A Domain Decomposition Method for Discretization of Multiscale Elliptic Problems by Discontinuous Galerkin Method

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Abstract. In this paper boundary value problems for second order elliptic equations with highly discontinuous coefficients are considered on a 2D polygonal region. The problems are discretized by a discontinuous Galerkin (DG) with finite element method (FEM) on triangular elements using piecewise linear functions.

The goal is to design and analyze a parallel algorithm for solving the discrete problem whose rate of convergence is independent of the jumps of the coefficients. The method discussed is an additive Schwarz method (ASM) which belongs to a class of domain decomposition methods and is one of the most efficient parallel algorithm for solving discretizations of PDEs.

It turns out that the convergence of the method presented here is almost optimal and only weakly depends on the jumps of coefficients. The suggested method is very well suited for parallel computations.

Keywords: Interior penalty method · Discontinuous Galerkin method · Elliptic equations with discontinuous coefficients · Finite element method · Additive Schwarz method

1 Introduction

We consider boundary value problems (BVPs) for second order elliptic equations with highly discontinuous coefficients posed on a 2D polygonal region. The problem is discretized by a discontinuous Galerkin (DG) method with FEM on triangular elements and piecewise linear functions, see [1,3], and references therein. The goal of this paper is to design and analyze a parallel algorithm for solving the discrete problem with rate of convergence independent of the jumps of coefficients.

The proposed algorithm is an additive Schwarz method (ASM) with overlaps and belongs to a class of domain decomposition methods and it is one of the most efficient parallel algorithms for solving discretizations of PDEs, see [5].

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In the paper the results obtained in [4] for continuous piecewise linear finite element discretization are extended to DG discretization. They are more general comparising to the results of [4].

The presented ASM is two-level with a special coarse space defined on large triangles of coarse triangulation, i.e. a multiscale coarse space. This is a space of continuous functions which are discrete harmonic on edges of the coarse triangles and inside of them in the sense of corresponding bilinear forms. The local spaces are defined in a standard way, on the fine triangulation, on extensions of coarse triangles; these spaces contain discontinuous functions. For literature on the topic see [4,5], and references therein.

It turns out that the convergence of the discussed ASM is dependent of the jumps of the coefficients on the boundary of coarse triangles only. For some distributions of jumps, the convergence of the ASM is also independent of these jumps.

The paper is organized as follows. In Sect. 2, differential and discrete problems are formulated. In Sect. 3, a two level ASM for solving the discrete problem is designed and analyzed. The main result is Theorem 5, which guarantees the optimality of the method. Section 4 is devoted to an implementation of the method discussed.

2 Differential and Discrete DG Problems

We consider the following elliptic problem:
Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \qquad \forall v \in H_0^1(\Omega) \tag{1}$$

where

$$a(u,v) = \int_\Omega \rho(x) \nabla u \cdot \nabla v dx, \qquad f(v) = \int_\Omega f v dx.$$

We assume that Ω is a polygonal region, $f \in L^2(\Omega)$ and $\rho(x) \geq \rho_0 > 0$, and $\rho \in L^\infty(\Omega)$. Under these assumptions problem (1) is well posed.

We will also assume that $\rho_0 \geq 1$. This can be fulfilled by scaling (1). It is used in the analysis of preconditioner discussed in Sect. 3.

Let $\mathcal{T}^h(\Omega)$ be a triangulation of Ω with triangular elements K_i and the mesh parameter h. It is constructed as a refinement of the coarse triangulation of Ω consisting of large triangles Ω_l of diameter $H_l, l = 1, \cdots, L, H_l = diam(\Omega_l)$ and $H = \max H_l$. The refinement procedure is repeated several times, where one step of the process is to split each triangle into four smaller ones, obtained by connecting the midpoints of its edges. Let $X_i(K_i)$ denote a space of linear functions on K_i and

$$X_h(\Omega) = \Pi_{i=1}^N X_i(K_i), \qquad \bar{\Omega} = \cup_{i=1}^N K_i,$$

be the space in which problem (1) is approximated. Note that $X_h(\Omega) \not\subset H^1(\Omega)$ and its elements do not vanish on $\partial\Omega$, in general.

