Constructions of Quadratic n-ary Hom-Nambu Algebras

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Abstract The aim of this paper is provide a survey on n-ary Hom-Nambu algebras and study quadratic n-ary Hom-Nambu algebras, which are n-ary Hom-Nambu algebras with an invariant, nondegenerate and symmetric bilinear forms that are also α-symmetric and β-invariant where α and β are twisting maps. We provide various constructions of quadratic n-ary Hom-Nambu algebras. Also is discussed their connections with representation theory and centroids.

1 Introduction

The main motivations to study n-ary algebras came firstly from Nambu mechanics [34] where a ternary bracket allows to use more than one hamiltonian and recently from string theory and M-branes which involve naturally an algebra with ternary operation called Bagger-Lambert algebra [11]. Also ternary operations appeared in the study of some quarks models see [22–24]. For more general theory and further results see references [5, 6, 15–17, 20, 21, 26, 27, 35, 38].

Algebras endowed with invariant nondegenerate symmetric bilinear form (scalar product) appeared also naturally in several domains in mathematics and physics. Such algebras were intensively studied for binary Lie and associative algebras. The main results are that called double extension given by Medina and Revoy [33] and T*-extension given by Bordemann [14]. These fundamental results were extended to

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n-ary algebras in [18]. The extension to Hom-setting for binary case was introduced and studied in [13]. For further results about Hom-type algebras, see refs [2–4, 25, 28, 30–32, 41, 42].

In this paper we summarize in the Sect. 2 definitions of n-ary Hom-Nambu algebras and recall the constructions using twisting principles and tensor product with n-ary algebras of Hom-associative type. In Sect. 3, we introduce the notion of quadratic n-ary Hom-Nambu algebra, generalizing the notion introduced for binary Hom-Lie algebras in [13]. A more general notion called Hom-quadratic n-ary Hom-Nambu algebra is introduced by twisting the invariance identity. In Sect. 4, we show that a quadratic n-ary Hom-Nambu algebra gives rise to a quadratic Hom-Leibniz algebra. A connection with representation theory is discussed in Sect. 5. We deal in particular with adjoint and coadjoint representations, extending the representation theory initiated in [13, 36]. Several procedures to built quadratic n-ary Hom-Nambu algebras are provided in Sect. 6. We use twisting principles, tensor product and T*-extension to construct quadratic n-ary Hom-Nambu algebras. Moreover we show that one may derive from quadratic n-ary Hom-Nambu algebra ones of increasingly higher arities and that under suitable assumptions it reduces to a quadratic $(n-1)$-ary Hom-Nambu algebra. Also real Faulkner construction is used to obtain ternary Hom-Nambu algebras. The last Sect. 7 is dedicated to introduce and study the centroids of n-ary Hom-Nambu algebras and their properties. We supply a construction procedure of quadratic n-ary Hom-Nambu algebras using elements of the centroid.

Most of the results concern n-ary Hom-Nambu algebras. Naturally they are valid and may be stated for n-ary Hom-Nambu-Lie algebras.

2 The *n*-ary Hom-Nambu Algebras

Throughout this paper, $\mathbb{K}$ is an algebraically closed field of characteristic zero, even though for most of the general definitions and results in the paper this assumption is not essential.

2.1 Definitions

In this section, we summarize definitions of n-ary Hom-Nambu algebras and n-ary Hom-Nambu-Lie algebras, introduced in [7], generalizing n-ary Nambu algebras and n-ary Nambu-Lie algebras (called also Filippov algebras or n-Lie algebras).

Definition 2.1 An *n-ary Hom-Nambu* algebra is a triple $(\mathcal{N}, [\cdot, \ldots, \cdot], \widetilde{\alpha})$ consisting of a vector space $\mathcal{N}$, an n-linear map $[\cdot, \ldots, \cdot] : \mathcal{N}^n \longrightarrow \mathcal{N}$ and a family $\widetilde{\alpha} = (\alpha_i)_{1\leq i\leq n-1}$ of linear maps $\alpha_i : \ \mathcal{N} \longrightarrow \mathcal{N}$, satisfying

$$\begin{aligned}&\big[\alpha_1(x_1),\ldots,\alpha_{n-1}(x_{n-1}),[y_1,\ldots,y_n]\big]\\&=\sum_{i=1}^{n}\big[\alpha_1(y_1),\ldots,\alpha_{i-1}(y_{i-1}),[x_1,\ldots,x_{n-1},y_i],\alpha_i(y_{i+1}),\ldots,\alpha_{n-1}(y_n)\big],\end{aligned}\quad (1)$$

for all $(x_1,\ldots,x_{n-1})\in\mathcal{N}^{n-1}$, $(y_1,\ldots,y_n)\in\mathcal{N}^n$.

The identity (1) is called *Hom-Nambu identity*.

Let $x=(x_1,\ldots,x_{n-1})\in\mathcal{N}^{n-1}$, $\widetilde{\alpha}(x)=(\alpha_1(x_1),\ldots,\alpha_{n-1}(x_{n-1}))\in\mathcal{N}^{n-1}$ and $y\in\mathcal{N}$. We define an adjoint map $ad(x)$ as a linear map on $\mathcal{N}$, such that

$$ad(x)(y)=[x_1,\ldots,x_{n-1},y]. \quad (2)$$

Then the Hom-Nambu identity (1) may be written in terms of adjoint map as

$$\begin{aligned}&ad(\widetilde{\alpha}(x))([x_n,\ldots,x_{2n-1}])\\&=\sum_{i=n}^{2n-1}[\alpha_1(x_n),\ldots,\alpha_{i-n}(x_{i-1}),ad(x)(x_i),\alpha_{i-n+1}(x_{i+1}),\ldots,\alpha_{n-1}(x_{2n-1})].\end{aligned}$$

Remark 2.1 When the maps $(\alpha_i)_{1\le i\le n-1}$ are all identity maps, one recovers the classical n-ary Nambu algebras. The Hom-Nambu Identity (1), for $n=2$, corresponds to Hom-Jacobi identity (see [29]), which reduces to Jacobi identity when $\alpha_1=id$.

Let $(\mathcal{N},[\cdot,\ldots,\cdot],\widetilde{\alpha})$ and $(\mathcal{N}',[\cdot,\ldots,\cdot]',\widetilde{\alpha}')$ be two n-ary Hom-Nambu algebras where $\widetilde{\alpha}=(\alpha_i)_{i=1,\ldots,n-1}$ and $\widetilde{\alpha}'=(\alpha'_i)_{i=1,\ldots,n-1}$. A linear map $f:\mathcal{N}\to\mathcal{N}'$ is an n-ary Hom-Nambu algebras *morphism* if it satisfies

$$\begin{aligned}f([x_1,\cdots,x_n])&=[f(x_1),\cdots,f(x_n)]'\\f\circ\alpha_i&=\alpha'_i\circ f\quad\forall i=1,\ldots,n-1.\end{aligned}$$

Definition 2.2 An n-ary Hom-Nambu algebra $(\mathcal{N},[\cdot,\ldots,\cdot],\widetilde{\alpha})$ where $\widetilde{\alpha}=(\alpha_i)_{1\le i\le n-1}$ is called *n-ary Hom-Nambu-Lie* algebra if the bracket is skew-symmetric that is

$$[x_{\sigma(1)},\ldots,x_{\sigma(n)}]=Sgn(\sigma)[x_1,\ldots,x_n],\quad\forall\sigma\in\mathcal{S}_n\ \text{and}\ \forall x_1,\ldots,x_n\in\mathcal{N}. \quad (3)$$

where $\mathcal{S}_n$ stands for the permutation group of n elements.

The condition (1) may be written using skew-symmetry property of the bracket as

$$\begin{aligned}&\big[\alpha_1(x_1),\ldots,\alpha_{n-1}(x_{n-1}),[y_1,\ldots,y_n]\big]\\&=\sum_{i=1}^{n}(-1)^{i+n}\big[\alpha_1(y_1),\ldots,\widehat{y_i},\ldots,\alpha_{n-1}(y_n),[x_1,\ldots,x_{n-1},y_i]\big],\end{aligned}\quad (4)$$

In the sequel we deal sometimes with a particular class of n-ary Hom-Nambu algebras which we call n-ary multiplicative Hom-Nambu algebras.

Definition 2.3 An *n-ary multiplicative Hom-Nambu algebra* (resp. *n-ary multiplicative Hom-Nambu-Lie algebra*) is an n-ary Hom-Nambu algebra (resp. n-ary Hom-Nambu-Lie algebra) $(\mathcal{N}, [\cdot, \dots, \cdot], \widetilde{\alpha})$ with $\widetilde{\alpha} = (\alpha_i)_{1 \leq i \leq n-1}$ where $\alpha_1 = \dots = \alpha_{n-1} = \alpha$ and satisfying

$$\alpha([x_1, \dots, x_n]) = [\alpha(x_1), \dots, \alpha(x_n)], \quad \forall x_1, \dots, x_n \in \mathcal{N}. \tag{5}$$

For simplicity, we will denote the n-ary multiplicative Hom-Nambu algebra as $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ where $\alpha : \mathcal{N} \to \mathcal{N}$ is a linear map. Also by misuse of language an element $x \in \mathcal{N}^n$ refers to $x = (x_1, \dots, x_n)$ and $\alpha(x)$ denotes $(\alpha(x_1), \dots, \alpha(x_n))$.

2.2 Constructions

In this section we recall the construction procedures by twisting principles. The first twisting principle, introduced for binary case in [39], was extend to n-ary case in [7]. The second twisting principle was introduced in [40]. Also we recall a construction by tensor product of symmetric totally n-ary Hom-associative algebra by an n-ary Hom-Nambu algebra given in [7].

The following Theorem gives a way to construct n-ary multiplicative Hom-Nambu algebras starting from a classical n-ary Nambu algebras and algebra endomorphisms.

Theorem 2.1 *([7]). Let $(\mathcal{N}, [\cdot, \dots, \cdot])$ be an n-ary Nambu algebra and $\rho : \mathcal{N} \to \mathcal{N}$ be an n-ary Nambu algebra endomorphism. Then $(\mathcal{N}, \rho \circ [\cdot, \dots, \cdot], \rho)$ is an n-ary multiplicative Hom-Nambu algebra.*

In the following we use the second twisting principal to generate new n-ary Hom-Nambu algebra starting from a given multiplicative n-ary Hom-Nambu algebra.

Theorem 2.2 *Let $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be a multiplicative n-ary Hom-Nambu algebra. Then $(\mathcal{N}, \alpha^{n-1} \circ [\cdot, \dots, \cdot], \alpha^n)$, for any integer n, is an n-ary multiplicative Hom-Nambu algebra.*

Example 2.1 ([7]). The polynomial algebra $\mathcal{N} = \mathbb{K}[x_1, x_2, x_3]$ of 3 variables x_1, x_2, x_3, with the bracket defined by the functional jacobian:

$$[f_1, f_2, f_3] = \begin{vmatrix} \frac{\delta f_1}{\delta x_1} & \frac{\delta f_1}{\delta x_2} & \frac{\delta f_1}{\delta x_3} \\ \frac{\delta f_2}{\delta x_1} & \frac{\delta f_2}{\delta x_2} & \frac{\delta f_2}{\delta x_3} \\ \frac{\delta f_3}{\delta x_1} & \frac{\delta f_3}{\delta x_2} & \frac{\delta f_3}{\delta x_3} \end{vmatrix},$$

is a ternary Nambu-Lie algebra. By considering a Nambu-Lie algebra endomorphism of such algebra, we construct a Hom-Nambu-Lie algebra on the polynomial algebra of 3 variables x_1, x_2, x_3.

Let $\gamma(x_1, x_2, x_3)$ be a polynomial or more general differentiable transformation of three variables mapping elements of $\mathcal{N}$ to elements of $\mathcal{N}$ and such that the determinant of the functional Jacobian $J(\gamma) = 1$. Any $\rho_\gamma : \mathcal{N} \to \mathcal{N}$, the composition transformation defined by $f \to f \circ \gamma$ for any $f \in \mathcal{N}$, defines an endomorphism of the ternary Nambu-Lie algebra given above. Therefore, for any such transformation γ, the triple $(\mathcal{N}, \rho_\gamma \circ [\cdot, \cdot, \cdot], \rho_\gamma)$ is a ternary Hom-Nambu-Lie algebra.

Now, we define the tensor product of two n-ary Hom-algebras and prove some results involving n-ary Hom-algebras of Lie type and Hom-associative type.

