Graded q-Differential Polynomial Algebra of Connection Form

Viktor Abramov and Olga Liivapuu

Abstract Given a graded associative unital algebra we construct a graded q-differential algebra, where q is a primitive Nth root of unity and prove that the generalized cohomologies of the corresponding N-complex are trivial. We construct a graded q-differential algebra of polynomials and introduce a notion of connection form. We find explicit formula for the curvature of connection form and prove Bianchi identity.

1 Introduction

An idea to generalize the concept of a differential module and to elaborate the corresponding algebraic structures by giving the basic property of differential $d^2 = 0$ a more general form $d^N = 0$, $N \geq 2$ seems to be very natural. Taking the equation $d^N = 0$ as a starting point one should choose a space where a calculus with $d^N = 0$ will be constructed. As a calculus with $d^N = 0$ may be considered as a generalization of $d^2 = 0$ and taking into account that there is an exterior calculus of differential forms with exterior differential $d^2 = 0$ on a smooth manifold one way to construct $d^N = 0$ is to take a smooth manifold and to consider objects on this manifold more general than the differentials forms. Our approach is based on q-deformed structures such as graded q-Leibniz rule, graded q-commutator, graded inner q-derivation, where q is a primitive Nth root of unity [1–6].

A notion of graded q-differential algebra was introduced in [7] (see also in [8–10]) and it may be viewed as a generalization of a graded differential algebra. Let us

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mention that a concept of graded q-differential algebra is closely related to the monoidal structure introduced in [11] for the category of N-complexes and it is proved in [12] that the monoids of the category of N-complexes can be identified as the graded q-differential algebras. It is well known that a connection and its curvature are basic elements of the theory of fiber bundles and they play an important role not only in a modern differential geometry but also in theoretical physics namely in a gauge field theory. A basic algebraic structure used in the theory of connections on modules is a graded differential algebra. A graded differential algebra is an algebraic model for the de Rham algebra of differential forms on a smooth manifold. Consequently considering a concept of graded q-differential algebra which is more general structure than a graded differential algebra we can develop a generalization of the theory of connections on modules. One of the aims of this paper is to present and study algebraic structures based on the relation $d^N = 0$ and to generalize a concept of connection and its curvature applying a concept of graded q-differential algebra to the theory of connections on modules.

In Sect. 2 we prove Theorem 2.1 which is very useful in the sense that we can construct various cochain N-complexes by means of this theorem. Theorem 2.1 asserts if their exist an element v of grading one of a graded associative unital algebra $\mathscr{A}$ which satisfies $v^N \in \mathscr{Z}(\mathscr{A})$, where $\mathscr{Z}(\mathscr{A})$ is the graded center of $\mathscr{A}$, then the inner graded q-derivation ad_v^q is N-differential. Next we prove that the generalized cohomologies of cochain N-complex of Theorem 2.1 are trivial. In Sect. 3 we give the definition of a graded q-differential algebra. We introduce the algebra of polynomials and endow it with the structure of graded q-differential algebra. We introduce two operators D, ∇ and the polynomials f_k, which are defined with the help of recurrent relation. We prove the Theorem 3.2 which give explicit power expansion formulae for the operator D and the polynomials f_k.

2 *N*-Complexes and Cohomologies

A concept of cohomology of a differential module or of a cochain complex with coboundary operator d is based on the quadratic nilpotency condition $d^2 = 0$. It is obvious that one can construct a generalization of a concept of cohomology of a cochain complex if the quadratic nilpotency $d^2 = 0$ is replaced by a more general nilpotency condition $d^N = 0$, where N is an integer satisfying $N \geq 2$.

Let $\mathscr{A} = \oplus_{k\in\mathbb{Z}_N}\mathscr{A}^k = \mathscr{A}^0 \oplus \mathscr{A}^1 \oplus \ldots \oplus \mathscr{A}^{N-1}$ be a $\mathbb{Z}_N$-graded associative unital $\mathbb{C}$-algebra whose identity element is denoted by $\mathbb{1}$. The subspace $\mathscr{A}^0 \subset \mathscr{A}$ of elements of grading zero is the subalgebra of an algebra $\mathscr{A}$. Since this subalgebra plays an important role in several structures related to a graded algebra $\mathscr{A}$ we will denote it by $\mathfrak{A}$, i.e. $\mathfrak{A} \equiv \mathscr{A}^0$. It is easy to see that each subspace $\mathscr{A}^i \subset \mathscr{A}$ of homogeneous elements of grading i is the $\mathfrak{A}$-bimodule. Hence in the case of a graded algebra $\mathscr{A}$ we have the set of $\mathfrak{A}$-bimodules $\mathscr{A}^0, \mathscr{A}^1, \mathscr{A}^2, \ldots, \mathscr{A}^{N-1}$. The graded subspace $\mathscr{Z}(\mathscr{A}) \subset \mathscr{A}$ generated by homogeneous elements $u \in \mathscr{A}^k$, which for any $v \in \mathscr{A}^l$ satisfy $uv = (-1)^{kl}vu$, is called a *graded center* of a graded algebra $\mathscr{A}$.

The derivation of degree m induced by an element $v \in \mathscr{A}^m$ will be denoted by

$$\mathrm{ad}_v(u) = [v, u] = v\,u - (-1)^{ml} uv, \tag{1}$$

where $u \in \mathscr{A}^l$. The graded derivation ad_v is called an *inner graded derivation* of an algebra $\mathscr{A}$.

