Harald Atmanspacher
Emmanuel Haven
Kirsty Kitto
Derek Raine (Eds.)

LNCS 8369

Quantum Interaction

7th International Conference, QI 2013
Leicester, UK, July 25–27, 2013
Selected Papers

Springer

Lecture Notes in Computer Science 8369

For further volumes:
http://www.springer.com/series/7407

Harald Atmanspacher · Emmanuel Haven
Kirsty Kitto · Derek Raine (Eds.)

Quantum Interaction

7th International Conference, QI 2013
Leicester, UK, July 25–27, 2013
Selected Papers

Springer

Editors
Harald Atmanspacher
ETH Zurich
Zurich
Switzerland

Emmanuel Haven
School of Management
University of Leicester
Leicester
UK

Kirsty Kitto
Queensland University of Technology
Brisbane, QLD
Australia

Derek Raine
Department of Physics and Astronomy
Centre for Interdisciplinary Science
University of Leicester
Leicester
UK

ISSN 0302-9743 ISSN 1611-3349 (electronic)
ISBN 978-3-642-54942-7 ISBN 978-3-642-54943-4 (eBook)
DOI 10.1007/978-3-642-54943-4
Springer Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014937282

LNCS Sublibrary: SL1 – Theoretical Computer Science and General Issues

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

At the time of writing this preface, it is now more than six years ago that the first Quantum Interaction conference took place at Stanford University. It is probably no exaggeration to claim that the group of people who originally set up this conference were barely thinking of the eventuality that this conference would develop into a series. But it did and we are proud to announce that this volume published by Springer in the LNCS series, gathers the output of accepted papers at the seventh Quantum Interaction conference which took place at the University of Leicester.

The themes of the Quantum Interaction conference series continue to revolve around four main subject pillars: (i) information processing/retrieval/semantic representation and logic; (ii) cognition and decision making; (iii) finance/economics and social structures and (iv) biological systems.

The accepted and refereed papers published in this volume are ordered according to the above four themes.

The outcome of the refereeing process this year allocated nearly 60 % of all accepted papers to oral presentations. The remaining accepted papers were put in a poster session which continued throughout the full duration of the conference.

As with other Quantum Interaction conferences, this year we were again very fortunate to be able to count on the contributions of outstanding keynote speakers: Professor Nelson (Department of Mathematics - Princeton University); Professor Abramsky (Department of Computer Science - University of Oxford) and Professor Hiley (Theoretical Physics Research Unit - Birkbeck – University of London).

It is impossible to realize an event like this without the input of many people. Surely, the speakers and keynote addresses are essential. We thank all the speakers for their contributions. The input of both the Steering Committee and the Programme Committee, whose members were so active in refereeing the papers (on time!!) was essential. We also want to thank in particular Lisa Brandt for support covering virtually all the administrative details involved in the organization of this conference. We also want to thank Cheryl Hurkett for her infinite patience in setting up the website and adapting it to our ad-hoc wishes. The Centre for Interdisciplinary Science and the School of Management and the University of Leicester conference services need also to be thanked for their precious support.

December 2013

Harald Atmanspacher
Emmanuel Haven
Kirsty Kitto
Derek Raine

Organization

Steering Committee

Peter Bruza	Queensland University of Technology, Australia
Trevor Cohen	University of Texas, USA
Bob Coecke	University of Oxford, UK
Ariane Lambert-Mogiliansky	Paris School of Economics, France
Dominic Widdows	Microsoft Bing, USA

General Chairs

Emmanuel Haven	University of Leicester, UK
Derek Raine	University of Leicester, UK

Program Committee Chairs

Harald Atmanspacher	Institute for Frontier Areas of Psychology - Freiburg, Germany and Collegium Helveticum - ETH Zürich, Switzerland
Kirsty Kitto	Queensland University of Technology, Australia

Programme Committee

Sachi Arafat	University of Glasgow, Scotland
Peter beim Graben	Humboldt-Universität zu Berlin, Germany
Peter Bruza	Queensland University of Technology, Australia
Jerome Busemeyer	University of Indiana-Bloomington, USA
Bob Coecke	University of Oxford, UK
Trevor Cohen	University of Texas, USA
Thomas Filk	University of Freiburg, Germany
Liane Gabora	University of British Columbia, Canada
Andrei Khrennikov	Linnaeaus University, Sweden
Ariane Lambert-Mogiliansky	Paris School of Economics, France
Bill Lawless	Paine College, USA
Massimo Melucci	University of Padua, Italy
Jian Yun Nie	Université de Montréal, Canada
Dusko Pavlovic	Royal Holloway, University of London, UK
Mehrnoosh Sadrzadeh	University of Oxford, UK
Sandro Sozzo	University of Leicester, UK

Giuseppe Vitiello	Università di Salerno, Italy
Dominic Widdows	Microsoft Bing, USA
Guido Zuccon	The Commonwealth Scientific and Industrial Research Organisation (CSIRO), Australia

Website

Cheryl Hurkett	University of Leicester, UK

Contents

Stochastic Mechanics of Particles and Fields

Edward Nelson(✉)

Department of Mathematics, Princeton University, Princeton, NJ 08544, USA
nelson@math.princeton.edu

The configuration space of a physical system is a differentiable manifold M. The state of a system is given by a point x in M and a tangent vector v at x, the velocity of the configuration. Call the tangent bundle TM of M the state space of the system (though the phrase "velocity phase space" is often used). A dynamical variable is a possibly time-dependent function on state space.

Newtonian Mechanics. The kinetic energy T of a system is given, using tensor notation and the summation convention, by

$$T = \frac{1}{2} m_{ij} v^i v^j \tag{1}$$

where m_{ij} is a Riemannian metric, called the mass tensor. Using the Riemannian connection we can define acceleration. If $x(t)$ is the configuration of the system at time t, then in local coordinates the acceleration a is given by $a^i = \dot{v}^i + \Gamma^i_{jk} v^j v^k$ (where the dot is the time derivative). A force F is a possibly time-dependent covector field. Now we can express Newton's law:

$$F_i = m_{ij} a^j \tag{2}$$

If F is given, we have the equations of motion for the system in state space TM, expressed in local coordinates by

$$\begin{aligned} \dot{x}^i &= v^i \\ \dot{v}^i &= m^{ij} F_j - \Gamma^i_{jk} v^i v^k \end{aligned} \tag{3}$$

defining a flow, at least a local flow, on TM.

Lagrangian Mechanics. A Lagrangian L is a dynamical variable with the dimensions of energy. Given a path X in configuration space M with velocity vector $\dot{X}$, define the action I by

$$I = \int_{t_0}^{t_1} L(X, \dot{X}, t)\, dt \tag{4}$$

For an isolated system, L is time-independent. Hamilton's principle of least action is that I should be stationary under variations of the path with t_0 and t_1 fixed. This leads to the Euler-Lagrange equation

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial v^i} = 0 \tag{5}$$

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 1–5, 2014.
DOI: 10.1007/978-3-642-54943-4_1,

Now let us put Newtonian and Lagrangian mechanics together. Let M be a Riemannian manifold with mass tensor m_{ij} and kinetic energy (1), and define the potential energy V by $L = T - V$.

Given a point x on a Riemannian manifold M, it is possible to choose normal coordinates (NC) at x so that the first derivatives of m_{ij} and the Christoffel symbols are 0 at x. Thus

$$\frac{\partial L}{\partial x^i} = -\frac{\partial V}{\partial x_i} \quad \text{(NC)} \tag{6}$$

By (5) and (6),

$$-\frac{\partial V}{\partial x^i} = \frac{d}{dt}\left(m_{ij}v^j - \frac{\partial V}{\partial v^i}\right) \quad \text{(NC)} \tag{7}$$

By (2),

$$\frac{d}{dt}(m_{ij}v^j) = F_i \quad \text{(NC)} \tag{8}$$

so that

$$F_i = -\frac{\partial V}{\partial x^i} + \frac{d}{dt}\frac{\partial V}{\partial v^i} \quad \text{(NC)} \tag{9}$$

But F_i is a dynamical variable, a function of position and velocity, so $\frac{\partial V}{\partial v^i}$ must be independent of the velocity. That is, the Lagrangian must be of the form

$$L = \frac{1}{2}m_{ij}v^i v^j - \phi + A_i v^i \tag{10}$$

This is a tensor equation, so it holds globally in general coordinates. Call ϕ the scalar potential and A_i the covector potential, and call a Lagrangian of the form (10) a basic Lagrangian.

Let $X(s, x, t)$ be the configuration at time s of the system starting at x at time t. Hamilton's principal function is

$$S(x, t) = -\int_t^{t_1} L\big(X(s, x, t), \dot{X}(s, x, t), s\big)\, ds \tag{11}$$

A second form of the principle of least action is that S be stationary when the flow is perturbed by a time-dependent vector field. This leads to the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla^i S - A^i)(\nabla_i S - A_i) + \phi = 0 \tag{12}$$

with the same equations of motion (3). When the covector potential A_i is 0, so that $\phi = V$, this becomes

$$\frac{\partial S}{\partial t} = \frac{1}{2}\nabla^i S \nabla_i S + V \tag{12'}$$

All of this is well known in classical deterministic mechanics, and an extended exposition is in Chap. II of [1].

Basic Stochasticization. The word "stochasticization" is cacophonous, but it is a more accurate description of the procedure in stochastic mechanics than "stochastic quantization", because the physics remains classical despite the appearance of $\hbar$. (Also, the latter phrase has two distinct meanings.) The basic equations of stochastic mechanics will be derived here under two simplifying assumptions (see [1] for the general case). First, take M to be $\mathbb{R}^n$ and the mass tensor to be a constant diagonal matrix giving the masses of the various particles making up the configuration. Then M is flat and the Christoffel symbols Γ^i_{jk} are 0. But tensor notation is still useful; for example, v^i is the velocity and v_i is the momentum, and the Laplacian $\Delta = \nabla^i \nabla_i$ has the appropriate mass coefficients in it. Second, take the covector potential A_i to be 0.

Let w be the Wiener process on M, the stochastic process of mean 0 characterized by

$$dw^i dw_i = \hbar dt + o(dt) \tag{13}$$

We postulate that the motion of the configuration is a Markov process governed by the stochastic differential equation

$$dX^i = b^i\big(X(t), t\big)dt + dw^i \tag{14}$$

where b^i is the forward velocity. Thus the fluctuations are of order $dt^{\frac{1}{2}}$, and with a value larger than $\hbar$ in (13) this postulate could be falsified by experiment, without violating the Heisenberg uncertainty principle.

Now let us compute the expected kinetic action of this process. Let $dt > 0$ and let $df = f(t + dt) - f(t)$ (the increment rather than the differential, which does not exist if f is not differentiable). From (14),

$$dX^i = \int_t^{t+dt} b^i\big(X(r), r\big)\, dr + dw^i \tag{15}$$

Apply this equation to itself, i.e. to $X(r)$, giving

$$\begin{aligned} dX^i &= \int_t^{t+dt} b^i\Big(X(t) + \int_t^r b(X(s), s)ds + w(r) - w(t), r\Big)\, dr + dw^i \\ &= b^i dt + \nabla_k b^i W^k + dw^i + O(dt^2) \end{aligned} \tag{16}$$

where

$$W^k = \int_t^{t+dt} [w^k(r) - w^k(t)]\, ds \tag{17}$$

From this it follows that

$$\frac{1}{2} dX^i dX_i = \frac{1}{2} b^i b_i dt^2 + b^i dw_i dt + \nabla_i b^i dt^2 + \frac{\hbar dt}{2} + o(dt^2) \tag{18}$$

Let $\mathbb{E}_t$ be the conditional expectation given the configuration at time t. *First miracle*: the term $b_i dw_i dt$ in (18) is singular, of order $dt^{\frac{3}{2}}$, but by the Markov property $\mathbb{E}_t b^i dw_i dt = b^i \mathbb{E}_t dw_i dt = 0$. Hence the expected energy is

$$\mathbb{E}_t \frac{1}{2} \frac{dX^i}{dt} \frac{dX_i}{dt} = \frac{1}{2} b^i b_i + \frac{1}{2} \nabla_i b^i + \frac{\hbar}{2dt} - V\big(X(t)\big) + o(1) \tag{19}$$

Second miracle: the singular term $\frac{\hbar}{2dt}$ in (19) is a constant not depending on the path, so it drops out when taking the variation—form the Riemann sum for the action, take the variation with the singular term dropping out, and then pass from the Riemann sum to the integral. The stochastic principal function is

$$S(x,t) = -\mathbb{E}_{x,t} \int_t^{t_1} \left(\frac{1}{2} b^i b_i + \frac{\hbar}{2} \nabla_i b^i - V \right) \big(X(s), s \big) ds \tag{20}$$

where $\mathbb{E}_{x,t}$ is the expectation conditioned by $X(t) = x$.

The definition of a Markov process is that given the present, the past and the future are conditionally independent. This is a time-symmetric notion. In addition to the forward velocity b^i there is the backward velocity b_*^i. The current velocity v^i and the osmotic velocity u^i are defined by

$$v^i = \frac{b^i + b_*^i}{2} \tag{21}$$

$$u^i = \frac{b^i - b_*^i}{2} \tag{22}$$

The osmotic velocity depends only on the time-dependent probability density ρ. Let

$$R = \frac{\hbar}{2} \log \rho \tag{23}$$

Then

$$u^i = \frac{1}{\hbar} \nabla^i R \tag{24}$$

Computation shows that

$$\frac{\partial S}{\partial t} + \frac{1}{2} \nabla^i S \nabla_i S + V - \frac{1}{2} \nabla^i R \nabla_i R - \frac{\hbar}{2} \Delta R = 0 \tag{25}$$

$$\frac{\partial R}{\partial t} + \nabla_i R \nabla^i S + \frac{\hbar}{2} \Delta S = 0 \tag{26}$$

Here (25) is the stochastic Hamilton-Jacobi equation. There is no deterministic analogue of (26) since $R = 0$ when $\hbar = 0$. These two coupled nonlinear partial differential equations determine the process X. *Third miracle*: with

$$\psi = e^{(R+iS)} \tag{27}$$

these equations are equivalent to the Schrödinger equation

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} \left(-\frac{1}{2} \Delta + V \right) \psi \tag{28}$$

This derivation is that of Guerra and Morato [2], but using the classical Lagrangian. The result extends to the general case, when there is a covector potential A_i and M is not necessarily flat; see [1].

