A Comparison of Leibniz and Lie Cohomology and Deformations

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Abstract In this talk we compare Leibniz and Lie algebra cohomology and deformations of a given Lie algebra. We get some sufficient conditions for not getting more Leibniz deformations just the Lie ones. These conditions are easy to verify. As an example, we describe the universal infinitesimal versal Leibniz deformation of the 4-dimensional diamond algebra.

1 Introduction

Leibniz algebras were introduced in [10] as a non antisymmetric version of Lie algebras. Lie algebras are special Leibniz algebras, and Pirashvili introduced [16] a spectral sequence, that, when applied to Lie algebras, measures the difference between the Lie algebra cohomology and the Leibniz cohomology. Lie algebras have deformations as Leibniz algebras and those are piloted by the adjoint Leibniz 2-cocycles. In the present talk, we focus on the second Leibniz cohomology groups $HL^2(\mathfrak{g}, \mathfrak{g})$, $HL^2(\mathfrak{g}, \mathbb{C})$ with adjoint and trivial representations of a complex Lie algebra $\mathfrak{g}$. We adopt a very elementary approach, to compare $HL^2(\mathfrak{g}, \mathfrak{g})$ and $HL^2(\mathfrak{g}, \mathbb{C})$ to $H^2(\mathfrak{g}, \mathfrak{g})$ and $H^2(\mathfrak{g}, \mathbb{C})$ respectively. In both cases, HL^2 is the direct sum of 3 spaces: $H^2 \oplus ZL_0^2 \oplus \mathscr{C}$ wher H^2 is the Lie algebra cohomology group, ZL_0^2 is the space of symmetric Leibniz 2-cocycles and $\mathscr{C}$ is a space of coupled Leibniz 2-cocycles, the nonzero elements of which have the property that their symmetric and antisymmetric parts are not Leibniz cocycles. Our comparison gives some useful practical information about the structure of Lie and Leibniz cocycles. As an example, we analyse the

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4-dimensional diamond algebra which is used to construct a Wess-Zumino-Witten model. We completely describe its universal infinitesimal Leibniz and Lie deformation by computing Massey products.

The talk is based on joint work with Mandal and Magnin [5].

2 Leibniz Cohomology and Deformations

Leibniz algebras were introduced by J.-L. Loday [10, 12]. Let $\mathbb{K}$ denote a field.

Definition 2.1 A Leibniz algebra is a $\mathbb{K}$-module L, equipped with a bracket operation that satisfies the Leibniz identity:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \quad \text{for } x, y, z \in L.$$

Any Lie algebra is automatically a Leibniz algebra, as in the presence of antisymmetry, the Jacobi identity is equivalent to the Leibniz identity. More examples of Leibniz algebras were given in [10–12], and recently Leibniz algebras are intesively studied.

Let L be a Leibniz algebra and M a representation of L. By definition, M is a $\mathbb{K}$-module equipped with two actions (left and right) of L,

$$[-, -] : L \times M \longrightarrow M \;\text{ and } [-, -] : M \times L \longrightarrow M \;\text{ such that}$$

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds, whenever one of the variables is from M and the two others from L. Define $CL^n(L; M) := \mathrm{Hom}_{\mathbb{K}}(L^{\otimes n}, M),\; n \geq 0$. Let

$$\delta^n : CL^n(L; M) \longrightarrow CL^{n+1}(L; M)$$

be a $\mathbb{K}$-homomorphism defined by

$$\begin{aligned}
&\delta^n f(x_1, \ldots, x_{n+1}) \\
&\quad := [x_1, f(x_2, \ldots, x_{n+1})] + \sum_{i=2}^{n+1} (-1)^i [f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}), x_i] \\
&\qquad + \sum_{1 \leq i < j \leq n+1} (-1)^{j+1} f(x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x}_j, \ldots, x_{n+1}).
\end{aligned}$$

Then $(CL^*(L; M), \delta)$ is a cochain complex, whose cohomology is called the cohomology of the Leibniz algebra L with coefficients in the representation M. The n-th cohomology is denoted by $HL^n(L; M)$. In particular, L is a representation of itself with the obvious action given by the bracket in L. The n-th cohomology of L with coefficients in itself is denoted by $HL^n(L; L)$.

Let S_n be the symmetric group. Recall that a permutation $\sigma \in S_{p+q}$ is called a (p, q)-shuffle, if $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$, and $\sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q)$. We denote the set of all (p, q)-shuffles in S_{p+q} by $Sh(p, q)$.

For $\alpha \in CL^{p+1}(L; L)$ and $\beta \in CL^{q+1}(L; L)$, define $\alpha \circ \beta \in CL^{p+q+1}(L; L)$ by

$$\begin{aligned}\alpha \circ \beta(x_1, \dots, x_{p+q+1}) \\ &= \sum_{k=1}^{p+1}(-1)^{q(k-1)}\{\sum_{\sigma \in Sh(q,p-k+1)} sgn(\sigma)\alpha(x_1, \dots, x_{k-1}, \beta(x_k, x_{\sigma(k+1)}, \dots, \\ &\qquad x_{\sigma(k+q)}), x_{\sigma(k+q+1)}, \dots, x_{\sigma(p+q+1)})\}.\end{aligned}$$

The graded cochain module $CL^*(L; L) = \bigoplus_r CL^r(L; L)$ equipped with the bracket defined by

$$[\alpha, \beta] = \alpha \circ \beta + (-1)^{pq+1}\beta \circ \alpha \quad \text{for } \alpha \in CL^{p+1}(L; L) \text{ and } \beta \in CL^{q+1}(L; L)$$

and the differential map d by $d\alpha = (-1)^{|\alpha|}\delta\alpha$ for $\alpha \in CL^*(L; L)$ is a differential graded Lie algebra. (Here $|\alpha|$ denotes the degree of the cochain α.)

Let now $\mathbb{K}$ a field of zero characteristic and the tensor product over $\mathbb{K}$ will be denoted by $\otimes$. We recall the notion of deformation of a Lie (Leibniz) algebra $\mathfrak{g}$ (L) over a commutative algebra with identity base A with a fixed augmentation $\varepsilon : A \to \mathbb{K}$ and maximal ideal $\mathfrak{M}$. Assume $dim(\mathfrak{M}^k/\mathfrak{M}^{k+1}) < \infty$ for every k (see [6]).