The discrete problem for (1) is of the form:
Find $u_h^* \in X_h(\Omega)$ such that

$$\hat{a}_h(u_h^*, v_h) = f(v_h), \qquad v_h \in X_h(\Omega), \tag{2}$$

where for $u, v \in X_h(\Omega), u = \{u_i\}_{i=1}^N, u_i \in X_i(K_i)$,

$$\hat{a}_h(u, v) = \sum_{i=1}^{N} \hat{a}_i(u, v), \qquad f(v) = \sum_{i=1}^{N} \int_{K_i} f v_i dx.$$

Since we use linear elements, we can assume without loss of generality that $\rho_{|K_i} = \rho_i$ is constant on K_i. Here

$$\hat{a}_i(u, v) = a_i(u, v) + s_i(u, v) + p_i(u, v),$$

$$a_i(u, v) = \int_{K_i} \rho_i \nabla u_i \cdot \nabla v_i dx,$$

$$s_i(u, v) = \sum_{E_{ij} \subset \partial K_i} \int_{E_{ij}} \omega_{ij} [n_i^T \rho_i \nabla u_i (v_j - v_i) + n_i^T \rho_i \nabla v_i (u_j - u_i)] \, ds,$$

$$p_i(u, v) = \sum_{E_{ij} \subset \partial K_i} \frac{\sigma}{h} \int_{E_{ij}} \gamma_{ij} (u_i - u_j)(v_i - v_j) \, ds$$

where $E_{ij} = E_{ji} = \partial K_i \cap \partial K_j, E_{ij} \subset \partial K_i$ and $E_{ji} \subset \partial K_j$; $n_i = n_{E_{ij}}$ is the unit normal vector to E_{ij} pointing from K_i to K_j;

$$\omega_{ij} \equiv \omega_{E_{ij}} = \frac{\rho_j}{\rho_i + \rho_j}, \qquad \omega_{ji} \equiv \omega_{E_{ji}} = \frac{\rho_i}{\rho_i + \rho_j}$$

and

$$\gamma_{ij} \equiv \gamma_{E_{ij}} = \frac{2\rho_i \rho_j}{\rho_i + \rho_j};$$

σ is a positive penalty parameter (sufficiently large, see below Lemma 1). For boundary egdes these definitions extend straightforwardly, setting for $E_{ij} \subset \partial\Omega$: $\omega_{ij} = 1$, $\omega_{ji} = 0$, $v_j = u_j = 0$ and $\gamma_{ij} = \rho_i$.

To analyze problem (2) we introduce some auxiliary bilinear forms and a broken norm. Let

$$d_h(u, v) = \sum_{i=1}^{N} d_i(u, v), \quad d_i(u, v) = a_i(u, v) + p_i(u, v) \tag{3}$$

and let the weighted broken norm in $X_h(\Omega)$ be defined by

$$\| u \|_{1,h}^2 \equiv d_h(u, u) = \sum_{i=1}^{N} \{ \| (\rho_i)^{1/2} \nabla u_i \|_{L^2(K_i)}^2 + \sum_{E_{ij} \subset \partial K_i} \frac{\sigma}{h} \gamma_{ij} \| u_i - u_j \|_{L^2(E_{ij})}^2 \}. \tag{4}$$

Lemma 1. *There exists $\sigma_0 > 0$ such that for $\sigma \geq \sigma_0$ there exist positive constants C_0 and C_1 independent of ρ_i and h such that for any $u \in X_h$ hold*

$$C_0 d_i(u,u) \leq \hat{a}_i(u,u) \leq C_1 d_i(u,u)$$

and

$$C_0 d_h(u,u) \leq \hat{a}(u,u) \leq C_1 d_h(u,u).$$

For the proof we refer the reader to [2]; see also [3] or [1].