Stochastic Mechanics of Particles. The wave function ψ describes the Markov process completely; $|\psi|^2$ is the time-dependent probability density ρ, $u^i = \nabla^i \Re \log \psi$, and $v^i = \nabla^i \Im \log \psi$. Stochastic mechanics has been developed by many people, especially in Italy and the US. There are discussions of energy, nodes, interference, bound states, statistics (Bose or Fermi), and spin in [1], together with references to the original work.

The original hope that stochastic mechanics would provide a realistic alternative to quantum mechanics has not been realized by the theory in its present form. This is because the Markov process lives on configuration space M, and a point in M may consist of widely separated particles in physical space. This leads to an unphysical nonlocality—instantaneous signaling between widely separated particles—if the trajectories of the process are regarded as physically real; see the discussion in [3].

Stochastic Mechanics of Fields. There are two motivations for applying stochastic mechanics to fields. One is that fields live on physical spacetime and nonlocality problems may be avoided. The other is that it may provide useful technical tools in constructive quantum field theory.

The strategy is to apply basic stochasticization to a basic field Lagrangian. So far as I know, this approach has not been tried before.

Consider a real scalar field ϕ on d-dimensional spacetime. Choose a spacelike hyperplane $\mathbb{R}^s$, where s, the number of space dimensions, is $d-1$. The configuration space is a set of scalar functions ϕ on $\mathbb{R}^s$. Denote a velocity vector by π and define the kinetic energy by

$$\int_{\mathbb{R}^s} [(\nabla\phi)^2 + \pi^2]\, dx_1 \ldots dx_s \tag{29}$$

Then the classical motion with zero potential energy satisfies the wave equation. Now we have the setup to apply stochastic mechanics, with a basic Lagrangian. There are problems both in the classical and quantum theories due to the infinite number of degrees of freedom in field theory. All I have to report at present is this plan for research. The hope is that making the Markov processes, rather than the quantum field, the focus of investigation will prove easier and more fruitful than the usual Hamiltonian approach of constructive quantum field theory (whether in Minkowski spacetime or via the Euclidean method).

References

1. Nelson, E.: Quantum Fluctuations. Princeton University Press, Princeton (1985). http://www.math.princeton.edu/~nelson/books/qf.pdf
2. Guerra, F., Morato, L.M.: Quantization of dynamical systems and stochastic control theory. Phys. Rev. D **27**(3), 1774–1786 (1983). http://prd.aps.org/abstract/PRD/v27/i8/p1774
3. Nelson, E.: Review of stochastic mechanics. J. Phys. Conf. Ser. **361** (2012) (EmerQuM 11: Emergent Quantum Mechanics 2011 (Hans von Foerster Congress) 10–13 November 2011, Vienna, Austria, ed. Gerhard Grö. http://iopscience.iop.org/1742-6596/361/1/012011/)

Quantum Mechanics: Harbinger of a Non-commutative Probability Theory?

B.J. Hiley(✉)

TPRU, Birkbeck, University of London, Malet Street, London WC1E 7HX, UK
b.hiley@bbk.ac.uk

Abstract. In this paper we discuss the relevance of the algebraic approach to quantum phenomena first introduced by von Neumann before he confessed to Birkoff that he no longer believed in Hilbert space. This approach is more general and allows us to see the structure of quantum processes in terms of non-commutative probability theory, a non-Boolean structure of the implicate order which contains Boolean sub-structures which accommodates the explicate classical world. We move away from mechanical 'waves' and 'particles' and take as basic what Bohm called a *structure process*. This enables us to learn new lessons that can have a wider application in the way we think of structures in language and thought itself.

1 Introduction

As Murry Gell-Mann [1] once wrote:-

> Quantum mechanics, that mysterious, confusing discipline, which none of us really understands but which we know how to use. It works perfectly, as far as we can tell, in describing physical reality, but it is a 'counter-intuitive' discipline, as social scientists would say. Quantum mechanics is not a theory, but rather a framework within which we believe a correct theory must fit.

The professional physicist still finds explaining exactly what the quantum formalism is telling us about Nature very difficult. We know that it is something to do with non-commutativity because the commutative algebra of functions used in classical physics is replaced by a non-commuting algebra of operators where the operators become 'observables' while their eigenvalues correspond to the values found in experiments. The formalism works, but what does it all mean? Do the problems arise simply because of the small scale nature of the phenomena, or because the spectacular behaviour of matter only occurs at low temperature, with no general consequences for the way think about the macroscopic world in general? Or is it pointing to something much more general which reaches down into the very being of our lives, providing a different paradigm that affects the way we think in general?

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 6–21, 2014.
DOI: 10.1007/978-3-642-54943-4_2,

Let me begin with a personal difficulty. I have always been puzzled by the contrast between the way quantum theory was originally introduced and the way we worry over it today. Quantum theory was introduced to explain two main phenomena, the stability of matter at room temperatures and the frequency spectrum of radiation coming from very hot objects like the sun. Classical mechanics provides us with no explanation of the stability of an atom, the stability of a molecule, the stability of a biological cell, the stability of a solid crystal, the stability of my desk and so on. We need quantum mechanics to provide the explanation. We also need quantum mechanics to explain the radiation black body radiation. Let me repeat, we needed quantum theory to explain the stability of large scale matter at room temperature and to explain effects emanating from very hot bodies. Contrast this with what we worry about now. We worry about the fragility of the coherence of the quantum state, we worry about schizophrenic cats, about the collapse of the wave function and we worry about the implications of quantum non-locality [2].

These are the details we physicists are concerned with. What we do not dispute is the novelty of the conceptual and mathematical form of the ideas that are involved. Perhaps the most radical notion is that we must give up reductionism with its view that ultimately the world must be analysed into elementary parts and the relations between these parts define what we perceive to be the world around us. To emphasise this failure of reductionism consider our latest attempts to find the ultimate constituents of the nucleon. Instead of finding simplicity, we find is a sea of activity which we analyse in terms of a multitude of partons comprising valence quarks, quark-antiquark pairs, gluons and perhaps even more [3]. There is no 'ultimon' in which we pin the solidity of the macroscopic world.

However we should not be surprised. Bohr talks about the "impossibility of any sharp separation between the behaviour of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear." [4]. In other words there exists a kind of wholeness in which "we are not dealing with an arbitrary renunciation of a more detailed analysis of atomic phenomena, but with a recognition that such an analysis is *in principle* excluded." [5].

Primas [7] goes further

> According to quantum mechanics the world is a whole, *a whole which cannot be made out of parts.* If ones agrees that quantum mechanics is a serious theory of matter then one cannot adopt the classical picture of physical reality with its traditional metaphysical presuppositions. In particular, the nonseparability and nonlocality of the material world and holistic features are not compatible with the ontology usually adopted in classical physics.

So where do we start? Fortunately at the level of atoms and molecules we do have what appear to be autonomous objects, but these objects have both classical and quantum properties. Macroscopic molecules behave like classical objects but they combine through quantum processes, where, then, is the description that has both classical and quantum features?

If we examine the mathematical structures we have, it appears as if classical mechanics is totally different from the mathematical description commonly used in quantum mechanics. The variables used in classical mechanics, state functions of the position and momentum, $f(x, p)$, have a product rule that is commutative

$$f(x, p) \cdot g(x, p) - g(x, p) \cdot f(x, p) = 0 \tag{1}$$

Whereas in quantum mechanics, x and p are replaced by operators that do not commute. Similarly state functions, the density matrices, ρ, used in the algebraic form of quantum mechanics, do not commute either. Here I am using density matrices to describe the state of a system in contrast to the usual wave function because it is a more general approach which reduces to the usual approach only in a special case, namely, when ρ is of rank one and idempotent, $\rho^2 = \rho$. Then we can write

$$\rho = \psi^* \psi$$

where ψ is the wave function that causes us so many interpretational difficulties. What is important to notice here is the states themselves need not commute, namely,

$$\rho_1 \rho_2 - \rho_2 \rho_1 \neq 0$$

This should be contrasted with the behaviour of classical state functions, Eq. (1). We immediately see that a fundamental mathematical difference between classical and quantum mechanics lies in difference between the commutative and non-commutative structures. We can already get a hint of how to combine these two aspects into a single structure if we ask, "Where do we see commutativity and non-commutativity occurring regularly in our everyday experiences?" Not in the relations of things, but in the order of *action*. For example, I cannot walk through a door without opening it first. The order of action is vital. For those of a philosophical turn of mind, it is Heraclitus not Democritus, it is Schelling and Fichte, not Kant that provide the clues. As Fichte [10] writes

> The question is whether philosophy should begin with the fact or an act (i.e., with a pure activity that presupposes no object, but instead, produces its own object, therefore with an acting that immediately becomes a deed).

In this paper I want to use the algebraic formalism, because it is through this formalism that the real novelty of the quantum ideas and the connection with process come through showing us that there is a radically new way of looking at all aspects of life that leads us to abandon the classical paradigm and to replace it by a richer paradigm in which *structure process* is basic [9].

2 Structure Process and Algebraic Order

When I started to work with David Bohm in the '60s he was thinking of how relativity and quantum theory could be brought together in a new way. To avoid the difficulties of a rigid object presents to relativity, Bohm introduced the notion of a structure process in which a set of discrete "space-like" elements undergo discrete or continuous changes as they move and unfold in a *process* of development. He argued that such a notion implies that the structural process as a whole, with its set of manifold relationships of partial order of discrete elements, is logically and existentially prior to the notion of a continuous space-time, in the sense that the latter is an abstraction from the former, representing a kind of approximate 'map' of the overall structure process. Thus the particles and fields, and indeed space-time itself were to be abstracted from this deeper process. His discussions were conceptual and philosophical. The question he left unanswered was how these notions could be developed into a coherent mathematical structure.

While I was thinking about these problems, I happened to come across two significant discussions. The first was an essay by Hankins [11] who was reviewing Hamilton's work in which he introduced a notion, "the algebra of pure time". Hamilton thought that "in algebra the relations which we first consider and compare, are relations between successive changing thing or thought". He then goes on "Relations between successive thoughts thus viewed as successes states of one more general and changing thought, are the primary relations of algebra". Note he uses 'thought' not material process. This raises the interesting notion that algebra is not only about material process as the physicist will believe; it has a much more general function, describing the order, not only of material relations, but also the order of thought. Thought is not subject to the order of space-time. There is no notion of locality. Could this algebraic notion of order take us beyond the order of space-time revealing new relations of the type we see in entangled states?

Then there was Grasmann's *Ausdehnungslehre* [12] that had a profound influence on Clifford's development of his algebra. It is this algebra that I have found extremely fascinating and which forms a basis of my recent work on what I call *Bohm's non-commutative dynamics* [13][1]. Grassmann introduced the notion of an *extensive* to carry the notion of a *continuous becoming*. We all experience one thought transforming into another, new thought. Is the new thought separate from the old thought? No. The old thought contains the potentiality of the new thought, while the new thought contains a trace of the old thought. Symbolically this is written as $[T_1, T_2]$ then succession can be captured through a groupoid multiplication rule

$$[T_1, T_i] \circ [T_j, T_3] = [T_1, T_3]; \qquad \text{only when } i = j. \tag{2}$$

As I have shown elsewhere [15], encapsulated in this idea is the notion of unfolding that is central to the notion of enfolding and enfolding that leads directly

[1] Unfortunately I have to make it clear that the spirit of the view Bohm and I were developing together had little in common with the proponents of the subject now called "Bohmian mechanics".

to the Heisenberg equation of motion. This is one of the equations that form the basis of the time development of quantum processes that we use implicitly throughout this paper.

Let us first start by explain how these ideas lead us to Clifford algebras [15]. Clifford [14], exploiting the ideas of Grassmann and Hamilton, introduced a multiplication rule, which he called *polar* multiplication, and which we now call *Clifford* multiplication. This follows from Eq. (2) together with

$$[T_1, T_2] = -[T_2, T_1]$$

Now one can easily show that the following rule is satisfied

$$[T_1, T_2] \circ [T_2, T_3] + [T_2, T_3] \circ [T_1, T_2] = 0$$

showing that the product in anti-commutative. Now let us consider the special case in which the thought turns into itself, that is $[T_1, T_1]$, the *idempotent* which formally satisfies

$$[T_1, T_1] \circ [T_1, T_1] = [T_1, T_1]$$

This shows that the thought is not static, but keeps on turning into itself. To show exactly how this structure produces the formal orthogonal Clifford algebra requires a little extra work which we will not need in this paper so we will simply refer to the original work of Clifford [14] or to Hiley [15] for a more extensive discussion of the ideas introduced here.

However one thing that I will mention here to complete the background is to explain where Hamilton fits in. Hamilton was interested in generalising the complex numbers, which would involve seeing how three mutually perpendicular two dimensional Argand planes can be fitted together into a three dimensional space. Recall that the complex number i can be regarded as a rotation through 90° in, say, the $x - y$ plane. How then do we combine this rotation with a 90°, rotation, say j, in the $x - z$ plain, and a 90°, say k, in the $y - z$ plain? Clearly $i^2 = j^2 = k^2 = -1$. Notice we have introduced three separate, but related 'square roots of -1'. What Clifford showed was that if you take $[T_0, T_1]$ to be a movement along the x-axis and $[T_0, T_2]$ to be a movement along the y-axis, then $[T_1, T_2]$ is a movement (i.e., rotation) taking T_1 into T_2. Clearly if you apply $[T_1, T_2]$ again you get $[T_1, T_2]^2$ which Clifford took to be -1, so that Hamilton's quaternions became a special case of a Clifford algebra.

3 Where Does Quantum Theory Fit in?

All of this mathematical structure was developed when classical mechanics was the only mechanics known. Imagine the surprise when nature threw up spin and the Pauli σ algebra and then Dirac showed that a relativistic generalisation required the relativistic electron depended on a set of anti-commuting γ-matrices. Both of these structures are examples arising in the tower of orthogonal Clifford algebras.

In an orthogonal Clifford algebra, $\mathcal{C}$, the rotations emerge as inner automorphisms defined by

$$A' = RAR^{-1}, \qquad \forall A \in \mathcal{C}.$$

where R is a set of invertible elements in $\mathcal{C}$. The multiplicative group, G of invertible elements R is called the Clifford group, which in the physics community is known as the spin group. The Clifford group gives us direct access to the double cover of the usual rotation group and the spinor comes "for free" as an element of a suitably chosen minimal left ideal. It was through a detailed study of orthogonal Clifford algebras that Hiley and Callaghan [16] extended the Bohm approach to relativistic particles with spin.