Definition 2.2 A deformation λ of a Lie algebra $\mathfrak{g}$ (or a Leibniz algebra L) with base $(A, \mathfrak{M})$, or simply with base A is an A-Lie algebra (or an A-Leibniz algebra) structure on the tensor product $A \otimes \mathfrak{g}$ (or $A \otimes L$) with the bracket $[,]_\lambda$ such that

$$\varepsilon \otimes id : A \otimes \mathfrak{g} \to \mathbb{K} \otimes \mathfrak{g} \quad (\text{or } \varepsilon \otimes id : A \otimes L \to \mathbb{K} \otimes L)$$

is an A-Lie algebra (A-Leibniz algebra) homomorphism.

A deformation of the Lie (Leibniz) algebra $\mathfrak{g}$ (L) with base A is called *infinitesimal*, or *first order*, if in addition to this, $\mathfrak{M}^2 = 0$. We call a deformation of *order* k, if $\mathfrak{M}^{k+1} = 0$. A deformation with base is called local if A is a local algebra over $\mathbb{K}$, which means A has a unique maximal ideal.

Suppose A is a complete local algebra ($A = \varprojlim_{n\to\infty} (A/\mathfrak{M}^n)$), where $\mathfrak{M}$ is the maximal ideal in A. Then a deformation of $\mathfrak{g}$ (L) with base A which is obtained as the projective limit of deformations of $\mathfrak{g}$ (L) with base $A/\mathfrak{M}^n$ is called a *formal deformation* of $\mathfrak{g}$ (L).

Definition 2.3 Suppose λ is a given deformation of L with base $(A, \mathfrak{M})$ and augmentation $\varepsilon : A \to \mathbb{K}$. Let A' be another commutative algebra with identity and a fixed augmentation $\varepsilon' : A' \to \mathbb{K}$. Suppose $\phi : A \to A'$ is an algebra homomorphism with $\phi(1) = 1$ and $\varepsilon' \circ \phi = \varepsilon$. Let $ker(\varepsilon') = \mathfrak{M}'$. Then the push-out $\phi_*\lambda$ is the deformation of L with base $(A', \mathfrak{M}')$ and bracket

$$[a_1{}' \otimes_A (a_1 \otimes l_1), a_2' \otimes_A (a_2 \otimes l_2)]_{\phi_* \lambda} = a_1' a_2' \otimes_A [a_1 \otimes l_1, a_2 \otimes l_2]_\lambda$$

where $a_1', a_2' \in A',\ a_1, a_2 \in A$ and $l_1, l_2 \in L$. Here A' is considered as an A-module by the map $a' \cdot a = a'\phi(a)$ so that

$$A' \otimes L = (A' \otimes_A A) \otimes L = A' \otimes_A (A \otimes L).$$

Definition 2.4 (see [2]) Let C be a complete local algebra. A formal deformation η of a Lie algebra $\mathfrak{g}$ (Leibniz algebra L) with base C is called versal, if

(i) for any formal deformation λ of $\mathfrak{g}$ (L) with base A there exists a homomorphism $f : C \to A$ such that the deformation λ is equivalent to $f_* \eta$;
(ii) if A satisfies the condition $\mathfrak{M}^2 = 0$, then f is unique.

Theorem 2.1 *If $H^2(\mathfrak{g}; \mathfrak{g})$ is finite dimensional, then there exists a of $\mathfrak{g}$ (similarly for L).*

Proof Follows from the general theorem of Schlessinger [17], like it was shown for Lie algebras in [2].

In [3] a construction for a versal deformation of a Lie algebra was given and it was generalized to Leibniz algebras in [6]. The computation of a specific Leibniz algebra example is given in [4].

3 Comparison of the Cohomology Spaces HL^2 and H^2 for a Lie Algebra

In [16] the relation between Chevalley-Eilenberg and Leibniz homology with coefficients in a right module is considered via a spectral sequence. The statements are valid in the cohomological version as well. As a corollary, one deduces

Proposition 3.1 [16] *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$ and M be a right $\mathfrak{g}$-module. If*

$$H_*(\mathfrak{g}, M) = 0, \textit{then } HL_*(\mathfrak{g}, M) = 0.$$

As the similar statement is true for cohomologies, it implies that rigid Lie algebras are Leibniz rigid as well.

Now we describe the Leibniz 2-cohomology spaces with the help of Lie 2-cohomology space of a Lie algebra $\mathfrak{g}$.

Recall that a symmetric bilinear form $B \in S^2\mathfrak{g}^*$ is invariant, i.e. $B \in \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$ if and only if $B([Z, X], Y) = -B(X, [Z, Y])$ for every $X, Y, Z \in \mathfrak{g}$. The Koszul map [9] $\mathscr{I} : \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}} \to \left(\bigwedge^3 \mathfrak{g}^*\right)^{\mathfrak{g}} \subset Z^3(\mathfrak{g}, \mathbb{C})$ is defined by $\mathscr{I}(B) = I_B$, with $I_B(X, Y, Z) = B([X, Y], Z)$ for every $X, Y, Z \in \mathfrak{g}$. Since the projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathscr{C}^2\mathfrak{g}$ induces an isomorphism

$$\varpi \ : \ \ker \mathscr{I} \to S^2 \left(\mathfrak{g}/\mathscr{C}^2\mathfrak{g}\right)^*,$$

(where $\mathscr{C}^2\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$), $\dim \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}} = \frac{p(p+1)}{2} + \dim \operatorname{Im} \mathscr{I}$, with $p = \dim H^1(\mathfrak{g}, \mathbb{C})$. For reductive $\mathfrak{g}$, $\dim \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}} = \dim H^3(\mathfrak{g}, \mathbb{C})$. Note also that the restriction of $\delta_{\mathbb{C}}$ to $\left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$ is $-\mathscr{I}$.

Definition 3.1 $\mathfrak{g}$ is said to be $\mathscr{I}$-null (resp. $\mathscr{I}$-exact) if $\mathscr{I} = 0$ (resp. $\operatorname{Im} \mathscr{I} \subset B^3(\mathfrak{g}, \mathbb{C})$).