Let A be a $\mathbb{K}$-vector space, μ be an n-linear map on A and $\eta_i, i \in \{1, \ldots, n-1\}$, be linear maps on A. A triple $(A, \mu, \widetilde{\eta} = (\eta_1, \ldots, \eta_{n-1}))$ is said to be a symmetric n-ary totally Hom-associative algebra over $\mathbb{K}$ if the following identities hold

$$\mu(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = \mu(a_1, \ldots, a_n), \ \forall \sigma \in \mathcal{S}_n, \tag{6}$$

$$\begin{aligned} &\mu(\mu(a_1, \ldots, a_n), \eta_1(a_{n+1}), \ldots, \eta_{n-1}(a_{2n-1})) \\ &= \mu(\eta_1(a_1), \mu(a_2, \ldots, a_{n+1}), \eta_2(a_{n+2}), \ldots, \eta_{n-1}(a_{2n-1})) \\ &= \cdots = \mu(\eta_1(a_1), \ldots, \eta_{n-1}(a_{n-1}), \mu(a_n, \ldots, a_{2n-1})), \end{aligned} \tag{7}$$

where $a_1, \ldots, a_{2n-1} \in A$.

Theorem 2.3 *Let $(A, \mu, \widetilde{\eta} = (\eta_1, \ldots, \eta_{n-1}))$ be a symmetric n-ary totally Hom-associative algebra and $(\mathcal{N}, [\cdot, \ldots, \cdot]_{\mathcal{N}}, \widetilde{\alpha})$ be an n-ary Hom-Nambu algebra. Then the tensor product $A \otimes \mathcal{N}$ carries a structure of n-ary Hom-Nambu algebra over $\mathbb{K}$ with respect to the n-linear operation defined by*

$$[a_1 \otimes x_1, \ldots, a_n \otimes x_n] = \mu(a_1, \ldots, a_n) \otimes [x_1, \ldots, x_n]_{\mathcal{N}}, \quad \textit{where} \quad x_l \in \mathcal{N}, a_l \in A, l \in \{1, \ldots, n\}, \tag{8}$$

and linear maps $\widetilde{\zeta} = (\zeta_1, \ldots, \zeta_{n-1})$ where $\zeta_i = \eta_i \otimes \alpha_i$, for $i \in \{1, \ldots, n-1\}$, defined by

$$\zeta_i(a \otimes x) = \eta_i(a) \otimes \alpha_i(x), \ \forall\, a \otimes x \in A \otimes \mathcal{N}. \tag{9}$$

Proof For $a_k \otimes x_k,\ b_l \otimes y_l \in A \otimes \mathcal{N}$, $1 \le k \le n-1$ and $1 \le l \le n$, we have

$$\begin{aligned} &[\zeta_1(a_1 \otimes x_1), \ldots, \zeta_{n-1}(a_{n-1} \otimes x_{n-1}), [b_1 \otimes y_1, \ldots, b_n \otimes y_n]] \\ &= \mu(\eta_1(a_1), \ldots, \eta_{n-1}(a_{n-1}), \mu(b_1, \ldots, b_n)) \otimes [\alpha_1(x_1), \ldots, \alpha_{n-1}(x_{n-1}), \\ &\quad [y_1, \ldots, y_n]_{\mathcal{N}}]_{\mathcal{N}}. \end{aligned}$$

The symmetry and totally associativity of A lead to

$$\begin{aligned}
&[\zeta_1(b_1 \otimes y_1), \ldots, [a_1 \otimes x_1, \ldots, a_{n-1} \otimes x_{n-1}, b_l \otimes y_l], \ldots, \zeta_{n-1}(b_n \otimes y_n)] \\
&= \mu(\eta_1(b_1), \ldots, \mu(a_1, \ldots, a_{n-1}, b_l), \ldots, \eta_{n-1}(b_{n-1})) \\
&\quad \otimes [[\alpha_1(y_1), \ldots, [x_1, \ldots, x_{n-1}, y_l]_{\mathcal{N}}, \ldots, \alpha_{n-1}(y_n)]_{\mathcal{N}} \\
&= \mu(\eta_1(a_1), \ldots, \eta_{n-1}(a_{n-1}), \mu(b_1, \ldots, b_n)) \\
&\quad \otimes [\alpha_1(y_1), \ldots, [x_1, \ldots, x_{n-1}, y_l]_{\mathcal{N}}, \ldots, \alpha_{n-1}(y_n)]_{\mathcal{N}}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{l=1}^{n}[\zeta_1(b_1 \otimes y_1), \ldots, [a_1 \otimes x_1, \ldots, a_{n-1} \otimes x_{n-1}, b_l \otimes y_l], \ldots, \zeta_{n-1}(b_n \otimes y_n)] \\
&= \mu(\eta_1(a_1), \ldots, \eta_{n-1}(a_{n-1}), \mu(b_1, \ldots, b_n)) \\
&\quad \otimes \Big(\sum_{l=1}^{n}[\alpha_1(y_1), \ldots, [x_1, \ldots, x_{n-1}, y_l]_{\mathcal{N}}, \ldots, \alpha_{n-1}(y_n)]_{\mathcal{N}}\Big) \\
&= \mu(\eta_1(a_1), \ldots, \eta_{n-1}(a_{n-1}), \mu(b_1, \ldots, b_n)) \otimes [\alpha_1(x_1), \ldots, \alpha_{n-1}(x_{n-1}), \\
&\qquad [y_1, \ldots, y_n]_{\mathcal{N}}]_{\mathcal{N}}.
\end{aligned}$$

Corollary 2.1 *Let (A, μ, η) be a multiplicative symmetric n-ary Hom-associative algebra (i.e. $\eta \circ \mu = \mu \circ \eta^{\otimes n}$) and $(\mathcal{N}, [\cdot, \ldots, \cdot]_{\mathcal{N}}, \widetilde{\alpha})$ be a multiplicative n-ary Hom-Nambu algebra. Then $A \otimes \mathcal{N}$ is a multiplicative n-ary Hom-Nambu algebra.*

Remark 2.2 Let $(A, \cdot)$ be a binary commutative associative algebra and $(\mathcal{N}, [\cdot, \ldots, \cdot]_{\mathcal{N}}, \widetilde{\alpha})$ be an n-ary Hom-Nambu algebra. Then the tensor product $A \otimes \mathcal{N}$ carries a structure of n-ary Hom-Nambu algebra over $\mathbb{K}$ with respect to the n-linear operation defined by

$$[a_1 \otimes x_1, \ldots, a_n \otimes x_n] = (a_1 \cdot \ldots \cdot a_n) \otimes [x_1, \ldots, x_n]_{\mathcal{N}}, \tag{10}$$

and linear maps $\widetilde{\zeta} = (\zeta_1, \ldots, \zeta_{n-1})$ where $\zeta_i = id \otimes \alpha_i$, for $i \in \{1, \ldots, n-1\}$, defined by

$$\zeta_i(a \otimes x) = a \otimes \alpha_i(x), \ \forall\, a \otimes x \in A \otimes \mathcal{N}. \tag{11}$$

3 Definitions and Examples of Quadratic n-ary Hom-Nambu Algebras

In this section we introduce a class of Hom-Nambu-Lie algebras which possess a scalar product (a nondegenerate symmetric bilinear form which is invariant). This class of algebras is important due to their appearance in a number of physical contexts. They were extensively studied in the case of Lie algebras and Lie superalgebras, see

[9, 10, 14, 33]. The study was extended to Hom-Lie algebras in [13]. See also [18] for 3-Lie algebras.

Definition 3.1 Let $(\mathcal{N}, [., \ldots, .], \widetilde{\alpha})$, $\widetilde{\alpha} = (\alpha_i)_{1\leq i\leq n-1}$, be an n-ary Hom-Nambu algebra and $B : \mathcal{N} \times \mathcal{N} \to \mathbb{K}$ be a nondegenerate symmetric bilinear form such that, for all $y, z \in \mathcal{N}$ and $x \in \wedge^{n-1}\mathcal{N}$

$$B([x_1, \cdots, x_{n-1}, y], z) + B(y, [x_1, \cdots, x_{n-1}, z]) = 0, \tag{12}$$
$$B(\alpha_i(y), z) = B(y, \alpha_i(z)), \quad \forall i \in \{1, \ldots, n-1\}. \tag{13}$$

The quadruple $(\mathcal{N}, [\cdot, \ldots, \cdot], \widetilde{\alpha}, B)$ is called quadratic n-ary Hom-Nambu algebra.

Remark 3.1 If $\alpha_i = Id$ for all $i \in \{1, \ldots, n-1\}$, we recover quadratic (metric) n-ary Nambu algebras.

Definition 3.2 An n-ary Hom-Nambu algebra $(\mathcal{N}, [., \ldots, .], \widetilde{\alpha})$, $\widetilde{\alpha} = (\alpha_i)_{1\leq i\leq n-1}$, is called *Hom-quadratic* if there exists a pair (B, β) where $B : \mathcal{N} \times \mathcal{N} \to \mathbb{K}$ is a nondegenerate symmetric bilinear form and $\beta \in End(\mathcal{N})$ a linear map satisfying

$$B([x_1, \cdots, x_{n-1}, y], \beta(z)) + B(\beta(y), [x_1, \cdots, x_{n-1}, z]) = 0, \tag{14}$$

We call the identity (14) the β-invariance of B. We recover the quadratic n-ary Hom-Nambu algebras when $\beta = id$ and the identity (12) is called the invariance of B. The tuple $(\mathcal{N}, [\cdot, \ldots, \cdot], \widetilde{\alpha}, B, \beta)$ denotes the Hom-quadratic n-ary Hom-Nambu algebra.

Example 3.1 We consider an example of ternary Hom-Nambu algebra given in [40]. Let V be a $\mathbb{K}$-module and $B : V^{\otimes 2} \to \mathbb{K}$ be a nondegenerate symmetric bilinear form. Suppose $\alpha : V \to V$ is an involution, that is $\alpha^2 = id$. Assume that α is B-symmetric, that is $B(\alpha(x), y) = B(x, \alpha(y))$ for all $x, y \in V$. We have also $B(\alpha(x), \alpha(y)) = B(\alpha^2(x), y) = B(x, y)$. Then for any scalar $\lambda \in \mathbb{K}$, the triple product

$$[x, y, z]_\alpha = \lambda(B(y, z)\alpha(x) - B(z, x)\alpha(y)) \quad \text{for all } x,\ y,\ z \in V. \tag{15}$$

gives a Hom-quadratic ternary Hom-Nambu algebra $(V, [., ., .]_\alpha, (\alpha, \alpha))$, α-invariant by the pair (B, α). Indeed, for $x, y, z, t \in V$

$$\begin{aligned} B([x, y, z]_\alpha, \alpha(t)) &= \lambda(B(B(y, z)\alpha(x) - B(z, x)\alpha(y), \alpha(t))) \\ &= \lambda(B(y, z)B(\alpha(x), \alpha(t)) - B(z, x)B(\alpha(y), \alpha(t))) \\ &= \lambda(B(y, z)B(x, t) - B(z, x)B(y, t)) \\ &= \lambda(B(\alpha(y), \alpha(z))B(x, t) - B(\alpha(z), \alpha(x))B(y, t)) \\ &= \lambda(B(\alpha(z), \alpha(y))B(x, t) - B(\alpha(z), \alpha(x))B(y, t)) \end{aligned}$$

$$= \lambda(B(\alpha(z), \alpha(y)B(x,t) - \alpha(x)By, t)))$$
$$= -\lambda(B(\alpha(z), \alpha(x)B(y,t) - \alpha(y)B(t,x)))$$
$$= -B(\alpha(z), [x, y, t]_\alpha).$$

Example 3.2 Let $(\mathcal{N}, [\cdot,\cdot,\cdot], (\alpha_1, \alpha_2))$ be a 3-dimensional ternary Hom-Nambu-Lie algebras, defined with respect to a basis $\{e_1, e_2, e_3\}$ of $\mathcal{N}$ by

$$[e_1, e_2, e_3] = e_1 + 2e_2 + e_3, \tag{16}$$

$$\alpha_1(e_1) = 0,\ \alpha_1(e_2) = \lambda e_1 + v e_2,\ \alpha_1(e_3) = \frac{\lambda}{2}e_1 + \frac{v}{2}e_2, \tag{17}$$

$$\alpha_2(e_1) = 0,\ \alpha_2(e_2) = 0,\ \alpha_2(e_3) = be_3, \tag{18}$$

where λ, v, b are parameters with $\lambda v \neq 0$. These 3-dimensional ternary algebras admit symmetric bilinear forms B given, with respect to the previous basis, by the following matrix

$$M = \begin{pmatrix} -\frac{v}{\lambda} & 1 & (\frac{v}{\lambda} - 2) \\ 1 & -\frac{\lambda}{v} & (2\frac{\lambda}{v} - 1) \\ (\frac{v}{\lambda} - 2) & (2\frac{\lambda}{v} - 1) & (4 - 4\frac{\lambda}{v} - \frac{v}{\lambda}) \end{pmatrix}$$

The ternary Hom-Nambu-Lie algebras $(\mathcal{N}, [\cdot,\cdot,\cdot], (\alpha_1, \alpha_2))$, with respect to B, is not quadratic because B is degenerate ($det(M) = 0$).