Now I want to draw your attention to another algebra which appeared in a classic paper by von Neumann [17]. Again here we are not specifically concerned with quantum processes, but we arrive at an algebra that plays a central role in quantum mechanics. Let us begin by considering the translations in an (x, p) symplectic (phase) space. We can write these translations as

$$\widehat{U}(\alpha) = \exp(i\alpha\widehat{P}); \quad \widehat{V}(\beta) = \exp(i\beta\widehat{X})$$

If the generator $\widehat{P}$ and $\widehat{X}$ are defined by the relations

$$\widehat{U}(\alpha)f(\widehat{X})\widehat{U}(\alpha)^{-1} = f(\widehat{X} + \alpha); \quad \widehat{V}(\beta)g(\widehat{P})V(\beta)^{-1} = g(\widehat{P} + \beta)$$

then $(\widehat{X}, \widehat{P})$ must satisfy the relation $[\widehat{X}, \widehat{P}] = i$ where $[\cdot, \cdot]$ is the usual commutator[2]. We follow von Neumann and write

$$\widehat{S}(\alpha, \beta) = \exp i(\alpha\widehat{P} + \beta\widehat{X})$$

Here $\widehat{S}(\alpha, \beta)$ is the generator of the Heisenberg group acting in the symplectic space.

It is often believed that the Heisenberg algebra is a sign that we have entered the quantum domain, but this is not true. Most of the exploration of the properties of this group are by people working in radar, which was, of course, designed to locate the position and speed of aircraft, hardly a quantum phenomena!

If we interpret $(\widehat{X}, \widehat{P})$ as the Hermitian operators used in the Hilbert space approach to quantum mechanics, then we see that the parameters (α, β) define a dual structure. This dual structure contains all the information contained in the Hilbert space formalism, but in a novel way. von Neumann shows there is a $1 - 1$ correspondence between the Hilbert space formalism and the functions $a(\alpha, \beta)$ through the relation

$$\widehat{A} = \int\int a(\alpha, \beta)\widehat{S}(\alpha, \beta)d\alpha d\beta. \tag{3}$$

[2] Of course position and momentum have different dimensions so we choose $x \leftrightarrow \widehat{X}$ and $p \leftrightarrow \epsilon\widehat{P}$. Note that we are not appealing to anything quantum mechanical at this stage. It is only in quantum mechanics that we write $\epsilon = 1/\hbar$.

To obtain expectation values that agree with those formed in standard quantum mechanics, we introduce an element ρ_s, the density matrix and form

$$f_\rho(\alpha,\beta) = Tr[\rho_s \widehat{S}(\alpha,\beta)]$$

so that

$$\langle \widehat{A} \rangle = \int\int a(\alpha,\beta) f_\rho(\alpha,\beta) d\alpha d\beta.$$

The form of this equation suggests that in the space defined by (α,β), expectation values can be found in the same way as they are found in standard commutative statistics, however in this case the variables (α,β) are non-commutative, as we will soon see, so we have a generalisation of ordinary statistics. This generalisation was suggested first by Moyal [19] who further brought out the physical meaning of the approach by identifying α with x and β with p, so that we have a non-commutative phase space. In more general terms, a non-commutative symplectic space. It is important to note that there exists a commutative sub-space which contains classical mechanics.

I have called the space spanned by (α,β) non-commutative, but in what sense? We have seen in Eq. (3) there is a relation between $\widehat{A} \leftrightarrow a(\alpha,\beta)$ so if the operators do not commute, this must be reflected in the product $a(\alpha,\beta) \star b(\alpha,\beta)$. Indeed von Neumann showed that it was necessary to introduce a new product

$$a(\alpha,\beta) \star b(\alpha,\beta) = \int\int e^{2i(\gamma\beta - \delta\alpha)} a(\gamma-\alpha, \delta-\beta) b(\alpha,\beta) d\alpha d\beta$$

Although this definition was introduced by von Neumann, it is now called a Moyal product because Moyal derived it in a more suitable form, namely,

$$a(x,p) \star b(x,p) = a(x,p) \exp[i\hbar(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_x \overrightarrow{\partial}_p)/2] b(x,p)$$

It is easy to show that this product gives

$$x \star p - p \star x = i\hbar$$

Since we have a non-commuting product, we can form two brackets; the Moyal bracket

$$\{a,b\}_{MB} = \frac{a \star b - b \star a}{i\hbar}$$

This corresponds to the commutator bracket in standard quantum mechanics, and the Baker bracket

$$\{a,b\}_{BB} = \frac{a \star b + b \star a}{2}$$

which corresponds to an anti-commutator.

The interesting result is that in the limit $O(\hbar^2)$, we find the Moyal bracket becomes the Poisson bracket of classical mechanics

$$\{a, b\}_{MB} = \{a, b\}_{PB} = O(\hbar^2) = [\partial_x a \partial_p b - \partial_p a \partial_x b]$$

While the Backer bracket becomes a commutative product

$$\{a, b\}_{BB} = ab + O(\hbar^2)$$

Thus we find classical mechanics appearing as a sub-algebra in the non-commutative symplectic algebra.

Just to confuse matters, the algebra that we have outlined above goes under several different names. Sometimes it is called the Weyl algebra [18], sometimes it is called the Moyal algebra [20], but, as we have shown, has it roots in the algebra introduced by von Neumann [17] in 1931. I prefer to follow Crumeyrolle [21] and call it the symplectic Clifford algebra because of its close relation with the orthogonal Clifford algebra. It is not appropriate to go into the details of the symplectic Clifford algebra in this paper. We will simply point out a couple of features that I hope will stimulate some interest in the structure.

Firstly the orthogonal Clifford algebra can be generated by the fermionic (Grassmann) creation and annihilation operators while the symplectic Clifford can be generated by the bosonic creation and annihilations operators. Secondly the orthogonal Clifford group is the spin group so that spin occurs naturally in the algebra. The symplectic Clifford group generates the double covering group of the symplectic group, the metaplectic group. It 'lives' in the covering space and accounts for phase properties like the Gouy effect [22, 23] and the Aharonov-Bohm effect [24].

The final point I would like to make is that both these algebras are geometric algebras in the sense that they describe deeper properties of the geometry of space-time that do not, *a priori*, depend on quantum theory. Rather quantum phenomena exploit these deeper properties of space-time. As these properties are global aspects of the underlying classical space-time, it should not be surprising to find non-local effects because there is no way of describing these global effects using only local Lie algebraic structures.

4 Non-commutative Probability

I am arguing here that these global structures are not merely properties of the material world. They have ramifications for all forms of activity, including the organising orders in thought. Recall that it was Hamilton and Grassmann thinking about the order of thought that led them to take algebraic structures seriously in the first place. The application of algebras has been very successful in the material word and it is only recently that people have been trying to apply the ideas in other areas releasing the formalism from the shackles of the quantum theory.

What has held people back from exploiting quantum algebras has been the thinking that the only way to deal with quantum-like phenomena is through the

Hilbert space formalism with its total dependence on the wave function. The algebraic approach to quantum phenomena, while capturing all aspects of the theory that have already been explored, is a more general formalism and requires a different mind-set from that used in the Hilbert space formalism. No one has worked harder to promote the algebraic approach than theoretical chemist, Hans Primas [7,8]. The general concept that he emphasises is that no matter how we mathematically analyse quantum phenomena, its essential feature is based on a non-commutative structures which he regards as needing a non-Boolean logic. The classical world is Boolean and we have yet to understand the radically different attitudes needed to comprehend a non-Boolean way of thinking.

The two algebras that we have introduced in this paper are both non-Boolean and are specific examples of what are called von Neumann algebras. Their apparent very different structure is because the orthogonal Clifford is a type I von Neumann algebra, while the symplectic Clifford is of type II. A discussion of a type I algebra, based on the symplectic structure has been carried out in Hiley and Monk [25] and in Bohm, Davies and Hiley [26]. I mention these technicalities because the difference in appearance of the two Clifford algebras might cause some uncertainty in the general discussion. In practice the precise differences are not important for the purpose of this paper. Both have idempotents and it is through the idempotents that we can see how the non-Boolean structure arises.

In our approach, the idempotent is the analogue of the projection operator in the standard approach and it is the projection operators that lead to a propositional logic, exploited by Birkhoff and von Neumann [28]. They were the first to suggest that quantum theory should be regarded as a new 'logic', quantum logic, which has now been developed into a formal structure [29]. However this has not been very popular amongst physicists and chemists because it has not led to any new ways of thinking about quantum phenomena. Nevertheless it is necessary to understand how this structure arises before we can establish a different point of view. First it is necessary to know that the structure of any von Neumann algebra is determined by its idempotents, or, if you like, its projection operators. Both idempotents and projection operators have two eigenvalues, $(0, 1)$, so we can regard them as propositions giving us a truth value. Because of the non-commutativity of idempotents, quantum theory gives rise to a non-Boolean propositional calculus [27].

The problem with this interpretation is that it appears to become epistemological, that is it has to do with questions that we ask of the physical system. As a physicist, I am interested in the ontology underlying quantum phenomena. I will follow Eddington [30] and introduce a *structural concept of existence* rather than relying on some metaphysical concept of a particle. Existence manifests itself in two ways–it either exists or it doesn't. Thus let us associate existence with an idempotent. This seems an eminently good notion that has very general applicability. For example if I consider who I am, where is the real me? My mind is constantly in turmoil, my body, my cells and even my bones are actively changing their constituents. I inhale and exhale, etc. I am in constant change, yet it is still me. I am an idempotent, constantly changing into myself. Yes, there are small changes over time, so that the idempotent can change over time, but time scales become crucial particularly at the sub-atomic scales, where 'particles' exist for very short times. The notion of relative stability becomes primary at this level and it is here that processes exhibit fleeting existences.

If we follow this route, then non-commuting structures throw up very interesting consequences. Idempotents do not necessarily commute which implies that when some processes are manifest, others can be completely undefined, so that we cannot even ask if they exist or not. But there is more. Since there exist inner automorphisms in the algebras, we can relate non-commuting idempotents so that, for example we can write $\epsilon' = A\epsilon A^{-1}$. In order to see what this means, let us consider a matrix representation of the structure[3]. Then

$$\epsilon'_{jj} = \sum_k A_{jk}\epsilon_{kk}A^{-1}_{kj}$$

Thus we see that each transformed ϵ'_{jj} contains contributions from *all* the idempotents in the set $\{\epsilon\}$. We have called this the *exploding transformation*[4]. What this transformation implies is that the idempotent ϵ'_{jj} contains contributions from all the idempotents $\{\epsilon\}$. In terms of existence, this means that when we make a transformation, it is not that the existing entity signified by one idempotent 'vanishes into thin air', as it were, but that it contributes and is active in the new idempotent.

If we only think in terms of classical materialism, then this idea makes little sense, but if we think of process, then it implies that everything is an undivided whole but within that totality there exist invariants, the invariants that give rise to quasi-local, semi-stable structures to which we give the name 'particle'. These semi-stable structures can come together and form even more stable structures through their mutual interaction. It is out of these stable structures that the classical world emerges. After all, as has already been pointed out, quantum mechanics was introduced to explain the stability of the macroscopic world.

Thus the individual 'particle' exists only in the background of the total process. As Primas puts it "The environment must never be left out of consideration" [6]. However it is actually stronger than that. Without the background there would be no invariant, there would be no particle. This is totally different from the classical view where we assume the particle exists *a priori* as an autonomous preexistent object.

5 Example of Non-commutative Probability in Quantum Mechanics

I want to continue by exploring the appearance of non-commutative probability in the von Neumann/Moyal algebraic approach. Let us follow Moyal and

[3] All type I von Neumann algebras have matrix representations.

[4] This is the structure used in the Huygens construction and hence the Feynman path integral method [31].

Feynman [32] and regarding $f_\rho(\alpha, \beta)$ as a probability measure even though it can take negative values[5]. Here we will identify $\alpha = x, \beta = p$ and take ρ to be the density matrix for a system in a pure state $\psi(x,t)$ so that

$$f_\psi(x,p) = \frac{1}{2\pi}\int \psi^*(x-\tau/2)e^{-ip\tau}\psi(x+\tau/2)d\tau.$$

This will be recognised as the Wigner function which is easily obtained from the two-point density matrix [35]. In this case the parameters (x,p) are the mean coordinates of what de Gosson calls a *quantum blob* [33].

Since the probability measure depends upon two variables, we can ask for the conditional expectation of, say, the momentum at a point x. Moyal [19] shows this is

$$\rho(x)\bar{p} = \int p f_\psi(x,p) = \left(\frac{1}{2i}\right)[(\partial_{x_1} - \partial_{x_2})\psi(x_1)\psi(x_2)]_{x_1=x_2=x}$$

If we write $\psi = Re^{iS}$ we find

$$\bar{p}(x,t) = \nabla S(x,t)$$

which is just the so called "guidance condition" used in the Bohm approach to quantum mechanics [38] only in this context it is not guiding anything. Here it is simply a conditional expectation value of the momentum.

Moyal also obtains an equation for the transport of this momentum. Starting from Heisenberg's equation of motion, Moyal finds

$$\partial_t(\rho\bar{p}_k) + \sum_i \partial_{x_i}(\rho p_k \partial_{x_i} H) + \rho\partial_{x_k} H = 0$$

Once again if we write $\psi = Re^{iS}$, we find

$$\frac{\partial}{\partial x_k}\left[\frac{\partial S}{\partial t} + H - \frac{\nabla^2\rho}{8m\rho}\right] = 0$$

Or

$$\frac{\partial S}{\partial t} + H - \frac{\nabla^2\rho}{8m\rho} = \frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 - \frac{1}{2m}\frac{\nabla^2 R}{R} = 0$$

where we have chosen the constant of integration to be zero. This equation is just the quantum Hamilton-Jacobi equation exploited in the Bohm approach. This equation is simply a conservation of energy equation where $\nabla^2 R/2mR$, is the quantum potential which is regarded as a new quality of energy. One can show that the quantum potential is related to the T_{jj} component of the energy-momentum tensor of the Schrödinger field [34].

By integrating $\int \nabla S dt$, a set of stream lines can be calculated as first shown in Philippidis, Dewdney and Hiley [37] for the classic two-slit interference pattern.

[5] See Bartlet [36] for a discussion of this point.

Other typical quantum situations are discussed in Bohm and Hiley [38], Holland [39] and Wyatt [40]. If we assume, with Bohm that the particle actually possesses this momentum, then we have the possibility of understanding the behaviour of individual particles. What we have done here is to represent the particle in a phase space spanned by the co-ordinates $(x, \bar{p})$ and have calculated an ensemble of 'trajectories' along which the particle could travel.