The notions of graded commutator and graded derivation of a graded algebra can be generalized within the framework of noncommutative geometry and the theory of quantum groups with the help of q-deformations. Let q be a primitive Nth root of unity. The *graded q-commutator* $[\,,\,]_q : \mathscr{A}^k \otimes \mathscr{A}^l \to \mathscr{A}^{k+l}$ is defined by

$$[u, v]_q = uv - q^{kl} vu. \tag{2}$$

A graded q-derivation of degree m of a graded algebra $\mathscr{A}$ is a linear mapping $\delta : \mathscr{A} \to \mathscr{A}$ of degree m with respect to graded structure of $\mathscr{A}$, i.e. $\delta : \mathscr{A}^i \to \mathscr{A}^{i+m}$, which satisfies the graded q-Leibniz rule

$$\delta(uv) = \delta(u)\,v + q^{ml} u\,\delta(v), \tag{3}$$

where u is a homogeneous element of grading l, i.e. $u \in \mathscr{A}^l$. In analogy with an inner graded derivation one defines an inner graded q-derivation of degree m of a graded algebra $\mathscr{A}$ associated to an element $v \in \mathscr{A}^m$ by the formula

$$\mathrm{ad}^q_v(u) = [v, u]_q = vu - q^{ml} uv, \tag{4}$$

where $u \in \mathscr{A}^l$.

A left K-module E is said to be an *N-differential module* if it is equipped with an endomorphism $d : E \to E$ which satisfies $d^N = 0$. An N-differential module E with N-differential d is said to be a *cochain N-complex of modules* or simply *N-complex* if E is a graded module $E = \oplus_{k\in\mathbb{Z}} E^k$ and its N-differential d has degree 1 with respect to a graded structure of E, i.e. $d : E^k \to E^{k+1}$.

We prove the following theorem which can be used to construct a cochain N-complex for a certain class of graded associative unital algebras (see also [8], p. 394).

Theorem 2.1 *Let $\mathscr{A} = \oplus_{k\in\mathbb{Z}_N} \mathscr{A}^k$ be a graded associative unital algebra and q be a primitive Nth root of unity. If there exists an element $v \in \mathscr{A}^1$ of grading one which satisfies the condition $v^N \in \mathscr{Z}(\mathscr{A})$ then the inner graded q-derivation $d = \mathrm{ad}^q_v$ of degree 1 is an N-differential and the sequence*

$$\mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \xrightarrow{d} \mathscr{A}^2 \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{A}^{N-1} \tag{5}$$

is the cochain N-complex.

Proof We begin the proof with a power expansion of d^k, where $1 \le k \le N$. Let u be a homogeneous element of an algebra $\mathscr{A}$ whose grading will be denoted by $|u|$. For the first values of $k = 1, 2, 3$ a straightforward computation gives

$$\begin{aligned} du &= [v, u]_q = vu - q^{|u|}uv, \\ d^2u &= [v, [v, u]_q]_q = v^2u - q^{|u|}[2]_q vuv + q^{2|u|+1}uv^2, \\ d^3u &= v^3u - q^{|u|}[3]_q v^2uv + q^{2|u|+1}[3]_q vuv^2 - q^{3|u|+3}uv^3. \end{aligned}$$

We state that for any $k \in \{1, 2, \dots, N\}$ and any homogeneous $u \in \mathscr{A}$ a power expansion of d^k has the form

$$d^k u = \sum_{i=0}^{k} (-1)^i p_i \begin{bmatrix} k \\ i \end{bmatrix}_q v^{k-i}\, uv^i, \tag{6}$$

where $p_i = q^{i|u|+\sigma(i)}$ and $\sigma(i) = \frac{i(i-1)}{2}$. We proof this statement by means of mathematical induction assuming that the above power expansion (6) for d^k is true and then showing that it has the same form for $k + 1$. Indeed we have

$$\begin{aligned} d^{k+1}u &= d(d^k u) = d\Big(\sum_{i=0}^{k} (-1)^i p_i \begin{bmatrix} k \\ i \end{bmatrix}_q v^{k-i} uv^i\Big) \\ &= \sum_{i=0}^{k} (-1)^i p_i \begin{bmatrix} k \\ i \end{bmatrix}_q (v^{k+1-i}uv^i - q^{|u|+k} v^{k-i} uv^{i+1}) \\ &= \sum_{i=0}^{k} (-1)^i p_i \begin{bmatrix} k \\ i \end{bmatrix}_q v^{k+1-i}uv^i - \sum_{i=0}^{k} (-1)^i q^{|u|+k} p_i \begin{bmatrix} k \\ i \end{bmatrix}_q v^{k-i}uv^{i+1} \\ &= v^{k+1}u + \sum_{i=1}^{k} (-1)^i p_i \begin{bmatrix} k \\ i \end{bmatrix}_q v^{k+1-i}uv^i \\ &\quad - \sum_{i=0}^{k-1} (-1)^i q^{|u|+k} p_i \begin{bmatrix} k \\ i \end{bmatrix}_q v^{k-i}uv^{i+1} - (-1)^k q^{|u|+k} p_k\, uv^{k+1} \\ &= v^{k+1}u + \sum_{i=1}^{k} (-1)^i p_i \begin{bmatrix} k \\ i \end{bmatrix}_q v^{k+1-i}uv^i \\ &\quad + \sum_{i=1}^{k} (-1)^i q^{|u|+k} p_{i-1} \begin{bmatrix} k \\ i-1 \end{bmatrix}_q v^{k-i}uv^{i+1} + (-1)^{k+1} q^{|u|+k} p_k\, uv^{k+1} \\ &= v^{k+1}u + \sum_{i=1}^{k} (-1)^i \Big(p_i \begin{bmatrix} k \\ i \end{bmatrix}_q + q^{|u|+k} p_{i-1} \begin{bmatrix} k \\ i-1 \end{bmatrix}_q\Big) v^{k+1-i}uv^i \\ &\quad + (-1)^{k+1} q^{|u|+k} p_k\, uv^{k+1}. \end{aligned}$$

Now the coefficients in the last expansion we can write as follows

$$p_i \begin{bmatrix} k \\ i \end{bmatrix}_q + q^{|u|+k} p_{i-1} \begin{bmatrix} k \\ i-1 \end{bmatrix}_q = p_i \Big(\begin{bmatrix} k \\ i \end{bmatrix}_q + q^{k+\sigma(i-1)-\sigma(i)} \begin{bmatrix} k \\ i-1 \end{bmatrix}_q \Big),$$

and making use of

$$\sigma(i-1) - \sigma(i) = \frac{(i-1)(i-2)}{2} - \frac{i(i-1)}{2} = 1 - i$$

and making use of well known recurrent relation for q-binomial coefficients we get

$$\begin{bmatrix} k \\ i \end{bmatrix}_q + q^{k+1-i} \begin{bmatrix} k \\ i-1 \end{bmatrix}_q = \begin{bmatrix} k+1 \\ i \end{bmatrix}_q .$$

As $p_{k+1} = q^{|u|+k} p_k$ we finally obtain

$$d^{k+1} u = \sum_{i=0}^{k+1} (-1)^i p_i \begin{bmatrix} k+1 \\ i \end{bmatrix}_q v^{k-i} u v^i,$$

and this ends the proof of the formula for power expansion of d^k.