Lemma 1 implies that the discrete problem (2) is well posed if the penalty parameter $\sigma \geq \sigma_0$. Below σ is fixed and assumed to satisfy the above condition.

The error bound is given by

Theorem 2. *Let u^* and u_h^* be the solutions of (1) and (2). For $u^*_{|K_i} \in H^2(K_i)$ holds*

$$\| u^* - u_h^* \|_{1,h}^2 \leq M h^2 \sum_{i=1}^{N} \rho_i |u^*|^2_{H^2(K_i)}$$

where M is independent of h, u^* and ρ_i.

The proof follows from Lemma 1; for details see, for example, [3].

3 ASM with a Multiscale Coarse Space

We design and analyze a two-level additive Schwarz method (ASM) for solving the discrete problem (2). For that the general theory of ASMs is used, see [5]. The decomposition of $X_h(\Omega)$ consists of the local spaces defined on subdomains extended from the coarse triangles Ω_l, and the global space of continuous discrete harmonic functions related to the coarse triangulation.

3.1 Decomposition of $X_h(\Omega)$

Let

$$X_h(\Omega) = V^{(0)}(\Omega) + \sum_{l=1}^{L} V^{(l)}(\Omega) \tag{5}$$

where $V^{(0)}(\Omega)$ is a coarse space while $V^{(l)}(\Omega), l = 1, \ldots L$, are local spaces associated with Ω_l. They are defined as follows. For $l = 1, \ldots, L$, Ω_l is extended to Ω_l' by adding triangles from the fine triangulation around $\partial\Omega_l$ which intersect $\partial\Omega_i$ by vertex and/or edge. In this way we get an overlapping partitioning of Ω,

$$\bar{\Omega} = \bigcup_{l=1}^{L} \bar{\Omega}_l'$$

with overlap $\delta_l \approx 2h$ defined as

$$\delta_l = dist(\partial\Omega_l' \setminus \partial\Omega, \partial \overset{o}{\Omega}_l \setminus \partial\Omega)$$

where $\overset{o}{\Omega}_l$ denotes the interior part of Ω_l which is not overlapped by any other Ω_p for $p \neq l$; see [5, p. 198] for figures which exemplify such decomposition.

The local spaces $V^{(l)}(\Omega)$ for $l = 1, \ldots, L$ are defined as

$$V^{(l)}(\Omega) = \{\{v_i\}_{i=1}^N \in X_h(\Omega) : v_i = 0 \text{ on } K_i \not\subset \bar{\Omega}_l'\}. \tag{6}$$

Thus $V^{(l)}(\Omega)$ is the restriction of $X_h(\Omega)$ to $\bar{\Omega}_l'$ and zero outside of $\bar{\Omega}_l'$.

The coarse space $V^{(0)}(\Omega)$ is defined in a special way. The functions in $V^{(0)}(\Omega)$ are going to be piecewise linear continuous on the fine triangulation and discrete harmonic on $\partial\Omega_l$ and in Ω_l. Let ν be the set of all vertices of $\bar{\Omega}_l$. With each $x^{(k)} \in \nu$, a function $\Phi_k(x)$ is associated with support on a union of coarse triangles Ω_l for which $x^{(k)}$ is a common vertex. On the set ν, we set $\Phi_k(x^{(k)}) = 1$ and $\Phi_k(x) = 0$ otherwise. Next we define Φ_k on the boundary of each Ω_l. Let $x^{(k)}$ be a vertex of Ω_l and let F_{lp} denote an edge of Ω_l shared with Ω_p, $F_{lp} = \partial\Omega_l \cap \partial\Omega_p$. Let $a_{\Omega_l}(\cdot,\cdot)$ be the restriction of $a(\cdot,\cdot)$ to $\bar{\Omega}_l$, i.e.