Alternately we could examine the conditional expectation value of x which is given by

$$\rho\bar{x} = \int x f_\phi(x,p)dx = \frac{1}{2i}[(\partial_{p_1} - \partial_{p_2})\phi^*(p_1)\phi(p_2)]_{p_1=p_2=p}$$

where $\phi(p)$ is the Fourier transformation of the wave function $\psi(x)$. Again if we write $\phi(p) = R(p)e^{iS(p)}$, we find

$$\bar{x}(p) = -\frac{\partial S_p}{\partial p}.$$

This relation replaces the so called guidance condition, but note here there is no way that this expression can be regarded as "guiding" anything. Nevertheless we construct a new phase space, this time with co-ordinates $(\bar{x}, p)$. One can also calculate stream lines in this space [41] and one finds that the streamlines are different in the two cases. The question is then how do we reconcile these differences?

5.1 Shadow Manifolds

In order to explain the appearance of these two phase spaces, we must recall the Gelfand-Naimark construction [43]. This construction requires us to think of the evolution of material processes an entirely new way. Rather than starting from an *a priori* given space-time with its preassigned topological and metrical properties upon which the algebraic structure that describes the evolution of the material process, we start from the algebraic structure and then abstract the properties of the underlying manifold. If the dynamical algebraic structure is commutative then the Gelfand-Naimark theorem tells us that there is a unique underlying manifold whose topological and metrical structure are determined by the dynamical algebra, In this case the points of the space are maximal two-sided ideals of the algebra so that the points of the space are part of the algebra itself. Thus the space of points are not separate entities but are part of the whole structure.

While this works for a commutative structure, one finds no unique underlying manifold if the algebra is non-commutative. The best one can do is to abstract out a set of *shadow manifolds*. The two phase spaces that we constructed above are examples of these shadow manifolds. This can be taken to be a rigorous mathematical statement of Bohr's principle of complementarity. It does not need to be considered as "wave-particle" duality, a notion, although popular, makes very little sense when carefully examined. In our view the non-commutative structure is the ontological structure which captures more clearly the notion of wholeness that Bohr felt was an essential feature of quantum phenomena. But if the structure is non-Boolean and you are trying to explain it in terms of a Boolean logic, then the two alternative structures arise merely because we are trying to project the process into an inappropriate descriptive form.

At this stage the idea is being discussed in terms of just two shadow spaces. However it can be shown that there could be many shadow spaces. For example the two spaces arise from the Fourier transform but mathematically we could use the fractional Fourier transformation, then we can obtain a family of shadow manifolds as shown in Brown [42].

If one wants to consider this in philosophical terms then the non-commutative algebra is essentially a description of the implicate order, while the shadow manifolds are merely the explicate orders [44]. The way I have tried to get this view across is to recall the gestalt effect revealed in pattern or drawing in which we can see two alternative figures. A typical well known example is the old lady-young lady image shown in Fig. 1.

Fig. 1. Old lady-young lady.

Here it is the observer that is trying to find some meaning in the drawing. Of course that does not mean that we descend into some form of subjectivism. The drawing is real, we are simply trying to make sense of it.

It is interesting to note that Primas [6], by recognising the holistic nature of quantum phenomena, also argues that phenomena or patterns have no *a priori* meaning. We provide the meaning in the same sense that we provide the meaning to Fig. 1. Naturally the meaning is subjective but the pattern or the phenomena is not. That is real; that is ontological. Primas goes on to argue that pattern recognition is a map from the non-Boolean world into a Boolean description. This is exactly what Bohm [44] was getting at with his implicate-explicate order. The explicate orders are Boolean accounts that emerge from the non-Boolean world of quantum phenomena.

6 Conclusion

We have taken the algebraic description of quantum phenomena to illustrate how non-commutative probability theory applies to the material world. But the general structure of the idea has a much wider application and holds even in the world of thought. We all experience the struggle to explicate our thoughts and feelings! We are forced into explanations that are precise, are Boolean, but our thought is not Boolean. We cannot give *one* view of reality, not because we, as humans, are limited in our in our abilities or that we are "clumsy" in the laboratory disturbing everything we try to explore. We are limited because nature is holistic and does not allow a reductionist view of nature except in a somewhat limited domain, limited but vital for our immediate survival. However the deeper lessons that we learn about material reality, hold even more so when it comes the mental world.

It is not that the mental world is separate from the physical world. They are both aspects of the same underlying structure process. I don't have the time here to discuss this further but this aspect has been eloquently argued by Bohm [45], an argument that I will not repeat here. I hope that this paper will begin to reddress those who argue that the lessons of quantum theory have nothing to teach us about these much deeper questions, particularly those addressing the relation between mind and matter. By brining out the deeper structure of the ideas, we do not waste the opportunity by being trapped in arguments that claim the brain is too hot and too wet for these ideas to be relevant.

References

1. Gell-Mann, M.: The nature of matter. In: Mulvey, J.H. (ed.) Wolfson College Lectures 1980, p. 169. Clarendon Press, Oxford (1981)
2. Ma, X.-S., Herbst, T., Scheidl, T., Wang, D., Kropatschek, S., Naylor, W., Wittmann, B., Mech, A., Kofler, J., Anisimova, E., Makarov, V., Jennewein, T., Ursin, R., Zeilinger, A.: Quantum teleportation over 143 kilometres using active feed-forward. Nature **489**, 269–273 (2012). doi:10.1038/nature11472
3. Flack, R.: Private communication
4. Bohr, N.: Atomic Physics and Human Knowledge, p. 39. Science Editions, New York (1961)
5. Bohr, N.: Atomic Physics and Human Knowledge, p. 62. Science Editions, New York (1961)
6. Primas, H.: Pattern recognition in molecular quantum mechanics'. Theoret. Chim. Acta (Berl.) **29**, 127–148 (1975)
7. Primas, H.: Realism and quantum mechanics. In: Pragwitz, D., Skyrms, B., Westerståhl, D. (eds.) Logic, Methodology and Philosophy of Science IX, pp. 609–631. Elsevier, Amsterdam (1994)
8. Primas, H., Müller-Herold, U.: Quantum-mechanical system theory: a unifying framework for observations and stochastic processes in quantum mechanics. Adv. Chem. Phys. **38**, 1–107 (2009)
9. Bohm, D.: Space, time, and the Quantum theory understood in terms of discrete structural process. In: Proceedings of the International Conference on Elementary Particles, Kyoto, pp. 252–287 (1965)
10. Fichte, J.G.: Introductions to the Wissenschaftslehr. p. 51, Translated by Breazeale, D. Hackett Press, Indianapolis (1994)

11. Hankins, T.L.: Algebra as pure time: William Rowan Hamilton and the foundations of algebra. In: Machamer, P.K., Turnbull, R.G. (eds.) Motion and Time, Space and Matter, pp. 327–359. Ohio State University Press, Columbus (1976)
12. Grassmann, H.G.: Math, Gesammeth, Phyk. Werke, Leipzig (1894) (English translation by Kannenberg, L.C.: A New Branch of Mathematics: The Ausdehnungslehre of 1844 and Other Works. Open Court, Chicago (1995))
13. Hiley, B.J.: Bohmian Non-commutative Dynamics: History and New Developments. arxiv:1303.6057
14. Clifford, W.K.: Applications of Grassmann's extensive algebra. Am. J. Maths. Pure Appl. **1**, 350–358 (1887)
15. Hiley, B.J.: Process, Distinction, Groupoids and Clifford Algebras: an Alternative View of the Quantum Formalism. In: Coecke, B. (ed.) New Structures for Physics. Lecture Notes in Physics, vol. 813, pp. 705–752. Springer, Heidelberg (2011)
16. Hiley, B.J., Callaghan, R.E.: Clifford algebras and the Dirac-Bohm quantum Hamilton-Jacobi Equation. Found. Phys. **42**, 192–208 (2012)
17. von Neumann, J.: Die Eindeutigkeit der Schrödingerschen Operatoren. Math. Ann. **104**, 570–87 (1931)
18. Folland, G.B.: Harmonic Analysis in Phase Space, vol. 122. Princeton University Press, Princeton (1989)
19. Moyal, J.E.: Quantum mechanics as a statistical theory. Proc. Camb. Phil. Soc. **45**, 99–123 (1949)
20. Zachos, C.K., Fairlie, D.B., Cuthright, T.L.: Overview of phase-space quantization. In: Zachos, C.K., Fairlie, D.B., Cuthright, T.L. (eds.) Quantum Mechanics in Phase Space, pp. 7–34. World Scientific, Singapore (2005)
21. Crumeyrolle, A.: Orthogonal and Symplectic Clifford Algebras : Spinor Structures. Kluwer, Dordrecht (1990)
22. Gouy, L.G.: Sur une propriété nouvelle des ondes lumineuses. C. R. Acad. Sci. (Paris) **110**, 1251–1253 (1890)
23. Siegman, A.E.: Lasers. Oxford University Press, Oxford (1986)
24. Aharonov, Y., Bohm, D.: Significance of electromagnetic potentials in the quantum theory. Phys. Rev. **115**, 485–491 (1959)
25. Hiley, B.J., Monk, N.: Quantum phase space and the discrete Weyl algebra. Mod. Phys. Lett. **8**, 3625–3633 (1993)
26. Bohm, D.J., Davies, P.G., Hiley, B.J.: Algebraic Quantum Mechanics and Pregeometry, Quantum Theory: Reconsiderations of Foundations, 3, Växjö, Sweden 2005. Adenier, G., Krennikov, A. Yu, Nieuwenhuizen, Theo, M., (ed.), pp. 314–24. AIP (2006)
27. Jauch, J.M.: Foundations of Quantum Mechanics. Addison-Wesley, Reading (1968)
28. Birkhoff, G., von Neumann, J.: The logic of quantum mechanics. Ann. Maths. **37**, 823–843 (1936)
29. Bell, J.L., Machover, M.: A Course of Mathematical Logic. North Holland, Amsterdam (1977)
30. Eddington, A.S.: The Philosophy of Physical Science, Ann Arbor Paperback. University of Michigan Press, Michigan (1958)
31. Feynman, R.P., Hibbs, A.R.: Quantum mechanics and path integrals: Emended Edition. Dover Publications, Mineola (2012)
32. Feynman, R.P.: Negative Probability. In: Hiley, B.J., Peat, F.D. (eds.) Quantum Implications: Essays in Honour of David Bohm, pp. 235–248. Routledge and Kegan Paul, London (1987)
33. de Gosson, M.: Quantum blobs. Found. Phys. **43**, 1–18 (2012)

34. Takabayasi, T.: Remarks on the formulation of quantum mechanics with classical pictures and on relations between linear scalar fields and hydrodynamical fields. Prog. Theo. Phys. **9**(3), 187–222 (1953)
35. Bohm, D., Hiley, B.J.: On a quantum algebraic approach to a generalised phase space. Found. Phys. **11**, 179–203 (1981)
36. Bartlett, M.S.: Negative probability. Math. Proc. Cam. Phil. Soc. **41**, 71–73 (1945)
37. Philippidis, C., Dewdney, D., Hiley, B.J.: Quantum interference and the quantum potential. Nuovo Cimento **52B**, 15–28 (1979)
38. Bohm, D., Hiley, B.J.: The Undivided Universe: An Ontological Interpretation of Quantum Theory. Routledge, London (1993)
39. Holland, P.R.: The Quantum Theory of Motion: An Account of the de Broglie-Bohm Causal Interpretation of Quantum Mechanics. Cambridge University Press, Cambridge (1995)
40. Wyatt, R.E.: Quantum Dynamics with Trajectories: Introduction to Quantum Hydrodynamics. Interdisciplinary Applied Mathematics. Springer, New York (2005)
41. Brown, M.R., Hiley, B.J.: Schrödinger revisited: an algebraic approach. arxiv:quant-ph/0005026 (2000)
42. Brown, M.R.: The symplectic and metaplectic groups in quantum mechanics and the Bohm interpretation, Ph.D. thesis, University of London (2004)
43. Varilly, J.C.: An Introduction to Noncommutative Geometry. European Mathematical Society Series of Lectures in Mathematics. European Mathematical Society, Zürich (2006)
44. Bohm, D.: Wholeness and the Implicate Order. Routledge, London (1980)
45. Bohm, D.: A new theory of the relationship between mind and matter. Philos. Psychol. **3**, 271–286 (1990)

Interference in Text Categorisation Experiments

Giorgio Maria Di Nunzio[1(✉)], Peter Bruza[2], and Laurianne Sitbon[2]

[1] Department of Information Engineering, University of Padua, Padua, Italy
dinunzio@dei.unipd.it

[2] Science and Engineering Faculty, Queensland University of Technology, Brisbane, Australia
{p.bruza,laurianne.sitbon}@qut.edu.au

Abstract. This article examines manual textual categorisation by human coders with the hypothesis that the law of total probability may be violated for difficult categories. An empirical evaluation was conducted to compare a one step categorisation task with a two step categorisation task using crowdsourcing. It was found that the law of total probability was violated. Both a quantum and classical probabilistic interpretations for this violation are presented. Further studies are required to resolve whether quantum models are more appropriate for this task.

1 Introduction

Automatic categorisation of documents is gathering interest from researchers in text processing and machine learning and the development of novel approaches to solve this problem has led to a number of annotated collections. Such a paradigm generally assumes that although there can be disagreement on a category label for a given document, each document labelled in a series is processed independently of others, and each potential category for a document is considered in isolation. Therefore the development and the evaluation of automatic categorisation algorithms is deeply dependent on this assumption. However it has been found that some categories may be problematic to such algorithms: that is they are difficult to label properly. In this paper, we conjecture that such a situation may arise from a default of the assumption of independence between documents and categories, and propose to study this from a decision making perspective.

The study of the relationship between perception, categorisation and decision making is an active research area since the beginning of 2000. Townsend et al. [1] introduced a new paradigm to study these type of interactions in order to test Markov models. The framework used is the following: (i) a stimulus is presented to a subject, (ii) the subject is then asked to categorise it, (iii) finally the subject is asked to make a decision. The research question investigated is whether the probability of a decision depends on the characteristics of the initial stimulus, or is the probability of the decision affected by the categorisation step? In other words, can a conscious cognitive process that does not impact the reality of the stimulus interfere with a decision about the stimulus? The hypothesis was that

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 22–33, 2014.
DOI: 10.1007/978-3-642-54943-4_3,

subjects would follow a Markov property, that is, the final decision is based on the categorisation step and not on the original stimulus.