Example 3.1 The $(2N + 1)$-dimensional complex Heisenberg Lie algebra $\mathscr{H}_N$ $(N \geqslant 1)$ with basis $(x_i)_{1 \leqslant i \leqslant 2N+1}$ and nonzero commutation relations (with anticommutativity) $[x_i, x_{N+i}] = x_{2N+1}$ $(1 \leqslant i \leqslant N)$ is $\mathscr{I}$-null, for any $B \in \left(S^2\mathscr{H}_N{}^*\right)^{\mathscr{H}_N}$, $B(x_i, x_{2N+1}) = B(x_i, [x_i, x_{N+i}]) = -B([x_i, x_i], x_{N+i}) = 0$ (similarly with x_{N+i} instead of x_i) $(1 \leqslant i \leqslant N)$, and $B(x_{2N+1}, x_{2N+1}) = B(x_{2N+1}, [x_1, x_{N+1}]) = -B([x_1, x_{2N+1}], x_{N+1}) = 0$.

If $\mathfrak{c}$ denotes the center of $\mathfrak{g}$, then $\mathfrak{c} \otimes \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$ is the space of invariant $\mathfrak{c}$-valued symmetric bilinear maps and we denote $F = Id \otimes \mathscr{I} : \mathfrak{c} \otimes \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}} \to C^3(\mathfrak{g}, \mathfrak{g}) = \mathfrak{g} \otimes \bigwedge^3 \mathfrak{g}^*$. Then $\operatorname{Im} F = \mathfrak{c} \otimes \operatorname{Im} \mathscr{I}$.

Theorem 3.1 *Let $\mathfrak{g}$ be any finite dimensional complex Lie algebra and $ZL_0^2(\mathfrak{g}, \mathfrak{g})$ (resp. $ZL_0^2(\mathfrak{g}, \mathbb{C})$) the space of symmetric adjoint (resp. trivial) Leibniz 2-cocycles.*

(i) $ZL_0^2(\mathfrak{g}, \mathfrak{g}) = \mathfrak{c} \otimes \ker \mathscr{I}$. *In particular,* $\dim ZL_0^2(\mathfrak{g}, \mathfrak{g}) = c\,\frac{p(p+1)}{2}$ *where* $c = \dim \mathfrak{c}$ *and* $p = \dim \mathfrak{g}/\mathscr{C}^2\mathfrak{g} = \dim H^1(\mathfrak{g}, \mathbb{C})$.
(ii) $ZL^2(\mathfrak{g}, \mathfrak{g}) \big/ \left(Z^2(\mathfrak{g}, \mathfrak{g}) \oplus ZL_0^2(\mathfrak{g}, \mathfrak{g})\right) \cong (\mathfrak{c} \otimes Im\, \mathscr{I}) \cap B^3(\mathfrak{g}, \mathfrak{g})$.
(iii) $HL^2(\mathfrak{g}, \mathfrak{g}) \cong H^2(\mathfrak{g}, \mathfrak{g}) \oplus (\mathfrak{c} \otimes \ker \mathscr{I}) \oplus \left((\mathfrak{c} \otimes Im\, \mathscr{I}) \cap B^3(\mathfrak{g}, \mathfrak{g})\right)$.
(iv) $ZL_0^2(\mathfrak{g}, \mathbb{C}) = \ker \mathscr{I}$.
(v) $ZL^2(\mathfrak{g}, \mathbb{C}) \big/ \left(Z^2(\mathfrak{g}, \mathbb{C}) \oplus ZL_0^2(\mathfrak{g}, \mathbb{C})\right) \cong Im \mathscr{I} \cap B^3(\mathfrak{g}, \mathbb{C})$.
(vi) $HL^2(\mathfrak{g}, \mathbb{C}) \cong H^2(\mathfrak{g}, \mathbb{C}) \oplus \ker \mathscr{I} \oplus \left(Im\, \mathscr{I} \cap B^3(\mathfrak{g}, \mathbb{C})\right)$.

Proof (i) The Leibniz 2-cochain space $CL^2(\mathfrak{g}, \mathfrak{g}) = \mathfrak{g} \otimes (\mathfrak{g}^*)^{\otimes 2}$ decomposes as $\left(\mathfrak{g} \otimes \bigwedge^2 \mathfrak{g}^*\right) \oplus \left(\mathfrak{g} \otimes S^2\, \mathfrak{g}^*\right)$ with $\mathfrak{g} \otimes S^2\, \mathfrak{g}^*$ the space of symmetric elements in $CL^2(\mathfrak{g}, \mathfrak{g})$. By definition of the Leibniz coboundary δ, one has for $\psi \in CL^2(\mathfrak{g}, \mathfrak{g})$ and $X, Y, Z \in \mathfrak{g}$

$$(\delta\psi)(X, Y, Z) = u + v + w + r + s + t \tag{1}$$

with $u = [X, \psi(Y, Z)]$, $v = [\psi(X, Z), Y]$, $w = -[\psi(X, Y), Z]$, $r = -\psi([X, Y], Z)$, $s = \psi(X, [Y, Z])$, $t = \psi([X, Z], Y)$. δ coincides with the usual coboundary operator on $\mathfrak{g} \otimes \bigwedge^2 \mathfrak{g}^*$. Now, let $\psi = \psi_1 + \psi_0 \in CL^2(\mathfrak{g}, \mathfrak{g})$, $\psi_1 \in \mathfrak{g} \otimes \bigwedge^2 \mathfrak{g}^*$, $\psi_0 \in \mathfrak{g} \otimes S^2\, \mathfrak{g}^*$.

Suppose $\psi \in ZL^2(\mathfrak{g}, \mathfrak{g}) : \delta\psi = 0 = \delta\psi_1 + \delta\psi_0 = d\psi_1 + \delta\psi_0$. Then $\delta\psi_0 = -d\psi_1 \in \mathfrak{g} \otimes \bigwedge^3 \mathfrak{g}^*$ is antisymmetric. Then permuting X and Y in formula (1) for ψ_0 yields $(\delta\psi_0)(Y, X, Z) = -v - u + w - r + t + s$. As $\delta\psi_0$ is antisymmetric, we get

$$w + s + t = 0. \tag{2}$$

Now, the circular permutation (X, Y, Z) in (1) for ψ_0 yields $(\delta\psi_0)(Y, Z, X) = -v - w + u - s - t + r$. Again, by antisymmetry,

$$v + w + s + t = 0, \tag{3}$$

i.e. $(\delta\psi_0)(X, Y, Z) = u + r$.