$$a_{\Omega_l}(u,v) = \sum_{K_i \subset \bar{\Omega}_l} (\rho_i \nabla u, \nabla v)_{L^2(K_i)} = (\rho^{(l)} \nabla u, \nabla v)_{L^2(\Omega_l)}, \tag{7}$$

where by definition $\rho^{(l)} = \rho_i$ on $K_i \subset \bar{\Omega}_l$. Then the restriction of $a_{\Omega_l}(\cdot,\cdot)$ to F_{lp} is defined as

$$a_{F_{lp}}(u,v) = (\rho_{lp} D_\tau u, D_\tau v)_{L^2(F_{lp})} \tag{8}$$

where D_τ is the tangential derivative and ρ_{lp}, on F_{lp}, is the harmonic average of the coefficients on Ω_l and Ω_p, i.e. $\rho_{lp} = 2\rho^{(l)}\rho^{(p)}/(\rho^{(l)} + \rho^{(p)})$. Note that in this way $\rho_{lp} = \rho_{pl}$. On F_{lp}, we define the values of Φ_k as the solution of the one-dimensional problem:

$$a_{F_{lp}}(\Phi_k, v) = 0 \qquad \forall v \in \overset{o}{V}_h(F_{lp}) \tag{9}$$

with Dirichlet boundary conditions $\Phi_k(x^{(k)}) = 1$ and $\Phi_k(x^{(m)}) = 0$ at the other end, $x^{(m)}$, of F_{lp}. Above, $\overset{o}{V}_h(F_{lp})$ is the set of piecewise linear continuous functions with zero values on ∂F_{lp}. In addition we set $\Phi_k(x) = 0$ on those edges of Ω_l which do not end at $x^{(k)}$.

Finally we extend Φ_k, already defined on $\partial\Omega_l$, into Ω_l as a discrete harmonic function in the sense of $a_{\Omega_l}(\cdot,\cdot)$, i.e.

$$\begin{cases} a_{\Omega_l}(\Phi_k, v) = 0, & \forall v \in \overset{o}{V}_h(\Omega_l) \\ \text{with } \Phi_k(x) \text{ on } \partial\Omega_l \text{ defined in (9).} \end{cases} \tag{10}$$

Here $v \in \overset{o}{V}_h(\Omega_l)$ is a set of piecewise linear continuous functions defined on $\bar{\Omega}_l$ with zero values on $\partial\Omega_l$.

Using these functions, the coarse space $V^{(0)}(\Omega)$ is defined as

$$V^{(0)} = span\{\Phi_k(x)\}_{x^{(k)} \in \nu}. \tag{11}$$

Of course $V^{(0)} \subset X_h(\Omega)$.

Remark 3. This space is called a multiscale coarse space and at the beginning it was used to obtain more accurate approximation. In [4], $V^{(0)}(\Omega)$ was used also as a coarse space in ASM for the conforming (continuous) finite element method in the case when the coefficients are piecewise constant across $\partial\Omega_l$, $l = 1, \dots, L$.

3.2 Inexact Solver

For $u^{(0)}, v^{(0)} \in V^{(0)}(\Omega)$, let

$$b_0(u^{(0)}, v^{(0)}) = d_h(u^{(0)}, v^{(0)}), \tag{12}$$

where $d_h(\cdot,\cdot)$ is defined in (3).

For $l = 1, \dots, L$ we set

$$b_l(u^{(l)}, v^{(l)}) = d_{\Omega'_l}(u^{(l)}, v^{(l)}), \quad u^{(l)}, v^{(l)} \in V^{(l)}(\Omega) \tag{13}$$

where for $u^{(l)} = \{u_i^{(l)}\}_{i=1}^N, \;\; v^{(l)} = \{v_i^{(l)}\}_{i=1}^N$

$$\begin{aligned} d_{\Omega'_l}(u^{(l)}, v^{(l)}) = & \sum_{K_i \subset \bar{\Omega}'_l} \{(\rho_i \nabla u_i^{(l)}, \nabla v_i^{(l)})_{L^2(K_i)} + \\ & + \sum_{E_{ij} \subset \partial K_i} \gamma_{ij} \frac{\sigma}{h} (u_j^{(l)} - u_i^{(l)}, v_j^{(l)} - v_i^{(l)})_{L^2(E_{ij})}\}. \end{aligned} \tag{14}$$

