Quantized Reduced Fusion Elements and Kostant's Problem

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Abstract We find a partial solution to the problem of Kostant concerning description of the locally finite endomorphisms of highest weight irreducible modules. The solution is obtained by means of its reduction to an extension of the quantization problem. While the classical quantization problem consists in finding $\star$-product deformations of the commutative algebras of functions, we consider the q-case when the initial object is already a noncommutative algebra.

1 Introduction

Let $\check{U}_q\mathfrak{g}$ be the quantized universal enveloping algebra "of simply connected type" [8] that corresponds to a finite dimensional split semisimple Lie algebra $\mathfrak{g}$. Let $L(\lambda)$ be the irreducible highest weight $\check{U}_q\mathfrak{g}$-module of highest weight λ. The aim of this paper is to show that for certain values of λ, the action map $\check{U}_q\mathfrak{g} \to \big(\operatorname{End} L(\lambda)\big)_{\text{fin}}$ is surjective. Here $\big(\operatorname{End} L(\lambda)\big)_{\text{fin}}$ stands for the locally finite part of $\operatorname{End} L(\lambda)$ with respect to the adjoint action of $\check{U}_q\mathfrak{g}$. For the Lie-algebraic case ($q = 1$), this problem

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is known as the classical Kostant's problem, see [6, 7, 12, 15, 16]. The complete answer to it is still unknown even in the $q = 1$ case. However, there are examples of λ for which the action map $U(\mathfrak{g}) \to \big(\mathrm{End}\, L(\lambda)\big)_{\mathrm{fin}}$ is not surjective. Such examples exist even in the case $\mathfrak{g}$ is of type A [17].

The main idea of our approach to Kostant's problem, both in the Lie-algebraic and quantum group cases, is that $\big(\mathrm{End}\, L(\lambda)\big)_{\mathrm{fin}}$ has two other presentations. First, it follows from the results of [11] that $\big(\mathrm{End}\, L(\lambda)\big)_{\mathrm{fin}}$ is canonically isomorphic to $\mathrm{Hom}_U\big(L(\lambda), L(\lambda) \otimes F\big)$, where U is $U(\mathfrak{g})$ (resp. $\check{U}_q\mathfrak{g}$), and F is the algebra of (quantized) regular functions on the connected simply connected algebraic group G corresponding to the Lie algebra $\mathfrak{g}$. In other words, F is spanned by matrix elements of finite dimensional representations of U with an appropriate multiplication.

One more presentation of the algebra $\big(\mathrm{End}\, L(\lambda)\big)_{\mathrm{fin}}$ comes from the fact that $\mathrm{Hom}_U\big(L(\lambda), L(\lambda) \otimes F\big)$ is isomorphic as a vector space to a certain subspace F' of F. The subspace F' can be equipped with a $\star$-multiplication obtained from the multiplication on F by applying the so-called reduced fusion element. Then $\big(\mathrm{End}\, L(\lambda)\big)_{\mathrm{fin}}$ is isomorphic as an algebra to F' with this new multiplication. For certain values of λ, the same $\star$-multiplication on F' can be defined by applying the universal fusion element, that yields the affirmative answer to Kostant's problem in such cases.

More exactly, consider the triangular decomposition $U = U^- U^0 U^+$. We have $L(\lambda) = M(\lambda)/K_\lambda \mathbf{1}_\lambda$, where $M(\lambda)$ is the corresponding Verma module, $\mathbf{1}_\lambda$ is the generator of $M(\lambda)$, and $K_\lambda \subset U^-$. Consider also the opposite Verma module $\widetilde{M}(-\lambda)$ with the lowest weight $-\lambda$ and the lowest weight vector $\widetilde{\mathbf{1}}_{-\lambda}$. Then its maximal U-submodule is of the form $\widetilde{K}_\lambda \cdot \widetilde{\mathbf{1}}_{-\lambda}$, where $\widetilde{K}_\lambda \subset U^+$. We have $F' = F[0]^{K_\lambda + \widetilde{K}_\lambda}$—the subspace of U^0-invariant elements of F annihilated by both K_λ and $\widetilde{K}_\lambda$. The $\star$-product on $F[0]^{K_\lambda + \widetilde{K}_\lambda}$ has the form

