\rangle} \\
&= (\widehat{\pi}_{n+1,n}(\nabla_x))\left(\left(\mathbf{d}_{n+1}(\gamma^{\langle i,n+1\rangle})\right)^{\langle n,n-1,\ldots,i+1,i\rangle}\right) \\
&= (\widehat{\pi}_{n+1,n}(\nabla_x))\,(\mathbf{d}_i(\gamma))
\end{aligned}$$

Thus we are done through.

Corollary 4.3 *Let* $\nabla_x^+, \nabla_x^- \in \hat{\mathbb{J}}^{D^{n+1}}(\pi)$ *with*

$$\widehat{\pi}_{n+1,n}(\nabla_x^+) = \widehat{\pi}_{n+1,n}(\nabla_x^-)$$

Then

$$\left(\mathrm{id}_E \otimes \mathscr{W}_{i_{D\{n+1\}_n \to D^{n+1}}}\right)\left(\nabla_x^+(\gamma)\right) = \left(\mathrm{id}_E \otimes \mathscr{W}_{i_{D\{n+1\}_n \to D^{n+1}}}\right)\left(\nabla_x^-(\gamma)\right)$$

for any $\gamma \in \left(M \otimes \mathscr{W}_{D^{n+1}}\right)_{\pi(x)}$.

Definition 4.2 The notion of a D^n-tangential *over* the bundle $\pi\colon E \to M$ *at* x is defined by induction on n. The notion of a D-tangential *over* the bundle $\pi\colon E \to M$ *at* x shall be identical with that of a D-pseudotangential *over* the bundle $\pi\colon E \to M$ *at* x . Now we proceed inductively. A D^{n+1}-pseudotangential

$$\nabla_x : \left(M \otimes \mathscr{W}_{D^{n+1}}\right)_{\pi(x)} \to \left(E \otimes \mathscr{W}_{D^{n+1}}\right)_x$$

over the bundle $\pi\colon E \to M$ at $x \in E$ is called a D^{n+1}-tangential *over* the bundle $\pi\colon E \to M$ *at* x if it acquiesces in the following two conditions:

1. $\widehat{\pi}_{n+1,n}(\nabla_x)$ is a D^n-tangential *over* the bundle $\pi\colon E \to M$ *at* x.
2. For any $\gamma \in (M \otimes \mathscr{W}_{D^n})_{\pi(x)}$, we have

$$\begin{aligned}
&\nabla_x\left(\left(\mathrm{id}_M \otimes \mathscr{W}_{(d_1,\ldots,d_n,d_{n+1})\in D^{n+1}\mapsto(d_1,\ldots,d_nd_{n+1})\in D^n}\right)(\gamma)\right) \\
&= \left(\mathrm{id}_E \otimes \mathscr{W}_{(d_1,\ldots,d_n,d_{n+1})\in D^{n+1}\mapsto(d_1,\ldots,d_nd_{n+1})\in D^{n+1}}\right)\left(\left(\widehat{\pi}_{n+1,n}(\nabla_x)\right)(\gamma)\right)
\end{aligned}$$

Notation 4.3 *We denote by* $\mathbb{J}_x^{D^n}(\pi)$ *the totality of* D^n*-tangentials* ∇_x *over the bundle* $\pi\colon E \to M$ *at* $x \in E$*. We denote by* $\mathbb{J}^{D^n}(\pi)$ *the set-theoretic union of* $\mathbb{J}_x^{D^n}(\pi)$*'s for all* $x \in E$*. In particular,* $\mathbb{J}^{D^0}(\pi) = \hat{\mathbb{J}}^{D^0}(\pi) = E$ *by convention and* $\mathbb{J}^D(\pi) = \hat{\mathbb{J}}^D(\pi)$ *by definition. By the very definition of* D^n*-tangential, the projections* $\widehat{\pi}_{n+1,n}$*:*

$\hat{\mathbb{J}}^{D^{n+1}}(\pi) \rightarrow \hat{\mathbb{J}}^{D^{n}}(\pi)$ are naturally restricted to mappings $\pi_{n+1,n}: \mathbb{J}^{D^{n+1}}(\pi) \rightarrow \mathbb{J}^{D^{n}}(\pi)$. Similarly for $\pi_n: \mathbb{J}^{D^{n}}(\pi) \rightarrow M$ and $\pi_{n,m}: \mathbb{J}^{D^{n}}(\pi) \rightarrow \mathbb{J}^{D^{m}}(\pi)$ with $m \leq n$.

It is easy to see that

Proposition 4.3 *Let m, n be natural numbers with $m \leq n$. Let $k_1, \ldots, k_m$ be positive integers with $k_1 + \cdots + k_m = n$. For any $\nabla_x \in \mathbb{J}^{D^n}(\pi)$, any $\gamma \in (M \otimes \mathscr{W}_{D^m})_{\pi(x)}$ and any $\sigma \in \mathbf{S}_n$, we have*

$$\nabla_x \left(\left(\mathrm{id}_M \otimes \mathscr{W}_{(d_1,\ldots,d_n)\in D^n \mapsto \left(d_{\sigma(1)}\ldots d_{\sigma(k_1)}, d_{\sigma(k_1+1)}\ldots d_{\sigma(k_1+k_2)}, \ldots, d_{\sigma(k_1+\ldots+k_{m-1}+1)}\ldots d_{\sigma(n)}\right)} \right) (\gamma) \right)$$
$$= \left(\mathrm{id}_E \otimes \mathscr{W}_{(d_1,\ldots,d_n)\in D^n \mapsto \left(d_{\sigma(1)}\ldots d_{\sigma(k_1)}, d_{\sigma(k_1+1)}\ldots d_{\sigma(k_1+k_2)}, \ldots, d_{\sigma(k_1+\ldots+k_{m-1}+1)}\ldots d_{\sigma(n)}\right)} \right)$$
$$\left(\left(\pi_{n,m}(\nabla_x) \right) (\gamma) \right)$$

Interestingly enough, any D^n-pseudotangential naturally gives rise to what might be called a $\mathbb{D}$-*pseudotangential* for any simplicial infinitesimal space $\mathbb{D}$ of dimension less than or equal to n.

Theorem 4.4 *Let n be a natural number. Let $\mathbb{D}$ be a simplicial infinitesimal space of dimension less than or equal to n. Any D^n-pseudotangential ∇_x over the bundle $\pi: E \rightarrow M$ at $x \in E$ naturally induces a mapping $\nabla_x^{\mathbb{D}}: (M \otimes \mathscr{W}_{\mathbb{D}})_{\pi(x)} \rightarrow (E \otimes \mathscr{W}_{\mathbb{D}})_x$ abiding by the following three conditions:*

1. *We have*
$$\left(\pi \otimes \mathrm{id}_{\mathscr{W}_{\mathbb{D}}}\right) \left(\nabla_x^{\mathbb{D}}(\gamma)\right) = \gamma$$
for any $\gamma \in (M \otimes \mathscr{W}_{\mathbb{D}})_{\pi(x)}$.
2. *We have*
$$\nabla_x^{\mathbb{D}}(\alpha \underset{i}{\cdot} \gamma) = \alpha \underset{i}{\cdot} \left(\nabla_x^{\mathbb{D}}(\gamma)\right)$$
for any $\alpha \in \mathbb{R}$ and any $\gamma \in (M \otimes \mathscr{W}_{\mathbb{D}})_{\pi(x)}$, where i is a natural number with $1 \leq i \leq \deg \mathbb{D}$.
3. *The diagram*
$$\begin{array}{ccc} (M \otimes \mathscr{W}_{\mathbb{D}})_{\pi(x)} & \rightarrow & (M \otimes \mathscr{W}_{\mathbb{D}})_{\pi(x)} \otimes \mathscr{W}_{D_m} \\ \nabla_x \downarrow & & \downarrow \nabla_x \otimes \mathrm{id}_{\mathscr{W}_{D_m}} \\ (E \otimes \mathscr{W}_{\mathbb{D}})_x & \rightarrow & (E \otimes \mathscr{W}_{\mathbb{D}})_x \otimes \mathscr{W}_{D_m} \end{array}$$
is commutative, where m is an arbitrary natural number, the upper horizontal arrow is
$$\mathrm{id}_M \otimes \mathscr{W}_{(d_1,\ldots,d_k,e)\in\mathbb{D}\times D_m \mapsto (d_1,\ldots,d_{i-1},ed_i,d_{i+1},\ldots d_k)\in\mathbb{D}},$$
and the lower horizontal arrow is
$$\mathrm{id}_E \otimes \mathscr{W}_{(d_1,\ldots,d_k,e)\in\mathbb{D}\times D_m \mapsto (d_1,\ldots,d_{i-1},ed_i,d_{i+1},\ldots d_k)\in\mathbb{D}}$$

with $k = \deg \mathbb{D}$ *and* $1 \leq i \leq k$.

If the simplicial infinitesimal space $\mathbb{D}$ *is symmetric, the induced mapping* $\nabla_x^{\mathbb{D}}$: $(M \otimes \mathcal{W}_{\mathbb{D}})_{\pi(x)} \to (E \otimes \mathcal{W}_{\mathbb{D}})_x$ *acquiesces in the following condition of symmetry besides the above ones:*

- *We have*

$$\nabla_x^{\mathbb{D}}(\gamma^\sigma) = (\nabla_x^{\mathbb{D}}(\gamma))^\sigma$$

for any $\sigma \in \mathbf{S}_k$ *and any* $\gamma \in (M \otimes \mathcal{W}_{\mathbb{D}})_{\pi(x)}$.

Proof For the sake of simplicity in description, we deal, by way of example, with the case that $n = 3$ and $\mathbb{D} = D\{3\}_2$, for which the standard quasi-colimit representation was given in the proof of Theorem 2.3. Therefore, giving $\gamma \in \left(M \otimes \mathcal{W}_{D\{3\}_2}\right)_{\pi(x)}$ is equivalent to giving $\gamma_{12}, \gamma_{13}, \gamma_{23} \in \left(M \otimes \mathcal{W}_{D^2}\right)_{\pi(x)}$ with $\mathbf{d}_2(\gamma_{12}) = \mathbf{d}_2(\gamma_{13})$, $\mathbf{d}_1(\gamma_{12}) = \mathbf{d}_2(\gamma_{23})$ and $\mathbf{d}_1(\gamma_{13}) = \mathbf{d}_1(\gamma_{23})$. By Proposition 4.2, we have

$$\begin{aligned}
&\mathbf{d}_2(\widehat{\pi}_{3,2}(\nabla_x)(\gamma_{12})) = \widehat{\pi}_{3,2}(\nabla_x)(\mathbf{d}_2(\gamma_{12})) = \widehat{\pi}_{3,2}(\nabla_x)(\mathbf{d}_2(\gamma_{13})) = \mathbf{d}_2(\widehat{\pi}_{3,2}(\nabla_x)(\gamma_{13})) \\
&\mathbf{d}_1(\widehat{\pi}_{3,2}(\nabla_x)(\gamma_{12})) = \widehat{\pi}_{3,2}(\nabla_x)(\mathbf{d}_1(\gamma_{12})) = \widehat{\pi}_{3,2}(\nabla_x)(\mathbf{d}_2(\gamma_{23})) = \mathbf{d}_2(\widehat{\pi}_{3,2}(\nabla_x)(\gamma_{23})) \\
&\mathbf{d}_1(\widehat{\pi}_{3,2}(\nabla_x)(\gamma_{13})) = \widehat{\pi}_{3,2}(\nabla_x)(\mathbf{d}_1(\gamma_{13})) = \widehat{\pi}_{3,2}(\nabla_x)(\mathbf{d}_1(\gamma_{23})) = \mathbf{d}_1(\widehat{\pi}_{3,2}(\nabla_x)(\gamma_{23})),
\end{aligned}$$

which determines a unique $\nabla_x^{D\{3\}_2}(\gamma) \in \left(E \otimes \mathcal{W}_{D\{3\}_2}\right)_x$ with

$$\begin{aligned}
\mathbf{d}_1(\nabla_x^{D\{3\}_2}(\gamma)) &= \widehat{\pi}_{3,2}(\nabla_x)(\gamma_{23}) \\
\mathbf{d}_2(\nabla_x^{D\{3\}_2}(\gamma)) &= \widehat{\pi}_{3,2}(\nabla_x)(\gamma_{13}) \\
\mathbf{d}_3(\nabla_x^{D\{3\}_2}(\gamma)) &= \widehat{\pi}_{3,2}(\nabla_x)(\gamma_{12}).
\end{aligned}$$

The proof that $\nabla_x^{D\{3\}_2}$: $\left(M \otimes \mathcal{W}_{D\{3\}_2}\right)_{\pi(x)} \to \left(E \otimes \mathcal{W}_{D\{3\}_2}\right)_x$ acquiesces in the desired four properties is safely left to the reader.

Remark 4.2 The reader should note that the induced mapping $\nabla_x^{\mathbb{D}}$ is defined in terms of the standard quasi-colimit representation of $\mathbb{D}$. The concluding corollary of this subsection will show that the induced mapping $\nabla_x^{\mathbb{D}}$ is independent of our choice of a quasi-colimit representation of $\mathbb{D}$ to a large extent, whether it is standard or not, as long as ∇ is not only a D^n-pseudotangential but also a D^n*-tangential.* We note in passing that $\hat{\pi}_{n,m}(\nabla)$ with $m \leq n$ is no other than $\nabla_x^{D^m}$.

Proposition 4.4 *Let* π': $P \to E$ *be another bundle with* $x \in P$. *If* $\nabla_{\pi'(x)}$ *is a* n*-tangential*$_2$ *over the bundle* π: $E \to M$ *at* $\pi'(x) \in E$ *and* ∇_x *is a* n*-tangential*$_2$ *over the bundle* π': $P \to E$ *at* $x \in E$, *then the composition* $\nabla_x \circ \nabla_{\pi'(x)}$ *is a* n*-tangential*$_2$ *over the bundle* $\pi \circ \pi'$: $P \to M$ *at* $x \in E$, *and* $\pi_{n,n-1}(\nabla_x \circ \nabla_{\pi'(x)}) = \pi_{n,n-1}(\nabla_x) \circ \pi_{n,n-1}(\nabla_{\pi'(x)})$ *provided that* $n \geq 1$.

Proof In case of $n = 0$, there is nothing to prove. It is easy to see that if $\nabla_{\pi'(x)}$ is a n-tangential$_2$ over the bundle π: $E \to M$ *at* $\pi'(x) \in E$ and ∇_x is a n-tangential$_2$

over the bundle $\pi'\colon P \to E$ *at* $x \in E$, then the composition $\nabla_x \circ \nabla_{\pi'(x)}$ is an n-pseudoconnection *over* the bundle $\pi\colon E \to M$ *at* $x \in P$. If $\nabla_{\pi'(x)}$ is a $(n+1)$-tangential$_2$ over the bundle $\pi\colon E \to M$ *at* $\pi'(x) \in E$ and ∇_x is a $(n+1)$-tangential$_2$ over the bundle $\pi'\colon P \to E$ *at* $x \in P$, then we have

$$\begin{aligned}\pi_{n+1,n}(\nabla_x \circ \nabla_{\pi'(x)}) &= \mathbf{d}_{n+1} \circ \nabla_x \circ \nabla_{\pi'(x)} \circ \mathbf{s}_{n+1} \\ &= \mathbf{d}_{n+1} \circ \nabla_x \circ \mathbf{s}_{n+1} \circ \mathbf{d}_{n+1} \circ \nabla_{\pi'(x)} \circ \mathbf{s}_{n+1} \\ &\text{[By Proposition 4.1]} \\ &= \pi_{n+1,n}(\nabla_x) \circ \pi_{n+1,n}(\nabla_{\pi'(x)})\end{aligned}$$

Therefore we have

$$\begin{aligned}&\nabla_x \circ \nabla_{\pi'(x)}((\mathrm{id}_M \otimes \mathscr{W}_{(d_1,\ldots,d_n,d_{n+1})\in D^{n+1} \mapsto (d_1,\ldots,d_n d_{n+1})\in D^n})(\gamma)) \\ &= \nabla_x \left(\nabla_{\pi'(x)} \left(\left(\mathrm{id}_M \otimes \mathscr{W}_{(d_1,\ldots,d_n,d_{n+1})\in D^{n+1} \mapsto (d_1,\ldots,d_n d_{n+1})\in D^n}\right)(\gamma)\right)\right) \\ &= \nabla_x \left(\left(\mathrm{id}_E \otimes \mathscr{W}_{(d_1,\ldots,d_n,d_{n+1})\in D^{n+1} \mapsto (d_1,\ldots,d_n d_{n+1})\in D^n}\right)\left(\pi_{n+1,n}(\nabla_{\pi'(x)})(\gamma)\right)\right) \\ &= \left(\mathrm{id}_P \otimes \mathscr{W}_{(d_1,\ldots,d_n,d_{n+1})\in D^{n+1} \mapsto (d_1,\ldots,d_n d_{n+1})\in D^n}\right)\left(\pi_{n+1,n}(\nabla_x)\left(\pi_{n+1,n}(\nabla_{\pi'(x)})(\gamma)\right)\right) \\ &= \left(\mathrm{id}_P \otimes \mathscr{W}_{(d_1,\ldots,d_n,d_{n+1})\in D^{n+1} \mapsto (d_1,\ldots,d_n d_{n+1})\in D^n}\right)\left(\pi_{n+1,n}(\nabla_x \circ \nabla_{\pi'(x)})\right)\end{aligned}$$

Thus we can prove by induction on n that if $\nabla_{\pi'(x)}$ is a n-tangential$_2$ over the bundle $\pi\colon E \to M$ *at* $\pi'(x) \in E$ and ∇_x is a n-tangential$_2$ *over* the bundle $\pi'\colon P \to E$ *at* $x \in E$, then the composition $\nabla_x \circ \nabla_{\pi'(x)}$ is a n-tangential$_2$ *over* the bundle $\pi \circ \pi'\colon P \to M$ *at* $x \in E$.