3.3 The Operator Equation

For $l = 0, 1, \dots, L$ let $T_l : X_h(\Omega) \to V^{(l)}(\Omega)$ be defined by

$$b_l(T_l u, v) = \hat{a}_h(u, v), \qquad v \in V^{(l)}(\Omega). \tag{15}$$

Note that $T_l u$ is defined uniquely for given $u \in X_h(\Omega)$ as the solution of local problems defined on Ω'_l for $l = 1, \dots, L$, and the global one for $l = 0$.
Let

$$T = T_0 + T_1 + \dots + T_L. \tag{16}$$

We replace (2) by the following operator equation

$$T u_h^* = g_h \tag{17}$$

where $g_h = \sum_{l=0}^L g_l, g_l \equiv T_l u_h^*$. Note that to compute g_l we do not need to know u_h^*, the solution of (1).

Problems (2) and (17) are equivalent, what follows from the theorem below.

To formulate the convergence theorem for the discussed ASM we have to introduce some notation. Let us for each Ω_l define

$$\bar{\rho}_l = \sup_{K_i \subset \Omega_l^h} \rho_i \tag{18}$$

where Ω_l^h is a union of triangles $K_i \subset \bar{\Omega}_l$ which intersect $\partial\Omega_l$ by vertex and/or edge.

Theorem 4. *The operator T defined in (16) satisfies $T = T^* > 0$. Moreover, for any $u \in X_h(\Omega)$ there holds*

$$C_0\beta^{-1}\hat{a}_h(u,u) \leq \hat{a}_h(Tu,u) \leq C_1\hat{a}_h(u,u) \tag{19}$$

where

$$\beta = \max_{l=1,\ldots,L} \bar{\rho}_l \frac{H_l}{h}\left(1+\log\frac{H_l}{h}\right)^2, \tag{20}$$

with $\bar{\rho}_l$ defined in (18), and C_0 and C_1 are positive constants independent of H, h and the jumps of $\rho(x)$.

Remark 5. The proof of Theorem 5 needs to check three key assumptions of abstract theory of ASMs, see for example the book [5]. For that we need several auxiliary lemmas, some of them are new. The proof is omitted here due to the limit of pages. It will be published elsewhere together with supporting numerical tests.

4 Implementation

Equation (17) can be solved efficiently by the conjugate gradient method. To simplify the presentation we discuss here the Richardson's method instead. The latter is of the form: given $u^{(0)}$, iterate for $n = 0, 1, \ldots$

$$u^{(n+1)} = u^{(n)} - \tau^*(Tu^{(n)} - g_h) \tag{21}$$

where we can set $\tau^* = 2/(C_1 + C_0\beta^{-1})$ according to Theorem 5. Since

$$r^{(n)} \equiv (Tu^{(n)} - g_h) = \sum_{l=0}^{L}(T_l u^{(n)} - g_h) = \sum_{l=0}^{L} T_l(u^{(n)} - u_h^*) \equiv \sum_{l=0}^{L} r_l^{(n)}, \tag{22}$$

we need to compute $r_l^{(n)} \equiv T_l(u^{(n)} - u_h^*)$ for $l = 0, 1, \ldots, L$. Note that these problems, see (15), are independent of each other, therefore, they can be solved in parallel. Problems for $l = 1, \ldots, L$ are local, and they are defined on Ω_l'.

The problem for $l = 0$ is global and it is defined on the coarse triangulation with piecewise linear continuous functions. The solution of the coarse problem requires finding the coarse basis functions $\{\Phi_k\}$ for all the vertices $x^{(k)}$ of the coarse triangles. This is a precomputation step and it should be carried out before starting the iterative process (21).

The above implementation shows that the proposed algorithm is very well suited for parallel computations.

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