$$f_1 \star_\lambda f_2 = \mu\left(J^{\mathrm{red}}(\lambda)(f_1 \otimes f_2)\right),$$

where μ is the multiplication on F, and the reduced fusion element $J^{\mathrm{red}}(\lambda) \in U^- \mathbin{\widehat{\otimes}} U^+$ is computed in terms of the Shapovalov form on $L(\lambda)$. Notice that for generic λ the element $J^{\mathrm{red}}(\lambda)$ is equal up to an U^0-part to the fusion element $J(\lambda)$ related to the Verma module $M(\lambda)$, see for example [4].

We also investigate limiting properties of $J(\lambda)$. In particular, for some values of λ_0 we can guarantee that $f_1 \star_\lambda f_2 \to f_1 \star_{\lambda_0} f_2$ as $\lambda \to \lambda_0$. Also, for any λ_0 having a "regularity property" of this kind, the action map $U \to (\mathrm{End}\, L(\lambda_0))_{\mathrm{fin}}$ is surjective. This gives the affirmative answer to the (quantum version of) Kostant's problem.

For some values of λ, the subspace $F[0]^{K_\lambda + \widetilde{K}_\lambda}$ is a subalgebra of $F[0]$, and can be considered as (a flat deformation of) the algebra of regular functions on some Poisson homogeneous space G/G_1. In those cases, the algebra $\big(F[0]^{K_\lambda + \widetilde{K}_\lambda}, \star_\lambda\big)$ is an equivariant quantization of the Poisson algebra of regular functions on G/G_1.

This paper is organized as follows. In Sect. 2 we recall the definition of the version of quantized universal enveloping algebra used in this paper, and some related constructions that will be useful in the sequel. In Sect. 3 we construct an isomorphism

$\mathrm{Hom}_U\big(L(\lambda), L(\lambda)\otimes F\big) \simeq F[0]^{K_\lambda+\widetilde{K}_\lambda}$ and, as a corollary, provide a construction of a star-product on $F[0]^{K_\lambda+\widetilde{K}_\lambda}$ in terms of the Shapovalov form on $L(\lambda)$. In Sect. 4 we study limiting properties of fusion elements and the corresponding star-products. Namely, in Sect. 4.1 we introduce the notion of a J-regular weight. In a neighborhood of a J-regular weight the fusion element behaves nicely, which allows one to give a solution to the Kostant's problem for such weights (see Theorem 4.1). We also provide non-trivial examples of J-regular weights. Finally, in Sect. 4.2 we apply limiting properties of fusion elements to quantize explicitly certain Poisson homogeneous spaces (see Theorem 4.4).

In this short version of the paper the proofs are omitted. A complete version with proofs will be published elsewhere.

2 Algebra $\check{U}_q\mathfrak{g}$

Let $\Bbbk$ be the field extension of $\mathbb{C}(q)$ by all fractional powers $q^{1/n}$, $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. We use $\Bbbk$ as the ground field.

Let (a_{ij}) a finite type $r \times r$ Cartan matrix. Let d_i be relatively prime positive integers such that $d_i a_{ij} = d_j a_{ji}$. For any positive integer k, define

$$[k]_i = \frac{q^{kd_i} - q^{-kd_i}}{q^{d_i} - q^{-d_i}}, \quad [k]_i! = [1]_i\,[2]_i\,\dots\,[k]_i.$$

The algebra $U = \check{U}_q\mathfrak{g}$ is generated by the elements t_i, t_i^{-1}, e_i, f_i, $i = 1, \dots, r$, subject to the relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1$$
$$t_i e_j t_i^{-1} = q^{d_i\delta_{ij}} e_j,$$
$$t_i f_j t_i^{-1} = q^{-d_i\delta_{ij}} f_j,$$

$$e_i f_j - f_j e_i = \delta_{ij}\,\frac{k_i - k_i^{-1}}{q^{d_i} - q^{-d_i}}, \quad \text{where } k_i = \prod_{j=1}^{r} t_j^{a_{ij}},$$
$$\sum_{m=0}^{1-a_{ij}} \frac{(-1)^m}{[m]_i!\,[1-a_{ij}-m]_i!} e_i^m e_j e_i^{1-a_{ij}-m} = 0 \text{ for } i \neq j,$$
$$\sum_{m=0}^{1-a_{ij}} \frac{(-1)^m}{[m]_i!\,[1-a_{ij}-m]_i!} f_i^m f_j f_i^{1-a_{ij}-m} = 0 \text{ for } i \neq j$$

Notice that $k_i e_j k_i^{-1} = q^{d_i a_{ij}} e_j$, $k_i f_j k_i^{-1} = q^{-d_i a_{ij}} f_j$.