The hypothesis was tested in an experiment, in which subjects were shown pictures of faces and asked to categorise them into one of two possible groups and thereafter asked to decide on a course of action. The two groups were defined by the following visual features: face shape (wide, narrow); lip thickness (thick, thin). The main procedure of the experiment was divided into two parts. In Part I, subjects were asked to:

1. categorise a picture of a face (usually indicated as the input signal S) in either group G (good guys) or group B (bad guys);
2. decide on an action A (act aggressively) or F (act friendly), knowing that one group (B in this case) is more hostile than the other.

In Part II, subjects only had to make the decision response, not both.[1] subjects were given information about the correct proportions of face shapes for each category (about 60 %) and they were given incentives for each correct answer.

The corresponding Markov model underpinning this experiment is given by:

$$P(A|S) = P(A|G)P(G|S) + P(A|B)P(B|S) \quad (1)$$

$$P(F|S) = P(F|G)P(G|S) + P(F|B)P(B|S) \quad (2)$$

It was found that the law of total probability was not satisfied: the probability of taking a decision without categorising the face (left hand side of Eqs. 1 and 2) was significantly different from the decision taken after the categorisation step (right hand side of Eqs. 1 and 2). This is the consequence of the fact that most individuals are moderately influenced by their categorisation when they have to make a decision.

The quantum interaction research community has presented evidence that such empirical violations of the law of total probability can be modelled with quantum-like models using an interference term. The interference term modifies the law of total probability and is derived from incompatibility between subspaces representing events. A quantum-like inference model for this experiment was proposed [2,3] and such models have been widely proposed in quantum cognition [4–8].

The aim of this article is to investigate whether the law of total probability is being violated in text categorisation, and if it is, to investigate whether it can be modelled using a quantum-like interference model.

2 Interference in Text Categorisation

Text categorisation is the task of deciding whether a piece of text belongs to one or more categories, given a set of specified categories [9]. This task has

[1] In the experiment, the reverse order, decide-then-categorise, and categorise only was investigated as well. We do not analyse the reverse order in this paper.

important applications in the real world, for example: news stories are typically organised by subject categories or geographical areas; academic papers are often classified by domains and research areas; email messages are classified into the two categories of spam and non-spam. Since manual text categorisation is a time consuming process, machine learning approaches are used to automate this task. For supervised learning algorithms and for the evaluation of the automatic classifiers, some ground truth is required. In general, the ground truth is a collection of textual documents sampled from the same population of documents that will be automatically classified. This collection is carefully annotated by experts of the topics (or the categories) covered by the documents.

The goal of a supervised machine learning algorithm is to learn a classifier based on the manually annotated documents. More formally, given a finite set of pre-defined categories $C = \{c_1, ..., c_i, ..., c_n\}$ and a finite set D of documents, the learning of a categorizer consists of the definition of a function for each category c_i of this kind: $\Phi_i : D \rightarrow [0, 1]$. Therefore, a categorizer returns a degree of membership of a document d for a category c_i.[2] However, when the decision to assign or not d to c_i has to be taken, a threshold t may be needed such that if $\Phi_i > t$, d is assigned to category c_i, otherwise it is not assigned to c_i. The function Φ_i takes on different meanings according to the learning method used: in the Naïve Bayes approach, it is defined in terms of the conditional probability that a document d belongs to the category c_i, $P(c_i|d)$. This approach of learning a classifier for each class is called binary categorisation. Given a set of categories C, there are $|C|$ independent binary categorisation problems where each document of D must be assigned either to category c_i or its complement $\bar{c}_i = C - c_i$.

In machine learning literature, it has been consistently found that it is harder to learn a classifier for some categories. Our hypothesis for this phenomenon is that when human annotators decide that a document belongs to category c_1, they are necessarily undecided about whether it belongs to category c_2. This indecision is the hallmark of incompatible subspaces, which have been shown not to adhere to the law of total probability [6,8].

The process of categorising documents is very close to the categorisation and decision making problem described in the Sect. 1. More specifically, the act of categorising a document is *both* a categorisation step, i.e., a document is categorised into category c, and a decision step, i.e., placing the document in the 'box' which contains the documents of category c. Therefore, the process of categorisation can be modelled with a Markov chain, as shown in Fig. 1, in the following way:

- given a set of categories C and a document d to categorise;
- step 1, categorise d in either category c_1 or $\bar{c}_1$;
- step i, categorise d in either c_i or $\bar{c}_i$, knowing that the document was previously categorised under either c_{i-1} or $\bar{c}_{i-1}$.

The hypothesis here is similar to the original experiment of [1]: a human expert would follow a Markov property, that is the decision at the i-th step is

[2] The image of Φ_i can be different, for example $\Phi_i : D \rightarrow \mathbb{R}$.

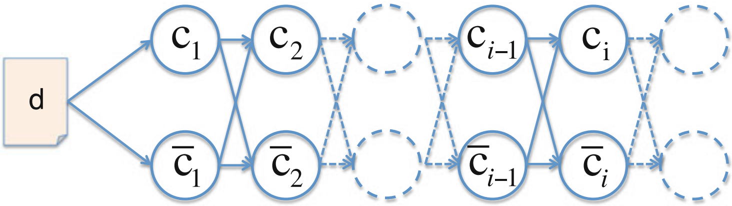


Fig. 1. The process of categorisation described by a Markov chain.

based on the categorisation information used at step $i-1$ and not on the features of the original document.

2.1 Categorisation Under Two Categories

In order to test the hypothesis using the face detection experimental framework, we contain the categorisation task to a task of deciding upon only two categories, $C = \{c_1, c_2\}$. In this new experiment, subjects will be asked to classify documents into either c_1 or not, and into either c_2 or not. The experiment is divided into two parts. In Part I, subjects will be asked to: (i) first, categorise a document in either category c_1 or $\bar{c}_1$; (ii) then, categorise the document under either c_2 or $\bar{c}_2$. In Part II, subjects will be asked to only decide whether the document is relevant for either c_2 or $\bar{c}_2$.

Given a document d, by the law of total probability, the probability of categorising d under c_2 or $\bar{c}_2$ is:

$$P(c_2|d) = P(c_2|c_1)P(c_1|d) + P(c_2|\bar{c}_1)P(\bar{c}_1|d) \quad (3)$$

$$P(\bar{c}_2|d) = P(\bar{c}_2|c_1)P(c_1|d) + P(\bar{c}_2|\bar{c}_1)P(\bar{c}_1|d) \quad (4)$$

In Part I of the experiment, the probabilities of the right hand side of the last two equations are estimated; while in Part II, the probabilities of the left hand side, $P(c_2|d)$ and $P(\bar{c}_2|d)$, are computed.

In order to be able to use the same formalism for analysing the outcomes as the face recognition experiment, the design of the text categorisation experiment must be as similar to it as practical. Therefore we need to carefully address the following issues: (1) the choice of features that describe the category and (2) the proportion of documents for each category.

In [1], and subsequently in [2], the features of the type of the faces of the two groups (good guy or bad guy) were clearly identified as a two dimensional space: roundness of the face (round or narrow), thickness of the lips (thick or thin). The face had to have neither any expression nor any hair style (all the faces were actually bald). Half of the faces were round with thin lips, half were narrow with thick lips. The size of these two features was slightly altered to have a range of different faces. The values of the size of the face and lips thickness was such that each group could be clearly identified. In this experiment, these two features were the only hint that could drive the judgement of the subject. Additionally, subjects are informed that although these are the general describing features of each category of face, they are neither necessary nor sufficient conditions. This is described further in the description of proportions of documents in categories.

For a text categorisation experiment, the situation is more complex. Text documents are high-dimensional objects since they are represented by words. Categories, in turn, can be described by many words or concepts that are related to the documents of the category. For this reason, the categorisation experiment cannot be limited to two dimensions.

The second issue concerns the proportion of documents between categories. The original experiment was designed to be completely symmetric, that is the altered faces were assigned to the two groups in the following way: round faces with thin lips had a 60 % chance of being assigned to the bad guy group; bad guys had a 70 % chance of being hostile. The other category had symmetric chances.

In the case of text categorisation, categories are very often unbalanced; therefore, building an experiment with the same number of positive and negative examples for each category would be quite unrealistic. In general, a document that presents the features of a category (we address the problem of the features in Sects. 2.2 and 2.4) belongs to that category; nevertheless, there are some documents that seem to cover a topic of a category but belong to the opposite category. We decided to manually select a reasonable sample of documents, by reasonable we mean comparable to the original experiment (34 pictures), from a standard benchmark such that: (i) a document may belong to both categories, to either c_1 or c_2, or to none; (ii) a document that belongs to a category (according to the classification given by the standard benchmark) may or may not contain the keywords used to summarise the category; (iii) a document that belongs neither to c_1 nor to c_2 (according to the classification given by the standard benchmark) may contain some of the keywords used to summarise the categories of interest.

2.2 Materials

The Reuters-21578 has been the most important benchmark for text categorisation since the 1990s. It consists of 21,578 news articles that appeared on the Reuters newswire in 1987. The articles were manually categorised by experts. By examining the description of the creation of the original dataset [10], it is possible to build some analogies with the Markov text categorisation process. For example, news stories had been hand-categorised for any of 72 categories available, but it was required to assign at most 6 of the 72 categories for each document. A story could be assigned one or more of these codes, or no code at all if none of the chosen six was appropriate.[3] The important step that clarifies the link between the Markov process and is that experts had to select from a

[3] The restriction to six codes was imposed to keep the effort required to build the dataset within certain budgetary limits.

set of categories for each document. Since, as humans, we need to process each decision in sequence, it is very likely that the experts had in mind a specific order of the 72 categories. Therefore, they were actually performing a sort of 72-step experiment, with the extra constraint of a maximum of 6 categories per news.

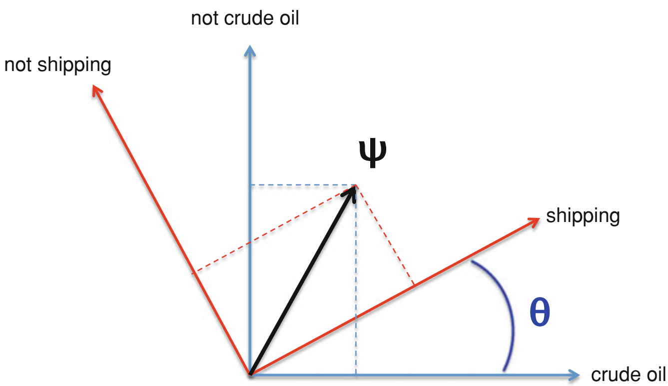


Fig. 2. Incompatibility between the two subspaces "shipping" and "crude oil". ψ denotes the cognitive state of the subject with respect to document d, the projections on the axis are the probabilities that the subject decides for one category or the other. θ measures the interference between the two subspaces.

Table 1. A breakdown of the 82 documents selected for the experiment.

Only ship	Only crude	Crude and ship	Not crude and not ship
20	14	29	19

There is also another important issue that reflects the problem of choosing the right features to describe categories. Quoting [10]: "Closer examination of the results, however, shows that the kind of errors made are quite different. Human errors stem mainly from inconsistent application of categories, especially the categories with the vaguest definitions, and from failing to specify all the categories when several should have been assigned to a story."

The process of manually categorising the Reuters collection presents many of the issues that we are addressing in this paper. For the two categories experiment, we select two Reuters categories: (1) a category about "shipping", and (2) a category about "crude oil". These two categories are good candidates for this pilot experiment because they have distinctive keywords that characterise each topic. However, there are documents that are about shipping crude oil and there are documents that mention ship and crude without being categorised in either category. For this reason, we hypothesise there may be incompatibility between the subspaces corresponding to "shipping" and the other corresponding to "crude oil" as depicted in Fig. 2. See Sect. 3 for further analyses about incompatibility.

After a careful manual selection, a total of 82 documents were selected. Table 1 shows the number of documents for each combination.

Table 2. Number of observation and number of subjects per group.

Number of HITS approved		500	1000	5000	Masters
Observations	One step task	796	790	734	681
	Two step task	802	813	808	829
# of Workers	One step task	30	42	48	45
	Two step task	32	62	38	42

2.3 Subjects

For this experiment, we used the Mechanical Turk[4] crowdsourcing platform for recruiting subjects. Collecting relevance judgements via crowdsourcing using MTurk has a demonstrated track record and has been shown to be of equal quality as that of lab-based experts [11,12]. For the purposes of this experiments, we decided that unit of work to be performed by a worker (a HIT (Human Intelligence Task)) would be based on the study of a single document (and either one or 2 categorisation questions). We followed a similar crowdsourcing methodology advocated by [13].

A variable that was manipulated was the "Number of HITS approved", that is the experience of the workers. In Mechanical Turk, there are seven levels: from 0 (anyone can participate) to "masters" (best workers). We ran the experiments with four levels of expertise to see whether there is any difference in the results between these groups. The levels are (the figure indicates the number of approved HITS): 500, 1000, 5000, and masters. For each experiment, we have a total number of observations equal to 82 documents times 10 observations per document. We report in Table 2 the exact number of observations after data cleansing (HIT not approved, inconsistent answers, workers who categorised the same documents more than once in the same experiment).

HITS were published in batches on Mechanical Turk from December 2012 until January 2013. For each level of expertise, the one step and the two step categorisation tasks were published at the same time. We started from the group of workers with at least 500 HITS approved. When all the HITS were completed for both tasks, we published the HITS for the workers of the next level. For workers at level masters it was a bit different, since we found masters to be more difficult to involve. For this reason, we had to publish the HITs more than once and add one cent to the base pay rate as an incentive (this is also why we have less observations for the one step categorisation).

[4] https://www.mturk.com/

2.4 Procedure

One Step Categorisation. During a one step categorisation HIT, a document was presented with the following layout:

- Description: The goal of this HIT is to decide which categories short news articles belong to.
- Instructions: (i) Read the news article carefully. (ii) The article consists of a title of one line (capital letters) and a body.
- Content: the actual content of one of the 82 REUTERS document is shown in a box. We showed only the tags TITLE and BODY of the original document.
- Decision: Decide whether the news is about "shipping" given to the following clues:
  - Such news often mention the size of the cargo being shipped, e.g., "tonnes" or related words such as "port";
  - Sometimes these news mention what is being shipped, e.g., "oil" or "crude";
  - Bear in mind, not all documents that contain such words necessarily belong to the category "shipping".