Theorem 4.5 *Let ∇ be a D^n-tangential over the bundle $\pi\colon E \to M$ at $x \in E$. Let $\mathbb{D}$ and $\mathbb{D}'$ be simplicial infinitesimal spaces of dimension less than or equal to n. Let χ be a monomial mapping from $\mathbb{D}$ to $\mathbb{D}'$. Let $\gamma \in \mathbf{T}_x^{\mathbb{D}'}(M)$. Then we have*

$$\nabla_{\mathbb{D}}((\mathrm{id}_M \otimes \mathscr{W}_\chi)(\gamma)) = (\mathrm{id}_E \otimes \mathscr{W}_\chi)(\nabla_{\mathbb{D}'}(\gamma))$$

Remark 4.3 The reader should note that the above far-flung generalization of Proposition 4.3 subsumes Proposition 4.2.

Proof In place of giving a general proof with formidable notation, we satisfy ourselves with an illustration. Here we deal only with the case that $\mathbb{D} = D^3$, $\mathbb{D}' = D(3)$ and χ is

$$\chi(d_1, d_2, d_3) = (d_1 d_2, d_1 d_3, d_2 d_3)$$

for any $(d_1, d_2, d_3) \in D^3$. We assume that $n \geq 3$. We note first that the monomial mapping $\chi\colon D^3 \to D(3)$ is the composition of two monomial mappings

$$\begin{aligned}\chi_1 &: D^3 \to D\{6; (1,2), (3,4), (5,6)\} \\ \chi_2 &: D\{6; (1,2), (3,4), (5,6)\} \to D(3)\end{aligned}$$

with

$$\chi_1(d_1, d_2, d_3) = (d_1, d_1, d_2, d_2, d_3, d_3)$$

for any $(d_1, d_2, d_3) \in D^3$ and

$$\chi_2(d_1, d_2, d_3, d_4, d_5, d_6) = (d_1 d_3, d_2 d_5, d_4 d_6)$$

for any $(d_1, d_2, d_3, d_4, d_5, d_6) \in D\{6; (1,2), (3,4), (5,6)\}$, while the former monomial mapping $\chi_1 \colon D^3 \to D\{6; (1,2), (3,4), (5,6)\}$ is in turn the composition of three monomial mappings

$$\begin{aligned}
&\chi_1^1 : D^3 \to D\{4; (1,2)\} \\
&\chi_1^2 : D\{4; (1,2)\} \to D\{5; (1,2), (3,4)\} \\
&\chi_1^3 : D\{5; (1,2), (3,4)\} \to D\{6; (1,2), (3,4), (5,6)\}
\end{aligned}$$

with

$$\chi_1^1(d_1, d_2, d_3) = (d_1, d_1, d_2, d_3)$$

for any $(d_1, d_2, d_3) \in D^3$,

$$\chi_1^2(d_1, d_2, d_3, d_4) = (d_1, d_2, d_3, d_3, d_4)$$

for any $(d_1, d_2, d_3, d_4) \in D\{4; (1,2)\}$ and

$$\chi_1^3(d_1, d_2, d_3, d_4, d_5) = (d_1, d_2, d_3, d_4, d_5, d_5)$$

for any $(d_1, d_2, d_3, d_4, d_5) \in D\{5; (1,2), (3,4)\}$. Therefore it suffices to prove that

$$\nabla\left(\left(\mathrm{id}_M \otimes \mathscr{W}_{\chi_1^1}\right)(\gamma')\right) = \left(\mathrm{id}_E \otimes \mathscr{W}_{\chi_1^1}\right)\left(\nabla_{D\{4;(1,2)\}}(\gamma')\right) \tag{2}$$

for any $\gamma' \in \left(M \otimes \mathscr{W}_{D\{4;(1,2)\}}\right)_{\pi(x)}$, that

$$\nabla_{D\{4;(1,2)\}}\left(\left(\mathrm{id}_M \otimes \mathscr{W}_{\chi_1^2}\right)(\gamma'')\right) = \left(\mathrm{id}_E \otimes \mathscr{W}_{\chi_1^2}\right)\left(\nabla_{D\{5;(1,2),(3,4)\}}(\gamma'')\right) \tag{3}$$

for any $\gamma'' \in \left(M \otimes \mathscr{W}_{D\{5;(1,2),(3,4)\}}\right)_{\pi(x)}$, that

$$\nabla_{D\{5;(1,2),(3,4)\}}\left(\left(\mathrm{id}_M \otimes \mathscr{W}_{\chi_1^3}\right)(\gamma''')\right) = \left(\mathrm{id}_E \otimes \mathscr{W}_{\chi_1^3}\right)\left(\nabla_{D\{6;(1,2),(3,4),(5,6)\}}(\gamma''')\right) \tag{4}$$

for any $\gamma''' \in \left(M \otimes \mathscr{W}_{D\{6;(1,2),(3,4),(5,6)\}}\right)_{\pi(x)}$, and that

$$\nabla_{D\{6;(1,2),(3,4),(5,6)\}}(\left(\mathrm{id}_M \otimes \mathscr{W}_{\chi_2}\right)(\gamma'''')) = \left(\mathrm{id}_E \otimes \mathscr{W}_{\chi_2}\right)\left(\nabla_{D(3)}(\gamma'''')\right) \tag{5}$$

for any $\gamma'''' \in \left(M \otimes \mathscr{W}_{D(3)}\right)_{\pi(x)} \mathbf{T}_x^{D(3)}(M)$. Since $D\{4; (1,2)\} = D(2) \times D^2$, it is easy to see that

$$\nabla\left(\left(\mathrm{id}_M \otimes \mathscr{W}_{\chi_1^1}\right)(\gamma')\right) = \nabla(\gamma'_1 \underset{1}{+} \gamma'_2) = \nabla(\gamma'_1) + \nabla(\gamma'_2)$$

where $\gamma'_1 = \gamma' \circ (i_1 \times \mathrm{id}_{D^2})$ and $\gamma'_2 = \gamma' \circ (i_2 \times \mathrm{id}_{D^2})$ with $i_1(d) = (d, 0) \in D(2)$ and $i_2(d) = (0, d) \in D(2)$ for any $d \in D$. On the other hand, we have

$$\left(\mathrm{id}_E \otimes \mathscr{W}_{\chi_1^1}\right)\left(\nabla_{D(4;(1,2))}(\gamma')\right) = \left(\mathrm{id}_E \otimes \mathscr{W}_{\chi_1^1}\right)\left(\mathbf{l}_{(\nabla(\gamma'_1),\nabla(\gamma'_2))}\right) = \nabla(\gamma'_1) + \nabla(\gamma'_2)$$

where $\mathbf{l}_{(\nabla(\gamma'_1),\nabla(\gamma'_2))}$ is the unique element of $E \otimes \mathscr{W}_{D(2)\times D^2}$ with

$$\left(\mathrm{id}_E \otimes \mathscr{W}_{i_1 \times \mathrm{id}_{D^2}}\right)\left(\mathbf{l}_{(\nabla(\gamma'_1),\nabla(\gamma'_2))}\right) = \nabla(\gamma'_1)$$

and

$$\left(\mathrm{id}_E \otimes \mathscr{W}_{i_2 \times \mathrm{id}_{D^2}}\right)\left(\mathbf{l}_{(\nabla(\gamma'_1),\nabla(\gamma'_2))}\right) = \nabla(\gamma'_2)$$

Thus we have established (2). By the same token, we can establish (3) and (4). In order to prove (5), it suffices to note that

$$\begin{aligned}&\left(\mathrm{id}_E \otimes \mathscr{W}_{i_{135}}\right)\left(\nabla_{D\{6;(1,2),(3,4),(5,6)\}}(\left(\mathrm{id}_M \otimes \mathscr{W}_{\chi_2}\right)(\gamma''''))\right)\\ &= \left(\mathrm{id}_E \otimes \mathscr{W}_{\chi_2 \circ i_{135}}\right)\left(\nabla_{D(3)}(\gamma'''')\right)\end{aligned}$$

together with the seven similar identities obtained from the above by replacing i_{135} by seven other $i_{jkl}\colon D^3 \to D\{6; (1,2), (3,4), (5,6)\}$ in the standard quasi-colimit representation of $D\{6; (1,2), (3,4), (5,6)\}$, where $i_{jkl}\colon D^3 \to D\{6; (1,2), (3,4), (5,6)\}$ $(1 \leq j < k < l \leq 6)$ is a mapping with $i_{jkl}(d_1, d_2, d_3) = (..., \underset{j}{d_1}, ..., \underset{k}{d_2}, ..., \underset{l}{d_3}, ...)$ (d_1, d_2 and d_3 are inserted at the j-th, k-th and l-th positions respectively, while the other components are fixed at 0). Its proof goes as follows. Since

$$\begin{aligned}&\left(\mathrm{id}_E \otimes \mathscr{W}_{i_{135}}\right)\left(\nabla_{D\{6;(1,2),(3,4),(5,6)\}}(\left(\mathrm{id}_M \otimes \mathscr{W}_{\chi_2}\right)(\gamma''''))\right)\\ &= \nabla(\left(\mathrm{id}_M \otimes \mathscr{W}_{\chi_2 \circ i_{135}}\right)(\gamma'''')),\end{aligned}$$

it suffices to show that

$$\nabla(\left(\mathrm{id}_M \otimes \mathscr{W}_{\chi_2 \circ i_{135}}\right)(\gamma'''')) = \left(\mathrm{id}_E \otimes \mathscr{W}_{\chi_2 \circ i_{135}}\right)\nabla_{D(3)}(\gamma'''')$$

However the last identity follows at once by simply observing that the mapping $\chi_2 \circ i_{135}\colon D^3 \to D(3)$ is the mapping

$$(d_1, d_2, d_3) \in D^3 \longmapsto (d_1 d_2, 0, 0) \in D(3),$$

which is the successive composition of the following three mappings:

$$(d_1, d_2, d_3) \in D^3 \longmapsto (d_1, d_2) \in D^2$$
$$(d_1, d_2) \in D^2 \longmapsto d_1 d_2 \in D$$
$$d \in D \longmapsto (d, 0, 0) \in D(3).$$

Corollary 4.4 *Let ∇ be a D^n-tangential over the bundle $\pi : E \to M$ at $x \in E$. Let $\mathbb{D}$ be a simplicially infinitesimal spaces of dimension less than or equal to n. Any nonstandard quasi-colimit representation of $\mathbb{D}$, if any mapping into $\mathbb{D}$ in the representation is monomial, induces the same mapping as $\nabla_{\mathbb{D}}$ (induced by the standard quasi-colimit representation of $\mathbb{D}$) by the method in the proof of Theorem 4.4.*

Proof It suffices to note that

$$\nabla_{D^m}\left(\left(\mathrm{id}_M \otimes \mathscr{W}_\chi\right)(\gamma)\right) = \left(\mathrm{id}_E \otimes \mathscr{W}_\chi\right)\left(\nabla_{\mathbb{D}}(\gamma)\right)$$

for any mapping $\chi : D^m \to \mathbb{D}$ in the given nonstandard quasi-colimit representation of $\mathbb{D}$, which follows directly from the above theorem.

5 The Third Approach to Jets

Definition 5.1 Let n be a natural number. A *D_n-pseudotangential over* the bundle $\pi : E \to M$ *at* $x \in E$ is a mapping

$$\nabla_x : \left(M \otimes \mathscr{W}_{D_n}\right)_{\pi(x)} \to \left(E \otimes \mathscr{W}_{D_n}\right)_x$$

abiding by the following two conditions:

1. We have
$$\left(\pi \otimes \mathrm{id}_{\mathscr{W}_{D_n}}\right)\left(\nabla_x(\gamma)\right) = \gamma$$
for any $\gamma \in \left(M \otimes \mathscr{W}_{D_n}\right)_{\pi(x)}$.
2. For any $\gamma \in \left(E \otimes \mathscr{W}_{D_n}\right)_x$ and any $\alpha \in \mathbb{R}$, we have
$$\nabla_x(\alpha\gamma) = \alpha\nabla_x(\gamma)$$
3. The diagram

$$\begin{array}{ccc}
\left(M \otimes \mathscr{W}_{D_n}\right)_{\pi(x)} & \xrightarrow{\mathrm{id}_M \otimes \mathscr{W}_{(d_1,d_2)\in D_n \times D_m \mapsto d_1 d_2 \in D_n}} & \left(M \otimes \mathscr{W}_{D_n}\right)_{\pi(x)} \otimes \mathscr{W}_{D_m} \\
\nabla_x \downarrow & & \downarrow \nabla_x \otimes \mathrm{id}_{\mathscr{W}_{D_m}} \\
\left(E \otimes \mathscr{W}_{D_n}\right)_x & \xrightarrow{\mathrm{id}_E \otimes \mathscr{W}_{(d_1,d_2)\in D_n \times D_m \mapsto d_1 d_2 \in D_n}} & \left(E \otimes \mathscr{W}_{D_n}\right)_x \otimes \mathscr{W}_{D_m}
\end{array}$$

commutes, where m is an arbitrary natural number.

Remark 5.1 The third condition in the above definition claims what is called infinitesimal linearity.

Notation 5.1 *We denote by* $\widehat{\mathbb{J}}_x^{D_n}(\pi)$ *the totality of* D_n*-pseudotangentials over the bundle* $\pi : E \to M$ *at* $x \in E$*. We denote by* $\widehat{\mathbb{J}}^{D_n}(\pi)$ *the set-theoretic union of* $\widehat{\mathbb{J}}_x^{D_n}(\pi)$*'s for all* $x \in E$*.*

It is easy to see that

Lemma 5.1 *The following diagram is an equalizer in the category of Weil algebras:*

$$\mathcal{W}_{D_n} \xrightarrow{\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n \mapsto d_1 d_2 \in D_n}} \mathcal{W}_{D_{n+1}\times D_n} \underset{\mathcal{W}_{(d_1,d_2,d_3)\in D_{n+1}\times D_{n+1}\times D_n \mapsto (d_1,d_2 d_3)\in D_{n+1}\times D_n}}{\overset{\mathcal{W}_{(d_1,d_2,d_3)\in D_{n+1}\times D_{n+1}\times D_n \mapsto (d_1 d_2,d_3)\in D_{n+1}\times D_n}}{\rightrightarrows}} \mathcal{W}_{D_{n+1}\times D_{n+1}\times D_n}$$

Proposition 5.1 *Let* ∇_x *be a* D_{n+1}*-pseudotangential overthe bundle* $\pi : E \to M$ *at* $x \in E$ *and* $\gamma \in \left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)}$*. Then th ere exists a unique* $\gamma' \in \left(E \otimes \mathcal{W}_{D_n}\right)_x$ *such that the composition of mappings*

$$\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \xrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n \mapsto d_1 d_2 \in D_n}} \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)} \otimes \mathcal{W}_{D_n}$$
$$\xrightarrow{\nabla_x \otimes \mathrm{id}_{\mathcal{W}_{D_n}}} \left(E \otimes \mathcal{W}_{D_{n+1}}\right)_x \otimes \mathcal{W}_{D_n} \tag{6}$$

applied to γ *results in*

$$\left(\mathrm{id}_E \otimes \mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n \mapsto d_1 d_2 \in D_n}\right)\left(\gamma'\right) \tag{7}$$