Now our aim is to show that the power expansion (6) implies $d^N u = 0$ for any $u \in \mathscr{A}$. Indeed making use of (6) we can express the Nth power of d as follows

$$d^N u = \sum_{i=0}^{N} (-1)^i p_i \begin{bmatrix} N \\ i \end{bmatrix}_q v^{k-i} u v^i. \tag{7}$$

Taking into account that q is a primitive Nth root of unity we get

$$\begin{bmatrix} N \\ i \end{bmatrix}_q = 0, \quad i \in \{1, 2, \ldots, N-1\}.$$

Hence the terms in (7), which are numbered with $i = 1, 2, \ldots, N-1$, vanish, and we are left with two terms

$$d^N u = v^N u + (-1)^N q^{\sigma(N)} u v^N.$$

As v^N is the element of grading zero (modulo N) of the graded center $\mathscr{Z}(\mathscr{A})$ we can rewrite the above formula as follows

$$d^N u = (1 + (-1)^N q^{\sigma(N)})\, u v^N, \quad \sigma(N) = \frac{N(N-1)}{2}.$$

In order to show that the multiplier in the above formula vanish for any $N \geq 2$ we consider separately two cases for N to be an odd or even positive integer. If N is an odd positive integer then the multiplier $1 + (-1)^N q^{\sigma(N)}$ vanish because in this case

$$1 + (-1)^N q^{\sigma(N)} = 1 - (q^N)^{\frac{N-1}{2}} = 0.$$

If N is an even positive integer then

$$1 + (-1)^N q^{\sigma(N)} = 1 + (q^{\frac{N}{2}})^{N-1} = 1 + (-1)^{N-1} = 0.$$

Hence for any $N \geq 2$ we have $d^N = 0$, and this ends the proof of the theorem. □

Let us fix a positive integer $m \in \{1, 2, \ldots, N-1\}$ and split up the Nth power of N-differential d as follows $d^N = d^m \circ d^{N-m}$. Then the nilpotency condition for N-differential can be written in the form $d^N = d^m \circ d^{N-m} = 0$ and this leads to possible generalization of a concept of cohomology. For each integer $1 \leq m \leq N-1$ one can define the submodules

$$Z_m(E) = \{x \in E : d^m x = 0\} \subset E, \tag{8}$$

$$B_m(E) = \{x \in E : \exists y \in E,\ x = d^{N-m} y\} \subset E. \tag{9}$$

From $d^N = 0$ it follows that $B_m(E) \subset Z_m(E)$. For each $m \in \{1, 2, \ldots, N-1\}$ the quotient module $H_m(E) := Z_m(E)/B_m(E)$ is said to be a *generalized homology of order m* of N-differential module E. The following lemma which is proved in [8] gives a very useful criteria for the triviality of the generalized cohomologies of an N-differential module.

Lemma 2.1 *Let E be an N-differential module over a ring $\mathbf{k}$, $N \geq 2$ be an integer and q be an element of $\mathbf{k}$ satisfying the conditions $[N]_q = 0$ and $[n]_q$ is invertible for any integer $1 \leq n \leq N-1$. If there is a module-endomorphism $h : E \to E$ satisfying $h \circ d - q\, d \circ h = Id_E$ then the generalized cohomologies of an N-differential module E are trivial, i.e. for any integer $1 \leq n \leq N-1$ it holds $H_n(E) = 0$.*

Based on this lemma we can prove that the generalized cohomologies of the cochain N-complex described in Theorem 2.1 are trivial. It is worth mentioning that the same argument is used in [10] to show that the generalized cohomologies of the N-differential module constructed by means of the algebra of $N \times N$-matrices $M_N(\mathbf{k})$ are trivial.

Theorem 2.2 *Let q be a primitive Nth root of unity, $\mathscr{A} = \oplus_{i \in \mathbb{Z}_N} \mathscr{A}^i$ be a graded associative unital algebra with an element $v \in \mathscr{A}^1$ satisfying $v^N = \lambda \mathbb{1}$, where $\lambda \neq 0$. Then the generalized cohomologies $H_n(\mathscr{A})$ of the cochain N-complex of Theorem 2.1*

$$\mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \xrightarrow{d} \mathscr{A}^2 \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{A}^{N-1} \tag{10}$$

with N-differential $d = \mathrm{ad}_v^q$, induced by an element v, are trivial, i.e. for any $n \in \{1, 2, \ldots, N-1\}$ we have $H_n(\mathscr{A}) = 0$.