From (2) and (3), $v = 0$. Applying twice the circular permutation (X, Y, Z) to v, we get first $w = 0$ and then $u = 0$. Hence $(\delta\psi_0)(X, Y, Z) = r = -\psi_0([X, Y], Z)$. Note first that $u = 0$ reads $[X, \psi_0(Y, Z)] = 0$. As X, Y, Z are arbitrary, ψ_0 is $\mathfrak{c}$-valued. Now the permutation of Y and Z changes r to $-t = s$ (from (3)). Again, by antisymmetry of $\delta\psi_0$, $r = t = -s$. As X, Y, Z are arbitrary, one gets $\psi_0 \in \mathfrak{c} \otimes \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$. Now $F(\psi_0) = -r = -\delta\psi_0 = d\psi_1 \in B^3(\mathfrak{g}, \mathfrak{g})$. Hence

$$\psi_0 \in ZL_0^2(\mathfrak{g}, \mathfrak{g}) \Leftrightarrow F(\psi_0) = 0 \Leftrightarrow \psi_1 \in Z^2(\mathfrak{g}, \mathfrak{g}) \Leftrightarrow \psi_0 \in \mathfrak{c} \otimes \ker \mathscr{I}.$$

Consider now the linear map $\Phi : ZL^2(\mathfrak{g}, \mathfrak{g}) \to F^{-1}(B^3(\mathfrak{g}, \mathfrak{g}))\big/ \ker F$ defined by $\psi \mapsto [\psi_0] \,(\mathrm{mod} \ker F)$. Φ is onto: for any $[\varphi_0] \in F^{-1}(B^3(\mathfrak{g}, \mathfrak{g}))\big/ \ker F$, $\varphi_0 \in \mathfrak{c} \otimes \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$, one has $F(\varphi_0) \in B^3(\mathfrak{g}, \mathfrak{g})$, hence $F(\varphi_0) = d\varphi_1$, $\varphi_1 \in C^2(\mathfrak{g}, \mathfrak{g})$, and then $\varphi = \varphi_0 + \varphi_1$ is a Leibniz cocycle such that $\Phi(\varphi) = [\varphi_0]$. Now $\ker \Phi = Z^2(\mathfrak{g}, \mathfrak{g}) \oplus ZL_0^2(\mathfrak{g}, \mathfrak{g})$, since condition $[\psi_0] = [0]$ reads $\psi_0 \in \ker F$ which is equivalent to $\psi \in Z^2(\mathfrak{g}, \mathfrak{g}) \oplus ZL_0^2(\mathfrak{g}, \mathfrak{g})$. Hence Φ yields an isomorphism $ZL^2(\mathfrak{g}, \mathfrak{g})\big/\left(Z^2(\mathfrak{g}, \mathfrak{g}) \oplus ZL_0^2(\mathfrak{g}, \mathfrak{g})\right) \cong F^{-1}(B^3(\mathfrak{g}, \mathfrak{g}))\big/ \ker F$. The latter is isomorphic to $\mathrm{Im}\, F \cap B^3(\mathfrak{g}, \mathfrak{g}) \cong (\mathfrak{c} \otimes \mathrm{Im}\, \mathscr{I}) \cap B^3(\mathfrak{g}, \mathfrak{g})$.

(ii) results from the invariance of $\psi_0 \in ZL_0^2(\mathfrak{g}, \mathfrak{g})$.
(iii) results immediately from (i) and (ii) since $BL^2(\mathfrak{g}, \mathfrak{g}) = B^2(\mathfrak{g}, \mathfrak{g})$ as the Leibniz differential on $CL^1(\mathfrak{g}, \mathfrak{g}) = \mathfrak{g}^* \otimes \mathfrak{g} = C^1(\mathfrak{g}, \mathfrak{g})$ coincides with the usual one.
(iv)-(vi) are similar.

Remark 3.1 Since $\ker \mathscr{I} \oplus \left(\mathrm{Im}\, \mathscr{I} \cap B^3(\mathfrak{g}, \mathbb{C})\right) \cong \ker h$ where h denotes $\mathscr{I}$ composed with the projection of $Z^3(\mathfrak{g}, \mathbb{C})$ onto $H^3(\mathfrak{g}, \mathbb{C})$, the result (vi) is the same as in [13].

Remark 3.2 Any supplementary subspace to $Z^2(\mathfrak{g}, \mathbb{C}) \oplus ZL_0^2(\mathfrak{g}, \mathbb{C})$ in $ZL^2(\mathfrak{g}, \mathbb{C})$ consists of coupled Leibniz 2-cocycles, i.e. the nonzero elements have the property that their symmetric and antisymmetric parts are not cocycles. To get such a supplementary subspace, pick any supplementary subspace W to $\ker \mathscr{I}$ in $\left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$ and take $\mathscr{C} = \left\{B + \omega\, ;\, B \in W \cap \mathscr{I}^{-1}(B^3(\mathfrak{g}, \mathbb{C})), I_B = d\omega\right\}$.

Definition 3.2 $\mathfrak{g}$ is said to be an adjoint (resp. trivial) ZL^2-uncoupling if

$$(\mathfrak{c} \otimes \mathrm{Im}\, \mathscr{I}) \cap B^3(\mathfrak{g}, \mathfrak{g}) = \{0\} \left(\text{resp. } \mathrm{Im}\, \mathscr{I} \cap B^3(\mathfrak{g}, \mathbb{C}) = \{0\}\right).$$

The class of adjoint ZL^2-uncoupling Lie algebras is rather extensive since it contains all zero-center Lie algebras and all $\mathscr{I}$-null Lie algebras. For non zero-center, adjoint ZL^2-uncoupling implies trivial ZL^2-uncoupling, since $\mathfrak{c} \otimes \big(\mathrm{Im}\,\mathscr{I} \cap B^3(\mathfrak{g}, \mathbb{C})\big) \subset (\mathfrak{c} \otimes \mathrm{Im}\,\mathscr{I}) \cap B^3(\mathfrak{g}, \mathfrak{g})$. The reciprocal holds obviously true for $\mathscr{I}$-exact Lie algebras. However we do not know if it holds true in general (e.g. we do not know of a nilpotent Lie algebra which is not $\mathscr{I}$-exact).