4 Relationship Between Quadratic *n*-ary Hom-Nambu-Lie Algebra and Quadratic Hom-Leibniz Algebra

In the context of Hom-Lie algebras one gets the class of Hom-Leibniz algebras (see [29]). A Hom-Leibniz algebra is a triple $(V, [\cdot,\cdot], \alpha)$ consisting of a linear space V, a bilinear map $[\cdot,\cdot] : V \times V \to V$ and a homomorphism $\alpha : V \to V$ with respect to $[\cdot,\cdot]$ satisfying

$$[\alpha(x), [y,z]] = [[x,y], \alpha(z)] + [\alpha(y), [x,z]] \tag{19}$$

We fix in the following notations. Let $(\mathcal{N}, [\cdot, \ldots, \cdot], \widetilde{\alpha})$ be an n-ary Hom-Nambu algebra, we define

• a linear map $L : \otimes^{n-1}\mathcal{N} \longrightarrow End(\mathcal{N})$ by

$$L(x) \cdot z = [x_1, \ldots, x_{n-1}, z], \tag{20}$$

for all $x = x_1 \otimes \ldots \otimes x_{n-1} \in \otimes^{n-1}\mathcal{N},\ z \in \mathcal{N}$ and extending it linearly to all $\otimes^{n-1}\mathcal{N}$. Notice that $L(x) \cdot z = ad(x)(z)$.

If the n-ary Hom-Nambu algebra $\mathcal{N}$ is multiplicative, then we define
• a linear map $\hat{\alpha} : \otimes^{n-1}\mathcal{N} \longrightarrow \otimes^{n-1}\mathcal{N}$ by

$$\hat{\alpha}(x) = \alpha(x_1) \otimes \ldots \otimes \alpha(x_{n-1}) \tag{21}$$

for all $x = x_1 \otimes \ldots \otimes x_{n-1} \in \otimes^{n-1}\mathcal{N}$,
• a bilinear map $[\ ,\ ]_\alpha : \otimes^{n-1}\mathcal{N} \times \otimes^{n-1}\mathcal{N} \longrightarrow \otimes^{n-1}\mathcal{N}$ defined by

$$[x, y]_\alpha = L(x) \bullet_\alpha y = \sum_{i=0}^{n-1} \big(\alpha(y_1), \ldots, L(x) \cdot y_i, \ldots, \alpha(y_{n-1})\big), \tag{22}$$

for all $x = x_1 \otimes \ldots \otimes x_{n-1} \in \otimes^{n-1}\mathcal{N},\ y = y_1 \otimes \ldots \otimes y_{n-1} \in \otimes^{n-1}\mathcal{N}$

Lemma 4.1 *Let $(\mathcal{N}, [\cdot, \ldots, \cdot], \widetilde{\alpha})$ be a multiplicative n-ary Hom-Nambu algebra then the map L satisfies*

$$L([x, y]_\alpha) \cdot \alpha(z) = L(\tilde{\alpha}(x)) \cdot \big(L(y) \cdot z\big) - L(\tilde{\alpha}(y)) \cdot \big(L(x) \cdot z\big) \tag{23}$$

for all $x, y \in \mathcal{L}(\mathcal{N}),\ z \in \mathcal{N}$.

If $(\mathcal{N}, [\cdot, \ldots, \cdot], \widetilde{\alpha})$ is a multiplicative n-ary Hom-Nambu-Lie algebra, we denote by $\mathcal{L}(\mathcal{N})$ the space $\wedge^{n-1}\mathcal{N}$ and we call it the fundamental set.

Proposition 4.1 *The triple $\big(\mathcal{L}(\mathcal{N}),\ [\ ,\ ]_\alpha,\ \hat{\alpha}\big)$, where $[\ ,\ ]_\alpha$ and $\hat{\alpha}$ are defined respectively in (21) and (22), is a Hom-Leibniz algebra.*

Remark 4.1 The invariance identity (12) of an n-ary Nambu algebra with respect to a bilinear form B can be written

$$B(L(x) \cdot y, z) + B(y, L(x) \cdot z) = 0, \tag{24}$$

and β-invariance identity (14) by

$$B(L(x) \cdot y, \beta(z)) + B(\beta(y), L(x) \cdot z) = 0, \tag{25}$$

Proposition 4.2 *Let $(\mathcal{N}, [\cdot, \ldots, \cdot], \alpha, B, \alpha)$ be a Hom-quadratic multiplicative Hom-Nambu-Lie algebra α-invariant and $\big(\mathcal{L}(\mathcal{N}),\ [\ ,\ ]_\alpha,\ \hat{\alpha}\big)$ be its associated Hom-Leibniz algebra, then the natural scalar product on $\mathcal{L}(\mathcal{N})$, $\widehat{B}$ defined by*

$$\widehat{B}(x, y) = B(x_1 \wedge \ldots \wedge x_{n-1}, y_1 \wedge \ldots \wedge y_{n-1}) = \prod_{i=0}^{n-1} B(x_i, y_i) \tag{26}$$

and later extending linearly to all of $\mathcal{L}(\mathcal{N})$, is $\tilde{\alpha}$-invariant. That is, for all $x, y, z \in \mathcal{L}(\mathcal{N})$:

$$\widehat{B}([z, x]_\alpha, \hat{\alpha}(y)) + \widehat{B}(\hat{\alpha}(x), [z, x]_\alpha) = 0. \tag{27}$$

Proof Let $x = (x_1, \dots, x_{n-1}),\ y = (y_1, \dots, y_{n-1})$ and let $z \in \mathcal{L}(\mathcal{N})$. Then using equation (27) we have

$$\begin{aligned}
\widehat{B}([z,x]_\alpha, \hat{\alpha}(y)) &= \widehat{B}(L(z) \bullet_\alpha x, \hat{\alpha}(y))\\
&= \sum_{i=0}^{n-1} \widehat{B}\big(\big(\alpha(x_1), \dots, L(z)\cdot x_i, \dots, \alpha(x_{n-1})\big), \big(\alpha(y_1), \dots, \alpha(y_{n-1})\big)\big)\\
&= \sum_{i=0}^{n-1} B(L(z)\cdot x_i, \alpha(y_i)) \prod_{j=0\ j\neq i}^{n-1} B(\alpha(x_j), \alpha(x_j))\\
&= -\sum_{i=0}^{n-1} B(\alpha(x_i), L(z)\cdot y_i) \prod_{j=0\ j\neq i}^{n-1} B(\alpha(x_j), \alpha(y_j))\\
&= -\sum_{i=0}^{n-1} \widehat{B}\big(\big(\alpha(x_1), \dots, \alpha(x_{n-1})\big), \big(\alpha(y_1), \dots, L(z)\cdot y_i, \dots, \alpha(y_{n-1})\big)\big)\\
&= -\widehat{B}(\hat{\alpha}(x), L(z) \bullet_\alpha y)\\
&= -\widehat{B}(\hat{\alpha}(x), [z,y]_\alpha).
\end{aligned}$$

5 Representations and Quadratic *n*-ary Hom-Nambu Algebras

In this Section we study in the general case the representation theory of n-ary Hom-Nambu algebras introduced for multiplicative n-ary Hom-Nambu algebras in [1]. We discuss in particular adjoint and coadjoint representations for quadratic n-ary Hom-Nambu algebras. The results obtained in this Section generalize those given for binary case in [13]. The representation theory of Hom-Lie algebras were independently studied in [36].

Definition 5.1 A representation of an n-ary Hom-Nambu algebra $(\mathcal{N}, [\cdot, \dots, \cdot], \tilde{\alpha})$ on a vector space V is a skew-symmetric multilinear map $\rho : \mathcal{N}^{n-1} \longrightarrow End(V)$, satisfying for $x, y \in \mathcal{N}^{n-1}$ the identity

$$\rho(\tilde{\alpha}(x))\circ\rho(y) - \rho(\tilde{\alpha}(y))\circ\rho(x) = \sum_{i=1}^{n-1} \rho(\alpha_1(x_1), \dots, L(y)\cdot x_i, \dots, \alpha_{n-2}(x_{n-1}))\circ \nu \tag{28}$$

where ν is an endomorphism on V. We denote this representation by a triple (V, ρ, μ).

Two representations (V, ρ, μ) and (V', ρ', μ') of $\mathcal{N}$ are *equivalent* if there exists $f : V \to V'$ an isomorphism of vector space such that $f(x \cdot v) = x \cdot' f(v)$ and $f \circ \nu = \nu' \circ f$ where $x \cdot v = \rho(x)(v)$ and $x \cdot' v' = \rho'(x)(v')$ for $x \in \mathcal{N}^{n-1}$, $v \in V$ and $v' \in V'$. Then V and V' are viewed as $\mathcal{N}^{n-1}$-modules.

Example 5.1 Let $(\mathcal{N}, [\cdot, \ldots, \cdot], \tilde{\alpha})$ be an n-ary Hom-Nambu-Lie algebra. The map L defined in (20) is a representation on $\mathcal{N}$, where the endomorphism v is the twist map α_{n-1}. The identity (28) is equivalent to Hom-Nambu identity (1). It is called the adjoint representation.

Proposition 5.1 *Let $(\mathcal{N}, [., \ldots,], \widetilde{\alpha})$ be an n-ary Hom-Nambu algebra and (V, ρ, v) be a representation of $\mathcal{N}$. The triple $(V^*, \rho^*, \widetilde{v})$, where $\rho^* : \mathcal{N}^{n-1} \to End(V^*)$ is given by $\rho^* = -{}^t\rho$ and $\mu^* : V^* \to V^*, \quad f \mapsto v^*(f) = f \circ v$, defines a representation of the n-ary Hom-Nambu-Lie algebra $(\mathcal{N}, [., \ldots,], \widetilde{\alpha})$ if and only if*

$$\rho(x)\circ\rho(\tilde{\alpha}(y))-\rho(y)\circ\rho(\tilde{\alpha}(x)) = \sum_{i=1}^{n-1} v\circ\rho(\alpha_1(x_1), \ldots, L(y)\cdot x_i, \ldots, \alpha_{n-2}(x_{n-1})) \tag{29}$$

Proof Let $f \in \mathcal{N}^*$, $x, y \in \mathcal{N}^{n-1}$ and $u \in \mathcal{N}$. We compute the right hand side of the identity (28)

$$\begin{aligned}
&\rho^*(\tilde{\alpha}(x)) \circ \rho^*(y)(f)(u) - \rho^*(\tilde{\alpha}(y)) \circ \rho^*(x)(f)(u)\\
&= (\rho^*(\tilde{\alpha}(x))(\rho^*(y)(f)) - \rho^*(\tilde{\alpha}(y))(\rho^*(x)(f)))(u)\\
&= -(\rho^*(y)(f)(\rho(\tilde{\alpha}(x)))(u)) + (\rho^*(x)(f)(\rho(\tilde{\alpha}(y)))(u))\\
&= f(\rho(y)(\rho(\tilde{\alpha}(x))(u))) - f(\rho(x)(\rho(\tilde{\alpha}(y))(u)))\\
&= f(\rho(y)(\rho(\tilde{\alpha}(x))(u)) - \rho(x)(\rho(\tilde{\alpha}(y))(u))).
\end{aligned}$$

In the other hand, the left hand side of (28) writes

$$\begin{aligned}
&(\sum_{i=1}^{n-1} \rho^*(\alpha_1(x_1), \ldots, L(y) \cdot x_i, \ldots, \alpha_{n-2}(x_{n-1})) \circ v^*(f))(u)\\
&= -\sum_{i=1}^{n-1}(v^*(f)(\rho(\alpha_1(x_1), \ldots, L(y) \cdot x_i, \ldots, \alpha_{n-2}(x_{n-1}))(u)))\\
&= -\sum_{i=1}^{n-1} f(v(\rho(\alpha_1(x_1), \ldots, L(y) \cdot x_i, \ldots, \alpha_{n-2}(x_{n-1}))(u)))\\
&= f(-\sum_{i=1}^{n-1} v(\rho(\alpha_1(x_1), \ldots, L(y) \cdot x_i, \ldots, \alpha_{n-2}(x_{n-1}))(u))).
\end{aligned}$$

Therefore we obtain the identity (29).

Corollary 5.1 *Let $(\mathcal{N}, L, \alpha_{n-1})$ be a representation of an n-ary Hom-Nambu algebra $(\mathcal{N}, [\cdot, \ldots, \cdot], \widetilde{\alpha})$. We define the map $\widetilde{L} : \mathcal{N}^{n-1} \to End(\mathcal{N}^*)$, for $x \in \mathcal{N}^{n-1}$, $f \in \mathcal{N}^*$ and $y \in \mathcal{N}$, by $(\widetilde{L}(x) \cdot f)(y) = -f(L(x) \cdot y)$.*
Then $(\mathcal{N}^*, \widetilde{L}, \alpha^*_{n-1})$ is a representation of $\mathcal{N}$ if and only if

$$L(x) \circ L(\tilde{\alpha}(y)) - L(y) \circ L(\tilde{\alpha}(x)) = \sum_{i=1}^{n-1} \alpha_{n-1} \circ L(\alpha_1(x_1), \ldots, L(y) \cdot x_i, \ldots, \alpha_{n-2}(x_{n-1})). \tag{30}$$

We establish now a connection between quadratic n-ary Hom-Nambu algebras and representation theory. We discuss coadjoint representations for quadratic n-ary Hom-Nambu algebras.