Proof Let us define the endomorphism h of the vector space of $\mathscr{A}$ as follows

$$h(u) = \frac{1}{(1-q)\,\lambda}\, v^{N-1}\, u,$$

where u is an element of an algebra $\mathscr{A}$. If u is a homogeneous element of a graded algebra $\mathscr{A}$ then $|h(u)| = |u| + N - 1$, where $|u|$ is the grading of an element u. For any homogeneous $u \in \mathscr{A}$ we have

$$\begin{aligned}
(h \circ d - q\, d \circ h)(u) &= h(du) - q\, d(h(u)) \\
&= h(\mathrm{ad}_v^q(u)) - \frac{q}{(1-q)\,\lambda}\, \mathrm{ad}_v^q(v^{N-1}u) \\
&= h([v, u]_q) - \frac{q}{(1-q)\,\lambda}\, [v, v^{N-1}]_q \\
&= h\,(v\,u - q^{|u|}uv) - \frac{q}{(1-q)\,\lambda}\, (v^N u - q^{|u|+N-1} v^{N-1} u\,v) \\
&= \frac{1}{(1-q)\,\lambda}\, v^N u - \frac{q^{|u|}}{(1-q)\,\lambda} v^{N-1} u\,v \\
&\quad - \frac{q}{(1-q)\,\lambda}\, v^N u + \frac{q^{|u|}}{(1-q)\,\lambda} v^{N-1} u\,v \\
&= \frac{(1-q)\,\lambda}{(1-q)\,\lambda}\, u = \mathrm{Id}_{\mathscr{A}}(u).
\end{aligned}$$

The endomorphism $h : \mathscr{A} \to \mathscr{A}$ of the vector space of an algebra $\mathscr{A}$ satisfies $h \circ d - q\, d \circ h = \mathrm{Id}_{\mathscr{A}}$ and it follows from Lemma 2.1 that the generalized cohomology of the cochain N-complex

$$\mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \xrightarrow{d} \mathscr{A}^2 \xrightarrow{d} \dots \xrightarrow{d} \mathscr{A}^{N-1}$$

are trivial. □

3 Graded q-Differential Algebras

In this section we use the cochain N-complex described in the Theorem 2.1 to construct a graded q-differential algebra which can be viewed as a natural generalization of the notion of graded differential. Then we will describe a graded q-differential polynomial algebra which arises in relation with a connection form which can be viewed as analog of connection form in a graded differential algebra introduced by Quillen in [13].

Definition 3.1 *A graded q-differential algebra* is a graded associative unital algebra $\mathscr{A} = \oplus_{k\in\mathbb{Z}} \mathscr{A}^k$ endowed with a linear mapping d of degree one such that the sequence

$$\cdots \xrightarrow{d} \mathscr{A}^{k-1} \xrightarrow{d} \mathscr{A}^{k} \xrightarrow{d} \mathscr{A}^{k+1} \xrightarrow{d} \cdots$$

is an N-complex with N-differential d satisfying the graded q-Leibniz rule

$$d(u\,v) = d(u)v + q^k u d(v), \tag{11}$$

where $u \in \mathscr{A}^k$, $v \in \mathscr{A}$.

It follows from Theorem 2.1

Theorem 3.1 *Let $\mathscr{A}$ be a graded associative unital algebra $\mathscr{A} = \oplus_k \mathscr{A}^k$, and q be a primitive Nth root of unity. If there exists an element of grading one $v \in \mathscr{A}^1$ which satisfies the condition $v^N \in \mathscr{Z}(\mathscr{A})$, where $\mathscr{Z}(\mathscr{A})$ is the graded center of $\mathscr{A}$, then the graded algebra $\mathscr{A}$ endowed with the inner graded q-derivation $d = \mathrm{ad}_v^q$ is a graded q-differential algebra (d is its N-differential).*

Indeed we can prove this theorem by taking into account that an inner graded q-derivation satisfies the graded q-Leibniz rule (3) and the inner graded q-derivation $d = \mathrm{ad}_v^q$, induced by an element of grading one $v \in \mathscr{A}^1$ such that $v^N \in \mathscr{Z}(\mathscr{A})$, is the N-differential of the cochain complex (Theorem 2.1)

$$\cdots \xrightarrow{d} \mathscr{A}^{k-1} \xrightarrow{d} \mathscr{A}^{k} \xrightarrow{d} \mathscr{A}^{k+1} \xrightarrow{d} \cdots .$$

Now we introduce a graded q-differential algebra of polynomials which arises in relation with an algebraic model of a connection form and this algebraic model is based on exterior calculus with differential satisfying $d^N = 0$. This algebra will be used in the next section in order to calculate the curvature of a connection form.

Let $\mathbb{N}_1 = \{i \in \mathbb{Z} : i \geq 1\}$ be the set of integers greater than or equal to one and $\{\mathfrak{d}, a_i\}_{i \in \mathbb{N}_1}$ be a set of variables. We consider the algebra of noncommutative polynomials $\mathfrak{P}_q[\mathfrak{d}, a]$ over $\mathbb{C}$ generated by the variables $\{\mathfrak{d}, a_i\}_{i \in \mathbb{N}_1}$ which are subjected to the commutation relations

$$\mathfrak{d} a_i = q^i\, a_i \mathfrak{d} + a_{i+1}, \qquad \forall i \in \mathbb{N}_1 \tag{12}$$

where q is any complex number different from zero. We denote the identity element of this algebra by $\not\Vdash$. Obviously we can split up the set of variables of the algebra $\mathfrak{P}_q[\mathfrak{d}, a]$ into two subsets $\{\mathfrak{d}\}$, $\{a_i\}_{i \in \mathbb{N}_1}$ which generate respectively the subalgebras $\mathfrak{P}_q[\mathfrak{d}] \subset \mathfrak{P}_q[\mathfrak{d}, a]$ and $\mathfrak{P}_q[a] \subset \mathfrak{P}_q[\mathfrak{d}, a]$. Hence the subalgebra $\mathfrak{P}_q[\mathfrak{d}]$ is generated by a single variable $\mathfrak{d}$, and the subalgebra $\mathfrak{P}_q[a]$ is freely generated by the variables $\{a_i\}_{i \in \mathbb{N}_1}$ because we do not assume any relation between variables a_i.