Corollary 3.1 (i) $HL^2(\mathfrak{g}, \mathfrak{g}) \cong H^2(\mathfrak{g}, \mathfrak{g}) \oplus (\mathfrak{c} \otimes \ker \mathscr{I})$ *if and only if* $\mathfrak{g}$ *is adjoint* ZL^2*-uncoupling.*

(ii) $HL^2(\mathfrak{g}, \mathbb{C}) \cong H^2(\mathfrak{g}, \mathbb{C}) \oplus \ker \mathscr{I}$ *if and only if* $\mathfrak{g}$ *is trivial* ZL^2*-uncoupling.*

Corollary 3.2 *For any Lie algebra* $\mathfrak{g}$ *with trivial center* $\mathfrak{c} = \{0\}$, $HL^2(\mathfrak{g}, \mathfrak{g}) = H^2(\mathfrak{g}, \mathfrak{g})$.

Remark 3.3 This fact also follows from the cohomological version of Theorem A in [16].

Proof Let $\mathfrak{g}$ be a Lie algebra and M be a right $\mathfrak{g}$-module. Consider the product map $m : \mathfrak{g} \otimes \Lambda^n \mathfrak{g} \longrightarrow \Lambda^{n+1}$ in the exterior algebra. This map yields an epimorphism of chain complexes

$$C_*(\mathfrak{g}, \mathfrak{g}) \longrightarrow C_*(\mathfrak{g}, \mathbb{K})[-1],$$

where $C_*(\mathfrak{g}, \mathbb{K})$ is the reduced chain complex:

$$C_0(\mathfrak{g}, \mathbb{K}) = 0, \quad C_i(\mathfrak{g}, \mathbb{K}) = C_i(\mathfrak{g}, \mathbb{K}) \text{ for } i > 0.$$

Define the reduced chain complex $CR_*(\mathfrak{g})$ such that $CR_*(\mathfrak{g}[1])$ is the kernel of the epimorphism $C_*(\mathfrak{g}, \mathfrak{g}) \longrightarrow C_*(\mathfrak{g}, \mathbb{K})[-1]$. Denote the cohomology of $CR_*(\mathfrak{g})$ by $HR_*(\mathfrak{g})$.

Let us recall Theorem A in [16]. It states that there exists a spectral sequence

$$E^2_{pq} = HR_p(\mathfrak{g} \otimes HL_q(\mathfrak{g}, M)) \Longrightarrow H^{rel}_{p+q}(\mathfrak{g}, M).$$

As the center of our Lie algebra is 0, it follows that $E^2_{00} = 0$, and so we get $H^{rel}_0(\mathfrak{g}, \mathfrak{g}) = 0$.

But then from the exact sequence in [16]

$$0 \leftarrow H_2(\mathfrak{g}, M) \leftarrow HL_2(\mathfrak{g}, M) \leftarrow H^{rel}_0(\mathfrak{g}, M) \leftarrow H_3(\mathfrak{g}, M) \leftarrow \ldots$$

we get

$$HL_2(\mathfrak{g}, M) = H_2(\mathfrak{g}, M).$$

Corollary 3.3 *For any reductive Lie algebra* $\mathfrak{g}$ *with center* $\mathfrak{c}$, $HL^2(\mathfrak{g}, \mathfrak{g}) = H^2(\mathfrak{g}, \mathfrak{g}) \oplus \left(\mathfrak{c} \otimes S^2\mathfrak{c}^*\right)$, *and* $\dim H^2(\mathfrak{g}, \mathfrak{g}) = \frac{c^2(c-1)}{2}$ *with* $c = \dim \mathfrak{c}$.

Proof $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{c}$ with $\mathfrak{s} = \mathscr{C}^2\mathfrak{g}$ semisimple. We first prove that $\mathfrak{g}$ is adjoint ZL^2-uncoupling. We have $\mathfrak{c} \otimes \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}} = \left(\mathfrak{c} \otimes \left(S^2\mathfrak{s}^*\right)^{\mathfrak{s}}\right) \oplus \left(\mathfrak{c} \otimes S^2\mathfrak{c}^*\right) = c\,\left(S^2\mathfrak{s}^*\right)^{\mathfrak{s}} \oplus c\,\left(S^2\mathfrak{c}^*\right)$. Suppose first $\mathfrak{s}$ simple. Then any bilinear symmetric invariant form on $\mathfrak{s}$ is some multiple of the Killing form K. Hence $\mathfrak{c} \otimes \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}} = c\,(\mathbb{C}K) \oplus c\,\left(S^2\mathfrak{c}^*\right)$. For any $\psi_0 \in \mathfrak{c} \otimes \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$, $F(\psi_0)$ is then some linear combination of copies of I_K. It is well-known, I_K is not a coboundary. Hence if we suppose that $F(\psi_0)$ is a coboundary, necessarily $F(\psi_0) = 0$. The Lie algebra $\mathfrak{g}$ is adjoint ZL^2-uncoupling when $\mathfrak{s}$ is simple. Now, if $\mathfrak{s}$ is not simple, $\mathfrak{s}$ can be decomposed as a direct sum $\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_m$ of simple ideals of $\mathfrak{s}$. Then $\left(S^2\mathfrak{s}^*\right)^{\mathfrak{s}} = \bigoplus_{i=1}^m \left(S^2\mathfrak{s}_i{}^*\right)^{\mathfrak{s}_i} = \bigoplus_{i=1}^m \mathbb{C}\,K_i$ (K_i Killing form of $\mathfrak{s}_i$.) The same reasoning then applies and shows that $\mathfrak{g}$ is adjoint ZL^2-uncoupling. From (ii) in Theorem 3.1, we have $ZL_0^2(\mathfrak{g}, \mathfrak{g}) = \mathfrak{c} \otimes S^2\mathfrak{c}^*$. Now, $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{c}$ with $\mathfrak{s} = \mathscr{C}^2\mathfrak{g}$ semisimple. The subalgebra $\mathfrak{s}$ can be decomposed as a direct sum $\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_m$ of ideals of $\mathfrak{s}$, hence of $\mathfrak{g}$. Then $H^2(\mathfrak{g}, \mathfrak{g}) = \bigoplus_{i=1}^m H^2(\mathfrak{g}, \mathfrak{s}_i) \oplus H^2(\mathfrak{g}, \mathfrak{c})$. As $\mathfrak{s}_i$ is a nontrivial $\mathfrak{g}$-module, $H^2(\mathfrak{g}, \mathfrak{s}_i) = \{0\}$ ([8], Prop. 11.4, page 154). So we get $H^2(\mathfrak{g}, \mathfrak{g}) = H^2(\mathfrak{g}, \mathfrak{c}) = c\,H^2(\mathfrak{g}, \mathbb{C})$. By the Künneth formula and Whitehead's lemmas,