Proof By dint of Lemma 5.1, it suffices to show that the composition of mappings

$$\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \xrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n \mapsto d_1 d_2 \in D_n}} \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)} \otimes \mathcal{W}_{D_n}$$
$$\xrightarrow{\nabla_x \otimes \mathrm{id}_{\mathcal{W}_{D_n}}} \left(E \otimes \mathcal{W}_{D_{n+1}}\right)_x \otimes \mathcal{W}_{D_n}$$
$$\xrightarrow{\mathrm{id}_E \otimes \mathcal{W}_{(d_1,d_2,d_3)\in D_{n+1}\times D_{n+1}\times D_n \mapsto (d_1,d_2 d_3)\in D_{n+1}\times D_n}}$$
$$\left(E \otimes \mathcal{W}_{D_{n+1}}\right)_x \otimes \mathcal{W}_{D_{n+1}\times D_n} \tag{8}$$

is equal to the composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}}\left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathcal{W}_{D_n}$$
$$\xrightarrow{\nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_n}}}\left(E\otimes\mathcal{W}_{D_{n+1}}\right)_x\otimes\mathcal{W}_{D_n}$$
$$\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{(d_1,d_2,d_3)\in D_{n+1}\times D_{n+1}\times D_n\mapsto(d_1d_2,d_3)\in D_{n+1}\times D_n}}$$
$$\left(E\otimes\mathcal{W}_{D_{n+1}}\right)_x\otimes\mathcal{W}_{D_{n+1}\times D_n} \tag{9}$$

Since $\otimes$ is a bifunctor, the diagram

$$\begin{array}{ccc}\left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathcal{W}_{D_n} & \rightarrow & \left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathcal{W}_{D_{n+1}\times D_n}\\ \nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_n}}\downarrow & & \downarrow\nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_{n+1}\times D_n}}\\ \left(E\otimes\mathcal{W}_{D_{n+1}}\right)_x\otimes\mathcal{W}_{D_n} & \rightarrow & \left(E\otimes\mathcal{W}_{D_{n+1}}\right)_x\otimes\mathcal{W}_{D_{n+1}\times D_n}\end{array}$$

commutes, where the upper horizontal arrow is

$$\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2,d_3)\in D_{n+1}\times D_{n+1}\times D_n\mapsto(d_1,d_2d_3)\in D_{n+1}\times D_n},$$

while the lower horizontal arrow is

$$\mathrm{id}_E\otimes\mathcal{W}_{(d_1,d_2,d_3)\in D_{n+1}\times D_{n+1}\times D_n\mapsto(d_1,d_2d_3)\in D_{n+1}\times D_n}.$$

Therefore the composition of mappings in (8) is equal to the composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}}\left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathcal{W}_{D_n}$$
$$\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2,d_3)\in D_{n+1}\times D_{n+1}\times D_n\mapsto(d_1,d_2d_3)\in D_{n+1}\times D_n}}\left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathcal{W}_{D_{n+1}\times D_n}$$
$$\xrightarrow{\nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_{n+1}\times D_n}}}\left(E\otimes\mathcal{W}_{D_{n+1}}\right)_x\otimes\mathcal{W}_{D_{n+1}\times D_n} \tag{10}$$

Since the composition of mappings

$$M\otimes\mathcal{W}_{D_n}\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}}M\otimes\mathcal{W}_{D_{n+1}\times D_n}$$
$$\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2,d_3)\in D_{n+1}\times D_{n+1}\times D_n\mapsto(d_1,d_2d_3)\in D_{n+1}\times D_n}}M\otimes\mathcal{W}_{D_{n+1}\times D_{n+1}\times D_n}$$

is trivially equal to the composition of mappings

$$M\otimes\mathcal{W}_{D_n}\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}}M\otimes\mathcal{W}_{D_{n+1}\times D_n}$$
$$\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2,d_3)\in D_{n+1}\times D_{n+1}\times D_n\mapsto(d_1d_2,d_3)\in D_{n+1}\times D_n}}M\otimes\mathcal{W}_{D_{n+1}\times D_{n+1}\times D_n},$$

the composition of mappings in (10) is equal to the composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}}\left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathcal{W}_{D_n}$$
$$\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2,d_3)\in D_{n+1}\times D_{n+1}\times D_n\mapsto(d_1d_2,d_3)\in D_{n+1}\times D_n}}\left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathcal{W}_{D_{n+1}\times D_n}$$
$$\xrightarrow{\nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_{n+1}\times D_n}}}\left(E\otimes\mathcal{W}_{D_{n+1}}\right)_x\otimes\mathcal{W}_{D_{n+1}\times D_n} \tag{11}$$

By dint of the third condition in Definition 5.1.1, the diagram

$$\begin{array}{ccc}\left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathcal{W}_{D_n} & \to & \left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathcal{W}_{D_{n+1}\times D_n}\\ \nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_n}}\downarrow & & \downarrow\nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_{n+1}\times D_n}}\\ \left(E\otimes\mathcal{W}_{D_{n+1}}\right)_x\otimes\mathcal{W}_{D_n} & \to & \left(E\otimes\mathcal{W}_{D_{n+1}}\right)_x\otimes\mathcal{W}_{D_{n+1}\times D_n}\end{array}$$

commutes, where the upper horizontal arrow is

$$\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2,d_3)\in D_{n+1}\times D_{n+1}\times D_n\mapsto(d_1d_2,d_3)\in D_{n+1}\times D_n},$$

and the lower horizontal arrow is

$$\mathrm{id}_E\otimes\mathcal{W}_{(d_1,d_2,d_3)\in D_{n+1}\times D_{n+1}\times D_n\mapsto(d_1d_2,d_3)\in D_{n+1}\times D_n}.$$

Therefore the composition of mappings in (11) is equal to the composition of mappings in (9), which completes the proof.

It is not difficult to see that

Proposition 5.2 *Given a D_{n+1}-pseudotangential ∇_x overthe bundle $\pi\colon E\to M$ at $x\in E$, the assignment $\gamma\in\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\mapsto\gamma'\in\left(E\otimes\mathcal{W}_{D_n}\right)_x$ in the above proposition, denoted by $\hat{\pi}_{n+1,n}(\nabla_x)$, is a D_n-pseudotangential over the bundle $\pi\colon E\to M$ at $x\in E$.*

Proof We have to verify the three conditions in Definition 5.1 concerning the mapping $\hat{\pi}_{n+1,n}(\nabla_x)\colon\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\to\left(E\otimes\mathcal{W}_{D_n}\right)_x$.

1. To see the first condition, it suffices to show that

$$\left(\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}\right)\circ\left(\pi\otimes\mathrm{id}_{\mathcal{W}_{D_n}}\right)\left(\left(\hat{\pi}_{n+1,n}(\nabla_x)\right)(\gamma)\right)=\gamma,$$

which is equivalent to

$$\left(\pi\otimes\mathrm{id}_{\mathcal{W}_{D_{n+1}\times D_n}}\right)\circ\left(\mathrm{id}_E\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}\right)\left(\left(\hat{\pi}_{n+1,n}(\nabla_x)\right)(\gamma)\right)=\gamma,$$

since $\otimes$ is a bifunctor. Therefore it suffices to show that the composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\underrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}}\left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathcal{W}_{D_n}$$
$$\underrightarrow{\nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_n}}}\left(E\otimes\mathcal{W}_{D_{n+1}}\right)_x\otimes\mathcal{W}_{D_n}\underrightarrow{\pi\otimes\mathrm{id}_{\mathcal{W}_{D_{n+1}\times D_n}}}\left(M\otimes\mathcal{W}_{D_{n+1}}\right)_x\otimes\mathcal{W}_{D_n}$$

applied to γ results in

$$\left(\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}\right)(\gamma),$$

which follows directly from the first condition in Definition 5.1.

2. To see the second, let us note first that the composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\underrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{d\in D_n\mapsto\alpha d\in D_n}}\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}$$
$$\underrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}}\left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathcal{W}_{D_n}$$

is equal to the composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\underrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}}\left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathcal{W}_{D_n}$$
$$\underrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto(\alpha d_1,d_2)\in D_{n+1}\times D_n}}\left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathcal{W}_{D_n}$$

Since ∇_x is a D_{n+1}-pseudotangential over the bundle $\pi: E\to M$ *at* $x\in E$, the diagram

$$\begin{array}{ccc}\left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathcal{W}_{D_n} & \rightarrow & \left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathcal{W}_{D_n}\\ \nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_n}}\downarrow & & \downarrow\nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_n}}\\ \left(E\otimes\mathcal{W}_{D_{n+1}}\right)_x\otimes\mathcal{W}_{D_n} & \rightarrow & \left(E\otimes\mathcal{W}_{D_{n+1}}\right)_x\otimes\mathcal{W}_{D_n}\end{array}$$

commutes, where the upper horizontal arrow is

$$\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto(\alpha d_1,d_2)\in D_{n+1}\times D_n},$$

while the lower horizontal arrow is

$$\mathrm{id}_E\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto(\alpha d_1,d_2)\in D_{n+1}\times D_n}.$$

Therefore the composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\underrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{d\in D_n\mapsto\alpha d\in D_n}}\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}$$
$$\underrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}}\left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathcal{W}_{D_n}$$
$$\underrightarrow{\nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_n}}}\left(E\otimes\mathcal{W}_{D_{n+1}}\right)_x\otimes\mathcal{W}_{D_n}$$

is equal to the composition of mappings

$$\left(M\otimes\mathscr{W}_{D_n}\right)_{\pi(x)}\xrightarrow{\mathrm{id}_M\otimes\mathscr{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}}\left(M\otimes\mathscr{W}_{D_{n+1}}\right)_{\pi(x)}\otimes\mathscr{W}_{D_n}$$
$$\xrightarrow{\nabla_x\otimes\mathrm{id}_{\mathscr{W}_{D_n}}}\left(E\otimes\mathscr{W}_{D_{n+1}}\right)_x\otimes\mathscr{W}_{D_n}\xrightarrow{\mathrm{id}_E\otimes\mathscr{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto(\alpha d_1,d_2)\in D_{n+1}\times D_n}}$$
$$\left(E\otimes\mathscr{W}_{D_{n+1}}\right)_x\otimes\mathscr{W}_{D_n}$$

The former composition of mappings applied to $\gamma\in\left(M\otimes\mathscr{W}_{D_n}\right)_{\pi(x)}$ results in

$$\left(\mathrm{id}_E\otimes\mathscr{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}\right)\left(\hat{\pi}_{n+1,n}(\nabla_x)(\alpha\gamma)\right),$$

while the latter composition of mappings applied to γ results in

$$\begin{aligned}&\left(\mathrm{id}_E\otimes\mathscr{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto(\alpha d_1,d_2)\in D_{n+1}\times D_n}\right)\circ\\&\left(\mathrm{id}_E\otimes\mathscr{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}\right)\left(\hat{\pi}_{n+1,n}(\nabla_x)(\gamma)\right)\\&=\left(\mathrm{id}_E\otimes\mathscr{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}\right)\left(\alpha\left(\hat{\pi}_{n+1,n}(\nabla_x)(\gamma)\right)\right).\end{aligned}$$

Therefore we have

$$\hat{\pi}_{n+1,n}(\nabla_x)(\alpha\gamma)=\alpha\left(\hat{\pi}_{n+1,n}(\nabla_x)(\gamma)\right)$$

3. To see the third, we have to show that the diagram

$$\begin{array}{ccc}\left(M\otimes\mathscr{W}_{D_n}\right)_{\pi(x)} & \xrightarrow{\mathrm{id}_M\otimes\mathscr{W}_{\mathbf{m}_{D_n\times D_m\to D_n}}} & \left(M\otimes\mathscr{W}_{D_n}\right)_{\pi(x)}\otimes\mathscr{W}_{D_m}\\ \hat{\pi}_{n+1,n}(\nabla_x)\downarrow & & \downarrow\hat{\pi}_{n+1,n}(\nabla_x)\otimes\mathrm{id}_{\mathscr{W}_{D_m}}\\ \left(E\otimes\mathscr{W}_{D_n}\right)_x & \xrightarrow{\mathrm{id}_E\otimes\mathscr{W}_{\mathbf{m}_{D_n\times D_m\to D_n}}} & \left(E\otimes\mathscr{W}_{D_n}\right)_x\otimes\mathscr{W}_{D_m}\end{array}\tag{12}$$

commutes, where m is an arbitrary natural number. Since the lower square of the diagram

$$\begin{array}{ccc}\left(M\otimes\mathscr{W}_{D_n}\right)_{\pi(x)} & \xrightarrow{\mathrm{id}_M\otimes\mathscr{W}_{\mathbf{m}_{D_n\times D_m\to D_n}}} & \left(M\otimes\mathscr{W}_{D_n}\right)_{\pi(x)}\otimes\mathscr{W}_{D_m}\\ \hat{\pi}_{n+1,n}(\nabla_x)\downarrow & & \downarrow\hat{\pi}_{n+1,n}(\nabla_x)\otimes\mathrm{id}_{\mathscr{W}_{D_n}}\\ \left(E\otimes\mathscr{W}_{D_n}\right)_x & \xrightarrow{\mathrm{id}_E\otimes\mathscr{W}_{\mathbf{m}_{D_n\times D_m\to D_n}}} & \left(E\otimes\mathscr{W}_{D_n}\right)_x\otimes\mathscr{W}_{D_m}\\ \mathrm{id}_E\otimes\mathscr{W}_{\mathbf{m}_{D_{n+1}\times D_n\to D_n}}\downarrow & & \downarrow\mathrm{id}_E\otimes\mathscr{W}_{\mathbf{m}_{D_{n+1}\times D_n\to D_n}\times\mathrm{id}_{D_m}}\\ \left(E\otimes\mathscr{W}_{D_{n+1}}\right)_x\otimes\mathscr{W}_{D_n} & \xrightarrow{\mathrm{id}_E\otimes\mathscr{W}_{\mathrm{id}_{D_{n+1}}\times\mathbf{m}_{D_n\times D_m\to D_n}}} & \left(E\otimes\mathscr{W}_{D_{n+1}}\right)_x\otimes\mathscr{W}_{D_n\times D_m}\end{array}\tag{13}$$

commutes, so that the commutativity of the diagram in (12) is equivalent to the commutativity of the outer square of the diagram in (13). .The composition of mappings

$$\left(M\otimes\mathscr{W}_{D_n}\right)_{\pi(x)}\xrightarrow{\hat{\pi}_{n+1,n}(\nabla_x)}\left(E\otimes\mathscr{W}_{D_n}\right)_x\xrightarrow{\mathrm{id}_E\otimes\mathscr{W}_{\mathbf{m}_{D_{n+1}\times D_n\to D_n}}}\left(E\otimes\mathscr{W}_{D_{n+1}}\right)_x\otimes\mathscr{W}_{D_n}$$

is equal to the composition of mappings

$$\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \underrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{\mathbf{m}_{D_{n+1}\times D_n \to D_n}}} \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)} \otimes \mathcal{W}_{D_n}$$
$$\underrightarrow{\nabla_x \otimes \mathrm{id}_{\mathcal{W}_{D_n}}} \left(E \otimes \mathcal{W}_{D_{n+1}}\right)_x \otimes \mathcal{W}_{D_n},$$

while the composition of mappings

$$\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \otimes \mathcal{W}_{D_m} \underrightarrow{\hat{\pi}_{n+1,n}(\nabla_x) \otimes \mathrm{id}_{\mathcal{W}_{D_m}}} \left(E \otimes \mathcal{W}_{D_n}\right)_x \otimes \mathcal{W}_{D_m}$$
$$\underrightarrow{\mathrm{id}_E \otimes \mathcal{W}_{\mathbf{m}_{D_{n+1}\times D_n \to D_n} \times \mathrm{id}_{D_m}}} \left(E \otimes \mathcal{W}_{D_{n+1}}\right)_x \otimes \mathcal{W}_{D_n \times D_m}$$

is equal to the composition of mappings

$$\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \otimes \mathcal{W}_{D_m} \underrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{\mathbf{m}_{D_{n+1}\times D_n \to D_n} \times \mathrm{id}_{D_m}}} \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)}$$
$$\otimes \mathcal{W}_{D_n \times D_m} \underrightarrow{\nabla_x \otimes \mathrm{id}_{\mathcal{W}_{D_n \times D_m}}} \left(E \otimes \mathcal{W}_{D_{n+1}}\right)_x \otimes \mathcal{W}_{D_n \times D_m}$$

It is easy to see that the diagram

$$\begin{array}{ccc}
\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} & \underrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{\mathbf{m}_{D_n \times D_m \to D_n}}} & \left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \otimes \mathcal{W}_{D_m} \\
\mathrm{id}_M \otimes \mathcal{W}_{\mathbf{m}_{D_{n+1}\times D_n \to D_n}} \downarrow & & \downarrow \mathrm{id}_M \otimes \mathcal{W}_{\mathbf{m}_{D_{n+1}\times D_n \to D_n} \times \mathrm{id}_{D_m}} \\
\left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)} \otimes \mathcal{W}_{D_n} & \underrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{\mathrm{id}_{D_{n+1}} \times \mathbf{m}_{D_n \times D_m \to D_n}}} & \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)} \otimes \mathcal{W}_{D_n \times D_m} \\
\nabla_x \otimes \mathrm{id}_{\mathcal{W}_{D_n}} \downarrow & & \downarrow \nabla_x \otimes \mathrm{id}_{\mathcal{W}_{D_n \times D_m}} \\
\left(E \otimes \mathcal{W}_{D_{n+1}}\right)_x \otimes \mathcal{W}_{D_n} & \underrightarrow{\mathrm{id}_E \otimes \mathcal{W}_{\mathrm{id}_{D_{n+1}} \times \mathbf{m}_{D_n \times D_m \to D_n}}} & \left(E \otimes \mathcal{W}_{D_{n+1}}\right)_x \otimes \mathcal{W}_{D_n \times D_m}
\end{array}$$

commutes, which implies that the outer square of the diagram in (13) commutes. This completes the proof.