At this point, the worker is asked whether the document is about "shipping" (Y/N).

Two Step Categorisation. During a two step categorisation HIT, a document is presented with the same initial layout as the one step categorisation. The difference is in the decision part:

- First Decision: Decide whether a news article belongs to the category "crude oil" or not, e.g., such news often mentions related words like "barrel" and/or "oil", but not all the documents that contain these words belong to the category "crude oil".
- Question: Is this document about crude oil? (Y/N)
- Second decision: Decide whether the news is about "shipping" or not, e.g., such news often mentions related words such as "tonnes" and/or "port" frequently but not all the documents that contain these words are about shipping. Bear in mind that documents about "crude oil" have the tendency to be about "shipping" too, but this is not always the case.
- Question: Is this document about shipping? (Y/N)?

2.5 Results

A breakdown of the results, in the same manner as [2], is given in Table 3. The first part of the table, indicated with 'Two step', presents the results of the estimate of the probabilities for each transition of the Markov model in the case of the two step classification task. The second part, 'One step', is the estimate of the probabilities of the one step classification task. The last column, 'Reuters', is the actual probability of category c_2 if we consider the Reuters tags. For each

Table 3. Experimental results Part I. Category c_1 corresponds to "crude oil", category c_2 to "shipping". We indicated in bold the values that are closer to the original Reuters classification.

Level	Two step						One step	Reuters
	Type	$P(c_1\|d)$	$P(c_2\|c_1)$	$P(\bar{c}_1\|d)$	$P(c_2\|\bar{c}_1)$	$P(c_2)$	$P(c_2)$	$P(c_2)$
500	c_1	0.7442	0.3250	0.2558	0.3364	**0.3279**	0.4466	0.3256
	$\bar{c}_1$	0.2231	0.4217	0.7769	0.4671	0.4570	**0.5173**	0.5323
1000	c_1	0.7459	0.3219	0.2541	0.3303	**0.3240**	0.4552	0.3240
	$\bar{c}_1$	0.2214	0.2588	0.7786	0.4950	0.4427	**0.5055**	0.5208
5000	c_1	0.7687	0.3647	0.2313	0.3434	**0.3598**	0.5431	0.3271
	$\bar{c}_1$	0.2263	0.4651	0.7737	0.5170	**0.5053**	0.6147	0.5263
Masters	c_1	0.7964	0.3551	0.2036	0.3667	**0.3575**	0.2514	0.3258
	$\bar{c}_1$	0.2558	0.4747	0.7442	0.5382	**0.5220**	0.3823	0.5323

level of expertise, we split the results according to what category the document given to the user belongs to, that is documents that belong to either category c_1 or not (this corresponds to showing pictures of good guys or bad guys).

Let us begin the analysis by observing the results of the first part of the two step categorisation task. In theory, the probability $P(c_1|d)$ of the first step should be very close to 1 (or close to 0 for documents that are not in c_1) because we know that we are giving the workers only the documents that belong to c_1 (or $\bar{c}_1$). Nevertheless, this probability never reaches 0.80 in either case and it is almost symmetric. This behaviour may indicate that there are some documents that are difficult to classify and the worker randomly assigns the documents to either c_1 or not.

The second part of the two step categorisation task allows us: (i) to estimate the marginal probability of c_2, and (ii) to compare it with the one computed in the one step categorisation task, and (iii) to compare it with the original Reuters manual categorisation. There is an interesting pattern that emerges from this part of the analysis which we highlighted in bold in Table 3. Unexperienced workers (with less than 5,000 HITS) tend to categorise the correct proportion of c_1 documents when they perform the two step categorisation task, while they over-estimate it in the case of the one step task. Conversely, they underestimate the proportions of $\bar{c}_i$ documents in the two step task, while they almost correlate perfectly with the Reuters' manual coders in the one step task. Experienced workers (more than 5,000 HITS) are very close to the correct proportions for both c_i and $\bar{c}_i$ when they perform a two-step categorisation task, while there is an over/under-estimation for the one step case.

These results suggest that a two step categorisation task may model the original Reuters labelling process faithfully. In fact, 6 out of 8 times the proportion of documents of category c_2 of the two-step experiment was very close to the Reuters classification, and it was almost perfectly matched in the case of more experienced users.

3 Discussion

The experiment shows that during manual text categorisation humans are affected by the order of the categories. Therefore, the assumption that each potential category for a document is considered in isolation is violated. Since we are interested in finding a mathematical framework for text categorisation that captures this effect, we begin this discussion by assuming that incompatibility exists in the document classification and the law of total probability is therefore being violated. By incompatibility, we mean that the joint probability between the categories in the two-step decision task cannot be formed.

The Naïve Bayes classifier is used as the focal point of discussion, which has the following form:

$$\Pr(c_1|d) \propto Pr(c_1)Pr(d|c_1) \tag{5}$$

The law of total probability with respect to category c_2 can be brought in as follows:

$$\Pr(c_1|d) \propto [\Pr(c_1, c_2) + \Pr(c_1, \bar{c_2})]Pr(d|c_1) \tag{6}$$

which we argue is violated, i.e.,

$$\Pr(c_1|d) \propto [\Pr(c_1, c_2) + \Pr(c_1, \bar{c_2}) + Intf]Pr(d|c_1) \tag{7}$$

where $Intf$ is an interference term. We can express this as follows in terms of the geometry of incompatible subspaces:

$$\begin{aligned}
\Pr(c_1) &= \|\mathbf{P_1}\psi\|^2 & (8)\\
&= \|(\mathbf{P_1} \cdot \mathbf{I})\psi\|^2 & (9)\\
&= \|(\mathbf{P_1} \cdot (\mathbf{P_2} + \mathbf{P_2^{\perp}})\psi\|^2 & (10)\\
&= \|\mathbf{P_1}\mathbf{P_2}\psi\|^2 + \|\mathbf{P_1}\mathbf{P_2^{\perp}}\psi\|^2 + Intf & (11)
\end{aligned}$$

where ψ denotes the cognitive state of the subject with respect to document d, $\|\mathbf{P_1}\psi\|^2$ is the probability that the subject decides for category c_1. Similarly, $\mathbf{P_2}$ and its dual $\mathbf{P_2}^{\perp}$ model that the subject decides for category c_2 (or not). In this way, the interference term can be integrated with the standard Naïve Bayes classification model.

The advantage of the quantum-like model is that it offers a clean mathematical framework which both motivates and defines the additional parameter $Intf$. From a machine learning point of view, the variable $Intf$ is a parameter of the model that can be estimated from past observations. If the misjudgements are consistent, this parameter may be used as a correction factor of the bias introduced by humans, hence we may have a new automatic classifier that predicts/corrects future decisions. In our case, we can fit the data almost perfectly in two of the experiments which are: (i) workers with at least 5,000 HITS and documents that do not belong to c_1, (ii) master workers and documents that

Table 4. Experimental results Part II. Fitting the data to the quantum model.

Level	Two step					
	Type	$P(c_1\|d)$	$P(c_2\|c_1)$	$P(\bar{c}_1\|d)$	$P(c_2\|\bar{c}_1)$	θ
5000	$\bar{c}_1$	0.23	0.48	0.77	0.52	1.3
Masters	$\bar{c}_1$	0.25	0.47	0.75	0.53	1.9

do not belong to c_1. Compare the fitted values of Table 4 to the estimates in Table 3.

A drawback of this model is that we introduce one constraint: the matrix of transition probabilities from one state to another (that is from one decision to another) must be double stochastic. However, in many experimental data, this condition is not met [14]. Even in the original experiment of faces [2], not all the results could be explained by a quantum model. In addition, some may question whether the law of total probability is actually being violated. In fact, a classical probabilistic model describes in the best possible way the evidence we have of an experiment. Consider the experiment with two questions. We have a sample space that is $\Omega_2 = \{(c_1, c_2,), (\bar{c}_1, c_2), (c_1, \bar{c}_2,), (\bar{c}_1, \bar{c}_2)\}$. In this space, the only way to compute the marginal probability of c_2 is, by following the classical axioms, $Pr_{\Omega_2}(c_2) = Pr_{\Omega_2}(c_1, c_2) + Pr_{\Omega_2}(\bar{c}_1, c_2)$.[5] Therefore, when we observe this experiment, we cannot claim a violation of the law of total probability.

The misunderstanding about this violation arises when we observe a second experiment, that is the one question experiment with a sample space is $\Omega_1 = \{c_2, \bar{c}_2\}$. At this point, the fact that the two probabilities are different, $Pr_{\Omega_1}(c_2) \neq Pr_{\Omega_2}(c_2)$, is nothing more than an 'unexpected' fact. It is like tossing one coin and computing the probability of heads and tails, then we toss the same coin together with another coin. We 'expect' to obtain the same probability of heads and tails because there is no reason to think otherwise. If we do not, it is not because the LTP is violated but because there is something we could not predict.[6]

4 Summary and Conclusions

The results of the pilot experiment presented in this paper accords with other similar studies in quantum cognition namely (1) Markov models are not sufficient to model multiple categorisation decisions, and (2) in some instances quantum-like interference models may be a superior model. The immediate consequences of such findings are that there is reason to believe that evaluation collection in text categorisation are biased and therefore provide a substandard account of human manual categorisation. What is more worrying there is that many

[5] The underscript Ω_2 reminds us in which space the probabilities are computed.

[6] For example, think about two loaded dice that contain a small magnet inside. Tossed separately, they work as fair dice; tossed together, the magnetic field influences the outcome.

categorisation algorithms base their processing (training for machine learning algorithms) on these biased versions of the truth. We propose that larger scale experiments should shed more light on the specifics of this bias and provide some insights into how interference terms should be uncovered.

References

1. Townsend, J., Silva, K., Spencer-Smith, J., Wenger, M.: Exploring the relations between categorization and decision making with regard to realistic face stimuli. Pragmatics Cogn. **8**, 83–105 (2000)
2. Busemeyer, J.R., Wang, Z., Lambert-Mogiliansky, A.: Empirical comparison of markov and quantum models of decision making. J. Math. Psychol. **53**, 423–433 (2009)
3. Pothos, E.M., Busemeyer, J.R.: A quantum probability explanation for violations of 'rational' decision theory. Proc. R. Soc. **276**, 2171–2178 (2009)
4. Aerts, D.: Quantum structure in cognition. J. Math. Psychol. **53**, 314–348 (2009)
5. Khrennikov, A.: Ubiquitous Quantum Structure. Springer, Berlin (2010)
6. Busemeyer, J., Pothos, E., Franco, R., Trueblood, J.: A quantum theoretical explanation for probability judgment errors. Psychol. Rev. **118**, 193–218 (2011)
7. Trueblood, J., Busemeyer, J.: A quantum probability explanation for order effects on inference. Cogn. Sci. **35**, 1518–1552 (2011)
8. Busemeyer, J., Bruza, P.: Quantum Cognition and Decision. Cambridge University Press, Cambridge (2012)
9. Sebastiani, F.: Machine learning in automated text categorization. ACM Comput. Surv. **34**, 1–47 (2002)
10. Hayes, P.J., Knecht, L.E., Cellio, M.J.: A news story categorization system. In: Proceedings of Association for Computational Linguistics, ALNC'88, Stroudsburg, PA, USA, pp. 9–17 (1988)
11. Alonso, O., Mizzaro, S.: Can we get rid of trec assessors? using mechanical turk for relevance assessment. In: SIGIR '09: Workshop on the Future of IR, Evaluation (2009)
12. Alonso, O., Baeza-Yates, R.: Design and implementation of relevance assessments using crowdsourcing. In: Clough, P., Foley, C., Gurrin, C., Jones, G.J.F., Kraaij, W., Lee, H., Mudoch, V. (eds.) ECIR 2011. LNCS, vol. 6611, pp. 153–164. Springer, Heidelberg (2011)
13. Grady, C., Lease, M.: Crowdsourcing document relevance assessment with mechanical turk. In: Proceedings of the NAACL HLT 2010 Workshop on Creating Speech and Language Data with Amazon's Mechanical Turk (CSLDAMT '10). Association for Computational Linguistics Stroudsburg, PA, USA, pp. 172–179 (2010)
14. Haven, E., Khrennikov, A.: Quantum Social Science. Cambridge University Press, Cambridge (2013)

Orthogonality and Orthography: Introducing Measured Distance into Semantic Space

Trevor Cohen[1(✉)], Dominic Widdows[2], Manuel Wahle[1], and Roger Schvaneveldt[3]

[1] University of Texas School of Biomedical Informatics at Houston, Houston, USA
trevor.cohen@uth.tmc.edu
[2] Microsoft Bing, Bellevue, WA, USA
[3] Arizona State University, Phoenix, USA

Abstract. This paper explores a new technique for encoding structured information into a semantic model, for the construction of vector representations of words and sentences. As an illustrative application, we use this technique to compose robust representations of words based on sequences of letters, that are tolerant to changes such as transposition, insertion and deletion of characters. Since these vectors are generated from the written form or orthography of a word, we call them 'orthographic vectors'. The representation of discrete letters in a continuous vector space is an interesting example of a Generalized Quantum model, and the process of generating semantic vectors for letters in a word is mathematically similar to the derivation of orbital angular momentum in quantum mechanics. The importance (and sometimes, the violation) of orthogonality is discussed in both mathematical settings. This work is grounded in psychological literature on word representation and recognition, and is also motivated by potential technological applications such as genre-appropriate spelling correction. The mathematical method, examples and experiments, and the implementation and availability of the technique in the Semantic Vectors package are also discussed.

Keywords: Distributional semantics · Orthographic similarity · Vector Symbolic Architectures

1 Introduction

The relationships between words, their representation in text, concepts in the mind, and objects in the real world, has been the source of inquiry over many centuries. Empirical, distributional paradigms have been shown to successfully derive human-like estimates of semantic distance from large text corpora, and recent developments in this area have mediated the enrichment of distributional models with structural information, such as the relative position of terms [1,2], and orthographic information describing the configuration of characters from which words are composed [3,4]. This paper extends these beginnings

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 34–46, 2014.
DOI: 10.1007/978-3-642-54943-4_4,

in three principal ways. First, we propose a new and very simple method for encoding structure into semantic vectors, using a quantization of the space between two extreme 'demarcator vectors'. This vector generation method performs well in experiments and has some key computational advantages. Second, the method is general enough to be applied within a wide range of Vector Symbolic Architectures (VSAs), including Circular Holographic Reduced Representations (CHRRs) that use complex vectors [5]. Thirdly, we investigate some higher-level compositions as well, demonstrating some early results with compositional representations for sentences.