Now our aim is to equip the algebra of polynomials $\mathfrak{P}_q[\mathfrak{d}, a]$ with a graded structure so that $\mathfrak{P}_q[\mathfrak{d}, a]$ will become a graded algebra. This can be done as follows: we assign grading zero to the identity element $\mathbb{1}$ of the algebra $\mathfrak{P}_q[\mathfrak{d}, a]$, grading one to the generator $\mathfrak{d}$ and grading i to a generator a_i, where $i \in \mathbb{N}_1$. Thus making use of previously defined notations we can describe the graded structure of generators of $\mathfrak{P}_q[\mathfrak{d}, a]$ by the formulae

$$|\mathbb{1}| = 0, \quad |\mathfrak{d}| = |a_1| = 1, \quad |a_i| = i, \quad i \geq 2. \tag{13}$$

As usual we extend this graded structure to the whole algebra $\mathfrak{P}_q[\mathfrak{d}, a]$ by defining the grading of any product of variables $\{\mathfrak{d}, a_i\}_{i\in\mathbb{N}_1}$ as the sum of gradings of its factors. It is easy to see that the algebra of polynomials $\mathfrak{P}_q[\mathfrak{d}, a]$ becomes the positively graded algebra. Hence we can write

$$\mathfrak{P}_q[\mathfrak{d}, a] = \oplus_{k\in\mathbb{N}}\mathfrak{P}_q^k[\mathfrak{d}, a],$$

where $\mathfrak{P}_q^k[\mathfrak{d}, a]$ is the subspace of homogeneous polynomials of grading k. It should be mentioned that the graded structure of $\mathfrak{P}[\mathfrak{d}, a]$ induces the graded structures of the subalgebras $\mathfrak{P}_q[\mathfrak{d}]$, $\mathfrak{P}_q[a]$ which are positively graded algebras as well. Clearly the positively graded algebra $\mathfrak{P}_q[\mathfrak{d}, a]$ becomes the $\mathbb{Z}_N$-graded algebra, where N any integer greater than 1, if we slightly modify the above described gradation by taking all gradings modulo N. Let us denote by Lin $\mathfrak{P}_q[a]$ the algebra of $\mathbb{C}$-endomorphisms of vector space of $\mathfrak{P}_q[a]$. Obviously Lin $\mathfrak{P}_q[a]$ is a graded algebra with gradation induced by the gradation of $\mathfrak{P}_q[a]$. Having defined the positively graded structure of the algebra $\mathfrak{P}_q[\mathfrak{d}, a]$ we can apply the notions of graded commutator and inner graded q-derivation described in the previous chapter to study the structure of $\mathfrak{P}_q[\mathfrak{d}, a]$. First of all we observe that the commutation relations (12) can be written by means of graded commutator and inner graded q-derivation in the form

$$[\mathfrak{d}, a_i]_q = a_{i+1}, \qquad \text{or} \qquad \mathrm{ad}_{\mathfrak{d}}^q(a_i) = a_{i+1}, \tag{14}$$

where $i \in \mathbb{N}_1$. This form of commutation relations suggests us to consider the inner graded q-derivation $\mathrm{ad}_{\mathfrak{d}}^q$ of the algebra $\mathfrak{P}_q[\mathfrak{d}, a]$ associated with a variable $\mathfrak{d}$. If we restrict $\mathrm{ad}_{\mathfrak{d}}^q$ to the subalgebra $\mathfrak{P}_q[a]$ we get the graded q-derivation of subalgebra $\mathfrak{P}_q[a]$ which we will denote by d, i.e.

$$d := \mathrm{ad}_{\mathfrak{d}}^q \,|_{\mathfrak{P}_q[a]}, \qquad d : \mathfrak{P}_q[a] \to \mathfrak{P}_q[a]. \tag{15}$$

Obviously d is a graded q-derivation of grading one of the $\mathbb{Z}_N$-graded algebra $\mathfrak{P}_q[a]$. From the commutation relations (14) it follows that

$$d(\mathbb{1}) = 0, \quad d(a_i) = a_{i+1},$$

for any $i \geq 1$. Let us define $D, \nabla \in \mathrm{Lin}\,\mathfrak{P}_q[a]$ of grading one and the polynomials $f_k \in \mathfrak{P}_q[a]$, where k is an integer greater than or equal to zero, by the formulae

$$D(P) = d(P) + a_1\, P, \tag{16}$$

$$\nabla(P) = d(P) + [a_1, P]_q, \tag{17}$$

$$f_0 = \mathbb{1},$$

$$f_1 = a_1,$$

$$f_k = D(f_{k-1}), \tag{18}$$

where $P \in \mathfrak{P}_q[a]$ is a homogeneous polynomial. We can write the linear mapping ∇ in the form $\nabla = \mathrm{ad}^q_{\mathfrak{d}+a_1}$ which clearly shows that ∇ is a inner graded q-derivation of the algebra $\mathfrak{P}_q[a]$. Hence for any polynomials $P, Q \in \mathfrak{P}_q[a]$, where P is homogeneous, it holds

$$D(PQ) = D(P)\,Q + q^{|P|} P\,d(Q), \tag{19}$$

$$\nabla(PQ) = \nabla(P)\; + q^{|P|} P\,\nabla(Q). \tag{20}$$

For the first values of k we calculate by means of the recurrent relation (18)

$$\begin{aligned}
f_2 &= a_2 + a_1^2,\\
f_3 &= a_3 + a_2\,a_1 + [2]_q\,a_1\,a_2 + a_1^3,\\
f_4 &= a_4 + a_3\,a_1 + [3]_q\,a_1 a_3 + [3]_q\,a_2^2\\
&\quad + a_2 a_1^2 + [3]_q\,a_1^2 a_2 + [2]_q\,a_1 a_2 a_1 + a_1^4,\\
f_5 &= a_5 + a_4 a_1 + [4]_q\,a_1 a_4 + [4]_q\,a_3 a_2\\
&\quad + \begin{bmatrix}4\\2\end{bmatrix}_q a_2 a_3 + a_3 a_1^2 + [3]_q\,a_2^2 a_1 + [4]_q\,a_2 a_1 a_2\\
&\quad + [2]_q\,[4]_q\,a_1 a_2^2 + \begin{bmatrix}4\\2\end{bmatrix}_q a_1^2 a_3 + [3]_q\,a_1 a_3 a_1\\
&\quad + [2]_q\,a_1 a_2 a_1^2 + [3]_q\,a_1^2 a_2 a_1 + a_2 a_1^3 + [4]_q\,a_1^3 a_2 + a_1^5.
\end{aligned} \tag{21}$$