Notation 5.2 *By the above proposition, we have the canonical projection* $\hat{\pi}_{n+1,n}$*:* $\widehat{\mathbb{J}}^{D_{n+1}}(\pi) \to \widehat{\mathbb{J}}^{D_n}(\pi)$ *so that, given* $\nabla_x \in \widehat{\mathbb{J}}_x^{D_{n+1}}(\pi)$ *and* $\gamma \in \left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)}$*, the composition of mappings in (6) applied to* γ *results in*

$$\left(\mathrm{id}_E \otimes \mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n \mapsto d_1 d_2 \in D_n}\right)\left(\hat{\pi}_{n+1,n}(\nabla_x)(\gamma)\right)$$

For any natural numbers n, m with $m \leq n$*, we define* $\hat{\pi}_{n,m}$*:* $\widehat{\mathbb{J}}^{D_n}(\pi) \to \widehat{\mathbb{J}}^{D_m}(\pi)$ *to be* $\hat{\pi}_{m+1,m} \circ \ldots \circ \hat{\pi}_{n,n-1}$.

Proposition 5.3 *Let* ∇_x *be a* D_{n+1}*-pseudotangential over the bundle* $\pi : E \to M$ *at* $x \in E$*. Then the diagram*

$$\begin{array}{ccc} \left(M\otimes\mathcal{W}_{D_{n+1}}\right)_{\pi(x)} & \xrightarrow{\nabla_x} & \left(E\otimes\mathcal{W}_{D_{n+1}}\right)_x \\ \widehat{\pi}_{n+1,n}\ \downarrow & & \downarrow\ \widehat{\pi}_{n+1,n} \\ \left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)} & \xrightarrow[\hat{\pi}_{n+1,n}(\nabla_x)]{} & \left(E\otimes\mathcal{W}_{D_n}\right)_x \end{array}$$

is commutative.

Proof It is easy to see that the following four diagrams are commutative:

$$\begin{array}{ccc} M\otimes\mathcal{W}_{D_{n+1}} & \xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_{n+1}\mapsto d_1d_2\in D_{n+1}}} & M\otimes\mathcal{W}_{D_{n+1}\times D_{n+1}} \\ \mathrm{id}_M\otimes\mathcal{W}_{\mathbf{i}_{D_n\subseteq D_{n+1}}}\ \downarrow & & \downarrow\ \mathrm{id}_M\otimes\mathcal{W}_{\mathbf{i}_{D_{n+1}\times D_n\subseteq D_{n+1}\times D_{n+1}}} \\ M\otimes\mathcal{W}_{D_n} & \xrightarrow[\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}]{} & M\otimes\mathcal{W}_{D_{n+1}\times D_n} \end{array}$$

$$\begin{array}{ccc} M\otimes\mathcal{W}_{D_{n+1}\times D_{n+1}} & \xrightarrow{\nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_{n+1}}}} & E\otimes\mathcal{W}_{D_{n+1}\times D_{n+1}} \\ \mathrm{id}_M\otimes\mathcal{W}_{\mathbf{i}_{D_{n+1}\times D_n\subseteq D_{n+1}\times D_{n+1}}}\ \downarrow & & \downarrow\ \mathrm{id}_E\otimes\mathcal{W}_{\mathbf{i}_{D_{n+1}\times D_n\subseteq D_{n+1}\times D_{n+1}}} \\ M\otimes\mathcal{W}_{D_{n+1}\times D_n} & \xrightarrow[\nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_n}}]{} & E\otimes\mathcal{W}_{D_{n+1}\times D_n} \end{array}$$

$$\begin{array}{ccc} M\otimes\mathcal{W}_{D_{n+1}} & \xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_{n+1}\mapsto d_1d_2\in D_{n+1}}} & M\otimes\mathcal{W}_{D_{n+1}\times D_{n+1}} \\ \nabla_x\ \downarrow & & \downarrow\ \nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_{n+1}}} \\ E\otimes\mathcal{W}_{D_{n+1}} & \xrightarrow[\mathrm{id}_E\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_{n+1}\mapsto d_1d_2\in D_{n+1}}]{} & E\otimes\mathcal{W}_{D_{n+1}\times D_{n+1}} \end{array}$$

[By the second condition in Definition 5.1]

$$\begin{array}{ccc} E\otimes\mathcal{W}_{D_{n+1}} & \xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_{n+1}\mapsto d_1d_2\in D_{n+1}}} & E\otimes\mathcal{W}_{D_{n+1}\times D_{n+1}} \\ \mathrm{id}_E\otimes\mathcal{W}_{\mathbf{i}_{D_n\subseteq D_{n+1}}}\ \downarrow & & \downarrow\ \mathrm{id}_E\otimes\mathcal{W}_{\mathbf{i}_{D_{n+1}\times D_n\subseteq D_{n+1}\times D_{n+1}}} \\ E\otimes\mathcal{W}_{D_n} & \xrightarrow[\mathrm{id}_E\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}]{} & E\otimes\mathcal{W}_{D_{n+1}\times D_n} \end{array}$$

Therefore the composition of mappings

$$\begin{aligned} &M\otimes\mathcal{W}_{D_{n+1}}\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{\mathbf{i}_{D_n\subseteq D_{n+1}}}}M\otimes\mathcal{W}_{D_n} \\ &\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}}M\otimes\mathcal{W}_{D_{n+1}\times D_n} \\ &=\left(M\otimes\mathcal{W}_{D_{n+1}}\right)\otimes\mathcal{W}_{D_n}\xrightarrow{\nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_n}}}\left(E\otimes\mathcal{W}_{D_{n+1}}\right)\otimes\mathcal{W}_{D_n} \\ &=E\otimes\mathcal{W}_{D_{n+1}\times D_n} \end{aligned}$$

is equal to the composition of mappings

$$\begin{aligned} &M\otimes\mathcal{W}_{D_{n+1}}\xrightarrow{\nabla_x}E\otimes\mathcal{W}_{D_{n+1}}\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{\mathbf{i}_{D_n\to D_{n+1}}}}E\otimes\mathcal{W}_{D_n} \\ &\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{(d_1,d_2)\in D_{n+1}\times D_n\mapsto d_1d_2\in D_n}}E\otimes\mathcal{W}_{D_{n+1}\times D_n} \end{aligned}$$

which yields the coveted result.

Corollary 5.1 *Let ∇_x be a D_{n+1}-pseudotangential over the bundle $\pi: E \to M$ at $x \in E$. For any $\gamma, \gamma' \in \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)}$, if*

$$\pi_{n+1,n}(\gamma) = \pi_{n+1,n}\left(\gamma'\right)$$

then

$$\pi_{n+1,n}(\nabla_x(\gamma)) = \pi_{n+1,n}\left(\nabla_x(\gamma')\right)$$

Proof By the above proposition, we have

$$\begin{aligned}\pi_{n+1,n}(\nabla_x(\gamma)) &= \hat{\pi}_{n+1,n}(\nabla_x)(\pi_{n+1,n}(\gamma)) \\ &= \hat{\pi}_{n+1,n}(\nabla_x)(\pi_{n+1,n}(\gamma')) = \pi_{n+1,n}(\nabla_x(\gamma')),\end{aligned}$$

which establishes the coveted proposition.

Definition 5.2 The notion of a D_n-*tangential over* the bundle $\pi: E \to M$ *at* $x \in E$ is defined inductively on n. The notion of a D_0-*tangential over* the bundle $\pi: E \to M$ *at* $x \in E$ and that of a D_1-*tangential over* the bundle $\pi: E \to M$ *at* $x \in E$ shall be identical with that of a D_0-pseudotangential *over* the bundle $\pi: E \to M$ *at* $x \in E$ and that of a D_1-pseudotangential *over* the bundle $\pi: E \to M$ *at* $x \in E$ respectively. Now we proceed by induction on n. A D_{n+1}-pseudotangential ∇_x: $\left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)} \to \left(E \otimes \mathcal{W}_{D_{n+1}}\right)_x$ *over* the bundle $\pi: E \to M$ *at* $x \in E$ is called a D_{n+1}-*tangential over* the bundle $\pi: E \to M$ *at* $x \in E$ if it acquiesces in the following two conditions:

1. $\hat{\pi}_{n+1,n}(\nabla_x)$ is a D_n-tangential *over* the bundle $\pi: E \to M$ *at* $x \in E$.
2. For any simple polynomial ρ of $d \in D_{n+1}$ with $l = \dim \rho$ and any $\gamma \in \left(M \otimes \mathcal{W}_{D_l}\right)_{\pi(x)}$, we have

$$\nabla_x(\gamma \circ \rho) = (\pi_{n+1,l}(\nabla_x)(\gamma)) \circ \rho$$

Notation 5.3 *We denote by $\mathbb{J}_x^{D_n}(\pi)$ the totality of D_n-tangentials over the bundle π: $E \to M$ at $x \in E$, while we denote by $\mathbb{J}^{D_n}(\pi)$ the totality of D_n-tangentials over the bundle $\pi: E \to M$. By the very definition of a D_n-tangential, the projection $\hat{\pi}_{n+1,n}$: $\widehat{\mathbb{J}}^{D_{n+1}}(\pi) \to \widehat{\mathbb{J}}^{D_n}(\pi)$ is naturally restricted to a mapping $\pi_{n+1,n}$: $\mathbb{J}^{D_{n+1}}(\pi) \to \mathbb{J}^{D_n}(\pi)$. Similarly for $\pi_{n,m}$: $\mathbb{J}^{D_n}(\pi) \to \mathbb{J}^{D_m}(\pi)$ with $m \leq n$.*

6 From the First Approach to the Second

Definition 6.1 Mappings φ_n: $\mathbf{J}^n(\pi) \to \mathbb{J}^{D^n}(\pi)$ $(n = 0, 1)$ shall be the identity mappings. We are going to define φ_n: $\mathbf{J}^n(\pi) \to \mathbb{J}^{D^n}(\pi)$ for any natural number n by induction on n. Let $x_n = \nabla_{x_{n-1}} \in \mathbf{J}^n(\pi)$ and $\nabla_{x_n} \in \mathbf{J}^{n+1}(\pi)$. We define $\varphi_{n+1}(\nabla_{x_n})$ as the composition of mappings

$$\begin{aligned}
&\left(M \otimes \mathcal{W}_{D^{n+1}}\right)_{\pi(x_n)} \\
&= ((M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}} \\
&\xrightarrow{\left\langle \pi_M^{M\otimes\mathcal{W}_{D^n}} \otimes \mathrm{id}_{\mathcal{W}_D}, \mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D} \right\rangle} \\
&(M \otimes \mathcal{W}_D)_{\pi(x_n)} \underset{M\otimes\mathcal{W}_D}{\times} ((M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}} \\
&\xrightarrow{\nabla_{x_n} \times \mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}} \\
&\left(\mathbf{J}^n(\pi) \otimes \mathcal{W}_D\right)_{x_n} \underset{M\otimes\mathcal{W}_D}{\times} ((M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}} \\
&\xrightarrow{\left(\varphi_n \otimes \mathrm{id}_{\mathcal{W}_D}\right) \times \mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}} \\
&\left(\mathbb{J}^{D^n}(\pi) \otimes \mathcal{W}_D\right)_{\varphi_n(x_n)} \underset{M\otimes\mathcal{W}_D}{\times} ((M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}} \\
&= \left(\left(\mathbb{J}^{D^n}(\pi) \underset{M}{\times} (M \otimes \mathcal{W}_{D^n})\right) \otimes \mathcal{W}_D\right)_{\varphi_n(x_n)\times(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}} \\
&\xrightarrow{\left((\nabla, \gamma) \in \mathbb{J}^{D^n}(\pi) \underset{M}{\times} (M \otimes \mathcal{W}_{D^n}) \mapsto \nabla(\gamma) \in E \otimes \mathcal{W}_{D^n}\right) \otimes \mathrm{id}_{\mathcal{W}_D}} \\
&((E \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_D)_{(E\otimes\mathcal{W}_{D^n})_{\pi_0(x_n)}} \\
&= \left(E \otimes \mathcal{W}_{D^{n+1}}\right)_{\pi_0(x_n)}
\end{aligned}$$

Surely we have to show that

Lemma 6.1 *We have*

$$\varphi_{n+1}(\nabla_{x_n}) \in \hat{\mathbb{J}}^{n+1}(\pi)$$

Proof We have to show that for any $\gamma \in \mathbf{T}^{n+1}_{\pi_n(x_n)}(M)$, any $\alpha \in \mathbb{R}$ and any $\sigma \in \mathbf{S}_{n+1}$, we have

$$\gamma = \left(\pi \otimes \mathrm{id}_{\mathcal{W}_{D^{n+1}}}\right) \circ \left(\varphi_{n+1}(\nabla_{x_n})\right)(\gamma) \tag{14}$$

$$\varphi_{n+1}(\nabla_{x_n})(\alpha \underset{i}{\cdot} \gamma) = \alpha \underset{i}{\cdot} \varphi_{n+1}(\nabla_{x_n})(\gamma) \quad (1 \leq i \leq n+1) \tag{15}$$

$$\varphi_{n+1}(\nabla_{x_n})(\gamma^\sigma) = (\varphi_{n+1}(\nabla_{x_n})(\gamma))^\sigma \tag{16}$$

We proceed by induction on n.

1. First we deal with (14). The mapping

$$\left(\pi \otimes \mathrm{id}_{\mathcal{W}_{D^{n+1}}}\right) \left(\varphi_{n+1}(\nabla_{x_n})\right)$$

is the composition of mappings

$$\begin{aligned}
&\left(M\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi(x_n)}\\
&=\left(\left(M\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D\right)_{\left(M\otimes\mathcal{W}_{D^n}\right)_{\pi(x_n)}}\\
&\xrightarrow{\left\langle\pi_M^{M\otimes\mathcal{W}_{D^n}}\otimes\mathrm{id}_{\mathcal{W}_D},\,\mathrm{id}_{\left(M\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D}\right\rangle}\\
&\left(M\otimes\mathcal{W}_D\right)_{\pi(x_n)}\underset{M\otimes\mathcal{W}_D}{\times}\left(\left(M\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D\right)_{\left(M\otimes\mathcal{W}_{D^n}\right)_{\pi(x_n)}}\\
&\xrightarrow{\nabla_{x_n}\times\mathrm{id}_{\left(M\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D}}\\
&\left(\mathbf{J}^n(\pi)\otimes\mathcal{W}_D\right)_{x_n}\underset{M\otimes\mathcal{W}_D}{\times}\left(\left(M\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D\right)_{\left(M\otimes\mathcal{W}_{D^n}\right)_{\pi(x_n)}}\\
&\xrightarrow{\left(\varphi_n\otimes\mathrm{id}_{\mathcal{W}_D}\right)\times\mathrm{id}_{\left(M\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D}}\\
&\left(\mathbb{J}^{D^n}(\pi)\otimes\mathcal{W}_D\right)_{\varphi_n(x_n)}\underset{M\otimes\mathcal{W}_D}{\times}\left(\left(M\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D\right)_{\left(M\otimes\mathcal{W}_{D^n}\right)_{\pi(x_n)}}\\
&=\left(\left(\mathbb{J}^{D^n}(\pi)\underset{M}{\times}\left(M\otimes\mathcal{W}_{D^n}\right)\right)\otimes\mathcal{W}_D\right)_{\varphi_n(x_n)\times\left(M\otimes\mathcal{W}_{D^n}\right)_{\pi(x_n)}}\\
&\xrightarrow{\left((\nabla,\gamma)\in\mathbb{J}^{D^n}(\pi)\underset{M}{\times}\left(M\otimes\mathcal{W}_{D^n}\right)\mapsto\nabla(\gamma)\in E\otimes\mathcal{W}_{D^n}\right)\otimes\mathrm{id}_{\mathcal{W}_D}}\\
&\left(\left(E\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D\right)_{\left(E\otimes\mathcal{W}_{D^n}\right)_{\pi_0(x_n)}}\\
&=\left(E\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi_0(x_n)}\xrightarrow{\pi\otimes\mathrm{id}_{\mathcal{W}_{D^{n+1}}}}\left(M\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi(x_n)}
\end{aligned}$$