The algebra U is a Hopf algebra with the comultiplication Δ, the counit ε, and the antipode σ given by

$$\begin{aligned} &\Delta(t_i) = t_i \otimes t_i, && \varepsilon(t_i) = 1,\ \sigma(t_i) = t_i^{-1} \\ &\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, && \varepsilon(e_i) = 0,\ \sigma(e_i) = -k_i^{-1} e_i \\ &\Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i, && \varepsilon(f_i) = 0,\ \sigma(f_i) = -f_i k_i. \end{aligned}$$

In what follows we will sometimes use the Sweedler notation for comultiplication.

Let U^0 be the subalgebra of U generated by the elements $t_1, \ldots, t_r, t_1^{-1}, \ldots, t_r^{-1}$. Let U^+ and U^- be the subalgebras generated respectively by the elements $e_1, \ldots, e_r$ and $f_1, \ldots, f_r$. We have a triangular decomposition $U = U^- U^0 U^+$. Denote by θ the involutive automorphism of U given by $\theta(e_i) = -f_i$, $\theta(f_i) = -e_i$, $\theta(t_i) = t_i^{-1}$. Notice that θ gives an algebra isomorphism $U^- \to U^+$. Set $\omega = \sigma\theta$, i.e., ω is the involutive antiautomorphism of U given by $\omega(e_i) = f_i k_i$, $\omega(f_i) = k_i^{-1} e_i$, $\omega(t_i) = t_i$.

Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of (a_{ij}) over $\mathbb{Q}$, that is, $\mathfrak{h}$ is (a rational form of) a Cartan subalgebra of the corresponding semisimple Lie algebra, $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset \mathfrak{h}^*$ the set of simple roots, $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_r^\vee\} \subset \mathfrak{h}$ the set of simple coroots. Let $\mathbf{R}$ be the root system, $\mathbf{R}_+$ the set of positive roots, and W the Weyl group.

Let $u_1, \ldots, u_r \in \mathfrak{h}$ be the simple coweights, i.e., $\langle \alpha_i, u_j \rangle = \delta_{ij}$. We denote by ρ the half sum of the positive roots.

Let

$$Q_+ = \sum_{\alpha \in \Pi} \mathbb{Z}_+ \alpha.$$

For $\lambda, \mu \in \mathfrak{h}^*$ we set $\lambda \geq \mu$ iff $\lambda - \mu \in Q_+$.

Denote by T the multiplicative subgroup generated by $t_1, \ldots, t_r$. Any $\lambda \in \mathfrak{h}^*$ defines a character $\Lambda : T \to \Bbbk$ given by $t_i \mapsto q^{d_i \langle \lambda, u_i \rangle}$. We will write $\Lambda = q^\lambda$. Notice that $q^\lambda(k_i) = q^{d_i \langle \lambda, \alpha_i^\vee \rangle}$. We extend q^λ to the subalgebra U^0 by linearity. We say that an element $x \in U$ is of weight λ if $txt^{-1} = q^\lambda(t)x$ for all $t \in T$.

For a U-module V, we denote by

$$V[\lambda] = \{v \in V \mid tv = q^\lambda(t)v \quad \text{for all } t \in T\}$$

the weight subspace of weight λ. We call the module V admissible if V is a direct sum of finite-dimensional weight subspaces $V[\lambda]$.

The Verma module $M(\lambda)$ over U with highest weight λ and highest weight vector $\mathbf{1}_\lambda$ is defined in the standard way:

$$M(\lambda) = U^- \mathbf{1}_\lambda, \qquad U^+ \mathbf{1}_\lambda = 0, \qquad t\mathbf{1}_\lambda = q^\lambda(t)\mathbf{1}_\lambda, \quad t \in T.$$

The map $U^- \to M(\lambda)$, $y \mapsto y\mathbf{1}_\lambda$ is an isomorphism of U^--modules.