Proposition 5.2 *Let $(\mathcal{N}, [\cdot, \ldots, \cdot], \widetilde{\alpha})$ be an n-ary Hom-Nambu algebra. If there exists $B : \mathcal{N} \times \mathcal{N} \to \mathbb{K}$ a bilinear form such that the quadruple $(\mathcal{N}, [\cdot, \ldots, \cdot], \widetilde{\alpha}, B)$ is a quadratic n-ary Hom-Nambu algebra then*

1. *$(\mathcal{N}^*, \widetilde{L}, \alpha^*_{n-1})$ is a representation of $\mathcal{N}$,*
2. *the representations $(\mathcal{N}, L, \alpha_{n-1})$ and $(\mathcal{N}^*, \widetilde{L}, \alpha^*_{n-1})$ are isomorphic.*

Proof To prove the first assertion, we should show that, for any $z \in \mathcal{N}$, we have

$$\begin{aligned} &L(x) \circ L(\tilde{\alpha}(y)) \cdot z - L(y) \circ L(\tilde{\alpha}(x)) \cdot z \\ &\quad = \sum_{i=1}^{n-1} \alpha_{n-1} \circ L(\alpha_1(x_1), \ldots, L(y) \cdot x_i, \ldots, \alpha_{n-2}(x_{n-1}))(z). \end{aligned} \tag{31}$$

Let $u \in \mathcal{N}$

$$\begin{aligned} &B(L(x) \circ L(\tilde{\alpha}(y)) \cdot z - L(y) \circ L(\tilde{\alpha}(x)) \cdot z, u) \\ &= B(L(x) \circ L(\tilde{\alpha}(y)) \cdot z, u) - (L(y) \circ L(\tilde{\alpha}(x)) \cdot z, u) \\ &= B(L(\tilde{\alpha}(y)) \cdot z, L(x) \cdot u) - (L(\tilde{\alpha}(x)) \cdot z, L(y) \cdot u) \\ &= B(z, L(\tilde{\alpha}(y)) \circ L(x) \cdot u) - (z, L(\tilde{\alpha}(x)) \circ L(y) \cdot u) \\ &= B(z, L(\tilde{\alpha}(y)) \circ L(x)(u) - L(\tilde{\alpha}(x)) \circ L(y)(u)). \end{aligned}$$

and

$$\begin{aligned} &B(\alpha_{n-1} \circ L(\alpha_1(x_1), \ldots, L(y) \cdot x_i, \ldots, \alpha_{n-2}(x_{n-1}))(z), u) \\ &= B(L(\alpha_1(x_1), \ldots, L(y) \cdot x_i, \ldots, \alpha_{n-2}(x_{n-1}))(z), \alpha_{n-1}(u)) \\ &= B(z, L(\alpha_1(x_1), \ldots, L(y) \cdot x_i, \ldots, \alpha_{n-2}(x_{n-1})) \circ \alpha_{n-1}(u)) \\ &= B(z, L(\alpha_1(x_1), \ldots, L(y) \cdot x_i, \ldots, \alpha_{n-2}(x_{n-1})) \circ \alpha_{n-1}(u)). \end{aligned}$$

Since B is bilinear, then

$$\begin{aligned} &B(\sum_{i=1}^{n-1} \alpha_{n-1} \circ L(\alpha_1(x_1), \ldots, L(y) \cdot x_i, \ldots, \alpha_{n-2}(x_{n-1}))(z), u) \\ &= B(z, \sum_{i=1}^{n-1} L(\alpha_1(x_1), \ldots, L(y) \cdot x_i, \ldots, \alpha_{n-2}(x_{n-1})) \circ \alpha_{n-1}(u)). \end{aligned}$$

Hence

$$\begin{aligned}
&B(L(x)\circ L(\tilde{\alpha}(y))\cdot z - L(y)\circ L(\tilde{\alpha}(x))\cdot z\\
&\quad -\sum_{i=1}^{n-1}\alpha_{n-1}\circ L(\alpha_1(x_1),\dots,L(y)\cdot x_i,\dots,\alpha_{n-2}(x_{n-1}))(z),u)\\
&= B(z, L(\tilde{\alpha}(y))\circ L(x)\cdot u - L(\tilde{\alpha}(x))\circ L(y)\cdot u\\
&\quad -\sum_{i=1}^{n-1}L(\alpha_1(x_1),\dots,L(y)\cdot x_i,\dots,\alpha_{n-2}(x_{n-1}))\circ\alpha_{n-1}(u))\\
&= 0.
\end{aligned}$$

Since B is nondegenerate then the identity (31) holds.

For the second assertion we consider the map $\psi : \mathcal{N} \to \mathcal{N}^*$ defined by $x \mapsto B(x,\cdot)$ which is bijective since B is nondegenerate and prove that it is also a module morphism.

6 Constructions of Quadratic *n*-ary Hom-Nambu Algebras

We provide in this section some key constructions of Hom-quadratic n-ary Hom-Nambu-Lie algebras. First we extend twisting principles, then the T^*-extension construction for Hom-quadratic n-ary Hom-Nambu-Lie algebras. Moreover, we show constructions involving tensor product of Hom-quadratic commutative Hom-associa-
tive algebra and Hom-quadratic Hom-Nambu-Lie algebra considered in Theorem 1.3.

6.1 Twisting Principles

Let $(\mathcal{N}, [\cdot,\dots,\cdot], B)$ be a quadratic n-ary Nambu algebras. We denote $Aut_S(\mathcal{N}, B)$ by the set of symmetric automorphisms of $\mathcal{N}$ with respect of B, that is automorphisms $f : \mathcal{N} \to \mathcal{N}$ such that $B(f(x), y) = B(x, f(y)), \forall x, y \in \mathcal{N}$.

Proposition 6.1 *Let $(\mathcal{N}, [\cdot,\dots,\cdot], B)$ be a quadratic n-ary Nambu algebra and $\rho \in Aut_S(\mathcal{N}, B)$.*

Then $(\mathcal{N}, [\cdot, \dots, \cdot]_\rho, \tilde{\rho}, B, \rho)$ *where*

$$[\cdot, \dots, \cdot]_\rho = \rho \circ [\cdot, \dots, \cdot] \tag{32}$$

is a Hom-quadratic n-ary Hom-Nambu algebra, and $(\mathcal{N}, [\cdot, \dots, \cdot]_\rho, \tilde{\rho}, B_\rho)$ *where*

$$B_\rho(x, y) = B(\rho(x), y) \tag{33}$$

is a quadratic n-ary Hom-Nambu algebra.

Proof Let $x = (x_1, \dots, x_{n-1}) \in \mathcal{N}^{\otimes n-1}$ et $y_1, y_2 \in \mathcal{N}$,

$$\begin{aligned} B([x_1, \dots, x_{n-1}, y_1]_\rho, \rho(y_2)) &= B([\rho(x_1), \dots, \rho(x_{n-1}), \rho(y_1)], \rho(y_2)) \\ &= -B(\rho(y_1), [\rho(x_1), \dots, \rho(x_{n-1}), \rho(y_2)]) \\ &= -B(\rho(y_1), \rho \circ [x_1, \dots, x_{n-1}, y_2]) \\ &= -B(\rho(y_1), [x_1, \dots, x_{n-1}, y_2]_\rho). \end{aligned}$$

In the other hand we have

$$\begin{aligned} B_\rho([x_1, \dots, x_{n-1}, y_1]_\rho, y_2) &= B(\rho[\rho(x_1), \dots, \rho(x_{n-1}), \rho(y_1)], y_2) \\ &= B([\rho(x_1), \dots, \rho(x_{n-1}), \rho(y_1)], \rho(y_2)) \\ &= -B(\rho(y_1), [\rho(x_1), \dots, \rho(x_{n-1}), \rho(y_2)]) \\ &= -B(\rho(y_1), \rho \circ [x_1, \dots, x_{n-1}, y_2]) \\ &= -B_\rho(y_1, [x_1, \dots, x_{n-1}, y_2]_\rho). \end{aligned}$$

Proposition 6.2 *Let* $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha, B)$ *be a quadratic multiplicative n-ary Hom-Nambu algebra. Then* $(\mathcal{N}, \alpha^{n-1} \circ [\cdot, \dots, \cdot], \alpha^n, B, \alpha^{n-1})$ *is a Hom-quadratic n-ary Hom-Nambu algebra and* $(\mathcal{N}, \alpha^{n-1} \circ [\cdot, \dots, \cdot], \alpha^n, B_\alpha)$*, where*

$$B_\alpha(x, y) = B(\alpha^{n-1}(x), y) = B(x, \alpha^{n-1}(y)), \tag{34}$$

is a quadratic n-ary Hom-Nambu algebra,

Proof Using the second twisting principle construction of Theorem 2.2, $(\mathcal{N}, \alpha^{n-1} \circ [\cdot, \dots, \cdot], \alpha^n)$ is a n-ary Hom-Nambu algebra. Let now $x_i, y, z \in \mathcal{N}$, $i \in \{1, \dots, n-1\}$, we have

$$B_\alpha(\alpha^{n-1}(y), z) = B(\alpha^{2n-2}(y), z) = B(\alpha^{n-1}(y), \alpha^{n-1}(z)) = B_\alpha(y, \alpha^{n-1}(z)).$$

In the other hand, we have

$$\begin{aligned} B_\alpha(\alpha^{n-1} \circ [x_1, \dots, x_{n-1}, y], z) &= B(\alpha^{n-1} \circ [x_1, \dots, x_{n-1}, y], \alpha^{n-1}(z)) \\ &= B([\alpha^{n-1}(x_1), \dots, \alpha^{n-1}(x_{n-1}), \alpha^{n-1}(y)], \alpha^{n-1}(z)) \\ &= -B(\alpha^{n-1}(y), [\alpha^{n-1}(x_1), \dots, \alpha^{n-1}(x_{n-1}), \alpha^{n-1}(z)]) \end{aligned}$$

$$= -B(\alpha^{n-1}(y), \alpha^{n-1} \circ [x_1, \ldots, x_{n-1}, z)]$$
$$= -B_\alpha(y, \alpha^{n-1} \circ [x_1, \ldots, x_{n-1}, z)].$$

Therefore B_α is invariant.

6.2 *T*-Extension of n-ary Hom-Nambu Algebras*

We provide here a construction of n-ary Hom-Nambu algebra $\mathscr{L}$ which is a generalization of the trivial T^*-extension introduced in [14, 33].

Theorem 6.1 *Let $(\mathscr{N}, [\cdot, \ldots, \cdot]_{\mathscr{N}}, B)$ be a quadratic n-ary Nambu-Lie algebra and $\mathscr{N}^*$ be the underlying dual vector space. The vector space $\mathscr{L} = \mathscr{N} \oplus \mathscr{N}^*$ equipped with the following product $[\cdot, \ldots, \cdot]_{\mathscr{L}} : \mathscr{L}^n \to \mathscr{L}$ given, for $u_i = x_i + f_i \in \mathscr{L}$ where $i \in \{1, \ldots, n\}$ by*

$$[u_1, \ldots, u_n]_{\mathscr{L}} = [x_1, \ldots, x_n]_{\mathscr{N}} + \sum_{i=1}^{n} (-1)^{i+n+1} f_i \circ L(x_1, \ldots, \widehat{x_i}, \ldots, x_n), \quad (35)$$

and a bilinear form

$$B_{\mathscr{L}} : \begin{array}{c} \mathscr{L} \times \mathscr{L} \longrightarrow \mathscr{L} \\ B_{\mathscr{L}}(x + f, y + g) = B(x, y) + f(y) + g(x) \end{array} \quad (36)$$

is a quadratic n-ary Nambu algebra.