It is easy to see that the composition of mappings

$$\begin{aligned}
&\left(\mathbf{J}^n(\pi)\otimes\mathcal{W}_D\right)_{x_n}\underset{M\otimes\mathcal{W}_D}{\times}\left(\left(M\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D\right)_{\left(M\otimes\mathcal{W}_{D^n}\right)_{\pi(x_n)}}\\
&\xrightarrow{\left(\varphi_n\otimes\mathrm{id}_{\mathcal{W}_D}\right)\times\mathrm{id}_{\left(M\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D}}\\
&\left(\mathbb{J}^{D^n}(\pi)\otimes\mathcal{W}_D\right)_{\varphi_n(x_n)}\underset{M\otimes\mathcal{W}_D}{\times}\left(\left(M\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D\right)_{\left(M\otimes\mathcal{W}_{D^n}\right)_{\pi(x_n)}}\\
&=\left(\left(\mathbb{J}^{D^n}(\pi)\underset{M}{\times}\left(M\otimes\mathcal{W}_{D^n}\right)\right)\otimes\mathcal{W}_D\right)_{\{\varphi_n(x_n)\}\times\left(M\otimes\mathcal{W}_{D^n}\right)_{\pi(x_n)}}\\
&\xrightarrow{\left((\nabla,\gamma)\in\mathbb{J}^{D^n}(\pi)\underset{M}{\times}\left(M\otimes\mathcal{W}_{D^n}\right)\mapsto\nabla(\gamma)\in E\otimes\mathcal{W}_{D^n}\right)\otimes\mathrm{id}_{\mathcal{W}_D}}\\
&\left(\left(E\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D\right)_{\left(E\otimes\mathcal{W}_{D^n}\right)_{\pi_0(x_n)}}\\
&=\left(E\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi_0(x_n)}\xrightarrow{\pi\otimes\mathrm{id}_{\mathcal{W}_{D^{n+1}}}}\left(M\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi(x_n)}
\end{aligned}$$

is no other than the canonical projection of

$$\left(\mathbf{J}^n(\pi)\otimes\mathcal{W}_D\right)_{x_n}\underset{M\otimes\mathcal{W}_D}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}$$

to the second factor $((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}$. It is also easy to see that the composition of mappings

$$\begin{aligned}
&((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}\\
&\xrightarrow{\left\langle\pi_M^{M\otimes\mathcal{W}_{D^n}}\otimes\mathrm{id}_{\mathcal{W}_D},\,\mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}\right\rangle}\\
&(M\otimes\mathcal{W}_D)_{\pi(x_n)}\underset{M\otimes\mathcal{W}_D}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}\\
&\xrightarrow{\nabla_{x_n}\times\mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}}\\
&\left(\mathbf{J}^n(\pi)\otimes\mathcal{W}_D\right)_{x_n}\underset{M\otimes\mathcal{W}_D}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}\\
&\xrightarrow{\left(\varphi_n\otimes\mathrm{id}_{\mathcal{W}_D}\right)\times\mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}}\\
&\left(\mathbb{J}^{D^n}(\pi)\otimes\mathcal{W}_D\right)_{\varphi_n(x_n)}\underset{M\otimes\mathcal{W}_D}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}
\end{aligned}$$

is

$$\begin{aligned}
&((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}\\
&\xrightarrow{\left\langle\left(\varphi_n\otimes\mathrm{id}_{\mathcal{W}_D}\right)\circ\nabla_{x_n}\circ\left(\pi_M^{M\otimes\mathcal{W}_{D^n}}\otimes\mathrm{id}_{\mathcal{W}_D}\right),\,\mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}\right\rangle}\\
&\left(\mathbb{J}^{D^n}(\pi)\otimes\mathcal{W}_D\right)_{\varphi_n(x_n)}\underset{M\otimes\mathcal{W}_D}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}.
\end{aligned}$$

Therefore (14) follows at once.

2. Now we deal with (15), the treatment of which is divided into two cases, namely, $i\leq n$ and $i=n+1$. Since both of them are almost trivial, they can safely be left to the reader.
3. Finally we must deal with (16), for which it suffices to consider only transpositions $\sigma=\langle i,i+1\rangle$ $(1\leq i\leq n)$. Here we deal only with the most difficult case of $\sigma=\langle n,n+1\rangle$. We consider the composition of mappings

$$\begin{aligned}
&\left(M\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi(x_n)}\xrightarrow{\gamma\in\left(M\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi(x_n)}\mapsto\gamma^{\langle n,n+1\rangle}\in\left(M\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi(x_n)}}\\
&\left(M\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi(x_n)}\\
&=((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}
\end{aligned}$$

$$\xrightarrow{\left\langle \pi_M^{M\otimes\mathcal{W}_{D^n}} \otimes \mathrm{id}_{\mathcal{W}_D}, \mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D} \right\rangle}$$
$$(M \otimes \mathcal{W}_D)_{\pi(x_n)} \underset{M\otimes\mathcal{W}_D}{\times} ((M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}$$
$$\xrightarrow{\nabla_{x_n} \times \mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}}$$
$$\left(\mathbf{J}^n(\pi) \otimes \mathcal{W}_D\right)_{x_n} \underset{M\otimes\mathcal{W}_D}{\times} ((M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}$$
$$\xrightarrow{\left(\varphi_n \otimes \mathrm{id}_{\mathcal{W}_D}\right) \times \mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}}$$
$$\left(\mathbb{J}^{D^n}(\pi) \otimes \mathcal{W}_D\right)_{\varphi_n(x_n)} \underset{M\otimes\mathcal{W}_D}{\times} ((M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}$$
$$= \left(\left(\mathbb{J}^{D^n}(\pi) \underset{M}{\times} (M \otimes \mathcal{W}_{D^n})\right) \otimes \mathcal{W}_D\right)_{\varphi_n(x_n)\times\left(M\otimes\mathcal{W}_{D^n}\right)_{\pi(x_n)}}$$
$$\xrightarrow{\left((\nabla, \gamma) \in \mathbb{J}^{D^n}(\pi) \underset{M}{\times} (M \otimes \mathcal{W}_{D^n}) \mapsto \nabla(\gamma) \in E \otimes \mathcal{W}_{D^n}\right) \otimes \mathrm{id}_{\mathcal{W}_D}}$$
$$((E \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_D)_{(E\otimes\mathcal{W}_{D^n})_{\pi_0(x_n)}}$$
$$= \left(E \otimes \mathcal{W}_{D^{n+1}}\right)_{\pi_0(x_n)} \tag{17}$$

By the very definition of φ_n, the composition of mappings

$$\left(\mathbf{J}^n(\pi) \otimes \mathcal{W}_D\right)_{x_n} \underset{M\otimes\mathcal{W}_D}{\times} ((M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}$$
$$\xrightarrow{\left(\varphi_n \otimes \mathrm{id}_{\mathcal{W}_D}\right) \times \mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}}$$
$$\left(\mathbb{J}^{D^n}(\pi) \otimes \mathcal{W}_D\right)_{\varphi_n(x_n)} \underset{M\otimes\mathcal{W}_D}{\times} ((M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}$$
$$= \left(\left(\mathbb{J}^{D^n}(\pi) \underset{M}{\times} (M \otimes \mathcal{W}_{D^n})\right) \otimes \mathcal{W}_D\right)_{\varphi_n(x_n)\times\left(M\otimes\mathcal{W}_{D^n}\right)_{\pi(x_n)}}$$
$$\xrightarrow{\left((\nabla, \gamma) \in \mathbb{J}^{D^n}(\pi) \underset{M}{\times} (M \otimes \mathcal{W}_{D^n}) \mapsto \nabla(\gamma) \in E \otimes \mathcal{W}_{D^n}\right) \otimes \mathrm{id}_{\mathcal{W}_D}}$$
$$((E \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_D)_{(E\otimes\mathcal{W}_{D^n})_{\pi_0(x_n)}}$$
$$= \left(E \otimes \mathcal{W}_{D^{n+1}}\right)_{\pi_0(x_n)}$$

is equivalent to the composition of mappings

$$\left(\mathbf{J}^n(\pi) \otimes \mathcal{W}_D\right)_{x_n} \underset{M\otimes\mathcal{W}_D}{\times} ((M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}$$

$$= \left(\mathbf{J}^n(\pi) \otimes \mathscr{W}_D\right)_{x_n} \underset{M\otimes\mathscr{W}_D}{\times}$$

$$\left(\left(\left(M \otimes \mathscr{W}_{D^{n-1}}\right) \otimes \mathscr{W}_D\right) \otimes \mathscr{W}_D\right)_{\left(\left(M\otimes\mathscr{W}_{D^{n-1}}\right)\otimes\mathscr{W}_D\right)_{\left(M\otimes\mathscr{W}_{D^{n-1}}\right)_{\pi(x_n)}}}$$

$$= \left(\left(\mathbf{J}^n(\pi) \underset{M}{\times} \left(\left(M \otimes \mathscr{W}_{D^{n-1}}\right) \otimes \mathscr{W}_D\right)\right) \otimes \mathscr{W}_D\right)_*$$

$$[* = x_n \times \left(\left(M \otimes \mathscr{W}_{D^{n-1}}\right) \otimes \mathscr{W}_D\right)_{\left(M\otimes\mathscr{W}_{D^{n-1}}\right)_{\pi(x_n)}}]$$

$$\xrightarrow{\left(\mathrm{id}_{\mathbf{J}^n(\pi)} \times \left\langle \pi_M^{M\otimes\mathscr{W}_{D^{n-1}}} \otimes \mathrm{id}_{\mathscr{W}_D}, \mathrm{id}_{\left(M\otimes\mathscr{W}_{D^{n-1}}\right)\otimes\mathscr{W}_D}\right\rangle\right) \otimes \mathrm{id}_{\mathscr{W}_D}}$$

$$\left(\left(\mathbf{J}^n(\pi) \underset{M}{\times} (M \otimes \mathscr{W}_D) \underset{M}{\times} \left(\left(M \otimes \mathscr{W}_{D^{n-1}}\right) \otimes \mathscr{W}_D\right)\right) \otimes \mathscr{W}_D\right)_*$$

$$[* = x_n \times \pi(x_n) \times \left(\left(M \otimes \mathscr{W}_{D^{n-1}}\right) \otimes \mathscr{W}_D\right)_{\left(M\otimes\mathscr{W}_{D^{n-1}}\right)_{\pi(x_n)}}]$$

$$\xrightarrow{\left(\left(\begin{array}{c}(\nabla, t) \in \mathbf{J}^n(\pi) \times (M \otimes \mathscr{W}_D) \mapsto \\ \nabla(t) \in \mathbf{J}^{n-1}(\pi) \otimes \mathscr{W}_D\end{array}\right) \times \mathrm{id}_{\left(\left(M\otimes\mathscr{W}_{D^{n-1}}\right)\otimes\mathscr{W}_D\right)}\right) \otimes \mathrm{id}_{\mathscr{W}_D}}$$

$$\left(\left(\left(\mathbf{J}^{n-1}(\pi) \otimes \mathscr{W}_D\right) \underset{M}{\times} \left(\left(M \otimes \mathscr{W}_{D^{n-1}}\right) \otimes \mathscr{W}_D\right)\right) \otimes \mathscr{W}_D\right)_*$$

$$[* = \left(\mathbf{J}^{n-1}(\pi) \otimes \mathscr{W}_D\right)_{\pi_{n-1}(x_n)} \times \left(\left(M \otimes \mathscr{W}_{D^{n-1}}\right) \otimes \mathscr{W}_D\right)_{\left(M\otimes\mathscr{W}_{D^{n-1}}\right)_{\pi(x_n)}}]$$

$$= \left(\left(\mathbf{J}^{n-1}(\pi) \underset{M}{\times} \left(M \otimes \mathscr{W}_{D^{n-1}}\right)\right) \otimes \mathscr{W}_{D^2}\right)_{\pi_{n-1}(x_n)\times\left(M\otimes\mathscr{W}_{D^{n-1}}\right)_{\pi(x_n)}}$$

$$\xrightarrow{\varphi_{n-1} \times \mathrm{id}_{(M\otimes\mathscr{W}_{D^n})\otimes\mathscr{W}_D}}$$

$$\left(\left(\mathbb{J}^{D^{n-1}}(\pi) \underset{M}{\times} \left(M \otimes \mathscr{W}_{D^{n-1}}\right)\right) \otimes \mathscr{W}_{D^2}\right)_{\pi_0(x_n)\times\left(M\otimes\mathscr{W}_{D^{n-1}}\right)_{\pi(x_n)}}$$

$$\xrightarrow{\left((\nabla, \gamma) \in \mathbb{J}^{D^{n-1}}(\pi) \times \left(M \otimes \mathscr{W}_{D^{n-1}}\right) \mapsto \nabla(\gamma) \in E \otimes \mathscr{W}_{D^{n-1}}\right) \otimes \mathrm{id}_{\mathscr{W}_{D^2}}}$$

$$\left(\left(E \otimes \mathscr{W}_{D^{n-1}}\right) \otimes \mathscr{W}_{D^2}\right)_{\left(E\otimes\mathscr{W}_{D^{n-1}}\right)_{\pi_0(x_n)}}$$

$$= \left(E \otimes \mathscr{W}_{D^{n+1}}\right)_{\pi_0(x_n)}$$

Therefore (17) is no other than the composition of mappings

$$\left(M \otimes \mathscr{W}_{D^{n+1}}\right)_{\pi(x_n)}$$

$$\xrightarrow{\gamma \in \left(M \otimes \mathscr{W}_{D^{n+1}}\right)_{\pi(x_n)} \mapsto \gamma^{\langle n,n+1\rangle} \in \left(M \otimes \mathscr{W}_{D^{n+1}}\right)_{\pi(x_n)}}$$

$$\left(M \otimes \mathscr{W}_{D^{n+1}}\right)_{\pi(x_n)}$$

$$= ((M \otimes \mathscr{W}_{D^n}) \otimes \mathscr{W}_D)_{(M\otimes\mathscr{W}_{D^n})_{\pi(x_n)}}$$

$$\xrightarrow{\left\langle \pi_M^{M\otimes\mathcal{W}_{D^n}} \otimes \mathrm{id}_{\mathcal{W}_D}, \mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D} \right\rangle}$$

$$(M \otimes \mathcal{W}_D)_{\pi(x_n)} \underset{M\otimes\mathcal{W}_D}{\times} ((M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}$$

$$\xrightarrow{\nabla_{x_n} \times \mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}}$$

$$\left(\mathbf{J}^n(\pi) \otimes \mathcal{W}_D\right)_{x_n} \underset{M\otimes\mathcal{W}_D}{\times} ((M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}$$

$$= \left(\mathbf{J}^n(\pi) \otimes \mathcal{W}_D\right)_{x_n} \underset{M\otimes\mathcal{W}_D}{\times}$$

$$\left(\left(\left(M \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathcal{W}_D\right) \otimes \mathcal{W}_D\right)_{((M\otimes\mathcal{W}_{D^{n-1}})\otimes\mathcal{W}_D)_{\left(M\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi(x_n)}}}$$

$$= \left(\left(\mathbf{J}^n(\pi) \underset{M}{\times} \left(\left(M \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathcal{W}_D\right)\right) \otimes \mathcal{W}_D\right)_*$$

$$[* = x_n \times \left(\left(M \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathcal{W}_D\right)_{\left(M\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi(x_n)}}]$$

$$\xrightarrow{\left(\mathrm{id}_{\mathbf{J}^n(\pi)} \times \left\langle \pi_M^{M\otimes\mathcal{W}_{D^{n-1}}} \otimes \mathrm{id}_{\mathcal{W}_D}, \mathrm{id}_{\left(M\otimes\mathcal{W}_{D^{n-1}}\right)\otimes\mathcal{W}_D} \right\rangle\right) \otimes \mathrm{id}_{\mathcal{W}_D}}$$

$$\left(\left(\mathbf{J}^n(\pi) \underset{M}{\times} (M \otimes \mathcal{W}_D) \underset{M}{\times} \left(\left(M \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathcal{W}_D\right)\right) \otimes \mathcal{W}_D\right)_*$$

$$[* = x_n \times \pi(x_n) \times \left(\left(M \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathcal{W}_D\right)_{\left(M\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi(x_n)}}]$$

$$\xrightarrow{\left(\left(\begin{array}{c} (\nabla, t) \in \mathbf{J}^n(\pi) \times (M \otimes \mathcal{W}_D) \mapsto \\ \nabla(t) \in \mathbf{J}^{n-1}(\pi) \otimes \mathcal{W}_D \end{array}\right) \times \mathrm{id}_{\left(\left(M\otimes\mathcal{W}_{D^{n-1}}\right)\otimes\mathcal{W}_D\right)}\right) \otimes \mathrm{id}_{\mathcal{W}_D}}$$

$$\left(\left(\left(\mathbf{J}^{n-1}(\pi) \otimes \mathcal{W}_D\right) \underset{M}{\times} \left(\left(M \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathcal{W}_D\right)\right) \otimes \mathcal{W}_D\right)_*$$