Each of these developments is related to quantum interaction as follows. Demarcator vector generation is similar in a sense to the derivation of orbital angular momentum values in quantum mechanics, in that they use the same mathematics. The application within many VSAs, including those over complex vector space, makes the new methods available within algebras that are particularly related to generalized quantum models. Finally, compositional methods are important in research on generalized quantum structures, and the way many levels of representation are combined seamlessly in the compositional models presented here should be of interest to researchers in the field.

2 Orthogonality in Distributional and Orthographic Models

Geometric methods of distributional semantics derive vector representations of terms from electronic text, such that terms that occur in similar contexts will have similar vectors [6,7]. While such models have been shown to approximate human performance on a number of cognitive tasks [8,9], they generally do not take into account structural elements of language, and consequently have been referred to, at times critically, as "bags of words" models. Emerging approaches to semantic space models have leveraged reversible vector transformations to encode additional layers of meaning into vector representations of terms and concepts. Examples include the encoding of the relative position of terms [1,10], syntactic information [11], and orthographic information [4].

The general approach used depends upon the generation of vector representations for terms, and associating reversible vector transformations with properties of interest. In accordance with terminology developed in [12], we will refer to the vector representations of atomic components such as terms as *elemental vectors*. Elemental vectors are constructed using a randomization procedure such that they have a high probability of being mutually orthogonal, or close-to-orthogonal. This adds robustness to the model, by making it highly improbable that elemental vectors would be confused with one another, despite the distortion that occurs during training. However, it also introduces the implicit assumption that elemental vectors are unrelated to one another, which means that models generated in this way must be composed of discrete elements.

This limitation notwithstanding, this approach has allowed for the integration of structured information into distributional models of meaning. From the perspective of cognitive psychology, this is desirable as it presents the possibility of a unified term representation that can account for a broad range of experimental phenomena. Recent work in this area has leveraged circular convolution to generate vectors representing the orthographic form of words [3], and integrate these with a geometric model of distributional semantics [4]. Vector representations of orthographic word form are generated by using circular convolution to generate bound products representing the component bigrams of the term concerned, including non-contiguous bigrams. Karchergis *et al* give the following example (⊛ indicates binding using circular convolution) [4]:

$$\begin{aligned} word =& w + o + r + d + w \circledast o + o \circledast r + r \circledast d \\ &+ w \circledast o + (w \circledast \_) \circledast r + (w \circledast \_) \circledast d + (w \circledast o) \circledast r + ((w \circledast \_) \circledast r) \circledast d \\ &+ (w \circledast \_) \circledast d + (o \circledast \_) \circledast d + r \circledast d + ((w \circledast o)\_) \circledast d + (o \circledast r) \circledast d \end{aligned}$$

The vector representation for the term "word" is generated by combining a set of vectors representing unigrams, bigrams, and trigrams of characters. It is a characteristic of the model employed that each of these vectors have a high probability of being mutually orthogonal, or close-to-orthogonal. So, for example, the vectors representing the trigrams $((w \circledast \_) \circledast r)$ and $((w \circledast o) \circledast r)$ will be dissimilar from one another. Consequently it is necessary to explicitly encode all of the n-grams of interest, including gapped trigrams (such as "w_ r") to provide flexibility. From the perspective of computational complexity, this is not ideal, as the number of representational units that must be generated and encoded is at least quadratic to the length of the sequence.

Rather than explicitly encoding character position precisely (with respect to some other character, or the term itself), alternative models of orthographic representation allow for a degree of uncertainty with respect to letter position. These approaches measure the relatedness between terms on the basis of the similarity between probability distributions assigned to the positions of each matching character [13,14], providing a more flexible measure of similarity. However, on account of the constraints we have discussed, only orthographic representations based on discrete bigrams or the exact position of characters have been combined with other sorts of distributional information in an attempt to generate a holistic representation to date [3,15]. Imposing near-orthogonality adds robustness, but also necessitates ignoring potentially useful information to do with structure, namely the proximity between character positions within a word. Consequently, we have selected the generation of orthographic representations as an example application through which to illustrate the utility of our approach.

The paper proceeds as follows. First, we present the mathematical language we will use to describe the operators provided by VSAs, a family of representational approaches based on reversible vector transformations [16]. Next, we will describe an approach we have developed through which the distance between elemental vectors, and hence bound products, can be predetermined. In the context of an illustrative application for orthographic modeling, we show that this approach permits the encoding of structural information to do with proximity, rather than absolute position, into a distributional model. We then discuss a relationship between this approach and quantum mechanics, and conclude with some experimental results and example applications.

3 Mathematical Structure and Methods

3.1 Vector Symbolic Architectures (VSAs)

The reversible vector transformations we have discussed are a distinguishing feature of a family of representational approaches collectively known as VSAs [16]. In our experiments the VSAs we will use are Kanerva's Binary Spatter Code (BSC), which uses binary vectors [17], and Plate's CHRR [5], which uses complex vectors where each dimension represents an angle between $-\pi$ and π, using the implementation developed in [10]. In addition, we will use an approach based on permutation of real vectors [2].

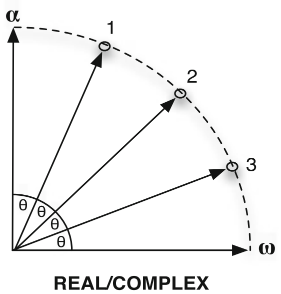
Binding is the primary operation facilitated by VSAs (in addition to standard operators for vector superposition and vector comparison). Binding is a multiplication-like operator through which two vectors are combined to form a third vector C that is *dissimilar from* either of its component vectors A and B. We will use the symbol "$\otimes$" for binding, and the symbol "$\oslash$" for the inverse of binding for the remainder of this paper. It is important that this operator be invertible: if $\mathrm{C} = \mathrm{A} \otimes \mathrm{B}$, then $\mathrm{A} \oslash \mathrm{C} = \mathrm{A} \oslash (\mathrm{A} \otimes \mathrm{B}) = \mathrm{B}$. In some models, this recovery may be approximate, but the robust nature of the representation guarantees that $\mathrm{A} \oslash \mathrm{C}$ is similar enough to B that B can easily be recognized as the best candidate for $\mathrm{A} \oslash \mathrm{C}$ in the original set of concepts. Thus the invertible nature of the bind operator facilitates the retrieval of the information it encodes.

In the case of the BSC, elemental vectors are initialized by randomly assigning 0 or 1 to each dimension with equal probability. Pairwise exclusive or (XOR) is used as a binding operator: $\mathrm{X} \otimes \mathrm{Y} = \mathrm{X}\ \mathrm{XOR}\ \mathrm{Y}$. As it is its own inverse, the binding and decoding processes are identical ($\otimes = \oslash$). For superposition, the BSC employs a majority vote: if the component vectors have more ones than zeros in a dimension, this dimension will have a value of one, with ties broken at random.

In CHRR, binding through circular convolution is accomplished by pairwise multiplication: $X \otimes Y = \{X_1 Y_1, X_2 Y_2, \ldots . X_{n-1} Y_{n-1}, X_n Y_n\}$, which is equivalent to addition of the phase angles of the circular vectors concerned. Binding is inverted by binding to the inverse of the vector concerned: $X \oslash Y = X \otimes Y^{-1}$, where the inverse of a vector is its complex conjugate. Elemental vectors are initialized by randomly assigning a phase angle to each dimension. Superposition is accomplished by pairwise addition of the unit circle vectors, and normalization of the result for each circular component. In the implementation used in our experiments, normalization occurs after training concludes, so the sequence in which superposition occurs is not relevant.

Our real vector implementation follows the approach developed by Sahlgren and his colleagues [2], and differs from the binary and complex implementations, in that elemental vectors are "bound" to permutations, rather than to other vectors. Elemental vectors are constructed by creating a high-dimensional zero vector (on the order of 1000 dimensions), and setting a small number of the dimensions of this vector (on the order of 10) to either +1 or −1 at random. The permutations utilized consist of shifting all of the elements of a given vector n positions to the right, where each value n is assigned to, or derived from, the information it is intended to encode. In the case of our orthographic model, this information consists of the character occurring in a particular position, so we have used the ASCII value of the character concerned as n. Binding is reversed by permuting all of the elements of the vector n positions to the left. Superposition is accomplished by adding the vectors concerned, and normalizing the result.

In all models, the "random" initiation of elemental vectors is rendered deterministic by seeding the random number generator with a hash value derived from a string or character of interest following the approach developed in [18]. This retains the property of near-orthogonality where desired, while ensuring that random overlap between elemental vectors is consistent across experiments.



P(1)

α	1	0	1	0
1	0.75	0.25	1	0
2	0.50	0.50	1	0
3	0.25	0.75	1	0
ω	0	1	1	0

BINARY

Fig. 1. Interpolation to generate five demarcator vectors. P(1) = probability of 1.

3.2 Measured Similarity

The first step in our approach involves generating a set of vectors that are a fixed distance apart, which we will refer to as *demarcator vectors* (D(position)), as illustrated in Fig. 1. The first pair of demarcator vectors are conventional elemental vectors $D(\alpha)$ and $D(\omega)$, constructed randomly such that they have a high probability of being mutually orthogonal or close-to-orthogonal. To ensure this with certainty, we render $D(\omega)$ orthogonal to $D(\alpha)$ using the quantum negation procedure, or its binary approximation, described in [19] and [20] respectively. The remaining demarcator vectors are generated by interpolation. In the continuous vector spaces, this is accomplished by subdividing the 90° angle between $D(\alpha)$ and $D(\omega)$ and generating the corresponding unit vectors. In binary vector space, this is accomplished by weighting the probability of assigning a 1 when $D(\alpha)$ and $D(\omega)$ disagree in accordance with the desired distance between the new demarcator vector and each of these extremes[1]. As the vectors representing

[1] The same random number sequence must be used for all vectors in a demarcator set, so that a consistent random value for each bit position is compared to the relevant thresholds.

adjacent numbers are approximately equidistant, the distances between vector pairs representing numbers the same distance apart should also be approximately equal (e.g. $sim(D(1), D(2)) \approx sim(D(2), D(3))$).

Table 1 illustrates the pairwise similarities between a set of five demarcator vectors constructed in this manner. In the binary case vectors of dimensionality 32,000 are used, in the complex case vectors of dimensionality 500 are used, and in the real case, vectors of dimensionality 1,000 are used. These dimensions were chosen so as to normalize the space requirements of the stored vectors across models, and were retained in our subsequent experiments. In all cases, the relatedness between demarcator vectors a fixed distance apart is approximately equal. For example, the similarity between all pairs of demarcator vectors two positions apart (e.g. 1 and 3) is approximately 0.5 in the binary implementation, and 0.71 in the complex and real vector implementations. In the binary implementation, the difference in relatedness is proportional to the difference in demarcator position. This is not the case in the complex or real implementations, where the drop in similarity between demarcator vectors becomes progressively steeper. This is an artifact of the metric used to measure similarity in each case. With binary vectors, $2 \times (0.5 -$ normalized Hamming distance) is used, but with continuous vectors the cosine distance metric is used. While a proportional decrement could be obtained by measuring the angle between complex vectors directly (or taking the arccos of this cosine value), we will retain the use of the cosine metric for our experiments.

Table 1. Pairwise similarity between demarcator vectors

BINARY						COMPLEX						REAL					
	α	**1**	**2**	**3**	ω		α	**1**	**2**	**3**	ω		α	**1**	**2**	**3**	ω
α	1.00	0.75	0.49	0.25	0.00	α	1.00	0.92	0.71	0.38	0.00	α	1.00	0.92	0.71	0.38	0.00
1	0.75	1.00	0.74	0.50	0.25	**1**	0.92	1.00	0.92	0.71	0.38	**1**	0.92	1.00	0.92	0.71	0.38
2	0.49	0.74	1.00	0.75	0.51	**2**	0.71	0.92	1.00	0.92	0.71	**2**	0.71	0.92	1.00	0.92	0.71
3	0.25	0.50	0.75	1.00	0.75	**3**	0.38	0.71	0.92	1.00	0.92	**3**	0.38	0.71	0.92	1.00	0.92
ω	0.00	0.25	0.51	0.75	1.00	ω	0.00	0.38	0.71	0.92	1.00	ω	0.00	0.38	0.71	0.92	1.00

3.3 Encoding Orthography

Using controlled degrees of non-orthogonality, we can encode information about the positions of letters in words into their vector representations. Like spatial encoding [14] and the overlap model [13], our approach is based upon measuring the difference in position between matching characters. This is accomplished by creating elemental vectors for characters, and binding them to demarcator vectors representing positions. For example, the orthographic vector for the term "word" is constructed as follows:

$$S(\text{word}) = E(\text{w}) \otimes D(1) + E(\text{o}) \otimes D(2) + E(\text{r}) \otimes D(3) + E(\text{d}) \otimes D(4)$$

As the elemental vectors for characters are mutually orthogonal or near-orthogonal, bound products derived from different characters will be orthogonal or near-orthogonal also. For example, in the complex vector space used to generate Table 1, $sim(E(\mathrm{w}) \otimes D(1), E(\mathrm{q}) \otimes D(1)) = 0$. Furthermore, the distance between bound products containing the same character will approximate the distance between their demarcator vectors. For example, in the real vector space used to generate Table 1, $sim(E(\mathrm{w}) \otimes D(1), E(\mathrm{w}) \otimes D(2)) = 0.92 = sim(D(1), D(2))$. Ultimately, the similarity between a pair of terms is derived from the distance between their matching characters. If this distance is generally low, the orthographic similarity between these terms will be high[2]. Thus, the models so generated are innately tolerant to variations such as transposition, insertion and deletion of sequence elements.

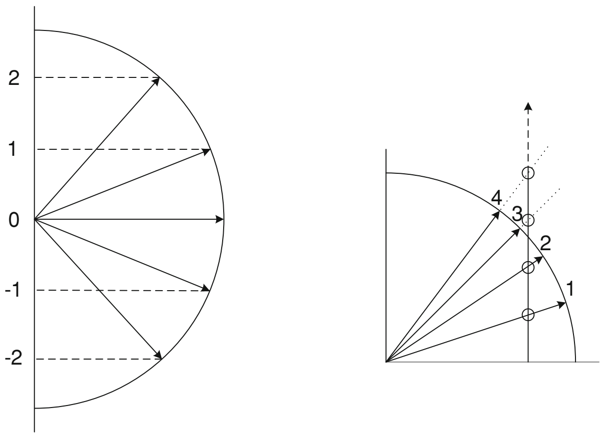


Fig. 2. Orbital angular momentum vectors, as derived from quantized states along a given axis (left), and a related strategy for encoding positions as vectors

4 Demarcator Vectors and Orbital Angular Momentum

Evenly distributing normalized vectors between two orthogonal vectors is one of many strategies we could adopt to generate demarcator vectors. To generalize this process, we can describe it as follows:

1. Construct a line in the vector space with a given starting point and direction.
2. Place demarcators along this line using some dividing strategy.
3. (Optional) Project demarcator vectors onto the unit circle to normalize them.