Proof $\star$) Set $u_i = x_i + f_i \in \mathscr{L}$ and $v_i = y_k + g_k \in \mathscr{L}$. We show the following Nambu identity on $\mathscr{L}$

$$[u_1, \ldots, u_{n-1}, [v_1, \ldots, v_n]_{\mathscr{L}}]_{\mathscr{L}} = \sum_{l=1}^{n} (-1)^{l+n} [v_1, \ldots, \widehat{v_l}, \ldots, v_n, [u_1, \ldots, u_{n-1}, v_l]_{\mathscr{L}}]_{\mathscr{L}}. \quad (37)$$

Let us compute first $[u_1, \ldots, u_{n-1}, [v_1, \ldots, v_n]_{\mathscr{L}}]_{\mathscr{L}}$. This is given by

$$\begin{aligned}
&[u_1, \ldots, u_{n-1}, [v_1, \ldots, v_n]_{\mathscr{L}}]_{\mathscr{L}} \\
&= [x_1, \ldots, x_{n-1}, [y_1, \ldots, y_n]_{\mathscr{N}}]_{\mathscr{N}} + \sum_{i=1}^{n-1} (-1)^{i+n+1} f_i \circ L(x_1, \ldots, \widehat{x_i}, \ldots, x_{n-1}, \\
&\quad [y_1, \ldots, y_n]_{\mathscr{N}}) \\
&\quad + \sum_{i=1}^{n} (-1)^{i+n} g_i \circ L(y_1, \ldots, \widehat{y_i}, \ldots, y_n) \circ L(x_1, \ldots, x_{n-1}).
\end{aligned}$$

Hence the right hand side of (37) gives, for any $l \in \{1, \ldots, n\}$

$$[v_1,\dots,\widehat{v_l},\dots,v_n,[u_1,\dots,u_{n-1},v_l]_{\mathscr{L}}]_{\mathscr{L}}=[y_1,\dots,\widehat{y_l},\dots,y_n,[x_1,\dots,x_{n-1},y_l]_{\mathscr{N}}]_{\mathscr{N}}$$
$$+\sum_{i=1}^{n-1}(-1)^{i+n}f_i\circ L(x_1,\dots,\widehat{x_i},\dots,x_{n-1},y_l)\circ L(y_1,\dots,\widehat{y_l},\dots,y_n)$$
$$+\sum_{i=1}^{n}(-1)^{i+n+1}g_i\circ L(y_1,\dots,\widehat{y_i},\dots,\widehat{y_l},\dots,y_n,[x_1,\dots,x_{n-1},y_l]_{\mathscr{N}})$$
$$+g_l\circ L(x_1,\dots,x_{n-1})\circ L(y_1,\dots,\widehat{y_l},\dots,y_n).$$

- Using the Nambu identity on $\mathscr{N}$, we obtain

$$[x_1,\dots,x_{n-1},[y_1,\dots,\widehat{y_l},\dots,y_n,z]_{\mathscr{N}}]_{\mathscr{N}}=$$
$$\sum_{i=1,i\neq l}^{n}(-1)^{i+n}[y_1,\dots,\widehat{y_i},\dots,\widehat{y_l},\dots,y_{n-1},[x_1,\dots,x_{n-1},y_i]_{\mathscr{N}},z]_{\mathscr{N}}$$
$$+[y_1,\dots,\widehat{y_l},\dots,y_{n-1},[x_1,\dots,x_{n-1},z]_{\mathscr{N}}]_{\mathscr{N}}.$$

Equivalently

$$\begin{aligned}&L(y_1,\dots,\widehat{y_l},\dots,y_n)\circ L(x_1,\dots,x_{n-1})\\ &=L(x_1,\dots,x_{n-1})\circ L(y_1,\dots,\widehat{y_l},\dots,y_n)\\ &\quad+\sum_{i=1}^{n-1}(-1)^{i+n+1}L(y_1,\dots,\widehat{y_i},\dots,\widehat{y_l},\dots,y_n,[x_1,\dots,x_{n-1},y_l]).\end{aligned}$$

Thus for any $l\in\{1,\dots n\}$

$$g_l\circ L(y_1,\dots,\widehat{y_l},\dots,y_n)\circ L(x_1,\dots,x_{n-1})-g_l\circ L(x_1,\dots,x_{n-1})\circ L(y_1,\dots,\widehat{y_l},\dots,y_n)$$
$$=\sum_{i=1}^{n-1}(-1)^{i+n}g_l\circ L(y_1,\dots,\widehat{y_i},\dots,\widehat{y_l},\dots,y_n,[x_1,\dots,x_{n-1},y_l]).$$

- In the other hand we show that, for $k\in\{1,\dots n\}$

$$-f_k\circ L(x_1,\dots,\widehat{x_k},\dots,x_{n-1},[y_1,\dots,y_n]_{\mathscr{N}})$$
$$=\sum_{i=1}^{n}(-1)^{i+n}f_k\circ L(x_1,\dots,\widehat{x_k},\dots,x_{n-1},y_i)\circ L(y_1,\dots,\widehat{y_i},\dots,y_n).$$

Using the Namby identity (37) on $\mathscr{N}$ and the invariance of B, we obtain

$$B([x_1,\dots,\widehat{x_k},\dots,x_n,[y_1,\dots,y_n]_{\mathscr{N}}]_{\mathscr{N}},z)=B(x_n,[x_1,\dots,\widehat{x_k},\dots,x_{n-1},\ [y_1,\dots,y_n]_{\mathscr{N}},z]_{\mathscr{N}}).$$

Hence

$$B(\sum_{i=1}^{n}(-1)^{i+n}[y_1,\dots,\widehat{y_i},\dots,y_n,[x_1,\dots,\widehat{x_k},\dots,x_n,y_i]_{\mathscr{N}}]_{\mathscr{N}},z)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}(-1)^{i+n} B([y_1,\dots,\widehat{y_i},\dots,y_n,[x_1,\dots,\widehat{x_k},\dots,x_n,y_i]_{\mathcal{N}}]_{\mathcal{N}},z)\\
&= -\sum_{i=1}^{n}(-1)^{i+n} B([x_1,\dots,\widehat{x_k},\dots,x_n,y_i]_{\mathcal{N}},[y_1,\dots,\widehat{y_i},\dots,y_n,z]_{\mathcal{N}})\\
&= -\sum_{i=1}^{n}(-1)^{i+n} B(x_n,[x_1,\dots,\widehat{x_k},\dots,x_{n-1},y_i,[y_1,\dots,\widehat{y_i},\dots,y_n,z]_{\mathcal{N}}]_{\mathcal{N}})\\
&= -B(x_n,\sum_{i=1}^{n}(-1)^{i+n}[x_1,\dots,\widehat{x_k},\dots,x_{n-1},y_i,[y_1,\dots,\widehat{y_i},\dots,y_n,z]_{\mathcal{N}}]_{\mathcal{N}}).
\end{aligned}
$$

Since B is nondegenerate, then

$$
\begin{aligned}
&-[x_1,\dots,\widehat{x_k},\dots,x_{n-1},[y_1,\dots,y_n]_{\mathcal{N}},z]_{\mathcal{N}}\\
&= \sum_{i=1}^{n}(-1)^{i+n}[x_1,\dots,\widehat{x_k},\dots,x_{n-1},y_i,[y_1,\dots,\widehat{y_i},\dots,y_n,z]_{\mathcal{N}}]_{\mathcal{N}},
\end{aligned}
$$

and equivalently

$$
\begin{aligned}
&-L(x_1,\dots,\widehat{x_k},\dots,x_{n-1},[y_1,\dots,y_n]_{\mathcal{N}})\\
&\quad= \sum_{i=1}^{n}(-1)^{i+n}L(x_1,\dots,\widehat{x_k},\dots,x_{n-1},y_i)\circ L(y_1,\dots,\widehat{y_i},\dots,y_n).
\end{aligned}
$$

Finally, the Nambu identity (37) is satisfied. Thus $(\mathcal{L},[\cdot,\dots,\cdot]_{\mathcal{L}})$ is an n-ary Nambu algebra.

Theorem 6.2 *Let $(\mathcal{N},[\cdot,\dots,\cdot]_{\mathcal{N}},B)$ be a quadratic n-ary Nambu-Lie algebra where $\alpha\in Aut_S(\mathcal{N},B)$ is an involution. Then $(\mathcal{L},[\cdot,\dots,\cdot]_{\Omega},\widetilde{\Omega},B_{\mathcal{L}},\Omega)$, where $\Omega:\mathcal{L}\to\mathcal{L}, x+f\to\Omega(x+f)=\alpha(x)+f\circ\alpha$ and $[\cdot,\dots,\cdot]_{\Omega}=\Omega\circ[\cdot,\dots,\cdot]_{\mathcal{L}}$, is a Hom-quadratic multiplicative n-ary Hom-Nambu algebra.*

Proof Let $x_1,\dots,x_n\in\mathcal{N}$ and $f_1,\dots,f_n\in\mathcal{N}^*$,

$$
\begin{aligned}
\Omega[x_1+f_1,\dots,x_n+f_n]_{\mathcal{L}} &= \alpha[x_1,\dots,x_n]_{\mathcal{N}}\\
&\quad+\sum_{i=1}^{n}(-1)^i f_i\circ L(x_1,\dots,\widehat{x_i},\dots,x_n)\circ\alpha,\\
&[\Omega(x_1+f_1),\dots,\Omega(x_n+f_n)]_{\mathcal{L}}\\
= [\alpha(x_1),\dots,\alpha(x_n)]_{\mathcal{N}} &+ \sum_{i=1}^{n}(-1)^i f_i\circ\alpha\circ L(\alpha(x_1),\dots,\widehat{x_i},\dots,\alpha(x_n)).
\end{aligned}
$$

That is for all $z\in\mathcal{N}$

$$
\begin{aligned}
\alpha\circ L(\alpha(x_1),\dots,\widehat{x_i},\dots,\alpha(x_n))(z) &= \alpha[\alpha(x_1),\dots,\widehat{x_i},\dots,\alpha(x_n),z]_{\mathcal{N}}\\
&= [\alpha^2(x_1),\dots,\widehat{x_i},\dots,\alpha^2(x_n),\alpha(z)]_{\mathcal{N}}
\end{aligned}
$$

$$
\begin{aligned}
&= [x_1, \ldots, \widehat{x_i}, \ldots, x_n, \alpha(z)]_{\mathscr{N}} \\
&= L(x_1, \ldots, \widehat{x_i}, \ldots, x_n) \cdot \alpha(z).
\end{aligned}
$$

Then $\Omega[x_1 + f_1, \ldots, x_n + f_n]_{\mathscr{L}} = [\Omega(x_1 + f_1), \ldots, \Omega(x_n + f_n)]_{\mathscr{L}}$.
In the following we show that Ω is symmetric with respect to $B_{\mathscr{L}}$.

Indeed, let $x, y \in \mathscr{N}$ and $f, h \in \mathscr{N}$

$$
\begin{aligned}
B_{\mathscr{L}}(\Omega(x + f), y + h) &= B_{\mathscr{L}}(\alpha(x) + f \circ \alpha, y + h) \\
&= B(\alpha(x), y) + f \circ \alpha(y) + h \circ \alpha(x) \\
&= B(x, \alpha(y)) + f \circ \alpha(y) + h \circ \alpha(x) \\
&= B_{\mathscr{L}}(x + f, \alpha(y) + h \circ \alpha) = B_{\mathscr{L}}(x + f, \Omega(y + h)).
\end{aligned}
$$

Thus, using Proposition 6.1, $(\mathscr{L}, [\cdot, \ldots, \cdot]_{\Omega}, \widetilde{\Omega}, B_{\mathscr{L}}, \Omega)$ is a Hom-quadratic multiplicative n-ary Hom-Nambu algebra. We have also that $(\mathscr{L}, [\cdot, \ldots, \cdot]_{\Omega}, \Omega, B_{\mathscr{L},\Omega})$, where $B_{\mathscr{L},\Omega}(u, v) = B_{\mathscr{L}}(\Omega(u), v)$, for all $u, v \in \mathscr{L}$, is a quadratic multiplicative n-ary Hom-Nambu algebra.

6.3 Tensor Product

Let $(A, \mu, \widetilde{\eta}, B_A, \beta_A)$ be a Hom-quadratic symmetric n-ary totally Hom-associative algebra, that is a symmetric n-ary totally Hom-associative algebra together with a symmetric nondegenerate form satisfying the following assertions

$$B_A(\eta_i(a), b) = B_A(a, \eta_i(b)), \text{ for all } i \in \{1, \ldots, n-1\} \tag{38}$$

$$
\begin{aligned}
&B_A(\mu(a_1, \ldots, a_{n-1}, b), \beta_A(c)) = B_A(\beta_A(a), \mu(a_1, \ldots, a_{n-1}, c)), \\
&\text{for all } a_i, b, c \in A,\ i \in \{1, \ldots, n-1\}.
\end{aligned} \tag{39}
$$

We discuss now the tensor product as in Proposition 5.1.

Theorem 6.3 *Let $(\mathscr{N}, [\cdot, \ldots, \cdot]_{\mathscr{N}}, \alpha, B_{\mathscr{N}}, \beta_{\mathscr{N}})$ be a Hom-quadratic n-ary Hom-Nambu algebra, then $(A \otimes \mathscr{N}, [\cdot, \ldots, \cdot], \zeta, \widetilde{B}, \omega)$, where*

$$\widetilde{B}(a \otimes x, b \otimes y) = B_A(a, b) B_{\mathscr{N}}(x, y), \tag{40}$$

$$\omega(a \otimes x) = \beta_A(a) \otimes \beta_{\mathscr{N}}(x), \tag{41}$$

is a Hom-quadratic n-ary Hom-Nambu algebra.

6.4 Hom-quadratic Hom-Nambu-Lie Algebras Induced by Hom-quadratic Hom-Lie Algebras

In [8] the authors provided a construction procedure of ternary Hom-Nambu-Lie algebras starting from a bilinear bracket of a Hom-Lie algebra and a trace function satisfying certain compatibility conditions including the twisting map.

The aim of this section is to prove that this procedure is still true for quadratic ternary Hom-Nambu-Lie algebra. First we recall the result in [8].