$$[* = \left(\mathbf{J}^{n-1}(\pi) \otimes \mathcal{W}_D\right)_{\pi_{n-1}(x_n)} \times \left(\left(M \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathcal{W}_D\right)_{\left(M\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi(x_n)}}]$$

$$= \left(\left(\mathbf{J}^{n-1}(\pi) \times \left(M \otimes \mathcal{W}_{D^{n-1}}\right)\right) \otimes \mathcal{W}_{D^2}\right)_{\pi_{n-1}(x_n)\times\left(M\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi(x_n)}}$$

$$\xrightarrow{\varphi_{n-1} \times \mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}}$$

$$\left(\left(\mathbb{J}^{D^{n-1}}(\pi) \underset{M}{\times} \left(M \otimes \mathcal{W}_{D^{n-1}}\right)\right) \otimes \mathcal{W}_{D^2}\right)_{\pi_0(x_n)\times\left(M\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi(x_n)}}$$

$$\xrightarrow{\left((\nabla, \gamma) \in \mathbb{J}^{D^{n-1}}(\pi) \times \left(M \otimes \mathcal{W}_{D^{n-1}}\right) \mapsto \nabla(\gamma) \in E \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathrm{id}_{\mathcal{W}_{D^2}}}$$

$$\left(\left(E \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathcal{W}_{D^2}\right)_{\left(E\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi_0(x_n)}}$$

$$= \left(E \otimes \mathcal{W}_{D^{n+1}}\right)_{\pi_0(x_n)}$$

On the other hand, the composition of mappings

$$
\begin{aligned}
&\left(M\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi(x_n)} \\
&=((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}} \\
&\underrightarrow{\left\langle\pi_M^{M\otimes\mathcal{W}_{D^n}}\otimes\mathrm{id}_{\mathcal{W}_D},\mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}\right\rangle} \\
&(M\otimes\mathcal{W}_D)_{\pi(x_n)}\underset{M\otimes\mathcal{W}_D}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}} \\
&\underrightarrow{\nabla_{x_n}\times\mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}} \\
&\left(\mathbf{J}^n(\pi)\otimes\mathcal{W}_D\right)_{x_n}\underset{M\otimes\mathcal{W}_D}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}} \\
&\underrightarrow{\left(\varphi_n\otimes\mathrm{id}_{\mathcal{W}_D}\right)\times\mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}} \\
&\left(\mathbb{J}^{D^n}(\pi)\otimes\mathcal{W}_D\right)_{\varphi_n(x_n)}\underset{M\otimes\mathcal{W}_D}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}} \\
&=\left(\left(\mathbb{J}^{D^n}(\pi)\underset{M}{\times}(M\otimes\mathcal{W}_{D^n})\right)\otimes\mathcal{W}_D\right)_{\varphi_n(x_n)\times(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}} \\
&\underrightarrow{\left((\nabla,\gamma)\in\mathbb{J}^{D^n}(\pi)\underset{M}{\times}(M\otimes\mathcal{W}_{D^n})\mapsto\nabla(\gamma)\in E\otimes\mathcal{W}_{D^n}\right)\otimes\mathrm{id}_{\mathcal{W}_D}} \\
&((E\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(E\otimes\mathcal{W}_{D^n})_{\pi_0(x_n)}} \\
&=\left(E\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi_0(x_n)} \\
&\underrightarrow{\gamma\in E\otimes\mathcal{W}_{D^{n+1}}\mapsto\gamma^{\langle n,n+1\rangle}\in E\otimes\mathcal{W}_{D^{n+1}}}\ \left(E\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi_0(x_n)}
\end{aligned}
$$

is the composition of mappings

$$
\begin{aligned}
&\left(M\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi(x_n)} \\
&=((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}} \\
&\underrightarrow{\left\langle\pi_M^{M\otimes\mathcal{W}_{D^n}}\otimes\mathrm{id}_{\mathcal{W}_D},\mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}\right\rangle} \\
&(M\otimes\mathcal{W}_D)_{\pi(x_n)}\underset{M\otimes\mathcal{W}_D}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}} \\
&\underrightarrow{\nabla_{x_n}\times\mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}} \\
&\left(\mathbf{J}^n(\pi)\otimes\mathcal{W}_D\right)_{x_n}\underset{M\otimes\mathcal{W}_D}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}} \\
&=\left(\mathbf{J}^n(\pi)\otimes\mathcal{W}_D\right)_{x_n}\underset{M\otimes\mathcal{W}_D}{\times}
\end{aligned}
$$

$$\left(\left(\left(M \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathcal{W}_D\right) \otimes \mathcal{W}_D\right)_{\left(\left(M\otimes\mathcal{W}_{D^{n-1}}\right)\otimes\mathcal{W}_D\right)_{\left(M\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi(x_n)}}}$$

$$= \left(\left(\mathbf{J}^n(\pi) \times \left(\left(M \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathcal{W}_D\right)\right) \otimes \mathcal{W}_D\right)_*$$

$$[* = x_n \times \left(\left(M \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathcal{W}_D\right)_{\left(M\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi(x_n)}}]$$

$$\xrightarrow{\left(\mathrm{id}_{\mathbf{J}^n(\pi)} \times \left\langle \pi_M^{M\otimes\mathcal{W}_{D^{n-1}}} \otimes \mathrm{id}_{\mathcal{W}_D}, \mathrm{id}_{\left(M\otimes\mathcal{W}_{D^{n-1}}\right)\otimes\mathcal{W}_D}\right\rangle\right) \otimes \mathrm{id}_{\mathcal{W}_D}}$$

$$\left(\left(\mathbf{J}^n(\pi) \underset{M}{\times} (M \otimes \mathcal{W}_D) \underset{M}{\times} \left(\left(M \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathcal{W}_D\right)\right) \otimes \mathcal{W}_D\right)_*$$

$$[* = x_n \times \pi(x_n) \times \left(\left(M \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathcal{W}_D\right)_{\left(M\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi(x_n)}}]$$

$$\frac{\left(\begin{pmatrix} (\nabla, t) \in \mathbf{J}^n(\pi) \times (M \otimes \mathcal{W}_D) \mapsto \\ \nabla(t) \in \mathbf{J}^{n-1}(\pi) \otimes \mathcal{W}_D \end{pmatrix} \times \mathrm{id}_{\left(\left(M\otimes\mathcal{W}_{D^{n-1}}\right)\otimes\mathcal{W}_D\right)}\right)}{\xrightarrow{\otimes \mathrm{id}_{\mathcal{W}_D}}}$$

$$\left(\left(\left(\mathbf{J}^{n-1}(\pi) \otimes \mathcal{W}_D\right) \underset{M\otimes\mathcal{W}_D}{\times} \left(\left(M \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathcal{W}_D\right)\right) \otimes \mathcal{W}_D\right)_*$$

$$[* = \left(\mathbf{J}^{n-1}(\pi) \otimes \mathcal{W}_D\right)_{\pi_{n-1}(x_n)} \underset{M\otimes\mathcal{W}_D}{\times} \left(\left(M \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathcal{W}_D\right)_{\left(M\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi(x_n)}}]$$

$$= \left(\left(\mathbf{J}^{n-1}(\pi) \times \left(M \otimes \mathcal{W}_{D^{n-1}}\right)\right) \otimes \mathcal{W}_{D^2}\right)_{\pi_{n-1}(x_n)\times\left(M\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi(x_n)}}$$

followed by the composition of mappings

$$\left(\left(\mathbf{J}^{n-1}(\pi) \times \left(M \otimes \mathcal{W}_{D^{n-1}}\right)\right) \otimes \mathcal{W}_{D^2}\right)_{\pi_{n-1}(x_n)\times\left(M\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi(x_n)}}$$

$$\xrightarrow{\varphi_{n-1} \times \mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}}$$

$$\left(\left(\mathbb{J}^{D^{n-1}}(\pi) \underset{M}{\times} \left(M \otimes \mathcal{W}_{D^{n-1}}\right)\right) \otimes \mathcal{W}_{D^2}\right)_{\pi_0(x_n)\times\left(M\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi(x_n)}}$$

$$\xrightarrow{\left((\nabla, \gamma) \in \mathbb{J}^{n-1}(\pi) \times \left(M \otimes \mathcal{W}_{D^{n-1}}\right) \mapsto \nabla(\gamma) \in E \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathrm{id}_{\mathcal{W}_{D^2}}}$$

$$\left(E \otimes \mathcal{W}_{D^{n-1}}\right) \otimes \mathcal{W}_{D^2} = \left(E \otimes \mathcal{W}_{D^{n+1}}\right)_{\pi_0(x_n)}$$

$$\xrightarrow{\gamma \in E \otimes \mathcal{W}_{D^{n+1}} \mapsto \gamma^{\langle n,n+1\rangle} \in E \otimes \mathcal{W}_{D^{n+1}}} \left(E \otimes \mathcal{W}_{D^{n+1}}\right)_{\pi_0(x_n)},$$

which is easily seen to be equivalent to the composition of mappings

$$\left(\left(\mathbf{J}^{n-1}(\pi) \times \left(M \otimes \mathcal{W}_{D^{n-1}}\right)\right) \otimes \mathcal{W}_{D^2}\right)_{\pi_{n-1}(x_n)\times\left(M\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi(x_n)}}$$

$$\xrightarrow{\mathrm{id}_{\mathbf{J}^{n-1}(\pi)\times\left(M\otimes\mathcal{W}_{D^{n-1}}\right)}\otimes\mathcal{W}_{(d_1.d_2)\in D^2\mapsto(d_2.d_1)\in D^2}}$$

$$\left(\left(\mathbf{J}^{n-1}(\pi)\times\left(M\otimes\mathcal{W}_{D^{n-1}}\right)\right)\otimes\mathcal{W}_{D^2}\right)_{\pi_{n-1}(x_n)\times\left(M\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi(x_n)}}$$

$$\xrightarrow{\varphi_{n-1}\times\mathrm{id}_{\left(M\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D}}$$

$$\left(\left(\mathbb{J}^{D^{n-1}}(\pi)\underset{M}{\times}\left(M\otimes\mathcal{W}_{D^{n-1}}\right)\right)\otimes\mathcal{W}_{D^2}\right)_{\pi_0(x_n)\times\left(M\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi(x_n)}}$$

$$\xrightarrow{\left((\nabla,\gamma)\in\mathbb{J}^{D^{n-1}}(\pi)\times\left(M\otimes\mathcal{W}_{D^{n-1}}\right)\mapsto\nabla(\gamma)\in E\otimes\mathcal{W}_{D^{n-1}}\right)\otimes\mathrm{id}_{\mathcal{W}_{D^2}}}$$

$$\left(\left(E\otimes\mathcal{W}_{D^{n-1}}\right)\otimes\mathcal{W}_{D^2}\right)_{\left(E\otimes\mathcal{W}_{D^{n-1}}\right)_{\pi_0(x_n)}}$$

$$=\left(E\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi_0(x_n)}$$

Therefore the desired result follows from the second condition in the item 3 of Notation 3.3.

Lemma 6.2 *The diagram*

$$\begin{array}{ccc}
\mathbf{J}^{n+1}(\pi) & \xrightarrow{\varphi_{n+1}} & \hat{\mathbb{J}}^{D^{n+1}}(\pi) \\
\pi_{n+1,n}\downarrow & & \downarrow\widehat{\pi}_{n+1,n} \\
\mathbf{J}^{n}(\pi) & \xrightarrow[\varphi_n]{} & \hat{\mathbb{J}}^{D^{n}}(\pi)
\end{array}$$

is commutative.

Proof Given $\nabla_{x_n}\in\mathbf{J}^{n+1}(\pi)$, $\left(\widehat{\pi}_{n+1,n}\circ\varphi_{n+1}\right)\left(\nabla_{x_n}\right)$ is, by the very definition of $\widehat{\pi}_{n+1,n}$, the composition of mappings

$$\left(M\otimes\mathcal{W}_{D^n}\right)_{\pi(x_n)}\xrightarrow{\mathbf{s}_{n+1}}\left(M\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi(x_n)}\xrightarrow{\varphi_{n+1}(\nabla_{x_n})}$$

$$\left(E\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi_0(x_n)}\xrightarrow{\mathbf{d}_{n+1}}\left(E\otimes\mathcal{W}_{D^n}\right)_{\pi_0(x_n)}$$

which is equivalent, by the very definition of $\varphi_{n+1}(\nabla_{x_n})$, to the composition of mappings

$$\left(M\otimes\mathcal{W}_{D^n}\right)_{\pi(x_n)}\xrightarrow{\mathbf{s}_{n+1}}\left(M\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi(x_n)}$$

$$=\left(\left(M\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D\right)_{\left(M\otimes\mathcal{W}_{D^n}\right)_{\pi(x_n)}}$$

$$\xrightarrow{\left\langle\pi_M^{M\otimes\mathcal{W}_{D^n}}\otimes\mathrm{id}_{\mathcal{W}_D},\,\mathrm{id}_{\left(M\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D}\right\rangle}$$

$$\left(\left(M\otimes\mathcal{W}_D\right)\underset{M\otimes\mathcal{W}_D}{\times}\left(\left(M\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D\right)\right)_{\{\pi(x_n)\}\times\left(M\otimes\mathcal{W}_{D^n}\right)_{\pi(x_n)}}$$

$$\xrightarrow{\nabla_{x_n}\times\mathrm{id}_{\left(M\otimes\mathcal{W}_{D^n}\right)\otimes\mathcal{W}_D}}$$

$$\left(\left(\mathbf{J}^n(\pi)\otimes\mathcal{W}_D\right)\underset{M\otimes\mathcal{W}_D}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)\right)_{\{\pi(x_n)\}\times(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}$$
$$\xrightarrow{\left(\varphi_n\otimes\mathrm{id}_{\mathcal{W}_D}\right)\times\mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}}$$
$$\left(\left(\mathbb{J}^{D^n}(\pi)\otimes\mathcal{W}_D\right)\underset{M\otimes\mathcal{W}_D}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)\right)_{\{\pi(x_n)\}\times(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}$$
$$=\left(\left(\mathbb{J}^{D^n}(\pi)\underset{M}{\times}(M\otimes\mathcal{W}_{D^n})\right)\otimes\mathcal{W}_D\right)_{\{\pi(x_n)\}\times(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}$$
$$\xrightarrow{\left((\nabla,\gamma)\in\mathbb{J}^{D^n}(\pi)\times(M\otimes\mathcal{W}_{D^n})\mapsto\nabla(\gamma)\in E\otimes\mathcal{W}_{D^n}\right)\otimes\mathrm{id}_{\mathcal{W}_D}}$$
$$((E\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(E\otimes\mathcal{W}_{D^n})_{\pi_0(x_n)}}$$
$$=\left(E\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi_0(x_n)}\xrightarrow{\mathbf{d}_{n+1}}(E\otimes\mathcal{W}_{D^n})_{\pi_0(x_n)}$$

This is easily seen to be equivalent to $\varphi_n(\pi_{n+1,n}\left(\nabla_{x_n}\right))$, which completes the proof.

Lemma 6.1 can be strengthened as follows:

Lemma 6.3 *We have*

$$\varphi_{n+1}(\nabla_{x_n})\in\mathbb{J}^{n+1}(\pi)$$

Proof With due regard to Lemmas 6.1 and 6.2, we have only to show that

$$\left(\varphi_{n+1}(\nabla_{x_n})\right)\circ\left(\mathrm{id}_M\otimes\mathcal{W}_{(d_1,\ldots,d_n,d_{n+1})\in D^{n+1}\mapsto(d_1,\ldots,d_nd_{n+1})\in D^n}\right)$$
$$=\left(\mathrm{id}_E\otimes\mathcal{W}_{(d_1,\ldots,d_n,d_{n+1})\in D^{n+1}\mapsto(d_1,\ldots,d_nd_{n+1})\in D^{n+1}}\right)\circ$$
$$\left(\widehat{\pi}_{n+1,n}(\varphi_{n+1}(\nabla_{x_n}))\right)\tag{18}$$

For $n=0$, there is nothing to prove. We proceed by induction on n. By the very definition of φ_{n+1}, the left-hand side of (18) is the composition of mappings

$$(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}$$
$$\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(d_1,\ldots,d_n,d_{n+1})\in D^{n+1}\mapsto(d_1,\ldots,d_nd_{n+1})\in D^n}}$$
$$\left(M\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi(x_n)}$$
$$=((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}$$
$$\xrightarrow{\left\langle\pi_M^{M\otimes\mathcal{W}_{D^n}}\otimes\mathrm{id}_{\mathcal{W}_D},\,\mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}\right\rangle}$$
$$(M\otimes\mathcal{W}_D)_{\pi(x_n)}\underset{M\otimes\mathcal{W}_D}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D)_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}$$
$$\xrightarrow{\nabla_{x_n}\times\mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}}$$

$$
\begin{aligned}
&\left(\mathbf{J}^{n}(\pi)\otimes\mathcal{W}_{D}\right)_{\pi(x_n)}\underset{M\otimes\mathcal{W}_{D}}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_{D})_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}\\
&\xrightarrow{\left(\varphi_n\otimes\mathrm{id}_{\mathcal{W}_D}\right)\times\mathrm{id}_{(M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_D}}\\
&\left(\mathbb{J}^{D^n}(\pi)\otimes\mathcal{W}_{D}\right)_{\pi(x_n)}\underset{M\otimes\mathcal{W}_{D}}{\times}((M\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_{D})_{(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}\\
&=\left(\left(\mathbb{J}^{D^n}(\pi)\underset{M}{\times}(M\otimes\mathcal{W}_{D^n})\right)\otimes\mathcal{W}_{D}\right)_{\{\pi(x_n)\}\times(M\otimes\mathcal{W}_{D^n})_{\pi(x_n)}}\\
&\xrightarrow{\left((\nabla,\gamma)\in\mathbb{J}^{D^n}(\pi)\times(M\otimes\mathcal{W}_{D^n})\mapsto\nabla(\gamma)\in E\otimes\mathcal{W}_{D^n}\right)\otimes\mathrm{id}_{\mathcal{W}_D}}\\
&((E\otimes\mathcal{W}_{D^n})\otimes\mathcal{W}_{D})_{(E\otimes\mathcal{W}_{D^n})_{\pi_0(x_n)}}\\
&=\left(E\otimes\mathcal{W}_{D^{n+1}}\right)_{\pi_0(x_n)}
\end{aligned}
$$

which is easily seen, by dint of Lemma 6.1, to be equivalent to the right-hand side of (18).