Getting a bit ahead we would like to point out that the polynomials f_k may be interpreted as the curvature of a connection if we view the generator a_1 as an algebraic model for a connection one form. Let us remind that if k is a positive integer then a composition of k is a representation of k as the sum of a sequence of strictly positive integers, and two sequences that differ in the order of their terms give different compositions of their sum while they define the same partition of k. For example if $k = 3$ then there are 4 compositions

$$3 = 3,\; 3 = 2 + 1,\; 3 = 1 + 2,\; 3 = 1 + 1 + 1.$$

Let Ψ_k be the set of all compositions of an integer k. We will write a composition of an integer k in the form of a sequence of strictly positive integers $\sigma = (i_1, i_2, \ldots, i_r)$, where $i_1 + i_2 + \cdots + i_r = k$. Let us denote

$$\begin{aligned}
k_1 &= i_1,\\
k_2 &= i_1 + i_2,\\
k_3 &= i_1 + i_2 + i_3,\\
&\ldots
\end{aligned}$$

$$k_{r-1} = i_1 + i_2 + \cdots + i_{r-1}.$$

It can be proved [14] that the number of all possible compositions of a positive integer k is 2^{k-1}, i.e. the set Ψ_k contains 2^{k-1} elements. The following theorem gives an explicit formula for the polynomials f_k:

Theorem 3.2 *For any integer $k \geq 2$ we have the following expansion of power of the operator D and the expansion of a polynomial f_k in terms of generators a_i:*

$$D^k = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q f_i\, d^{k-i},$$
$$f_k = \sum_{\sigma\in\Psi_k} \begin{bmatrix} k_2-1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} k_3-1 \\ k_2 \end{bmatrix}_q \cdots \begin{bmatrix} k-1 \\ k_{r-1} \end{bmatrix}_q a_{i_1} a_{i_2} \ldots a_{i_r},$$

where $\sigma = (i_1, i_2, \ldots, i_r)$ is a composition of an integer k.

Proof We will prove the expansion formulae of this theorem by the method of mathematical induction. In order to prove the expansion of power of the operator D by means of mathematical induction we begin with the base case and show that this formula holds when k is equal to 1. This is true because

$$D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q f_0\, d + \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q f_1 = d + a_1.$$

Next step in the proof is an inductive step, i.e. we assume that the expansion formula holds for some integer $k > 1$ and show that it also holds when $k+1$ is substituted for k. Indeed we have

$$\begin{aligned}
D^{k+1} &= D(D^k) = D\Big(\sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q f_i\, d^{k-i}\Big) \\
&= \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q \Big(D(f_i)\, d^{k-i} + q^i f_i d^{k+1-i}\Big) \\
&= \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q \Big(f_{i+1}\, d^{k-i} + q^i f_i d^{k+1-i}\Big) \\
&= f_{k+1} + \sum_{i=0}^{k-1} \begin{bmatrix} k \\ i \end{bmatrix}_q f_{i+1}\, d^{k-i} + q^i \sum_{i=1}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q f_i\, d^{k+1-i} + d^{k+1} \\
&= f_{k+1} + \sum_{i=1}^{k} \begin{bmatrix} k \\ i-1 \end{bmatrix}_q f_i\, d^{k+1-i} + q^i \sum_{i=1}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q f_i\, d^{k+1-i} + d^{k+1}
\end{aligned}$$

$$= f_{k+1} + \sum_{i=1}^{k}\Big(\begin{bmatrix} k \\ i-1 \end{bmatrix}_q + q^i \begin{bmatrix} k \\ i \end{bmatrix}_q\Big) f_i\, d^{k+1-i} + d^{k+1}$$
$$= f_{k+1} + \sum_{i=1}^{k} \begin{bmatrix} k+1 \\ i \end{bmatrix}_q f_i\, d^{k+1-i} + d^{k+1}$$
$$= \sum_{i=0}^{k+1} \begin{bmatrix} k+1 \\ i \end{bmatrix}_q f_i\, d^{k+1-i}.$$

Thus the expansion of power of the operator D is proved. Now if we apply the both sides of the proved formula to a_1 we obtain

$$f_{k+1} = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q f_i\, a_{k+1-i}, \tag{22}$$

and this is the recurrent formula for the polynomials f_k which we will use in the second part of the present proof in order to prove the expansion formula for f_k.

We start the proof of the expansion formula for a polynomial f_k with the base case when $k = 2$. In this case there are two compositions $2 = 2,\ 2 = 1 + 1$. Hence we have

$$f_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q a_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q a_1^2 = a_2 + a_1^2.$$

Comparing this result with the first formula in (21) we see that in the case when $k = 2$ the expansion formula for f_k is correct. The next step is an inductive step, i.e. we assume that the expansion formula holds for some positive integer $k > 2$ and show that it also holds when $k + 1$ is substituted for k. Let us consider the sum

$$\sum_{\sigma\in\Psi_{k+1}} \begin{bmatrix} k_2-1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} k_3-1 \\ k_2 \end{bmatrix}_q \cdots \begin{bmatrix} k \\ k_r \end{bmatrix}_q a_{i_1} a_{i_2} \ldots a_{i_{r+1}}, \tag{23}$$

where $\sigma = (i_1, i_2, \ldots, i_r, i_{r+1})$ is a composition of an integer $k+1$. Hence $i_1 + \cdots + i_r + i_{r+1} = k+1$. Our aim is to show that this sum is equal to the polynomial f_{k+1}. Let us fix an integer $i \in \{0, 1, \ldots, k\}$ and a generator a_{k+1-i}. It is clear that if we select the compositions of an integer $k+1$ which have the form $(i_1, i_2, \ldots, i_r, k+1-i)$, i.e. the last integer of each composition is previously fixed integer $k+1-i$, and we remove in each composition the last integer then the set of compositions $(i_1, i_2, \ldots, i_r)$ is the set of all compositions of an integer i, i.e. $\{(i_1, i_2, \ldots, i_r)\} = \Psi_i$. Indeed we have