$$
\begin{aligned}
H^2(\mathfrak{g}, \mathbb{C}) &= \left(H^2(\mathfrak{s}, \mathbb{C}) \otimes H^0(\mathfrak{c}, \mathbb{C})\right) \oplus \left(H^1(\mathfrak{s}, \mathbb{C})\right. \\
&\quad \left.\otimes H^1(\mathfrak{c}, \mathbb{C})\right) \oplus \left(H^0(\mathfrak{s}, \mathbb{C}) \otimes H^2(\mathfrak{c}, \mathbb{C})\right) \\
&= H^0(\mathfrak{s}, \mathbb{C}) \otimes H^2(\mathfrak{c}, \mathbb{C}) \\
&= \mathbb{C} \otimes H^2(\mathfrak{c}, \mathbb{C}).
\end{aligned}
$$

Hence

$$
\dim H^2(\mathfrak{g}, \mathfrak{g}) = \frac{c^2(c-1)}{2}.
$$

4 The Diamond Algebra

The 4-dimensional complex solvable "diamond" Lie algebra $\mathfrak{d}$ has basis (x_1, x_2, x_3, x_4) and nonzero commutation relations (with anticommutativity)

$$
[x_1, x_2] = x_3,\ [x_1, x_3] = -x_2,\ [x_2, x_3] = x_4. \tag{4}
$$

The relations show that $\mathfrak{d}$ is an extension of the one-dimensional abelian Lie algebra $\mathbb{C}x_1$ by the Heisenberg algebra $\mathfrak{n}_3$ with basis x_2, x_3, x_4. It is also known as the Nappi-Witten Lie algebra [14] or the central extension of the Poincaré Lie algebra in two dimensions. It is a solvable quadratic Lie algebra, as it admits a nondegenerate bilinear symmetric invariant form. Because of these properties, it plays an important role in conformal field theory.

We can use $\mathfrak{d}$ to construct a Wess-Zumino-Witten model, which describes a homogeneous four-dimensional Lorentz-signature space time [14].

It is easy to check that $\mathfrak{d}$ is $\mathscr{I}$-exact. In fact, one verifies that all other solvable 4-dimensional Lie algebras are $\mathscr{I}$-null (for a list, see e.g. [15]).

Consider $\mathfrak{d}$ as Leibniz algebra with basis $\{e_1,\ e_2,\ e_3,\ e_4\}$ over $\mathbb{C}$. Define a bilinear map $[\ ,\ ] : \mathfrak{d} \times \mathfrak{d} \longrightarrow \mathfrak{d}$ by $[e_2, e_3] = e_1$, $[e_3, e_2] = -e_1$, $[e_2, e_4] = e_2$, $[e_4, e_2] = -e_2$, $[e_3, e_4] = e_2 - e_3$ and $[e_4, e_3] = e_3 - e_2$, all other products of basis elements being 0.

We get a basis satisfying the usual commutation relations (4) by letting

$$x_1 = ie_4, \quad x_2 = e_3, \quad x_3 = i(-e_2 + e_3), \quad x_4 = ie_1. \tag{5}$$

One should mention that even though these two forms are equivalent over $\mathbb{C}$, they represent the two nonisomorphic real forms of the complex diamond algebra.

We found that by considering Leibniz algebra deformations of $\mathfrak{d}$ one gets more structures. Indeed it gives not only extra structures but also keeps the Lie structures obtained by considering Lie algebra deformations. To get the precise deformations we need to consider the cohomology groups.

We compute cohomologies necessary for our purpose. Let us use the simpler notation L for the diamond algebra. First consider the Leibniz cohomology space $HL^2(L; L)$. Our computation consists of the following steps:

(i) determine a basis of the space of cocycles $ZL^2(L; L)$,
(ii) determine a basis of the coboundary space $BL^2(L; L)$,
(iii) determine a basis of the quotient space $HL^2(L; L)$.

(i) Let $\psi \in ZL^2(L; L)$. Then $\psi : L \otimes L \longrightarrow L$ is a linear map and $\delta\psi = 0$, where

$$\begin{aligned}\delta\psi(e_i, e_j, e_k) = [e_i, \psi(e_j, e_k)] + [\psi(e_i, e_k), e_j] - [\psi(e_i, e_j), e_k] - \psi([e_i, e_j], e_k)\\ +\psi(e_i, [e_j, e_k]) + \psi([e_i, e_k], e_j) \text{ for } 0 \le i, j, k \le 4.\end{aligned}$$

Suppose $\psi(e_i, e_j) = \sum_{k=1}^{4} a_{i,j}^k e_k$ where $a_{i,j}^k \in \mathbb{C}$; for $1 \le i, j, k \le 4$. Since $\delta\psi = 0$, equating the coefficients of e_1, e_2, e_3 and e_4 in $\delta\psi(e_i, e_j, e_k)$ we get the following relations:

(i) $a_{1,1}^1 = a_{1,1}^2 = a_{1,1}^3 = a_{1,1}^4 = a_{1,2}^1 = a_{1,2}^3 = a_{1,2}^4 = 0$;
(ii) $a_{1,3}^4 = a_{1,4}^3 = a_{1,4}^4 = a_{2,1}^1 = a_{2,1}^3 = a_{2,1}^4 = a_{2,2}^1 = a_{2,2}^2 = a_{2,2}^3 = a_{2,2}^4 = 0$;
(iii) $a_{3,1}^4 = a_{3,3}^2 = a_{3,3}^3 = a_{3,3}^4 = a_{4,1}^3 = a_{4,1}^4 = a_{4,4}^2 = a_{4,4}^3 = a_{4,4}^4 = 0$;
(iv) $a_{1,2}^2 = -a_{2,1}^2 = a_{1,3}^2 = -a_{1,3}^3 = -a_{3,1}^2 = a_{3,1}^3$;
(v) $a_{1,3}^1 = -a_{3,1}^1 = a_{1,4}^2 = -a_{4,1}^2$;
(vi) $a_{2,3}^3 = -a_{3,2}^3 = -a_{2,4}^4 = a_{4,2}^4$; $a_{2,3}^4 = -a_{3,2}^4$; $a_{2,3}^2 = -a_{3,2}^2$;
(vi) $a_{2,4}^1 = -a_{4,2}^1$; $a_{2,4}^2 = -a_{4,2}^2$; $a_{2,4}^3 = -a_{4,2}^3$;

(vii) $a^1_{3,4} = -a^1_{4,3};\ a^2_{3,4} = -a^2_{4,3};\ a^3_{3,4} = -a^3_{4,3};\ a^4_{3,4} = -a^4_{4,3}$

(ix) $a^3_{3,4} = (a^1_{14} - a^2_{24});\ \ a^4_{3,4} = (a^2_{14} + a^2_{23})$

(x) $a^1_{33} = \frac{1}{2}(a^1_{23} + a^1_{32});\ a^1_{41} = -(a^1_{14} + a^1_{23} + a^1_{32})$.

Therefore, in terms of the ordered basis $\{e_i \otimes e_j\}_{1\le i,j\le 4}$ of $L \otimes L$ and $\{e_i\}_{1\le i\le 4}$ of L, the transpose of the matrix corresponding to ψ is of the form

$$M^t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ x_2 & x_1 & -x_1 & 0 \\ x_3 & x_2 & 0 & 0 \\ 0 & -x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x_4 & x_5 & x_6 & x_7 \\ x_8 & x_9 & x_{10} & -x_6 \\ -x_2 & -x_1 & x_1 & 0 \\ x_{11} & -x_5 & -x_6 & -x_7 \\ \frac{1}{2}(x_4 + x_{11}) & 0 & 0 & 0 \\ x_{12} & x_{13} & (x_3 - x_9) & (x_2 + x_5) \\ -(x_4 + x_3 + x_{11}) & -x_2 & 0 & 0 \\ -x_8 & -x_9 & -x_{10} & x_6 \\ -x_{12} & -x_{13} & -(x_3 - x_9) & -(x_2 + x_5) \\ x_{14} & 0 & 0 & 0 \end{pmatrix}.$$

where $x_1 = a^2_{1,2};\ x_2 = a^1_{1,3};\ x_3 = a^1_{1,4};\ x_4 = a^1_{2,3};\ x_5 = a^2_{2,3};\ x_6 = a^3_{2,3};$ $x_7 = a^4_{2,3};\ x_8 = a^1_{2,4};\ x_9 = a^2_{2,4}; x_{10} = a^3_{2,4};\ x_{11} = a^1_{3,2};\ x_{12} = a^1_{3,4};$ $x_{13} = a^2_{3,4}$ and $x_{14} = a^1_{4,4}$

are in $\mathbb{C}$. Let $\phi_i \in ZL^2(L;L)$ for $1 \le i \le 14$, be the cocycle with $x_i = 1$ and $x_j = 0$ for $i \ne j$ in the above matrix of ψ. It is easy to check that $\{\phi_1, \dots, \phi_{14}\}$ forms a basis of $ZL^2(L;L)$.

(ii) Let $\psi_0 \in BL^2(L;L)$. We have $\psi_0 = \delta g$ for some 1-cochain $g \in CL^1(L;L) =$ Hom $(L;L)$. Suppose the matrix associated to ψ_0 is the same as the above matrix M.

Let $g(e_i) = a^1_i e_1 + a^2_i e_2 + a^3_i e_3 + a^4_i e_4$ for $i = 1, 2, 3, 4$. The matrix associated to g is given by

$$(a^j_i)_{i,j=1,\dots,4}$$

From the definition of the coboundary we get

$$\delta g(e_i, e_j) = [e_i, g(e_j)] + [g(e_i), e_j] - \psi([e_i, e_j])$$

for $0 \le i, j \le 4$. If we write out the transpose matrix of

$$\delta g,$$

and compare it with M (since $\psi_0 = \delta g$ is also a cocycle in $CL^2(L; L)$), we conclude that the transpose matrix of ψ_0 is of the form

$$M^t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ x_2 & x_1 & -x_1 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & -x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x_4 & x_5 & x_6 & x_1 \\ x_8 & x_9 & x_{10} & -x_6 \\ -x_2 & -x_1 & x_1 & 0 \\ -x_4 & -x_5 & -x_6 & -x_1 \\ 0 & 0 & 0 & 0 \\ x_{12} & x_{13} & -x_9 & (x_2 + x_5) \\ 0 & -x_2 & 0 & 0 \\ -x_8 & -x_9 & -x_{10} & x_6 \\ -x_{12} & -x_{13} & x_9 & -(x_2 + x_5) \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\phi_i' \in BL^2(L; L)$ for $i = 1, 2, 4, 5, 6, 8, 9, 10, 12, 13$ be the coboundary with $x_i = 1$ and $x_j = 0$ for $i \neq j$ in the above matrix of ψ_0. It follows that $\{\phi_1', \phi_2', \phi_4', \phi_5', \phi_6', \phi_8', \phi_9', \phi_{10}', \phi_{12}', \phi_{13}'\}$ forms a basis of the coboundary space $BL^2(L; L)$.

(iii) It is straightforward to check that

$$\{[\phi_3], [\phi_7], [\phi_{11}], [\phi_{14}]\}$$

span $HL^2(L; L)$ where $[\phi_i]$ denotes the cohomology class represented by the cocycle ϕ_i.