Thus we have established the mappings $\varphi_n:\mathbf{J}^n(\pi)\to\mathbb{J}^{D^n}(\pi)$.

7 From the Second Approach to the Third

The principal objective in this section is to define a mapping $\psi_n:\mathbb{J}^{D^n}(\pi)\to\mathbb{J}^{D_n}(\pi)$. Let us begin with

Proposition 7.1 *Let ∇_x be a D^n-pseudotangential over the bundle $\pi: E\to M$ at $x\in E$ and $\gamma\in\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}$. Then there exists a unique $\gamma'\in\left(E\otimes\mathcal{W}_{D_n}\right)_x$ such that*

$$
\begin{aligned}
&\nabla_x(\left(\mathrm{id}_M\otimes\mathcal{W}_{(d_1,\ldots,d_n)\in D^n\longmapsto(d_1+\ldots+d_n)\in D_n}\right)(\gamma))\\
&=\left(\mathrm{id}_E\otimes\mathcal{W}_{(d_1,\ldots,d_n)\in D^n\longmapsto(d_1+\ldots+d_n)\in D_n}\right)\left(\gamma'\right)
\end{aligned}
$$

Proof This stems easily from the following simple lemma.

Lemma 7.1 *The diagram*

$$
\mathcal{W}_{D_n}\xrightarrow{\mathcal{W}_{(d_1,\ldots,d_n)\in D^n\longmapsto(d_1+\ldots+d_n)\in D_n}}\mathcal{W}_{D^n}\quad\begin{matrix}\xrightarrow{\mathcal{W}_{\tau_1}}\\ \vdots\\ \xrightarrow{\mathcal{W}_{\tau_i}}\\ \vdots\\ \xrightarrow{\mathcal{W}_{\tau_{n-1}}}\end{matrix}\quad\mathcal{W}_{D^n}
$$

is a limit diagram in the category of Weil algebras, where $\tau_i : D^n \to D^n$ *is the mapping permuting the* i*-th and* $(i+1)$*-th components of* D^n *while fixing the other components.*

Notation 7.1 *We will denote by* $\widehat{\psi}_n(\nabla_x)(\gamma)$ *the unique* γ' *in the above proposition, thereby getting a function* $\widehat{\psi}_n(\nabla_x) : \left(M \otimes \mathscr{W}_{D_n}\right)_{\pi(x)} \to \left(E \otimes \mathscr{W}_{D_n}\right)_x$.

Proposition 7.2 *For any* $\nabla_x \in \widehat{\mathbb{J}}_x^{D^n}(\pi)$*, we have* $\widehat{\psi}_n(\nabla_x) \in \widehat{\mathbb{J}}_x^{D_n}(\pi)$.

Proof We have to verify the three conditions in Definition 5.1 concerning the mapping $\widehat{\psi}_n(\nabla_x) : \left(M \otimes \mathscr{W}_{D_n}\right)_{\pi(x)} \to \left(E \otimes \mathscr{W}_{D_n}\right)_x$.

1. To see the first condition, it suffices to show that

$$\left(\mathrm{id}_M \otimes \mathscr{W}_{(d_1,\ldots,d_n)\in D^n \longmapsto (d_1+\ldots+d_n)\in D_n}\right)(\gamma)$$
$$= \left(\mathrm{id}_E \otimes \mathscr{W}_{(d_1,\ldots,d_n)\in D^n \longmapsto (d_1+\ldots+d_n)\in D_n}\right)\left(\left(\pi \otimes \mathrm{id}_{\mathscr{W}_{D_n}}\right)\left(\widehat{\psi}_n(\nabla_x)(\gamma)\right)\right),$$

which follows from

$$\left(\mathrm{id}_M \otimes \mathscr{W}_{(d_1,\ldots,d_n)\in D^n \longmapsto (d_1+\ldots+d_n)\in D_n}\right)\left(\left(\pi \otimes \mathrm{id}_{\mathscr{W}_{D_n}}\right)\left(\widehat{\psi}_n(\nabla_x)(\gamma)\right)\right)$$
$$= \left(\pi \otimes \mathrm{id}_{\mathscr{W}_{D^n}}\right)\left(\left(\mathrm{id}_E \otimes \mathscr{W}_{(d_1,\ldots,d_n)\in D^n \longmapsto (d_1+\ldots+d_n)\in D_n}\right)\left(\widehat{\psi}_n(\nabla_x)(\gamma)\right)\right)$$

[By the bifunctionality of $\otimes$]

$$= \left(\pi \otimes \mathrm{id}_{\mathscr{W}_{D^n}}\right)\left(\nabla_x\left(\left(\mathrm{id}_M \otimes \mathscr{W}_{(d_1,\ldots,d_n)\in D^n \longmapsto (d_1+\ldots+d_n)\in D_n}\right)(\gamma)\right)\right)$$

[By the very definition of $\widehat{\psi}_n(\nabla_x)$]

$$= \left(\mathrm{id}_M \otimes \mathscr{W}_{(d_1,\ldots,d_n)\in D^n \longmapsto (d_1+\ldots+d_n)\in D_n}\right)(\gamma)$$

2. Now we are going to deal with the second condition. It is easy to see that the composition of mappings

$$\left(M \otimes \mathscr{W}_{D_n}\right)_{\pi(x)} \xrightarrow{\mathrm{id}_M \otimes \mathscr{W}_{(\alpha\cdot)_{D_n}}} \left(M \otimes \mathscr{W}_{D_n}\right)_{\pi(x)} \xrightarrow{\mathscr{W}_{(d_1,\ldots,d_n)\in D^n \longmapsto (d_1+\ldots+d_n)\in D_n}} \left(M \otimes \mathscr{W}_{D^n}\right)_{\pi(x)}$$

is equivalent to the composition of mappings

$$\left(M \otimes \mathscr{W}_{D_n}\right)_{\pi(x)} \xrightarrow{\mathscr{W}_{(d_1,\ldots,d_n)\in D^n \longmapsto (d_1+\ldots+d_n)\in D_n}} \left(M \otimes \mathscr{W}_{D^n}\right)_{\pi(x)} \xrightarrow{\mathrm{id}_M \otimes \mathscr{W}_{\left(\underset{1}{\alpha\cdot}\right)_{D^n}}} \left(M \otimes \mathscr{W}_{D^n}\right)_{\pi(x)} \ldots \xrightarrow{\mathrm{id}_M \otimes \mathscr{W}_{\left(\underset{n}{\alpha\cdot}\right)_{D^n}}} \left(M \otimes \mathscr{W}_{D^n}\right)_{\pi(x)},$$

while the composition of mappings

$$(M\otimes\mathcal{W}_{D^n})_{\pi(x)}\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{\left(\alpha\underset{1}{\cdot}\right)_{D^n}}}(M\otimes\mathcal{W}_{D^n})_{\pi(x)}\ldots\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{\left(\alpha\underset{n}{\cdot}\right)_{D^n}}}(M\otimes\mathcal{W}_{D^n})_{\pi(x)}\xrightarrow{\nabla_x}(E\otimes\mathcal{W}_{D^n})_x$$

is equivalent to the composition of mappings

$$(M\otimes\mathcal{W}_{D^n})_{\pi(x)}\xrightarrow{\nabla_x}(E\otimes\mathcal{W}_{D^n})_x\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{\left(\alpha\underset{1}{\cdot}\right)_{D^n}}}(E\otimes\mathcal{W}_{D^n})_x\ldots\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{\left(\alpha\underset{n}{\cdot}\right)_{D^n}}}(E\otimes\mathcal{W}_{D^n})_x$$

Therefore the composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{(\alpha\cdot)_{D_n}}}\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\xrightarrow{\mathcal{W}_{(d_1,\ldots,d_n)\in D^n\longmapsto(d_1+\ldots+d_n)\in D_n}}(M\otimes\mathcal{W}_{D^n})_{\pi(x)}\xrightarrow{\nabla_x}(E\otimes\mathcal{W}_{D^n})_x$$

is equivalent to the composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\xrightarrow{\mathcal{W}_{(d_1,\ldots,d_n)\in D^n\longmapsto(d_1+\ldots+d_n)\in D_n}}(M\otimes\mathcal{W}_{D^n})_{\pi(x)}\xrightarrow{\nabla_x}(E\otimes\mathcal{W}_{D^n})_x\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{\left(\alpha\underset{1}{\cdot}\right)_{D^n}}}(E\otimes\mathcal{W}_{D^n})_x\ldots\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{\left(\alpha\underset{n}{\cdot}\right)_{D^n}}}(E\otimes\mathcal{W}_{D^n})_x,$$

which should be equivalent in turn to

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\xrightarrow{\widehat{\psi}_n(\nabla_x)}\left(E\otimes\mathcal{W}_{D_n}\right)_x\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{(d_1,\ldots,d_n)\in D^n\longmapsto(d_1+\ldots+d_n)\in D_n}}(E\otimes\mathcal{W}_{D^n})_x\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{\left(\alpha\underset{1}{\cdot}\right)_{D^n}}}(E\otimes\mathcal{W}_{D^n})_x\ldots\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{\left(\alpha\underset{n}{\cdot}\right)_{D^n}}}(E\otimes\mathcal{W}_{D^n})_x$$

Since the composition of mappings

$$\left(E\otimes\mathcal{W}_{D_n}\right)_x\xrightarrow{\mathcal{W}_{(d_1,\ldots,d_n)\in D^n\longmapsto(d_1+\ldots+d_n)\in D_n}}(E\otimes\mathcal{W}_{D^n})_x\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{\left(\alpha\underset{1}{\cdot}\right)_{D^n}}}(E\otimes\mathcal{W}_{D^n})_x\ldots\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{\left(\alpha\underset{n}{\cdot}\right)_{D^n}}}(E\otimes\mathcal{W}_{D^n})_x$$

is equivalent to the composition of mappings

$$\left(E\otimes\mathcal{W}_{D_n}\right)_x\ \underrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{(\alpha\cdot)_{D_n}}}\ \left(E\otimes\mathcal{W}_{D_n}\right)_x\ \underrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{(d_1,...,d_n)\in D^n\longmapsto(d_1+...+d_n)\in D_n}}$$
$$(E\otimes\mathcal{W}_{D^n})_x\ ,$$

the coveted result follows.

3. We are going to deal with the third condition. We have to show that the diagram

$$\begin{array}{ccc}\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)} & \underrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{\mathbf{m}_{D_n\times D_m\to D_n}}} & \left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\otimes\mathcal{W}_{D_m}\\ \widehat{\psi}_n(\nabla_x)\downarrow & & \downarrow\widehat{\psi}_n(\nabla_x)\otimes\mathrm{id}_{\mathcal{W}_{D_m}}\\ \left(E\otimes\mathcal{W}_{D_n}\right)_x & \underrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{\mathbf{m}_{D_n\times D_m\to D_n}}} & \left(E\otimes\mathcal{W}_{D_n}\right)_x\otimes\mathcal{W}_{D_m}\end{array} \tag{19}$$

commutes. It is easy to see that the diagram

$$\begin{array}{ccc}\left(E\otimes\mathcal{W}_{D_n}\right)_x & \mathrm{id}_E\otimes\mathcal{W}_{+_{D^n\to D_n}} & (E\otimes\mathcal{W}_{D^n})_x\\ \mathrm{id}_E\otimes\mathcal{W}_{\mathbf{m}_{D_n\times D_m\to D_n}}\downarrow & & \downarrow\mathrm{id}_E\otimes\mathcal{W}_\eta\\ \left(E\otimes\mathcal{W}_{D_n}\right)_x\otimes\mathcal{W}_{D_m} & \mathrm{id}_E\otimes\mathcal{W}_{+_{D^n\to D_n}\times\mathrm{id}_{D_m}} & (E\otimes\mathcal{W}_{D^n})_x\otimes\mathcal{W}_{D_m}\end{array}$$

commutes, where η stands for

$$(d_1,...,d_n,e)\in D^n\times D_m\longmapsto(d_1e,...,d_ne)\in D^n$$

so that the commutativity of the diagram in (19) is equivalent to the commutativity of the outer square of the diagram

$$\begin{array}{ccc}\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)} & \underrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{\mathbf{m}_{D_n\times D_m\to D_n}}} & \left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\otimes\mathcal{W}_{D_m}\\ \widehat{\psi}_n(\nabla_x)\downarrow & & \downarrow\widehat{\psi}_n(\nabla_x)\otimes\mathrm{id}_{\mathcal{W}_{D_m}}\\ \left(E\otimes\mathcal{W}_{D_n}\right)_x & \underrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{\mathbf{m}_{D_n\times D_m\to D_n}}} & \left(E\otimes\mathcal{W}_{D_n}\right)_x\otimes\mathcal{W}_{D_m}\\ \mathrm{id}_E\otimes\mathcal{W}_{+_{D^n\to D_n}}\downarrow & & \downarrow\mathrm{id}_E\otimes\mathcal{W}_{+_{D^n\to D_n}\times\mathrm{id}_{D_m}}\\ (E\otimes\mathcal{W}_{D^n})_x & \underrightarrow{\mathrm{id}_E\otimes\mathcal{W}_\eta} & (E\otimes\mathcal{W}_{D^n})_x\otimes\mathcal{W}_{D_m}\end{array} \tag{20}$$

where $+_{D^n\to D_n}$ stands for

$$(d_1,...,d_n)\in D^n\longmapsto(d_1+...+d_n)\in D_n$$

The composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\ \underrightarrow{\widehat{\psi}_n(\nabla_x)}\ \left(E\otimes\mathcal{W}_{D_n}\right)_x\ \underrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{+_{D^n\to D_n}}}\ (E\otimes\mathcal{W}_{D^n})_x$$

is equal to the composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\ \underrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{+_{D^n\to D_n}}}\ (M\otimes\mathcal{W}_{D^n})_{\pi(x)}\ \underrightarrow{\nabla_x}\ (E\otimes\mathcal{W}_{D^n})_x$$

while the composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\otimes\mathcal{W}_{D_m}\xrightarrow{\widehat{\psi}_n(\nabla_x)\otimes\mathrm{id}_{\mathcal{W}_{D_m}}}\left(E\otimes\mathcal{W}_{D_n}\right)_x\otimes\mathcal{W}_{D_m}$$
$$\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{+_{D^n\to D_n}}\times\mathrm{id}_{D_m}}(E\otimes\mathcal{W}_{D^n})_x\otimes\mathcal{W}_{D_m}$$

is equal to the composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{+_{D^n\to D_n}}\times\mathrm{id}_{D_m}}(M\otimes\mathcal{W}_{D^n})_{\pi(x)}\xrightarrow{\nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_m}}}$$
$$(E\otimes\mathcal{W}_{D^n})_x\otimes\mathcal{W}_{D_m}$$

Since the diagram

$$\begin{array}{ccc}
\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)} & \xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{\mathbf{m}_{D_n\times D_m\to D_n}}} & \left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\otimes\mathcal{W}_{D_m}\\
\mathrm{id}_M\otimes\mathcal{W}_{+_{D^n\to D_n}}\downarrow & & \downarrow\mathrm{id}_M\otimes\mathcal{W}_{+_{D^n\to D_n}}\times\mathrm{id}_{D_m}\\
(M\otimes\mathcal{W}_{D^n})_{\pi(x)} & \xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_\eta} & (M\otimes\mathcal{W}_{D^n})_{\pi(x)}\otimes\mathcal{W}_{D_m}\\
\nabla_x\downarrow & & \downarrow\nabla_x\otimes\mathrm{id}_{\mathcal{W}_{D_m}}\\
(E\otimes\mathcal{W}_{D^n})_x & \xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_\eta} & (E\otimes\mathcal{W}_{D^n})_x\otimes\mathcal{W}_{D_m}
\end{array}$$

commutes, the outer square of the diagram in (20) commutes. This completes the proof.