$$i_1 + i_2 + \cdots + i_r + k + 1 - i = k + 1,$$

which implies $i_1 + i_2 + \cdots + i_r = i$. Consequently if we select in the sum (23) all terms with $i_{r+1} = k+1-i$ (i.e. containing a generator a_{k+1-i} at the end of a product of generators) then we get the sum

$$\sum_{\sigma\in\Psi_{k+1}} \begin{bmatrix} k_2-1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} k_3-1 \\ k_2 \end{bmatrix}_q \dots \begin{bmatrix} k \\ i \end{bmatrix}_q a_{i_1}a_{i_2}\dots a_{i_r}a_{k+1-i}, \tag{24}$$

where the sum is taken over the compositions of integer $k+1$ which have the form $\sigma = (i_1, i_2, \dots, i_r, k+1-i) \in \Psi_{k+1}$. We would like to point out that the product of binomial coefficients of each term in this sum contains the factor

$$\begin{bmatrix} k \\ i \end{bmatrix}_q .$$

Hence we can write the sum (24) as follows

$$\begin{bmatrix} k \\ i \end{bmatrix}_q \Big(\sum_{\tau\in\Psi_i} \begin{bmatrix} k_2-1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} k_3-1 \\ k_2 \end{bmatrix}_q \dots \begin{bmatrix} i-1 \\ k_{r-1} \end{bmatrix}_q a_{i_1}a_{i_2}\dots a_{i_r}\Big)\, a_{k+1-i},$$

where $\tau = (i_1, i_2, \dots, i_r) \in \Psi_i$ and the sum is taken over all compositions of integer i. Now we make use of the assumption of an inductive step that the expansion formula for a polynomial f_m holds for each integer $m \in \{1, 2, \dots, k\}$. Hence the sum in the previous formula is equal to f_i, i.e

$$\sum_{\tau\in\Psi_i} \begin{bmatrix} k_2-1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} k_3-1 \\ k_2 \end{bmatrix}_q \dots \begin{bmatrix} i-1 \\ k_{r-1} \end{bmatrix}_q a_{i_1}a_{i_2}\dots a_{i_r} = f_i.$$

Thus the sum (24) is equal to

$$\begin{bmatrix} k \\ i \end{bmatrix}_q f_i\, a_{k+1-i},$$

and summing up all these terms with respect to i we get the sum (23). Consequently the sum (23) we started with is equal to the sum

$$\sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q f_i\, a_{k+1-i},$$

which in turn is equal to f_{k+1} (see the recurrent relation (22)). This ends the proof. □

We remind a reader that the parameter q which plays an important role in the structure of the algebra $\mathfrak{P}_q[\mathfrak{d}, a]$ is any complex number different from zero. Now we will study the structure of the algebra of polynomials $\mathfrak{P}_q[\mathfrak{d}, a]$ at a primitive Nth root of unity, i.e. we assume q to be a primitive Nth root of unity. We may expect that in this case the infinite set of variables $\{\mathfrak{d}, a_1, a_2, \dots\}$ is "cut off" and we get an algebra whose vector space is finite dimensional. Indeed we can prove the following proposition:

Proposition 3.1 *Let* $\mathfrak{P}_q[\mathfrak{d}, a]$ *be the algebra of polynomials generated by the set of variables* $\{\mathfrak{d}, a_i\}_{i\in\mathbb{N}_1}$ *which obey the commutation relations (12). If we assume that* q *is a primitive* N*th root of unity and the variable* $\mathfrak{d}$ *is subjected to the additional relation* $\mathfrak{d}^N = \lambda \cdot \mathbb{1}$*, where* λ *is a complex number, then for any integer* $k > N$ *a variable* a_k *vanishes, i.e. the algebra* $\mathfrak{P}_q[\mathfrak{d}, a]$ *is generated by the finite set of variables* $\{\mathfrak{d}, a_k\}_{k=1}^N$ *which obey the relations*

$$\begin{aligned} \mathfrak{d}a_1 &= q\, a_1\mathfrak{d} + a_2, \\ \mathfrak{d}a_2 &= q^2\, a_2\mathfrak{d} + a_3, \\ &\cdots \\ \mathfrak{d}a_{N-1} &= q^{N-1}\, a_{N-1}\mathfrak{d} + a_N, \\ \mathfrak{d}a_N &= a_N\mathfrak{d}, \\ \mathfrak{d}^N &= \lambda\cdot\mathbb{1}. \end{aligned} \tag{25}$$

The graded q*-derivation* $d = \mathrm{ad}_{\mathfrak{d}}^q : \mathfrak{P}_q[a] \to \mathfrak{P}_q[a]$ *associated to variable* $\mathfrak{d}$ *is an* N*-differential, i.e.* $d^N = 0$*, and the sequence*

$$\cdots \xrightarrow{d} \mathfrak{P}_q^{i-1}[a] \xrightarrow{d} \mathfrak{P}_q^{i}[a] \xrightarrow{d} \mathfrak{P}_q^{i+1}[a] \xrightarrow{d} \cdots$$

is a cochain N*-complex. The graded algebra* $\mathfrak{P}_q[a]$ *equipped with the* N*-differential* d *is a graded* q*-differential algebra.*

Proof We suppose that the algebra of polynomials is equipped with the $\mathbb{Z}_N$-gradation as it was explained earlier (13). It easily follows from the commutation relations of the algebra $\mathfrak{P}_q[\mathfrak{d}, a]$ that for any integer $k \geq 2$ we have

$$a_{k+1} = d^k(a_1),$$

where $d = \mathrm{ad}_{\mathfrak{d}}^q$ is the graded q-derivation associated with a variable $\mathfrak{d}$. Making use of the expansion of power of graded q-derivation used in the proof of Theorem 2.1 we obtain

$$a_{k+1} = d^k(a_1) = (\mathrm{ad}_{\mathfrak{d}}^q)^k(a_1) = \sum_{i=0}^{k} (-1)^i\, p_i \begin{bmatrix} k \\ i \end{bmatrix}_q \mathfrak{d}^{k-i}\, u\, \mathfrak{d}^i.$$