Definition 6.1 Let $(V, [\cdot,\cdot])$ be a binary algebra and $\tau : V \to \mathbb{K}$ be a linear form. The trilinear map $[\cdot,\cdot,\cdot]_\tau : V \times V \times V \to V$ is defined as

$$[x, y, z]_\tau = \tau(x)[y, z] + \tau(y)[z, x] + \tau(z)[x, y]. \tag{42}$$

Remark 6.1 If the bilinear multiplication $[\cdot,\cdot]$ is skew-symmetric, then the trilinear map $[\cdot,\cdot,\cdot]_\tau$ is skew-symmetric as well.

Theorem 6.4 *([8]). Let $(V, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra and $\gamma : V \longrightarrow V$ be a linear map. Furthermore, assume that τ is a trace function on V fulfilling*

$$\tau(\alpha(x))\tau(y) = \tau(x)\tau(\alpha(y)), \tag{43}$$
$$\tau(\gamma(x))\tau(y) = \tau(x)\tau(\gamma(y)), \tag{44}$$
$$\tau(\alpha(x))\gamma(y) = \tau(\gamma(x))\alpha(y), \tag{45}$$

for all $x, y \in V$. Then $(V, [\cdot,\cdot,\cdot]_\tau, (\alpha, \gamma))$ is a ternary Hom-Nambu-Lie algebra, and we say that it is induced by $(V, [\cdot,\cdot], \alpha)$.

Proposition 6.3 *Let $(V, [\cdot,\cdot], \alpha, B, \beta)$ be a Hom-quadratic Hom-Lie algebra satisfying*

$$B(\alpha(x), y) = B(x, \alpha(y)), \tag{46}$$
$$B(\gamma(x), y) = B(x, \gamma(y)), \tag{47}$$
$$\tau(x)B(\beta(y), z) - \tau(y)B(\beta(x), z) = 0 \quad \textit{for all } x, y, z \in V. \tag{48}$$

Then $(V, [\cdot,\cdot,\cdot]_\tau, (\alpha, \gamma), B, \beta)$ is a Hom-quadratic ternary Hom-Nambu-Lie algebra.

Proof Let $x_1, x_2, y_1, y_2 \in V$

$$B([x_1, x_2, y_1]_\tau, \beta(y_2)) = \tau(x_1)B([x_2, y_1], \beta(y_2)) - \tau(x_2)B([x_1, y_1], \beta(y_2)) + \tau(y_1)B([x_1, x_2], \beta(y_2)).$$

$$B(\beta(y_1), [x_1, x_2, y_2]_\tau) = \tau(x_1)B(\beta(y_1), [x_2, y_2]) - \tau(x_2)B(\beta(y_1), [x_1, y_2]) - \tau(y_2)B(\beta(y_1), [x_1, x_2]).$$

Since B is symmetric, then

$$B([x_1, x_2, y_1]_\tau, \beta(y_2)) + B(\beta(y_1), [x_1, x_2, y_2]_\tau) = 0.$$

6.5 Quadratic Hom-Nambu Algebras of Higher Arities

The purpose of this section is to observe that every Hom-quadratic multiplicative n-ary Hom-Nambu algebra gives rise to a sequence of quadratic multiplicative Hom-Nambu algebras of increasingly higher arities. The construction of this sequence was given first in [40].

Theorem 6.5 *Let $(\mathcal{N}, [\cdot, \ldots, \cdot], \alpha, B, \beta)$ be a Hom-quadratic multiplicative n-ary Hom-Nambu algebra. Define the $(2n-1)$- ary product*

$$[x_1, \ldots, x_{2n-1}]^{(1)} = [[x_1, \ldots, x_n], \alpha(x_{n+1}), \ldots, \alpha(x_{2n-1})] \quad \textit{for } x_i \in V \tag{49}$$

Then $\mathcal{N}^1 = (\mathcal{N}, [\cdot, \ldots, \cdot]^{(1)}, \alpha^2, B, \beta')$, where $\beta' = \beta\alpha$, is a Hom-quadratic multiplicative $(2n-1)$-ary Hom-Nambu algebra.

Proof For the proof of the $(2n-1)$-ary Hom-Nambu identity and the multiplicativity for $\mathcal{N}^1$, see [40].

Let $x_1, \ldots, x_{2n-2}, y_1, y_2 \in \mathcal{N}$

$$\begin{aligned}
B([x_1, \ldots, x_{2n-2}, y_1]^{(1)}, \beta'(y_2)) &= B([[x_1, \ldots, x_n], \alpha(x_{n+1}), \ldots, \alpha(x_{2n-2}), \alpha(y_1)],\\
&\quad \beta(\alpha(y_2)))\\
&= -B(\beta(\alpha(y_1)), [[x_1, \ldots, x_n], \alpha(x_{n+1}),\\
&\quad \ldots, \alpha(x_{2n-2}), \alpha(y_2)])\\
&= -B(\beta'(y_1), [x_1, \ldots, x_{2n-2}, y_2]^{(1)}).
\end{aligned}$$

Hence

$$B([x_1, \ldots, x_{2n-2}, y_1]^{(1)}, \beta'(y_2)) + B(\beta'(y_1), [x_1, \ldots, x_{2n-2}, y_2]^{(1)}) = 0.$$

Corollary 6.1 *Let $(\mathcal{N}, [\cdot, \ldots, \cdot], \alpha, B, \beta)$ be a Hom-quadratic multiplicative n-ary Hom-Nambu algebra. For $k \geq 1$ define the $(2^k(n-1)+1)$- ary product $[\cdot, \ldots, \cdot]^{(k)}$ inductively by setting $[\cdot, \ldots, \cdot]^{(0)} = [\cdot, \ldots, \cdot]$ and*

$$\begin{aligned}
&[x_1, \ldots, x_{2^k(n-1)+1}]^{(k)}\\
&= [[x_1, \ldots, x_{2^{k-1}(n-1)+1}]^{(k-1)}, \alpha^{2^{k-1}}(x_{2^{k-1}(n-1)+2}), \ldots, \alpha^{2^{k-1}}(x_{2^k(n-1)+1})]^{(k-1)}
\end{aligned} \tag{50}$$

for all $x_i \in \mathcal{N}$.

Then $\mathcal{N}^k = (\mathcal{N}, [\cdot, \dots, \cdot]^{(k)}, \alpha^{2^k}, B, \beta')$*, where* $\beta' = \beta\alpha^{2^{k-1}}$*, is a Hom-quadratic multiplicative* $(2^k(n-1)+1)$*-ary Hom-Nambu algebra.*

6.6 Quadratic Hom-Nambu Algebras of Lower Arities

The purpose of this section is to observe that, under suitable assumptions, a quadratic n-ary Hom-Nambu algebra with $n \geq 3$ reduces to a quadratic $(n-1)$-ary Hom-Nambu algebra. We use the construction given in [40].

Theorem 6.6 *Let* $n \geq 3$ *and* $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha = (\alpha_1, \dots, \alpha_{n-1}), B, \beta)$ *be a Hom-quadratic* n*-ary Hom-Nambu algebra. Suppose* $a \in \mathcal{N}$ *satisfies*

$$\alpha_1(a) = a \;\; \textit{and} \;\; [a, x_1, \dots, x_{n-2}, a] = 0 \;\; \textit{for all} \;\; x_i \in \mathcal{N}.$$

Then $\mathcal{N}_a = (\mathcal{N}, [\cdot, \dots, \cdot]_a, \alpha_a = (\alpha_2, \dots, \alpha_{n-1}), B, \beta)$*, where*

$$[x_1, \dots, x_{n-1}]_a = [a, x_1, \dots, x_{n-1}] \;\; \textit{for all} \;\; x_i \in \mathcal{N},$$

is a Hom-quadratic $(n-1)$*-ary Hom-Nambu algebra.*

Proof Using [40], $\mathcal{N}_a = (\mathcal{N}, [\cdot, \dots, \cdot]_a, \alpha_a = (\alpha_2, \dots, \alpha_{n-1}))$ is an $(n-1)$-ary Hom-Nambu algebra.

Let $x_1, \dots, x_{n-2}, y_1, y_2 \in \mathcal{N}$, then

$$\begin{aligned} B([x_1, \dots, x_{n-2}, y_1]_a, \beta(y_2)) &= B([a, x_1, \dots, x_{n-2}, y_1], \beta(y_2)) \\ &= -B(\beta(y_1), [a, x_1, \dots, x_{n-2}, y_2]) \\ &= -B(\beta(y_1), [x_1, \dots, x_{n-2}, y_2]_a). \end{aligned}$$

Hence

$$B([x_1, \dots, x_{n-2}, y_1]_a, \beta(y_2)) + B(\beta(y_1), [x_1, \dots, x_{n-2}, y_2]_a) = 0.$$

Corollary 6.2 *Let* $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha = (\alpha_1, \dots, \alpha_{n-1}), B, \beta)$ *be a Hom-quadratic* n*-ary Hom-Nambu algebra, with* $n \geq 3$*. Suppose for some* $k \in \{1, \dots, n-2\}$ *there exist* $a_i \in L$ *for* $1 \leq i \leq k$ *satisfying*

$$\alpha_i(a_i) = a_i \;\; \textit{for} \;\; 1 \leq i \leq k$$

and

$$[a_1, \dots, a_j, x_{j+1}, \dots, x_{n-1}, a_j] = 0 \;\; \textit{for} \;\; 1 \leq j \leq k \;\; \textit{and all} \;\; x_l \in \mathcal{N}.$$

Then $\mathcal{N}_k = (\mathcal{N}, [\cdot, \dots, \cdot]_k, \alpha_k = (\alpha_{k+1}, \dots, \alpha_{n-1}), B, \beta)$*, where*

$$[x_1, \ldots, x_{n-1}]_k = [a_1, \ldots, a_k, x_k, \ldots, x_{n-1}] \;\; \textit{for all} \;\; x_i \in \mathcal{N}$$

is a Hom-quadratic $(n-k)$*-ary Hom-Nambu algebra.*

6.7 Ternary Nambu Algebras Arising from the Real Faulkner Construction

Let $(\mathfrak{g}, [\cdot, \cdot], B)$ be a real finite-dimensional quadratic Lie algebra and $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. We denote by $\langle -, - \rangle$ the dual pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

For all $x \in \mathfrak{g}$ and $f \in \mathfrak{g}^*$ we define an element $\phi(x \otimes f) \in \mathfrak{g}$ by

$$B(y, \phi(x \otimes f)) = \langle [y, x], f \rangle = f([y, x]) \text{ for all } y \in \mathfrak{g}. \tag{51}$$

Extending ϕ linearly, defines a $\mathfrak{g}$-equivariant map $\phi : \mathfrak{g} \otimes \mathfrak{g}^* \longrightarrow \mathfrak{g}$, which is surjective. To lighten the notation we will write $\phi(x, f)$ for $\phi(x \otimes f)$ in the sequel. The $\mathfrak{g}$-equivariance of ϕ is equivalent to

$$[\phi(x, f), \phi(y, g)] = \phi([\phi(x, f), y], g) + \phi(y, \phi(x, f) \cdot g), \tag{52}$$

for all $x, y \in \mathfrak{g}$ and $f, g \in \mathfrak{g}^*$, where $\phi(x, f) \cdot g$ is defined by

$$\langle y, \phi(x, f) \cdot g \rangle = -\langle [y, \phi(x, f)], g \rangle, \text{ for all } y \in \mathfrak{g}. \tag{53}$$

The fundamental identity (52) suggests defining a bracket on $\mathfrak{g} \otimes \mathfrak{g}^*$ by

$$[x \otimes f, y \otimes g] = [\phi(x, f), y] \otimes g + y \otimes \phi(x, f) \cdot g. \tag{54}$$

Proposition 6.4 *([19]). The bracket* (54) *turns* $\mathfrak{g} \otimes \mathfrak{g}^*$ *into a Leibniz algebra.*

Proposition 6.5 *Let* $\alpha \in Aut_S(B, \mathfrak{g})$ *be an involution, then* $(\mathfrak{g} \otimes \mathfrak{g}^*, [., .]_\Omega, \Omega, B_\Omega)$, *where*

$$\Omega(x \otimes f) = \alpha(x) \otimes f \circ \alpha, \tag{55}$$

$$[x \otimes f, y \otimes g]_\Omega = \Omega \circ [x \otimes f, y \otimes g], \tag{56}$$

$$B_\Omega(x \otimes f, y \otimes g) = \langle \alpha(x), g \rangle \langle \alpha(y), f \rangle, \tag{57}$$

is a multiplicative quadratic Hom-Leibniz algebra.