Set $U_+^\pm = \operatorname{Ker} \varepsilon|_{U^\pm}$ and denote by $x \mapsto (x)_0$ the projection $U \to U^0$ along $U_+^- \cdot U + U \cdot U_+^+$. For any $\lambda \in \mathfrak{h}^*$ consider $\pi_\lambda : U^+ \otimes U^- \to \Bbbk$, $\pi_\lambda(x \otimes y) =$

$q^{\lambda}((\sigma(x)y)_0)$, and $\mathbb{S}_\lambda : U^-\otimes U^- \to \Bbbk$, $\mathbb{S}_\lambda(x\otimes y) = \pi_\lambda(\theta(x)\otimes y) = q^{\lambda}((\omega(x)y)_0)$. We call $\mathbb{S}_\lambda$ the Shapovalov form on U^- corresponding to λ. We can regard $\mathbb{S}_\lambda$ as a bilinear form on $M(\lambda)$.

Set

$$K_\lambda = \{y \in U^- \mid \pi_\lambda(x \otimes y) = 0 \quad \text{for all } x \in U^+\},$$

$$\widetilde{K}_\lambda = \{x \in U^+ \mid \pi_\lambda(x \otimes y) = 0 \quad \text{for all } y \in U^-\}.$$

Clearly, K_λ is the kernel of $\mathbb{S}_\lambda$, $\widetilde{K}_\lambda = \theta(K_\lambda)$. Notice that $K(\lambda) = K_\lambda \cdot \mathbf{1}_\lambda$ is the largest proper submodule of $M(\lambda)$, and $L(\lambda) = M(\lambda)/K(\lambda)$ is the irreducible U-module with highest weight λ. Denote by $\overline{\mathbf{1}}_\lambda$ the image of $\mathbf{1}_\lambda$ in $L(\lambda)$.

Let $F = \Bbbk[G]_q$ be the quantized algebra of regular functions on a connected simply connected algebraic group G that corresponds to the Cartan matrix (a_{ij}) (see [8, 14]). We can consider F as a Hopf subalgebra in the dual Hopf algebra $U^\star$. We will use the left and right regular actions of U on F defined respectively by the formulae $(\overrightarrow{a}\, f)(x) = f(xa)$ and $(f\, \overleftarrow{a})(x) = f(ax)$. Notice that F is a sum of finite-dimensional admissible U-modules with respect to both regular actions of U (see [14]).

3 Star Products and Fusion Elements

3.1 Algebra of Intertwining Operators

Let us denote by $U_{\text{fin}} \subset U$ the subalgebra of locally finite elements with respect to the right adjoint action of U on itself. We will use similar notation for any (right) U-module.

For any (left) U-module M we equip F with the left regular U-action and consider the space $\mathrm{Hom}_U(M, M \otimes F)$. For any $\varphi, \psi \in \mathrm{Hom}_U(M, M \otimes F)$ define

$$\varphi * \psi = (\mathrm{id} \otimes \mu) \circ (\varphi \otimes \mathrm{id}) \circ \psi, \tag{1}$$

where μ is the multiplication in F. We have $\varphi * \psi \in \mathrm{Hom}_U(M, M \otimes F)$, and this definition equips $\mathrm{Hom}_U(M, M \otimes F)$ with a unital associative algebra structure.

Consider the map $\Phi : \mathrm{Hom}_U(M, M \otimes F) \to \mathrm{End}\, M$, $\varphi \mapsto u_\varphi$, defined by $u_\varphi(m) = (\mathrm{id} \otimes \varepsilon)(\varphi(m))$; here $\varepsilon(f) = f(1)$ is the counit in F. Consider U_{fin}, $\mathrm{Hom}_U(M, M \otimes F)$ and $\mathrm{End}\, M$ as right U-module algebras: U_{fin} via right adjoint action, $\mathrm{Hom}_U(M, M \otimes F)$ via right regular action on F (i.e., $(\varphi \cdot a)(m) = (\mathrm{id} \otimes \overleftarrow{a})(\varphi(m))$), and $\mathrm{End}\, M$ in a standard way (i.e., $u \cdot a = \sum_{(a)} \sigma(a_{(1)})_M u a_{(2)\,M}$). Then $\mathrm{Hom}_U(M, M \otimes F)_{\text{fin}} = \mathrm{Hom}_U(M, M \otimes F)$, and $\Phi : \mathrm{Hom}_U(M, M \otimes F) \longrightarrow (\mathrm{End}\, M)_{\text{fin}}$ is an isomorphism of right U-module algebras (see [11, Proposition 6]).