Consequently if q is a primitive Nth root of unity, $\mathfrak{d}$ satisfies $\mathfrak{d}^N = \lambda \cdot \mathbb{1}$ and $k = N$ then making use of the same arguments as in the proof of Theorem 2.1) we conclude that all terms of the sum at the right-hand side of the above expansion formula vanish. Consequently we have $a_{N+1} = a_{N+2} = \ldots = 0$ and this ends the proof. □

It is well known that locally a connection of a vector bundle can be described with the help of matrix-valued 1-form. From an algebraic point of view this matrix-valued 1-form is an element of degree one of differential algebra of matrix-valued differential forms, where differential is identified with exterior differential and graduation is induced by degree of differential form. Hence an algebraic model for a connection can be constructed if we take a differential algebra $\mathscr{A}$ (over $\mathbb{C}$) and consider an element of degree one of this algebra A calling it connection form. Then a covariant differential induced by this connection form is the operator $\nabla = d + A$, and the curvature is the element of degree 2 given by $F = dA + A^2 = dA + \frac{1}{2}[A, A]$, where $[,]$ is the graded commutator of $\mathscr{A}$. This approach was proposed by Quillen in [13]. Following this approach we introduce a notion of N-connection form which particularly gives a connection form if $N = 2$. Let us denote by $\mathfrak{P}_q[\mathfrak{d}, a]$ the finite dimensional graded algebra generated by $\{\mathfrak{d}, a_k\}_{k=1}^N$ which obey relations (26) and by $\mathfrak{P}_q[a]$ the graded q-differential algebra generated by $\{a_k\}_{k=1}^N$ with N-differential d. Now we give the following definition:

Definition 3.2 The generator a_1 of $\mathbb{Z}_N$-graded q-differential algebra $\mathfrak{P}_q[a]$ will be referred to as an N-connection form and the algebra $\mathfrak{P}_q[a]$ will be referred to as an algebra of N-connection form. The operator $D = d + a_1 : \mathfrak{P}_q[a] \to \mathfrak{P}_q[a]$ will be called a covariant N-differential, and the polynomial f_N, whose explicit power expansion formula given in (3.2), will be called the curvature of N-connection form a_1.

Proposition 3.2 *If $\mathfrak{P}_q[a]$ is the algebra of N-connection form and d is its N-differential then the Nth power of the covariant N-differential D is the operator of multiplication by the curvature of N-connection form f_N.*

Proof The proof of this proposition is based on the first expansion formula proved in the Theorem 3.2. Indeed we can expand an Nth power of the covariant N-differential D into the sum of products of polynomials f_i and the powers of the N-differential d as follows

$$D^N = \sum_{i=0}^{N} \begin{bmatrix} N \\ i \end{bmatrix}_q f_i \, d^{N-i}.$$

As q is a primitive Nth root of unity this expansion can be essentially simplified in the case $k = N$ if we take into account that all q-binomial coefficients with $i \in \{1, 2, \dots, N-1\}$ vanish. The first term of this expansion also vanishes because d is the N-differential. Hence for any polynomial $P \in \mathfrak{P}_q[a]$ we have

$$D^N(P) = f_N \cdot P,$$

and this ends the proof. □

Proposition 3.3 *If $\mathfrak{P}_q[a]$ is the algebra of connection form and f_N is the curvature of connection form then the curvature satisfies the identity*

$$\nabla(f_N) = 0. \tag{26}$$

Proof Let us remind a reader that $\nabla = d + \mathrm{ad}^q_{a_1}$. We prove this proposition by means of the recurrent relation for polynomials f_k

$$f_{k+1} = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q f_i\, a_{k+1-i}.$$

Substituting N for k in the above relation we obtain

$$f_{N+1} = \sum_{i=0}^{N} \begin{bmatrix} N \\ i \end{bmatrix}_q f_i\, a_{N+1-i}. \tag{27}$$

As q is a primitive Nth root of unity we have

$$\begin{bmatrix} N \\ i \end{bmatrix}_q = 0,$$

for any integer $i \in \{1, 2, \ldots, N-1\}$. Consequently there are only two terms with non-zero q-binomial coefficients (labeled by $i = 0, N$) at the right-hand side of the relation (27) and

$$f_{N+1} = f_0\, a_{N+1} + f_N\, a_1.$$

The first term at the right-hand side of the above formula is also zero because of $a_{N+1} = 0$ (Proposition 3.1). Hence

$$\begin{aligned} 0 &= f_{N+1} - f_N\, a_1 = D(f_N) - f_N\, a_1 \\ &= d(f_N) + a_1 f_N - f_N\, a_1 = d(f_N) + [a_1, f_N]_q = (d + \mathrm{ad}^q_{a_1})(f_N) = \nabla(f_N). \end{aligned}$$

□

The identity (26) is an analogue of Bianchi identity for the curvature of N-connection form. It is worth mentioning that we can write the Bianchi identity for the curvature of N-connection form (26) in a different way if we consider the covariant N-differential D and the curvature f_N as the linear operators $D, f_N : \mathfrak{P}_q[a] \to \mathfrak{P}_q[a]$, i.e. $D, f_N \in \mathrm{Lin}\,\mathfrak{P}_q[a]$, where f_N is the operator of multiplication by f_N (we denote it by the same symbol as the curvature f_N in order not to make the notations very complicated). Then the Bianchi identity may be written in the form

$$[D, f_N]_q = 0.$$

Indeed

$$\begin{aligned}[D, f_N]_q &= D \circ f_N - f_N \circ D \\ &= d(f_N) + f_N \circ d + a_1 f_N - f_N \circ d - f_N\, a_1 \\ &= d(f_N) + [a_1, f_N]_q = \nabla(f_N) = 0.\end{aligned}$$

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