Proof Let $x, y, z \in \mathcal{N}$, $f, g, h \in \mathcal{N}^*$. Using (51) and (53), we have

$$\begin{aligned}B(y,\alpha(\phi(x\otimes f))) &= B(\alpha(y),\phi(x\otimes f))\\ &= \langle[\alpha(y),x],f\rangle\\ &= \langle\alpha([y,\alpha(x)]),f\rangle\\ &= \langle[y,\alpha(x)],f\circ\alpha\rangle\\ &= B(y,\phi(\alpha(x)\otimes f\circ\alpha)),\end{aligned}$$

and

$$\begin{aligned}\langle y,(\phi(x,f)\cdot g)\circ\alpha\rangle &= \langle\alpha(y),\phi(x,f)\cdot g\rangle\\ &= -\langle[\alpha(y),\phi(x,f)],g\rangle\\ &= -\langle\alpha([y,\alpha(\phi(x,f))]),g\rangle\\ &= -\langle[y,\alpha(\phi(x,f))],g\circ\alpha\rangle\\ &= -\langle[y,\phi(\alpha(x),f\circ\alpha)],g\circ\alpha\rangle\\ &= \langle y,\phi(\alpha(x),f\circ\alpha)\cdot(g\circ\alpha)\rangle.\end{aligned}$$

Thus, we obtain the following identity

$$\begin{aligned}\alpha(\phi(x\otimes f)) &= \phi(\alpha(x)\otimes f\circ\alpha),\\ (\phi(x,f)\cdot g)\circ\alpha &= \phi(\alpha(x),f\circ\alpha)\cdot(g\circ\alpha).\end{aligned}$$

Therefore

$$\begin{aligned}&\Omega([x\otimes f,y\otimes g]) = \alpha([\phi(x,f),y])\otimes g\circ\alpha+\alpha(y)\otimes(\phi(x,f)\cdot g)\circ\alpha\\ &= [\alpha(\phi(x,f)),\alpha(y)]\otimes g\circ\alpha+\alpha(y)\otimes(\phi(x,f)\cdot g)\circ\alpha\\ &= [\phi(\alpha(x)\otimes f\circ\alpha),\alpha(y)]\otimes g\circ\alpha+\alpha(y)\otimes\phi(\alpha(x),f\circ\alpha)\cdot(g\circ\alpha)\\ &= [\alpha(x)\otimes f\circ\alpha,\alpha(y)\otimes g\circ\alpha]\\ &= [\Omega(x\otimes f),\Omega(y\otimes g)].\end{aligned}$$

Thus, $\Omega([x\otimes f,y\otimes g]) = [\Omega(x\otimes f),\Omega(y\otimes g)]$. Then using Theorem 2.1, $(\mathfrak{g}\otimes\mathfrak{g}^*,[.,.]_\Omega,\Omega)$ is a multiplicative Hom-Leibniz algebra.

Since α is an involution, then

$$\begin{aligned}&B_\Omega([x\otimes f,y\otimes g]_\Omega,z\otimes h)\\ &= B_\Omega(\alpha([\phi(x,f),y])\otimes g\circ\alpha,z\otimes h)+B_\Omega(\alpha(y)\otimes(\phi(x,f)\cdot g)\circ\alpha,z\otimes h)\\ &= \langle[\phi(x,f),y],h\rangle\langle\alpha(z),g\circ\alpha\rangle+\langle y,h\rangle\langle\alpha(z),(\phi(x,f)\cdot g)\circ\alpha\rangle\\ &= \langle[\phi(x,f),y],h\rangle\langle z,g\rangle+\langle y,h\rangle\langle z,\phi(x,f)\cdot g\rangle\\ &= \langle[\phi(x,f),y],h\rangle\langle z,g\rangle-\langle y,h\rangle\langle[z,\phi(x,f)],g\rangle\\ &= -(\langle[z,\phi(x,f)],g\rangle\langle y,h\rangle-\langle z,g\rangle\langle[\phi(x,f),y],h\rangle)\\ &= -B_\Omega(y\otimes g,[x\otimes f,z\otimes h]_\Omega).\end{aligned}$$

Finally, the bilinear form B_Ω is symmetric nondegenerate and invariant. Then $(\mathfrak{g} \otimes \mathfrak{g}^*, [.,.]_\Omega, \Omega, B_\Omega)$ is a multiplicative quadratic Hom-Leibniz algebra.

The inner product on $\mathfrak{g}$ sets up an isomorphism $\flat : \mathfrak{g} \to \mathfrak{g}^*$ of $\mathfrak{g}$-modules, defined by $x^* = \flat(x) = B(x, \cdot)$.

The map ϕ defined by equation (51) induces a map $T : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$, by $T(x \otimes y) = \phi(x \otimes y^*)$. In other words, for all $x, y, z \in \mathfrak{g}$, we have

$$B(T(x \otimes y), z) = B([z, x], y),$$

whence

$$T(x \otimes y) = -T(y \otimes x).$$

This means that T factors through a map also denoted $T : \wedge^2\mathfrak{g} \to \mathfrak{g}$.

Using T we can define a ternary bracket on $\mathfrak{g}$ by

$$[x, y, z] := [T(x \otimes y), z] \tag{58}$$

and $(\mathfrak{g}, [.,.,.], B)$ is a quadratic ternary Nambu algebra.

Proposition 6.6 *Let $(\mathfrak{g}, [\cdot,\cdot], B)$ be a real finite-dimensional quadratic Lie algebra and $\alpha \in Aut_S(B, \mathfrak{g})$ be an involution. Then $(\mathfrak{g}, \alpha \circ [\cdot,\cdot,\cdot], (\alpha, \alpha), B_\alpha)$, where the bracket is defined in 58 and $B_\alpha(x, y) = B(\alpha(x), y)$, is a quadratic multiplicative ternary Hom-Nambu algebra.*

7 Centroids, Derivations and Quadratic *n*-ary Hom-Nambu Algebras

In this section, we first generalize to n-ary Hom-Nambu algebras the notion of centroid and its properties given in [12]. We also generalize to Hom setting the connections between centroid elements and derivations. Finally we construct quadratic n-ary Hom-Nambu algebras involving elements of the centroid of n-ary Nambu algebras.

7.1 Centroids of n-ary Hom-Nambu Algebras

Definition 7.1 Let $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be a multiplicative n-ary Hom-Nambu algebra and $End(\mathcal{N})$ be the endomorphism algebra of $\mathcal{N}$. Then the following subalgebra of $End(\mathcal{N})$

$$Cent(\mathcal{N}) = \{\theta \in End(\mathcal{N}) : \theta[x_1, \dots, x_n] = [\theta x_1, \dots, x_n],\ \forall x_i \in \mathcal{N}\} \tag{59}$$

is said to be the centroid of the n-ary Hom-Nambu algebra.

The definition is the same for classical case of n-ary Nambu algebra. We may also consider the same definition for any n-ary Hom-Nambu algebra.

Now, let $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be a multiplicative n-ary Hom-Nambu algebra. We denote by α^k, where $\alpha \in End(\mathcal{N})$, the k-times composition of α. We set in particular $\alpha^{-1} = 0$ and $\alpha^0 = Id$.

Definition 7.2 An α^k-centroid of a multiplicative n-ary Hom-Nambu algebra $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ is a subalgebra of $End(\mathcal{N})$ denoted $Cent_{\alpha^k}(\mathcal{N})$, given by

$$Cent_{\alpha^k}(\mathcal{N}) = \{\theta \in End(\mathcal{N}) : \theta[x_1, \dots, x_n] = [\theta x_1, \alpha^k(x_2) \dots, \alpha^k(x_n)],\ \forall x_i \in \mathcal{N}. \tag{60}$$

We recover the definition of the centroid when $k = 0$.

If $\mathcal{N}$ is a multiplicative n-ary Hom-Nambu-Lie algebra, then it is a simple fact that

$$\theta[x_1, \dots, x_n] = [\alpha^k(x_1), \dots, \theta x_p, \dots, \alpha^k(x_n)],\ \ \forall\ p \in \{1, \dots, n\}.$$

Lemma 7.1 *Let $(\mathcal{N}, [\cdot, \dots, \cdot])$ be an n-ary Nambu-Lie algebra. If $\theta \in Cent(\mathcal{N})$, then for $x_1, \dots, x_n \in \mathcal{N}$*

1. $[\theta^{p_1}x_1, \dots, \theta^{p_n}x_n] = \theta^{p_1+\dots+p_n}[x_1, \dots, x_n],\ \forall p_1, \dots, p_n \in \mathbb{N}$,
2. $[\theta^{p_1}x_1, \dots, \theta^{p_n}x_n] = Sgn(\sigma)[\theta^{p_1}x_{\sigma(1)}, \dots, \theta^{p_n}x_{\sigma(n)}],\ \forall p_1, \dots, p_n \in \mathbb{N}$ *and* $\forall \sigma \in \mathcal{S}_n$.

Proof Let $\theta \in Cent(\mathcal{N})$, $x_1, \dots, x_n \in \mathcal{N}$ and $1 \leq p \leq n$, we have

$$[\theta^p x_1, \dots, x_n] = \theta[\theta^{p-1}x_1, \dots, x_n] = \dots = \theta^p[x_1, \dots, x_n].$$

Also, observe that for any $k \in \{1, \dots, n\}$

$$\begin{aligned}[][x_1, \dots, \theta^p x_k, \dots, x_n] &= -[\theta^p x_k, x_2, \dots, x_1, \dots, x_n] \\ &= -\theta^p[x_k, x_2, \dots, x_1, \dots, x_n] = \theta^p[x_1, \dots, x_k, \dots, x_n].\end{aligned}$$

Then, similarly we have

$$[\theta^{p_1}x_1, \dots, \theta^{p_n}x_n] = \theta^{p_n}[\theta^{p_1}x_1, \dots, \theta^{p_{n-1}}x_{n-1}, x_n] = \dots = \theta^{p_1+\dots+p_n}[x_1, \dots, x_n].$$

The second assertion is a consequence of previous calculations and the skew-symmetry of $[\cdot, \dots, \cdot]$.

Proposition 7.1 *Let $(\mathcal{N}, [\cdot, \dots, \cdot])$ be an n-ary Nambu-Lie algebra and $\theta \in Cent(\mathcal{N})$.*

Let us fix p and set for any $x_1, \dots, x_n \in \mathcal{N}$

$$\{x_1, \dots, x_n\}_p = [\theta x_1, \dots, \theta x_{p-1}, \theta x_p, x_{p+1}, \dots, x_n]. \tag{61}$$

Then $(\mathcal{N}, \{\cdot, \dots, \cdot\}_p, \widetilde{\theta} = (\theta, \dots, \theta))$ *is an n-ary Hom-Nambu-Lie algebra.*

Proof For $\theta \in Cent(\mathcal{N})$ and $p \in \{1, \dots, n\}$, we have

$$\begin{aligned}\{\theta x_1, \dots, \theta x_{n-1}, \{y_1, \dots, y_n\}_p\}_p &= [\theta^2 x_1, \dots, \theta^2 x_p, \dots, \theta x_{n-1}, [\theta y_1, \dots, \theta y_p, \dots, y_n]]\\ &= [\theta^2 x_1, \dots, \theta^2 x_p, \dots, \theta x_{n-1}, \theta^p [y_1, \dots, y_n]]\\ &= \theta^{2p+n-1}([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]).\end{aligned}$$

In the other hand we have

$$\begin{aligned}&\sum_{k=0}^{n} \{\theta y_1, \dots, \{x_1, \dots, x_{n-1}, y_k\}_p, \dots, \theta y_n\}_p\\ &= \sum_{k=0}^{p} \{\theta y_1, \dots, \{x_1, \dots, x_{n-1}, y_k\}_p, \dots, \theta y_n\}_p\\ &\quad + \sum_{k=p}^{n} \{\theta y_1, \dots, \{x_1, \dots, x_{n-1}, y_k\}_p, \dots, \theta y_n\}_p\\ &= \sum_{k=0}^{p} [\theta^2 y_1, \dots, \theta[\theta x_1, \dots, \theta x_p, \dots, x_{n-1}, y_k], \dots, \theta^2 y_p, \dots, \theta y_n]\\ &\quad + \sum_{k=0}^{p} [\theta^2 y_1, \dots, \theta^2 y_p, \dots, [\theta x_1, \dots, \theta x_p, \dots, x_{n-1}, y_k], \dots, \theta y_n]\\ &= \sum_{k=0}^{p} \theta^{2p+n-1} [y_1, \dots, [x_1, \dots, x_{n-1}, y_k], \dots, y_n]\\ &\quad + \sum_{k=0}^{p} \theta^{2p+n-1} [2y_1, \dots, [x_1, \dots, x_{n-1}, y_k], \dots, y_n]\\ &= \theta^{2p+n-1}(\sum_{k=0}^{n} [y_1, \dots, [x_1, \dots, x_{n-1}, y_k], \dots, y_n]).\end{aligned}$$

Therefore the Hom-Nambu identity with respect to the bracket $[\cdot, \dots, \cdot]$ leads to the Hom-Nambu identity for $\{\cdot, \dots, \cdot\}_l$. The skew-symmetry is proved by second assertion of Lemma 7.1.

7.2 Centroids and Derivations of n-ary Hom-Nambu Algebras

Let $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be a multiplicative n-ary Hom-Nambu-Lie algebra.

Definition 7.3 For any $k \geq 1$, we call $D \in End(\mathcal{N})$ an α^k-*derivation* of the multiplicative n-ary Hom-Nambu-Lie $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ if D and α commute and we have

$$D[x_1, \ldots, x_n] = \sum_{i=1}^{n} [\alpha^k(x_1), \ldots, \alpha^k(x_{i-1}), D(x_i), \alpha^k(x_{i+1}), \ldots, \alpha^k(x_n)]. \quad (62)$$

We denote by $Der_{\alpha^k}(\mathcal{N})$ the set of α^k-derivations.