Thus $\dim(HL^2(L; L)) = 4$.

The representative cocycles of the cohomology classes forming a basis of $HL^2(L; L)$ are given explicitely as the following.

(1) $\phi_3 : \phi_3(e_1, e_4) = e_1,\ \phi_3(e_4, e_1) = -e_1;\ \phi_3(e_3, e_4) = e_3;\ \phi_3(e_4, e_3) = -e_3;$
(2) $\phi_7 : \phi_7(e_2, e_3) = e_4,\ \phi_7(e_3, e_2) = -e_4;$
(3) $\phi_{11} : \phi_{11}(e_3, e_2) = e_1,\ \phi_{11}(e_3, e_3) = \frac{1}{2}e_1,\ \phi_{11}(e_4, e_1) = -e_1;$
(4) $\phi_{14} : \phi_{14}(e_4, e_4) = e_1.$

Here ϕ_3 and ϕ_7 are skew-symmetric, so $\phi_i \in Hom(\Lambda^2 L; L) \subset Hom(L^{\otimes 2}; L)$ for $i = 3$ and 7.

Consider $\mu_i = \mu_0 + t\phi_i$ for $i = 3, 7, 11, 14$, where μ_0 denotes the original bracket in L.

This gives 4 non-equivalent infinitesimal deformations of the Leibniz bracket μ_0 with μ_3 and μ_7 giving the Lie algebra structure on the factor space $L[[t]]/ < t^2 >$.

Now we have to compute the nontrivial Massey brackets which give relations on the base of the miniversal deformation.

Let us start to compute the nonzero brackets $[\phi_i, \phi_i]$ which are the obstructions to extending infinitesimal deformations. We find

$$[\phi_3, \phi_3] = 0, \quad [\phi_7, \phi_7] = 0.$$

That means that these two infinitesimal Lie deformations can be extended. In fact, they can be extended to real Lie deformations as follows.

We give the new nonzero Lie brackets (and their anticommutative analogue).

The first deformation

$$\begin{aligned} [e_2, e_3]_t &= e_1 + te_4 \\ [e_2, e_4]_t &= e_2 \\ [e_3, e_4]_t &= e_2 - e_3 \end{aligned}$$

is isomorphic to $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$ for every nonzero value of t, see [7].

The second deformation represents a 2-parameter projective family $d(\lambda, \mu)$, for which each projective parameter (λ, μ) defines a nonisomorphic Lie algebra (in fact, the diamond algebra is a member of this family with $(\lambda, \mu) = (1, -1)$):

$$\begin{aligned} [e_2, e_3]_{\lambda,\mu} &= e_1 \\ [e_2, e_4]_{\lambda,\mu} &= \lambda e_2 \\ [e_3, e_4]_{\lambda,\mu} &= e_2 + \mu e_3 \\ [e_1, e_4]_{\lambda,\mu} &= (\lambda + \mu) e_1. \end{aligned}$$

Furthermore, we also have $[\phi_{14}, \phi_{14}] = 0$ which means that ϕ_{14} defines a real Leibniz deformation:

$$\begin{aligned} [e_2, e_3]_t &= e_1 \\ [e_2, e_4]_t &= e_2 \\ [e_3, e_4]_t &= e_2 - e_3 \\ [e_4, e_4]_t &= te_1. \end{aligned}$$

We note that this Leibniz algebra is not nilpotent.

For the bracket $[\phi_{11}, \phi_{11}]$ we get a nonzero 3-cocycle, so the infinitesimal Leibniz deformation with infinitesimal part being ϕ_{11} can not be extended even to the next order. That means it gives a relation on the base of the versal deformation.

The nontrivial mixed brackets $[\phi_i, \phi_j]$ also determine relations on the base of the versal deformation.

Among the six possible cases $[\phi_3, \phi_{11}]$, $[\phi_3, \phi_{14}]$ and $[\phi_{11}, \phi_{14}]$ are nontrivial 3-cocycles, the others are represented by 3-coboundaries.

Thus we need to check the Massey 3-brackets which are defined, namely
$< \phi_3, \phi_3, \phi_7 >, < \phi_3, \phi_7, \phi_7 >, < \phi_7, \phi_7, \phi_{11} >,$
$< \phi_7, \phi_7, \phi_{14} >, < \phi_7, \phi_{14}, \phi_{14} >.$

In these five possible Massey 3-brackets, only $< \phi_3, \phi_3, \phi_7 >$ is represented by nontrivial cocycle.

So we now proceed to compute the possible Massey 4-brackets. We get that four of them are nontrivial:
$< \phi_3, \phi_7, \phi_7, \phi_{11} >, < \phi_3, \phi_7, \phi_7, \phi_{14} >,$
$< \phi_7, \phi_7, \phi_{14}, \phi_{11} >, < \phi_7, \phi_7, \phi_{14}, \phi_{14} >.$

At the next step, we get that all the Massey 5-brackets which are defined are trivial.

So we can write the universal infinitesimal Leibniz deformation of our Lie algebra:

$$
\begin{aligned}
&[e_1, e_2]_v = [e_2, e_1]_v = [e_1, e_3]_v = [e_3, e_1]_v = 0,\\
&[e_1, e_4]_v = te_1, \quad [e_4, e_1]_v = -(t+u)e_1,\\
&[e_2, e_3]_v = e_1 + se_4, \quad [e_3, e_2]_v = (u-1)e_1 - se_4,\\
&[e_2, e_4]_v = e_2, \quad [e_4, e_2] = -e_2,\\
&[e_3, e_4]_v = e_2 + (t-1)e_3, \quad [e_4, e_3]_v = -e_2 + (1-t)e_3,\\
&[e_1, e_1]_v = [e_2, e_2]_v = 0, \quad [e_3, e_3]_v = 1/2ue_1,\\
&[e_4, e_4]_v = we_1.
\end{aligned}
$$

With the nontrivial Massey brackets and the identification $t = \phi_3, s = \phi_7, u = \phi_{11}, w = \phi_{14}$, we get that the base of the infinitesimal deformation is

$$\mathbb{C}[[t, s, u, w]]/\{u^2, tu, tw, uw; t^2 s; ts^2 u, ts^2 w, s^2 uw, s^2 w^2\}.$$

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