Proposition 7.3 *The diagram*

$$\begin{array}{ccc}
\hat{\mathbb{J}}_x^{D^{n+1}}(\pi) & \xrightarrow{\widehat{\psi}_{n+1}} & \hat{\mathbb{J}}_x^{D_{n+1}}(\pi)\\
\widehat{\pi}_{n+1,n}\downarrow & & \downarrow\widehat{\pi}_{n+1,n}\\
\hat{\mathbb{J}}_x^{D^n}(\pi) & \overrightarrow{\widehat{\psi}}_n & \hat{\mathbb{J}}_x^{D_n}(\pi)
\end{array}$$

commutes.

Proof Given $\nabla_x\in\hat{\mathbb{J}}_x^{D^{n+1}}(\pi)$, the composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\xrightarrow{\widehat{\pi}_{n+1,n}\left(\widehat{\psi}_{n+1}(\nabla_x)\right)}\left(E\otimes\mathcal{W}_{D_n}\right)_x\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{\mathbf{m}_{D_n\times D_n\to D_n}}}$$
$$\left(E\otimes\mathcal{W}_{D_n}\right)_x\otimes\mathcal{W}_{D_n}\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{+_{D^n\to D_n}}\times\mathrm{id}_{D_n}}(E\otimes\mathcal{W}_{D^n})_x\otimes\mathcal{W}_{D_n}\tag{21}$$

is equivalent to the composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\xrightarrow{\widehat{\pi}_{n+1,n}\left(\widehat{\psi}_{n+1}(\nabla_x)\right)}\left(E\otimes\mathcal{W}_{D_n}\right)_x\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{\mathbf{m}_{D_{n+1}\times D_n\to D_n}}}$$
$$\left(E\otimes\mathcal{W}_{D_{n+1}}\right)_x\otimes\mathcal{W}_{D_n}\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{+_{D^{n+1}\to D_{n+1}}}\times\mathrm{id}_{D_n}}\left(E\otimes\mathcal{W}_{D^{n+1}}\right)_x$$

$$\otimes \mathcal{W}_{D_n} \xrightarrow{\mathbf{d}_{n+1} \otimes \mathrm{id}_{\mathcal{W}_{D_n}}}$$
$$(E \otimes \mathcal{W}_{D^n})_x \otimes \mathcal{W}_{D_n}$$

which is in turn equivalent to the composition of mappings

$$\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \xrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{\mathbf{m}_{D_{n+1} \times D_n \to D_n}}} \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)}$$
$$\otimes \mathcal{W}_{D_n} \xrightarrow{\widehat{\psi}_{n+1} (\nabla_x) \otimes \mathcal{W}_{\mathrm{id}_{\mathcal{W}_{D_n}}}} \left(E \otimes \mathcal{W}_{D_{n+1}}\right)_x \otimes \mathcal{W}_{D_n} \xrightarrow{\mathrm{id}_E \otimes \mathcal{W}_{+_{D^{n+1} \to D_{n+1}}} \times \mathrm{id}_{D_n}}$$
$$\left(E \otimes \mathcal{W}_{D^{n+1}}\right)_x \otimes \mathcal{W}_{D_n} \xrightarrow{\mathbf{d}_{n+1} \otimes \mathrm{id}_{\mathcal{W}_{D_n}}} (E \otimes \mathcal{W}_{D^n})_x \otimes \mathcal{W}_{D_n}$$

This is to be supplanted by the composition of mappings

$$\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \xrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{\mathbf{m}_{D_{n+1} \times D_n \to D_n}}} \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)} \otimes \mathcal{W}_{D_n}$$
$$\xrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{+_{D^{n+1} \to D_{n+1}}} \times \mathrm{id}_{D_n}} \left(M \otimes \mathcal{W}_{D^{n+1}}\right)_{\pi(x)} \otimes \mathcal{W}_{D_n} \xrightarrow{\nabla_x \otimes \mathrm{id}_{\mathcal{W}_{D_n}}}$$
$$\left(E \otimes \mathcal{W}_{D^{n+1}}\right)_x \otimes \mathcal{W}_{D_n} \xrightarrow{\mathbf{d}_{n+1} \otimes \mathrm{id}_{\mathcal{W}_{D_n}}} (E \otimes \mathcal{W}_{D^n})_x \otimes \mathcal{W}_{D_n},$$

which is in turn equivalent to the composition of mappings

$$\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \xrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{\mathbf{m}_{D_{n+1} \times D_n \to D_n}}} \left(M \otimes \mathcal{W}_{D_{n+1}}\right)_{\pi(x)}$$
$$\otimes \mathcal{W}_{D_n} \xrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{+_{D^{n+1} \to D_{n+1}}} \times \mathrm{id}_{D_n}} \left(M \otimes \mathcal{W}_{D^{n+1}}\right)_{\pi(x)} \otimes \mathcal{W}_{D_n} \xrightarrow{\mathbf{d}_{n+1} \otimes \mathrm{id}_{\mathcal{W}_{D_n}}}$$
$$(M \otimes \mathcal{W}_{D^n})_{\pi(x)} \otimes \mathcal{W}_{D_n} \xrightarrow{\widehat{\pi}_{n+1,n} (\nabla_x) \otimes \mathrm{id}_{\mathcal{W}_{D_n}}} (E \otimes \mathcal{W}_{D^n})_x \otimes \mathcal{W}_{D_n}$$

by Proposition 4.2. This is to be supplanted by the composition of mappings

$$\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \xrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{\mathbf{m}_{D_n \times D_n \to D_n}}} \left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)}$$
$$\otimes \mathcal{W}_{D_n} \xrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{+_{D^n \to D_n}} \times \mathrm{id}_{D_n}} (M \otimes \mathcal{W}_{D^n})_{\pi(x)} \otimes \mathcal{W}_{D_n} \xrightarrow{\widehat{\pi}_{n+1,n} (\nabla_x) \otimes \mathrm{id}_{\mathcal{W}_{D_n}}}$$
$$(E \otimes \mathcal{W}_{D^n})_x \otimes \mathcal{W}_{D_n},$$

which is equivalent to the composition of mappings

$$\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \xrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{\mathbf{m}_{D_n \times D_n \to D_n}}} \left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)}$$
$$\otimes \mathcal{W}_{D_n} \xrightarrow{\widehat{\psi}_n \left(\widehat{\pi}_{n+1,n} (\nabla_x)\right) \otimes \mathrm{id}_{\mathcal{W}_{D_n}}} \left(E \otimes \mathcal{W}_{D_n}\right)_x \otimes \mathcal{W}_{D_n} \xrightarrow{\mathrm{id}_E \otimes \mathcal{W}_{+_{D^n \to D_n}} \times \mathrm{id}_{D_n}}$$
$$(E \otimes \mathcal{W}_{D^n})_x \otimes \mathcal{W}_{D_n}$$

This is really equivalent to the composition of mappings

$$\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \xrightarrow{\widehat{\psi}_n\left(\widehat{\pi}_{n+1,n}\left(\nabla_x\right)\right)} \left(E \otimes \mathcal{W}_{D_n}\right)_x \xrightarrow{\mathrm{id}_E \otimes \mathcal{W}_{\mathbf{m}_{D_n \times D_n \to D_n}}} \left(E \otimes \mathcal{W}_{D_n}\right)_x \otimes \mathcal{W}_{D_n} \xrightarrow{\mathrm{id}_E \otimes \mathcal{W}_{+_{D^n \to D_n}} \times \mathrm{id}_{D_n}} E \otimes \mathcal{W}_{D^n \times D_n} \tag{22}$$

This just established fact that the composition of mappings in (21) and that in (22) are equivalent implies the coveted result at once. This completes the proof.

Proposition 7.4 *Let $\mathbb{D}$ be a simplicial infinitesimal space of dimension n and degree m. Let ∇_x be a D^n-pseudotangential over the bundle $\pi : E \to M$ at $x \in E$ and $\gamma \in \left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)}$. Then the composition of mappings*

$$\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \xrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{+_{\mathbb{D} \to D_n}}} (M \otimes \mathcal{W}_{\mathbb{D}})_{\pi(x)} \xrightarrow{\nabla_x^{\mathbb{D}}} (E \otimes \mathcal{W}_{\mathbb{D}})_x$$

is equivalent to the composition of mappings

$$\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \xrightarrow{\widehat{\psi}_n(\nabla_x)} \left(E \otimes \mathcal{W}_{D_n}\right)_x \xrightarrow{\mathrm{id}_E \otimes \mathcal{W}_{+_{\mathbb{D} \to D_n}}} (E \otimes \mathcal{W}_{\mathbb{D}})_x$$

Proof Let $i : D^k \to \mathbb{D}$ be any mapping in the standard quasi-colimit representation of $\mathbb{D}$. The composition of mappings

$$\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \xrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{+_{\mathbb{D} \to D_n}}} (M \otimes \mathcal{W}_{\mathbb{D}})_{\pi(x)} \xrightarrow{\nabla_x^{\mathbb{D}}} (E \otimes \mathcal{W}_{\mathbb{D}})_x \xrightarrow{\mathrm{id}_E \otimes \mathcal{W}_i} \left(E \otimes \mathcal{W}_{D^k}\right)_x \tag{23}$$

is equivalent, by dint of Theorem 4.5, to the composition of mappings

$$\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \xrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{\mathbf{i}_{D_k \to D_n}}} \left(M \otimes \mathcal{W}_{D_k}\right)_{\pi(x)} \xrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{+_{D^k \to D_k}}} \left(M \otimes \mathcal{W}_{D^k}\right)_{\pi(x)} \xrightarrow{\nabla_x^{D^k}} \left(E \otimes \mathcal{W}_{D^k}\right)_x ,$$

which is in turn equivalent, by the very definition of $\widehat{\psi}_k$, to the composition of mappings

$$\left(M \otimes \mathcal{W}_{D_n}\right)_{\pi(x)} \xrightarrow{\mathrm{id}_M \otimes \mathcal{W}_{\mathbf{i}_{D_k \to D_n}}} \left(M \otimes \mathcal{W}_{D_k}\right)_{\pi(x)} \xrightarrow{\widehat{\psi}_k\left(\nabla_x^{D^k}\right)} \left(E \otimes \mathcal{W}_{D_k}\right)_x \xrightarrow{\mathrm{id}_E \otimes \mathcal{W}_{+_{D^k \to D_k}}} \left(E \otimes \mathcal{W}_{D^k}\right)_x .$$

This is indeed equivalent, by dint of Proposition 7.3, to the composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\underrightarrow{\widehat{\psi}_n(\nabla_x)}\left(E\otimes\mathcal{W}_{D_n}\right)_x\underrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{\mathbf{i}_{D_k\to D_n}}}\left(E\otimes\mathcal{W}_{D_k}\right)_x$$
$$\underrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{+_{D^k\to D_k}}}\left(E\otimes\mathcal{W}_{D^k}\right)_x,$$

which is in turn equivalent to the composition of mappings

$$\left(M\otimes\mathcal{W}_{D_n}\right)_{\pi(x)}\underrightarrow{\widehat{\psi}_n(\nabla_x)}\left(E\otimes\mathcal{W}_{D_n}\right)_x\underrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{+_{\mathbb{D}\to D_n}}}(E\otimes\mathcal{W}_{\mathbb{D}})_x$$
$$\underrightarrow{\mathrm{id}_E\otimes\mathcal{W}_i}\left(E\otimes\mathcal{W}_{D^k}\right)_x \tag{24}$$

The just established fact that the composition of mappings in (23) and that in (24) are equivalent implies the coveted result at once. This completes the proof.

Theorem 7.1 *For any* $\nabla_x\in\mathbb{J}_x^{D^n}(\pi)$*, we have* $\widehat{\psi}_n(\nabla_x)\in\mathbb{J}_x^{D_n}(\pi)$.

Proof In view of Proposition 7.2, it suffices to show that $\widehat{\psi}_n(\nabla_x)$ satisfies second the condition in Definition 5.2. Here we deal only with the case that $n=3$ and the simple polynomial ρ at issue is $d\in D_3\longmapsto d^2\in D$, leaving the general case safely to the reader. Since

$$(d_1+d_2+d_3)^2=2(d_1d_2+d_1d_3+d_2d_3)$$

for any $(d_1,d_2,d_3)\in D^3$, we have the commutative diagram

$$\begin{array}{ccc} D^3 & \overset{\chi}{\rightarrow} & D(6)\\ +_{D^3\to D_3}\downarrow & & \downarrow +_{D(6)\to D}\\ D_3 & \underset{\rho}{\rightarrow} & D\end{array} \tag{25}$$

where χ stands for the mapping

$$(d_1,d_2,d_3)\in D^3\mapsto(d_1d_2,d_1d_3,d_2d_3,d_1d_2,d_1d_3,d_2d_3)\in D(6)$$

Then the composition of mappings

$$(M\otimes\mathcal{W}_D)_{\pi(x)}\underrightarrow{\mathrm{id}_M\otimes\mathcal{W}_\rho}\left(M\otimes\mathcal{W}_{D_3}\right)_{\pi(x)}\underrightarrow{\widehat{\psi}_3(\nabla_x)}\left(E\otimes\mathcal{W}_{D_3}\right)_x$$
$$\underrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{+_{D^3\to D_3}}}\left(E\otimes\mathcal{W}_{D^3}\right)_x$$

is equivalent, by the very definition of $\widehat{\psi}_3$, to the composition of mappings

$$(M\otimes\mathcal{W}_D)_{\pi(x)}\underrightarrow{\mathrm{id}_M\otimes\mathcal{W}_\rho}\left(M\otimes\mathcal{W}_{D_3}\right)_{\pi(x)}\underrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{+_{D^3\to D_3}}}\left(M\otimes\mathcal{W}_{D^3}\right)_{\pi(x)}$$
$$\underrightarrow{\nabla_x}\left(E\otimes\mathcal{W}_{D^3}\right)_x$$

which is in turn equivalent to the composition of mappings

$$(M\otimes\mathcal{W}_D)_{\pi(x)}\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{+_{D(6)\to D}}}\left(M\otimes\mathcal{W}_{D(6)}\right)_{\pi(x)}\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_\chi}\left(M\otimes\mathcal{W}_{D^3}\right)_{\pi(x)}$$
$$\xrightarrow{\nabla_x}\left(E\otimes\mathcal{W}_{D^3}\right)_x$$

with due regard to the commutative diagram in (25). By Theorem 4.5, this is equivalent to the composition of mappings

$$(M\otimes\mathcal{W}_D)_{\pi(x)}\xrightarrow{\mathrm{id}_M\otimes\mathcal{W}_{+_{D(6)\to D}}}\left(M\otimes\mathcal{W}_{D(6)}\right)_{\pi(x)}\xrightarrow{\nabla_x^{D(6)}}\left(E\otimes\mathcal{W}_{D(6)}\right)_x$$
$$\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_\chi}\left(E\otimes\mathcal{W}_{D^3}\right)_x$$

which is in turn equivalent by Proposition 7.4 to the composition of mappings

$$(M\otimes\mathcal{W}_D)_{\pi(x)}\xrightarrow{\widehat{\psi}_1(\pi_{3,1}(\nabla_x))}(E\otimes\mathcal{W}_D)_x\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{+_{D(6)\to D}}}\left(E\otimes\mathcal{W}_{D(6)}\right)_x$$
$$\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_\chi}\left(E\otimes\mathcal{W}_{D^3}\right)_x$$

Since

$$\widehat{\psi}_1(\widehat{\pi}_{3,1}(\nabla_x))=\widehat{\pi}_{3,1}(\widehat{\psi}_3(\nabla_x))$$

by Proposition 7.3 and the commutativity of the diagram (25), this is equivalent to the composition of mappings

$$(M\otimes\mathcal{W}_D)_{\pi(x)}\xrightarrow{\pi_{3,1}(\widehat{\psi}_3(\nabla_x))}(E\otimes\mathcal{W}_D)_x\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_\rho}\left(E\otimes\mathcal{W}_{D_3}\right)_x$$
$$\xrightarrow{\mathrm{id}_E\otimes\mathcal{W}_{+_{D^3\to D_3}}}\left(E\otimes\mathcal{W}_{D^3}\right)_x,$$

which completes the proof.

Notation 7.3 *Thus the mapping* $\widehat{\psi}_n\colon\widehat{\mathbb{J}}^{D^n}(\pi)\to\widehat{\mathbb{J}}^{D_n}(\pi)$ *is naturally restricted to a mapping* $\psi_n\colon\mathbb{J}^{D^n}(\pi)\to\mathbb{J}^{D_n}(\pi)$.

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