For $x = (x_1, \ldots, x_{n-1}) \in \mathcal{N}^{n-1}$ satisfying $\alpha(x) = x$ and $k \geq 1$, we define the map $ad_k(x) \in End(\mathcal{N})$ by

$$ad_k(x)(y) = [x_1, \ldots, x_{n-1}, \alpha^k(y)] \;\; \forall y \in \mathcal{N}. \quad (63)$$

The map $ad_k(x)$ is an α^{k+1}-derivation, that we call inner α^{k+1}-derivation. We denote by $Inn_{\alpha^k}(\mathcal{N})$ the space generated by all inner α^{k+1}-derivations.

Set $Der(\mathcal{N}) = \bigoplus_{k \geq -1} Der_{\alpha^k}(\mathcal{N})$ and $Inn(\mathcal{N}) = \bigoplus_{k \geq -1} Inn_{\alpha^k}(\mathcal{N})$.

Lemma 7.2 *For $D \in Der_{\alpha^k}(\mathcal{N})$ and $D' \in Der_{\alpha^{k'}}(\mathcal{N})$, where $k + k' \geq -1$, we have $[D, D'] \in Der_{\alpha^{k+k'}}(\mathcal{N})$, where the commutator $[D, D']$ is defined as usual.*

Now, we define a linear map $\varsigma : Der_{\alpha^k}(\mathcal{N}) \to Der_{\alpha^{k+1}}(\mathcal{N})$ by $\varsigma(D) = \alpha \circ D$. Since the elements of $Der_{\alpha^k}(\mathcal{N})$ and α commute then ς is in the centroid of the Lie algebra $(Der(\mathcal{N}), [\cdot, \cdot])$.

Hence, using Proposition 7.1 we have

Proposition 7.2 *Let $(\mathcal{N}, [\cdot, \ldots, \cdot], \alpha)$ be a multiplicative n-ary Hom-Nambu-Lie algebra. The triple $(Der(\mathcal{N}), [\cdot, \cdot]_\varsigma, \varsigma)$, where the bracket is defined by $[\cdot, \cdot]_\varsigma = \varsigma \circ [\cdot, \cdot]$, is a Hom-Lie algebra.*

Proposition 7.3 *Let $(\mathcal{N}, [\cdot, \ldots, \cdot], \alpha)$ be a multiplicative n-ary Hom-Nambu-Lie algebra. If $D \in Der_{\alpha^k}(\mathcal{N})$ and $\theta \in Cent_{\alpha^{k'}}(\mathcal{N})$, then $\theta D \in Der_{\alpha^{k+k'}}(\mathcal{N})$.*

Proof Let $x_1, \ldots, x_n \in \mathcal{N}$ then

$$\begin{aligned} \theta D([x_1, \ldots, x_n]) &= \sum_{i=1}^{n} \theta[\alpha^k(x_1), \ldots, D(x_i), \ldots, \alpha^k(x_n)] \\ &= \sum_{i=1}^{n} [\alpha^{k+k'}(x_1), \ldots, \theta D(x_i), \ldots, \alpha^{k+k'}(x_n)]. \end{aligned}$$

Thus θD is an α^k-derivation.

Now we define the notion of central derivation. Let $(\mathcal{N}, [\cdot, \ldots, \cdot], \alpha)$ be a multiplicative n-ary Hom-Nambu-Lie algebra. We set $Z(\mathcal{N}) = \{x \in \mathcal{N} : [x, y_1, \ldots, y_{n-1}] = 0, \; \forall\, y_1, \ldots, y_{n-1} \in \mathcal{N}\}$, the center of the n-ary Hom-Nambu-Lie algebra.

Definition 7.4 Let $\varphi \in End(\mathcal{N})$, then φ is said to be a central derivation if $\varphi(\mathcal{N}) \subset Z(\mathcal{N})$ and $\varphi([\mathcal{N}, \dots, \mathcal{N}]) = 0$.

The set of all central derivations of $\mathcal{N}$ is denoted by $C(\mathcal{N})$.

Notice that an α^k-derivation φ is a central derivation if $\varphi(\mathcal{N}) \subset Z(\mathcal{N})$.

Theorem 7.1 *Let $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be a multiplicative n-ary Hom-Nambu-Lie algebra. Let D in $Der_{\alpha^k}(\mathcal{N})$ and θ in $Cent(\mathcal{N})$ such that $[\theta, \alpha] = 0$, then we have*

1. *$[D, \theta]$ is in the α^k-centroid of $\mathcal{N}$,*
2. *if $[D, \theta]$ is a central derivation then $D\theta$ is an α^k-derivation of $\mathcal{N}$.*

Proof (1) Let $D \in Der_{\alpha^k}(\mathcal{N}), \theta \in Cent(\mathcal{N})$ and $x_1, \dots, x_n \in \mathcal{N}$ we have

$$\begin{aligned}
D\theta([x_1, \dots, x_n]) &= D([\theta x_1, \dots, x_n]) \\
&= [D\theta x_1, \dots, \alpha^k(x_n)] + \sum_{i=2}^{n} [\alpha^k(\theta x_1), \dots, D(x_i), \dots, \alpha^k(x_n)] \\
&= [D\theta x_1, \dots, \alpha^k(x_n)] + \sum_{i=2}^{n} [\theta\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_n)] \\
&= [D\theta x_1, \dots, \alpha^k(x_n)] + \sum_{i=2}^{n} [\alpha^k(x_1), \dots, \theta D(x_i), \dots, \alpha^k(x_n)] \\
&= [D\theta x_1, \dots, \alpha^k(x_n)] + \theta D([x_1, \dots, x_n]) - [\theta D x_1, \dots, \alpha^k(x_n)].
\end{aligned}$$

Then

$$(D\theta - \theta D)([x_1, \dots, x_n]) = [(D\theta - \theta D)x_1, \alpha^k(x_2), \dots, \alpha^k(x_n)].$$

That is, $[D, \theta] = D\theta - \theta D \in Cent_k(\mathcal{N})$.

(2) Using Proposition 6.3, θD is an α^k-derivation and since $[D, \theta]$ is a α^k-derivation, then $D\theta = [D, \theta] + \theta D$ is also an α^k-derivation.

Let A be a $\mathbb{K}$-vector space, μ be an n-linear map on A and η be a linear map on A. Let (A, μ, η) be a multiplicative symmetric n-ary totally Hom-associative algebra. The η^k-centroid $Cent_{\eta^k}(A)$ of A is defined by

$$Cent_{\eta^k}(A) = \{f \in End(A) : \ f(\mu(a_1, \dots, a_n)) = \mu(f(a_1), \eta^k(a_2), \dots, \eta^k(a_n))\},$$

for all $a_i \in A$ and $i \in \{1, \dots, n\}$. The set of η^k-derivation, $Der_{\eta^k}(A)$, is a subset of $End(A)$ defined by $\varphi \in End(A)$ such that

$$\varphi(\mu(a_1, \dots, a_n)) = \sum_{i=1}^{n} \mu(\eta^k(a_1) \dots, \eta^k(a_{i-1}), \varphi(a_i), \eta^k(a_{i+1}), \dots, \eta^k(a_n))\},$$

for all $a_i \in A$.

Theorem 7.2 *Let (A, μ, η) be a multiplicative symmetric n-ary Hom-associative algebra and $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be a multiplicative n-ary Hom-Nambu-Lie algebra, then we have the following assertion*

- *If $f \in Cent_{\eta^k}(A)$ and $\theta \in Cent_{\alpha^k}(\mathcal{N})$, then $f \otimes \theta$ is in the ζ^k-centroid, where $\zeta^k = \eta^k \otimes \alpha^k$, of the Hom-Nambu-Lie algebra $A \otimes \mathcal{N}$ defined in 2.3.*
- *If $f \in Cent_{\eta^k}(A)$ and $D \in Der_{\alpha^k}(\mathcal{N})$, then $f \otimes D$ is a ζ^k-derivation of the Hom-Nambu-Lie algebra $A \otimes \mathcal{N}$.*

Proof Let $a_i \in A$, $x_i \in \mathcal{N}$ where $i \in \{1, \dots, n\}$ and f be a η^k-centroid on A.
- If $\theta \in Cent_{\alpha^k}(\mathcal{N})$, then

$$\begin{aligned}
&(f \otimes \theta)([a_1 \otimes x_1, \dots, a_n \otimes x_n]) = (f \otimes \theta)(\mu(a_1, \dots, a_n) \otimes [x_1, \dots, x_n]_{\mathcal{N}}) \\
&= \mu(f(a_1), \eta^k(a_2), \dots, \eta^k(a_n)) \otimes [\theta(x_1), \alpha^k(x_2), \dots, \alpha^k(x_n)]_{\mathcal{N}} \\
&= [(f \otimes \theta)(a_1 \otimes x_1), \zeta^k(a_2 \otimes x_2), \dots, \zeta^k(a_n \otimes x_n)].
\end{aligned}$$

Thus $f \otimes \theta$ is in the ζ^k-centroid of $A \otimes \mathcal{N}$.
- If $D \in Der_{\alpha^k}(\mathcal{N})$, then

$$\begin{aligned}
&(f \otimes D)([a_1 \otimes x_1, \dots, a_n \otimes x_n]) = f \otimes D((a_1 \cdot \dots \cdot a_n) \otimes [x_1, \dots, x_n]) \\
&= \mu(f(a_1), \eta^k(a_2), \dots, \eta^k(a_n)) \otimes \sum_{i=1}^{n} [\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_n)] \\
&= \sum_{i=1}^{n} \mu(\eta^k(a_1), \dots, f(a_i), \dots, \eta^k(a_n)) \otimes [\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_n)]_{\mathcal{N}} \\
&= \sum_{i=1}^{n} [\zeta^k(a_1 \otimes x_1), \dots, (f \otimes D)(a_i \otimes x_i), \dots, \zeta^k(a_n \otimes x_n)].
\end{aligned}$$

Therefore $f \otimes D$ is a ζ^k-derivation of $A \otimes \mathcal{N}$.

7.3 Centroids and Quadratic n-ary Hom-Nambu Algebras

Let $\theta \in Cent(\mathcal{N})$ such that θ is invertible and symmetric with respect to B (i.e. $B(\theta x, y) = B(x, \theta y)$). We set

$$Cent_S(\mathcal{N}) = \{\theta \in Cent(\mathcal{N}) : \theta \text{ symmetric with respect to } B\}.$$

Theorem 7.3 *Let $(\mathcal{N}, [\cdot, \dots, \cdot], B)$ be a quadratic n-ary Nambu-Lie algebra and $\theta \in Cent_S(\mathcal{N})$ such that θ is invertible. We consider a bilinear form B_θ defined by*

$$\begin{aligned} B_\theta : \mathcal{N} \times \mathcal{N} &\longrightarrow \mathbb{K} \\ (x, y) &\longmapsto B(\theta x, y). \end{aligned}$$

Then, $(\mathcal{N}, \{\cdot, \ldots, \cdot\}_l, (\theta, \ldots, \theta), B_\theta)$ *is a quadratic n-ary Hom-Nambu-Lie algebra.*

Proof It easy to proof that B_θ is symmetric and nondegenerate.

We have also θ is symmetric with respect to B_θ, indeed

$$B_\theta(\theta x, y) = B(\theta^2 x, y) = B(\theta x, \theta y) = B_\theta(x, \theta y).$$

The invariance of B_θ is given by, set $l \in \{1, \ldots, n-1\}$

$$\begin{aligned} B_\theta(\{x_1, \ldots, x_{n-1}, y\}_l, z) &= B_\theta([\theta x_1, \ldots, \theta x_l, \ldots, x_{n-1}, y], z) \\ &= B(\theta[\theta x_1, \ldots, \theta x_l, \ldots, x_{n-1}, y], z) \\ &= B([\theta^2 x_1, \ldots, \theta x_l, \ldots, x_{n-1}, y], z) \\ &= -B(y, [\theta^2 x_1, \ldots, \theta x_l, \ldots, x_{n-1}, z]) \\ &= -B_\theta(y, [\theta x_1, \ldots, \theta x_l, \ldots, x_{n-1}, z]) \\ &= -B_\theta(y, \{x_1, \ldots, x_{n-1}, z\}_l). \end{aligned}$$

In the other hand, when $l = n$ we have $[\theta x_1, \ldots, \theta x_n] = [\theta^2 x_1, \ldots, \theta x_{n-1}, x_n]$ then it's a consequence of the previous calculations.

Therefore $(\mathcal{N}, \{\cdot, \ldots, \cdot\}_l, (\theta, \ldots, \theta), B_\theta)$ is a quadratic n-ary Hom-Nambu-Lie algebra.

Notice that B_θ is also an invariant scalar product of the n-ary Nambu-Lie algebra $\mathcal{N}$.

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