Now we apply this to $M = M(\lambda)$ and $M = L(\lambda)$. Since $U_{\text{fin}} \to (\mathrm{End}\, M(\lambda))_{\text{fin}}$ is surjective (see [8, 9]), we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_U(M(\lambda), M(\lambda)\otimes F) & \longrightarrow & \mathrm{Hom}_U(L(\lambda), L(\lambda)\otimes F) \\ \downarrow \Phi_{M(\lambda)} & & \downarrow \Phi_{L(\lambda)} \\ (\mathrm{End}\, M(\lambda))_{\mathrm{fin}} & \longrightarrow & (\mathrm{End}\, L(\lambda))_{\mathrm{fin}} \end{array}$$

(see [11, Proposition 9]).

For any $\varphi \in \mathrm{Hom}_U(L(\lambda), L(\lambda)\otimes F)$ the formula

$$\varphi(\overline{\mathbf{1}}_\lambda) = \overline{\mathbf{1}}_\lambda \otimes f_\varphi + \sum_{\mu<\lambda} v_\mu \otimes f_\mu,$$

where v_μ is of weight μ, defines a map

$$\Theta : \mathrm{Hom}_U(L(\lambda), L(\lambda)\otimes F) \to F[0], \qquad \varphi \mapsto f_\varphi.$$

Theorem 3.1 *Θ is an embedding, and its image equals $F[0]^{K_\lambda+\widetilde{K}_\lambda}$.*

3.2 Reduced Fusion Elements

In this subsection we describe $\Theta^{-1} : F[0]^{K_\lambda+\widetilde{K}_\lambda} \to \mathrm{Hom}_U(L(\lambda), L(\lambda)\otimes F)$ explicitly in terms of the Shapovalov form. Recall that we can regard $\mathbb{S}_\lambda$ as a bilinear form on $M(\lambda)$. Denote by $\overline{\mathbb{S}}_\lambda$ the corresponding bilinear form on $L(\lambda)$. For any $\beta \in Q_+$ denote by $\overline{\mathbb{S}}_\lambda^\beta$ the restriction of $\overline{\mathbb{S}}_\lambda$ to $L(\lambda)[\lambda-\beta]$. Let $y_\beta^i \cdot \overline{\mathbf{1}}_\lambda$ be an arbitrary basis in $L(\lambda)[\lambda-\beta]$, where $y_\beta^i \in U^-[-\beta]$.

Take $f \in F[0]^{K_\lambda+\widetilde{K}_\lambda}$ and set $\varphi = \Theta^{-1}(f)$,

$$\varphi(\overline{\mathbf{1}}_\lambda) = \sum_{\beta\in Q_+}\sum_i y_\beta^i \overline{\mathbf{1}}_\lambda \otimes f^{\beta,i}.$$

For $\beta = 0$ we have $y_\beta^i = 1$ and $f^{\beta,i} = f$.

Proposition 3.1 $f^{\beta,i} = \sum_j \left(\overline{\mathbb{S}}_\lambda^\beta\right)_{ij}^{-1} \overrightarrow{\theta\left(y_\beta^j\right)} f.$

For any $\lambda \in \mathfrak{h}^*$ consider

$$J^{\mathrm{red}}(\lambda) = \sum_{\beta\in Q_+}\sum_{i,j} \left(\overline{\mathbb{S}}_\lambda^\beta\right)_{ij}^{-1} y_\beta^i \otimes \theta\left(y_\beta^j\right). \tag{2}$$

One can regard $J^{\mathrm{red}}(\lambda)$ as an element in a certain completion of $U^-\otimes U^+$.

Remark 3.1 This element $J^{\mathrm{red}}(\lambda)$ is not uniquely defined (e.g., because $U^- \to L(\lambda)$ has a kernel), but this does not affect our further considerations.

Remark 3.2 For $f \in F[0]^{K_\lambda + \widetilde{K}_\lambda}$ and $\varphi = \Theta^{-1}(f)$ one has $\varphi(\bar{\mathbf{1}}_\lambda) = J^{\mathrm{red}}(\lambda)(\bar{\mathbf{1}}_\lambda \otimes f)$.

Let us define an associative product $\star_\lambda$ on $F[0]^{K_\lambda + \widetilde{K}_\lambda}$ by means of Θ, i.e., for any $f_1, f_2 \in F[0]^{K_\lambda + \widetilde{K}_\lambda}$ we define $f_1 \star_\lambda f_2 = \Theta(\varphi_1 * \varphi_2)$, where $\varphi_1 = \Theta^{-1}(f_1)$, $\varphi_2 = \Theta^{-1}(f_2)$, and $*$ is the product on $\mathrm{Hom}_U(L(\lambda), L(\lambda) \otimes F)$ given by (1). By this definition, we get a right U-module algebra $(F[0]^{K_\lambda + \widetilde{K}_\lambda}, \star_\lambda)$.

Theorem 3.2 *We have*

$$f_1 \star_\lambda f_2 = \mu\left(\overrightarrow{J^{\mathrm{red}}(\lambda)}(f_1 \otimes f_2)\right). \tag{3}$$

Remark 3.3 Theorem 23 together with results of [11] implies that $\mathrm{Hom}_U(L(\lambda), L(\lambda) \otimes F)$, $(\mathrm{End}\, L(\lambda))_{\mathrm{fin}}$, and $\left(F[0]^{K_\lambda + \widetilde{K}_\lambda}, \star_\lambda\right)$ are isomorphic as right Hopf module algebras over U.

4 Limiting Properties of the Fusion Element

We say that $\lambda \in \mathfrak{h}^*$ is *generic* if $\langle \lambda + \rho, \beta^\vee \rangle \notin \mathbb{N}$ for all $\beta \in \mathbf{R}_+$. In this case $L(\lambda) = M(\lambda)$, and we set $J(\lambda) = J^{\mathrm{red}}(\lambda)$. Notice that $J(\lambda)$ up to a U^0-part equals the fusion element related to the Verma module $M(\lambda)$ (see, e.g., [4]).

4.1 Regularity

Let $\lambda_0 \in \mathfrak{h}^*$. Since $J(\lambda)$ is invariant w. r. to $\tau(\theta \otimes \theta)$ (where τ is the tensor permutation), one can easily see that the following conditions on λ_0 are equivalent: 1) for any U^--module M the family of operators $J(\lambda)^M : M \otimes F[0]^{\widetilde{K}_{\lambda_0}} \to M \otimes F$ naturally defined by $J(\lambda)$ is regular at $\lambda = \lambda_0$, 2) for any U^+-module N the family of operators $J(\lambda)_N : F[0]^{K_{\lambda_0}} \otimes N \to F \otimes N$ naturally defined by $J(\lambda)$ is regular at $\lambda = \lambda_0$. We will say that λ_0 is *J-regular* if these conditions are satisfied. Clearly, any generic λ_0 is J-regular.

The following theorem collects some general properties of J-regular weights. In particular, for J-regular weights the answer to Kostant's question is affirmative.

Theorem 4.1 *Assume that $\lambda_0 \in \mathfrak{h}^*$ is J-regular. Then*

(1) $F[0]^{K_{\lambda_0}} = F[0]^{\widetilde{K}_{\lambda_0}} = F[0]^{K_{\lambda_0} + \widetilde{K}_{\lambda_0}}$,
(2) the natural map $\mathrm{Hom}_U(M(\lambda_0), M(\lambda_0) \otimes F) \to \mathrm{Hom}_U(L(\lambda_0), L(\lambda_0) \otimes F)$ *is surjective,*

(3) (Kostant's problem) *the action map* $U_{\text{fin}} \to (\text{End}\, L(\lambda_0))_{\text{fin}}$ *is surjective,*
(4) *for any* $f, g \in F[0]^{K_{\lambda_0}}$ *we have* $\overrightarrow{J(\lambda)}(f \otimes g) \to \overrightarrow{J^{\text{red}}(\lambda_0)}(f \otimes g)$ *as* $\lambda \to \lambda_0$*,*
(5) *for any* $f, g \in F[0]^{K_{\lambda_0}}$ *we have* $f_1 \star_\lambda f_2 \to f_1 \star_{\lambda_0} f_2$ *as* $\lambda \to \lambda_0$*.*

The following two theorems provide examples of J-regular weights.

Theorem 4.2 *Let* $\alpha \in \mathbf{R}_+$*. Consider* $\lambda_0 \in \mathfrak{h}^*$ *that satisfies* $\langle \lambda_0 + \rho, \alpha^\vee \rangle \in \mathbb{N}$*,* $\langle \lambda_0 + \rho, \beta^\vee \rangle \notin \mathbb{N}$ *for all* $\beta \in \mathbf{R}_+ \setminus \{\alpha\}$*. Then* λ_0 *is* J*-regular.*

Theorem 4.3 *Let* $\Gamma \subset \Pi$*. Consider* $\lambda_0 \in \mathfrak{h}^*$ *that satisfies* $\langle \lambda_0 + \rho, \alpha_i^\vee \rangle \in \mathbb{N}$ *for all* $\alpha_i \in \Gamma$*,* $\langle \lambda_0 + \rho, \beta^\vee \rangle \notin \mathbb{N}$ *for all* $\beta \in \mathbf{R}_+ \setminus \text{Span}\, \Gamma$*. Then* λ_0 *is* J*-regular.*

4.2 Application to Poisson Homogeneous Spaces

Let $\Gamma \subset \Pi$. Assume that $\lambda \in \mathfrak{h}^*$ is such that $\langle \lambda, \alpha^\vee \rangle = 0$ for all $\alpha \in \Gamma$, and $\langle \lambda + \rho, \beta^\vee \rangle \notin \mathbb{N}$ for all $\beta \in \mathbf{R}_+ \setminus \text{Span}\, \Gamma$. By Theorem 4.3, λ is J-regular. In particular, $F[0]^{K_\lambda + \widetilde{K}_\lambda} = F[0]^{K_\lambda}$.

In what follows it will be more convenient to write F_q, J_q, and $K_{q,\lambda}$ instead of F, J, and K_λ. We will also need the classical limits $F_1 = \lim_{q \to 1} F_q$ and $K_{1,\lambda} = \lim_{q \to 1} K_{q,\lambda}$. They can be defined in the same way as in ([8], Sects. 3.4.5 and 3.4.6).

Clearly, F_1 is the algebra of regular functions on the connected simply connected group G, whose Lie algebra is $\mathfrak{g}$. Let $\mathfrak{k}$ be a reductive subalgebra of $\mathfrak{g}$ which contains $\mathfrak{h}$ and is defined by Γ, K the corresponding subgroup of G, and $F(G/K)$ the algebra of regular functions on the homogeneous space G/K. According to [11, Theorem 33], we have $F(G/K) = F_1[0]^{K_{1,\lambda}}$. Therefore we get

Proposition 4.1 $\lim_{q \to 1} F_q[0]^{K_{q,\lambda}} = F(G/K)$.

Furthermore, since $F_q[0]^{K_{q,\lambda}}$ is a Hopf module algebra over U, G/K is a Poisson homogeneous space over G equipped with the Poisson-Lie structure defined by the Drinfeld-Jimbo classical r-matrix $r_0 = \sum_{\alpha \in \mathbf{R}_+} e_\alpha \wedge e_{-\alpha}$.

All such structures on G/K were described in [10]. It follows from [10] that any such Poisson structure on G/K is uniquely determined by an an intermediate Levi subalgebra $\mathfrak{n}$ satisfying $\mathfrak{k} \subset \mathfrak{n} \subset \mathfrak{g}$ and some $\lambda \in \mathfrak{h}^*$ which satisfies certain conditions, in particular, $\langle \lambda, \alpha^\vee \rangle = 0$ for $\alpha \in \Gamma$ and $\langle \lambda, \beta^\vee \rangle \notin \mathbb{Z}$ for $\beta \in \text{Span}\, \Gamma_{\mathfrak{n}} \setminus \text{Span}\, \Gamma$. Here $\Gamma_{\mathfrak{n}}$ is the set of simple roots defining $\mathfrak{n}$.

Now we can describe the Poisson bracket on G/K defined by $\star_\lambda$-multiplication on $F_q[0]^{K_{q,\lambda}}$.

Theorem 4.4 *Assume that* $\langle \lambda_0, \alpha^\vee \rangle = 0$ *for* $\alpha \in \Gamma$ *and* $\langle \lambda_0, \beta^\vee \rangle \notin \mathbb{Z}$ *for* $\beta \in \mathbf{R}_+ \setminus \text{Span}\, \Gamma$*. Then the classical limit of* $(F_q[0]^{K_{q,\lambda_0}}, \star_{\lambda_0})$ *is the algebra* $F(G/K)$ *of regular functions on* G/K *equipped with the Poisson homogeneous structure defined by* $\mathfrak{n} = \mathfrak{g}$ *and* λ_0*.*

Notice that an analogous result for simple Lie algebras of classical type was obtained in [18] using reflection equation algebras.

Proposition 4.1 and Theorem 4.4 suggest a conjecture which we formulate below.

Let G be a connected Poisson affine algebraic group, $\mathfrak{g}$ the corresponding Lie bialgebra with the co-bracket δ, X a Poisson homogeneous G-variety, Y an affine Zariski open dense subset of X. Consider the Poisson algebra $F(Y)$ of regular functions on Y. Let $U_q\mathfrak{g}$ be a quantized universal enveloping algebra corresponding to $\mathfrak{g}$.

Conjecture 4.1 *There exists a Hopf module algebra over $U_q\mathfrak{g}$ whose classical limit is $F(Y)$.*

Let us show another example which confirms this conjecture. Consider the case $X = G$. Let $D(\mathfrak{g})$ be the classical double of $\mathfrak{g}$. According to [2], Poisson G-homogeneous structures on G are in one-to-one correspondence with Largangian subalgebras of $D(\mathfrak{g})$ transversal to $\mathfrak{g} \subset D(\mathfrak{g})$. Consider such a Lagrangian subalgebra $\mathfrak{l} \subset D(\mathfrak{g})$, which corresponds to a certain Poisson G-homogeneous structure on G. It is well known [1] that $\mathfrak{l}$ also induces a new Poisson-Lie structure on G, which differs from the original one by a so-called classical twist. Hence we obtain a new Lie bialgebra structure δ_1 on the Lie algebra $\mathfrak{g}$.

The following conjecture was made in [12] and later published in [13].

Conjecture 4.2 *There exists an element T in a certain completion of $(U_q\mathfrak{g})^{\otimes 2}$ which satisfies*

$$T^{12}(\Delta \otimes \mathrm{id})(T) = T^{23}(\mathrm{id} \otimes \Delta)(T) \tag{4}$$

and $(\varepsilon \otimes \mathrm{id})(T) = (\mathrm{id} \otimes \varepsilon)(T) = 1$ such that the Hopf algebra $U_{q,T}\mathfrak{g}$ quantizes $(\mathfrak{g}, \delta_1)$. Here $U_{q,T}\mathfrak{g}$ and $U_q\mathfrak{g}$ are isomorphic as algebras, and the co-multiplication on $U_{q,T}\mathfrak{g}$ is given by $\Delta_T(a) = T\Delta(a)T^{-1}$.

This conjecture was proved in [3, 5].

Now let $F_q(G)$ be the restricted dual of $U_q\mathfrak{g}$. It is well known that $F_q(G)$ quantizes $F(G)$. Let us equip $F_q(G)$ with a new product defined by $f_1 \star_T f_2 = \mu\left(\overrightarrow{T}(f_1 \otimes f_2)\right)$. According to (4), $\star_T$ is associative. Hence we get

Corollary 4.1 *The algebra $(F_q(G), \star_T)$ is a Hopf module algebra over $U_q\mathfrak{g}$ which quantizes the Poisson homogeneous structure on G defined by $\mathfrak{l}$.*

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