[2] For terms of different lengths, we elected to construct a set of demarcator vectors for each term. So while $D(\alpha)$ and $D(\omega)$ will be identical, the demarcator for a particular position may differ. It would also be possible to use identical demarcator vectors (by generating a set large enough to accommodate the longest term), which may be advantageous for some tasks.

Two examples of this strategy are illustrated in Fig. 2. In the example on the left, the generating line is the vertical axis, the dividing strategy is to mark points at even intervals along this axis, and the projection strategy is to project orthogonally from the vertical axis onto the unit sphere[3]. In the example on the right, we have chosen a generating line parallel to the vertical axis, marked points at even intervals, and projected onto the unit sphere using standard vector normalization. The left-hand strategy will be familiar to some readers: this is precisely the way *orbital angular momentum* states are generated in quantum mechanics. We refer the reader to a text on quantum mechanics for this derivation, e.g., [21, Ch 14]: the process includes solving a wave equation in three dimensions, exploring the angular momentum operator and the commutator relations between its components, and noting that each point on the axis can be mapped to many on the outer sphere (this ambiguity corresponding to the fact that measuring the component of angular momentum along one axis must leave the component along the other axes undetermined according to the Uncertainty Principle). The important point for this discussion is that it is quite standard to derive non-orthogonal vectors for states in this fashion, not only in generalized quantum structures but in quantum mechanics itself. The underlying spherical harmonic functions involved in angular momentum are orthogonal to one another under pairwise integration, but lead to several possible non-orthogonal angular momentum vectors.

The ambiguity of the projection onto the unit sphere in the angular momentum model in more than two dimensions is problematic, and we expect the strategy on the right to be simpler in practice. Note also that both strategies do not distribute vectors evenly around the unit sphere: for example, in the right-hand strategy, the vectors representing positions 3 and 4 are closer to each other than the vectors representing positions 1 and 2. Such flexibility to vary these pairwise similarities between positions along a string is a desirable property, because changes at the beginning of a word may be more significant than changes in the middle [13]. We note also that in our current implementation in the Semantic Vectors package, this generalized strategy works well for real-valued and complex-valued vectors, but is as yet underspecified for binary-valued vectors because (for example) 'the point half-way between A and B' is multiply defined using Hamming distance [22]. We are currently investigating appropriate strategies to bring binary vectors into this generalized description.

5 Applications: Orthography, Morphology, Sentence Similarity

Table 2 provides examples of nearest-neighbor search based on orthographic similarity in real, complex and binary vector spaces derived from the widely-used Touchstone Applied Science Associates, Inc. (TASA) corpus. Terms in the corpus that occurred between 5 and 15,000 times were represented as candidates for

[3] In this example we have drawn negative and positive positions, though in practice we have only experimented with nonnegative positions so far.

retrieval. The dimensionality of the real, complex and binary vector spaces concerned was 1000, 500 and 32,000 respectively. Our approach successfully recovers orthographically related terms, including terms containing substrings of the original term (“**dominic**” vs. “con**domini**um”); insertions (“**orthography**” vs. “**ortho**photo**graphy**”); substitutions (“**an**g**ular**” vs. “**an**n**ular**”) and transposition of characters (“**w**ah**le**” vs. “**w**ha**le**”). While not shown in the table on account of space constraints, the models produced similar sets of results for the same cue term.

Table 2. Orthographic similarity

REAL				COMPLEX		BINARY			
CUE	dominic	CUE	orthography	CUE	orthogonality	CUE	angular	CUE	wahle
0.92	dominican	0.92	orthophotography	0.86	orthogonally	0.73	agranular	0.94	whale
0.89	dominion	0.93	photography	0.85	orthogonal	0.70	annular	0.61	awhile
0.88	demonic	0.91	chromatography	0.84	orthodontia	0.67	angularly	0.60	while
0.85	dominions	0.90	orthographic	0.82	ornithology	0.66	gabular	0.60	whales
0.85	condominium	0.90	choreography	0.82	ornithologist	0.66	inaugural	0.60	whaley

Table 3. Comparison with benchmark conditions from [15]. 1X = original dimensionality (used for Table 2). 2X = twice original dimensionality. nb = no binding. ED $= 1 - \frac{\text{edit distance}}{\text{combined length}}$.

	BINARY			COMPLEX			REAL		
	1X	2X	2Xnb	1X	2X	2Xnb	1X	2X	ED
Stability	✓	✓	✓	✓	✓	✓	✓	✓	✓
Edge effects							✓		
Local TL	✓	✓	✓	✓	✓	✓	✓	✓	✓
Global TL									✓
Distal TL									
Compound TL	✓	✓		✓	✓		✓	✓	
Distinct RP	✓	✓	✓	✓	✓	✓	✓	✓	✓
Repeated RP	✓	✓		✓	✓	✓	✓	✓	✓

While we would be hesitant to propose the simple model of orthographic representation we have developed as a cognitive model of lexical coding, it is interesting to note that it does conform to the majority of a set of constraints abstracted from lexical priming data by Hannagan and his colleagues [15]. We will describe these constraints in brief here, but refer the interested reader to [15] for further details. The constraints are as follows: (1) **Stability**: a string should be most similar to itself (sim ≥ 0.95); (2) **Edge Effects**: substitutions at the edges of strings should be more disruptive; (3) **Local Translocations (TL)**: transposing adjacent characters should be less disruptive than substituting both of them; (4) **Global TL**: transposing *all* adjacent characters should be maximally disruptive (5) **Distal TL**: transposing non-adjacent characters should be more disruptive than substituting one, but less than substituting both; (6) **Compound TL**: TL and substitution should be more disruptive than substitution alone; (7) **Distinct Relative Position (RP)**: removing some characters should preserve some similarity; and (8) **Repeated RP**: removing a repeated or non-repeated letter should be equally disruptive[4]. Each constraint is accompanied by a set of test cases, consisting of paired strings, and the degree to which a model meets the constraint is determined from the estimated similarities between these pairs, and the relationships between them.

The extent to which the our models meet these constraints is shown in Table 3. Estimates based on all models consistently meet all constraints aside from those related to Edge Effects, and the Global and Distal TL constraints (in the latter case this is due to the fact that translocation of characters one position apart is *less* disruptive than substituting one of these characters). This represents a better fit to these constraints than comparable models based on letter distribution only (labeled "nb", or not bound). It also represents a better fit than the majority of approaches evaluated against this benchmark previously [15, 23], providing motivation for the further evaluation of a more developed model in the future. While the real model appeared to meet the edge effect related constraint at its original dimensionality, this finding did not hold at higher dimensionality, and was most likely produced by random overlap. This is not surprising given that our current model does not address edge effects. As suggested in Sect. 4, one way to address this issue would be to increase the distance between peripheral demarcator vectors, a customization we plan to evaluate in future work.

Table 4. Combining orthographic and semantic similarity

REAL				COMPLEX		BINARY			
CUE	think	CUE	bring	CUE	eat	CUE	catch	CUE	write
0.57	intend	0.68	bringing	0.44	eats	0.191	catching	0.180	writes
0.56	know	0.62	brings	0.43	ate	0.184	caught	0.173	writer
0.51	thinks	0.53	brought	0.39	meat	0.181	catches	0.172	rewritten
0.51	thinking	0.52	ring	0.36	eaten	0.176	watch	0.172	wrote
0.51	want	0.51	burning	0.36	restate	0.176	teach	0.171	reread

The results in Table 4 were obtained by superposing the orthographic vector for each term from the TASA corpus with a semantic vector for the same term generated using the permutation-based approach described in [2], with a 2+2

[4] As the randomization procedure makes it very unlikely that the estimates of similarity between any two pairs will be identical, we have considered a difference of ≤ 0.05 to be approximately equal. This mirrors the relaxed constraint that ≥ 0.95 is approximately identical used by Hannagan and his colleagues for the stability constraint [15, 23].

sliding window. As anticipated by previous work combining orthographic and semantic relatedness [24,25], the examples suggest that this model is able to find associations between morphologically related terms, including those between English verb roots and past tense forms are related by non-affixal patterns such as "**br**ing:**br**ou**g**ht". The combination of semantics and shared characters is evident in other examples, such as "**t**h**in**k:**int**end". However, this sensitivity to morphological similarity comes at a cost of introducing false similarity when common letter patterns do not have a semantic significance.

Table 5. Retrieval of Sentences from the TASA Corpus in complex (first two examples) and binary (second two examples) spaces

CUE	the greater the force of the air the louder the sounds
0.86	that is the smaller the wavelength the greater the energy of the radiation
0.86	the greater the amplitude the greater the amount of energy in the wave
0.85	the deeper the level of processing the stronger the trace and the better the memory
0.84	the darker the blue the deeper the water
CUE	these four quantum numbers are used in describing electron behavior
0.353	these numbers are important in chemistry
0.3433	these are usually used in the home
0.316	what punctuation marks are used in these four sentences
0.315	these are abbreviations that sometimes are used in written directions

6 Conclusion

In this paper we have introduced a novel approach through which the near-orthogonality of elemental vectors is deliberately violated to introduce measured similarity into semantic space. While illustrated primarily through orthographic modeling, the approach is general in nature and can be applied in any situation in which a representation of sequence that is tolerant to variation is desired. Furthermore, this approach may mediate the generation of holistic representations combining distributional and spatial information, a direction we plan to explore in future work. To facilitate further experimentation, our real, binary and complex orthographic vector implementations have been released as components of the open source Semantic Vectors package [27,28].

Acknowledgments. This research was supported by US National Library of Medicine grant R21 LM010826. We would like to thank Lance DeVine, for the CHRR implementation used in this research, and Tom Landauer for providing the TASA corpus.

References

1. Jones, M.N., Mewhort, D.J.K.: Representing word meaning and order information in a composite holographic lexicon. Psychol. Rev. **114**, 1–37 (2007)
2. Sahlgren, M., Holst, A.,Kanerva, P.: Permutations as a means to encode order in word space. In: Proceedings of the 30th Annual Meeting of the Cognitive Science Society (CogSci'08), July 23–26, Washington D.C., USA (2008)
3. Cox, G.E., Kachergis, G., Recchia, G., Jones, M.N.: Toward a scalable holographic word-form representation. Behav. Res. Methods **43**, 602–615 (2011)
4. Kachergis, G., Cox, G.E., Jones, M.N.: Orbeagle: integrating orthography into a holographic model of the lexicon. In: Honkela, T. (ed.) ICANN 2011, Part I. LNCS, vol. 6791, pp. 307–314. Springer, Heidelberg (2011)
5. Plate, T.A.: Holographic Reduced Representation: Distributed Representation for Cognitive Structures. CSLI Publications, Stanford (2003)
6. Cohen, T., Widdows, D.: Empirical distributional semantics: methods and biomedical applications. J. Biomed. Inf. **42**, 390–405 (2009)
7. Turney, P.D., Pantel, P.: From frequency to meaning: vector space models of semantics. J. Artif. Intell. Res. **37**(1), 141–188 (2010)
8. Landauer, T., Dumais, S.: A solution to plato's problem: the latent semantic analysis theory of acquisition, induction, and representation of knowledge. Psychol. Rev. **104**, 211–240 (1997)
9. Burgess, C., Livesay, K., Lund, K.: Explorations in context space: words, sentences, discourse. Discourse Process. **25**, 211–257 (1998)
10. De Vine, L., Bruza, P.: Semantic oscillations: Encoding context and structure in complex valued holographic vectors. In: Proceedings of AAAI Fall Symposium on Quantum Informatics for Cognitive Social, and Semantic Processes (2010)
11. Basile, P., Caputo, A., Semeraro, G.: Encoding syntactic dependencies by vector permutation. In: Proceedings of the EMNLP 2011 Workshop on GEometrical Models of Natural Language Semantics, GEMS, vol. 11, pp. 43–51 (2011)
12. Cohen, T., Widdows, D., Schvaneveldt, R., Rindflesch, T.: Finding schizophrenia's prozac: emergent relational similarity in predication space. In: Song, D., Melucci, M., Frommholz, I., Zhang, P., Wang, L., Arafat, S. (eds.) QI 2011. LNCS, vol. 7052, pp. 48–59. Springer, Heidelberg (2011)
13. Gomez, P., Ratcliff, R., Perea, M.: The overlap model: a model of letter position coding. Psychol. Rev. **115**, 577–600 (2008)
14. Davis, C.J.: The spatial coding model of visual word identification. Psychol. Rev. **117**(3), 713 (2010)
15. Hannagan, T., Dupoux, E., Christophe, A.: Holographic string encoding. Cogn. Sci. **35**(1), 79–118 (2011)
16. Gayler, R.W.: Vector symbolic architectures answer jackendoff's challenges for cognitive neuroscience. In: Slezak, P. (ed.) ICCS/ASCS International Conference on Cognitive Science, pp. 133–138. University of New South Wales, Sydney (2004)
17. Kanerva, P.: Binary spatter-coding of ordered k-tuples. In: Artificial Neural Networks – ICANN, vol. 96, pp. 869–873 (1996)
18. Wahle, M., Widdows, D., Herskovic, J.R., Bernstam, E.V., Cohen, T.: Deterministic binary vectors for efficient automated indexing of MEDLINE/PubMed abstracts. In: AMIA Annual Symposium Proceedings **2012**, pp. 940–949, November 2012
19. Widdows, D., Peters, S.: Word vectors and quantum logic experiments with negation and disjunction. In: Proceedings of 8th Mathematics of Language Conference, Bloomington, Indiana (2003)

20. Cohen, T., Widdows, D., De Vine, L., Schvaneveldt, R., Rindflesch, T.C.: Many paths lead to discovery: analogical retrieval of cancer therapies. In: Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.) QI 2012. LNCS, vol. 7620, pp. 90–101. Springer, Heidelberg (2012)
21. Bohm, D.: Quantum Theory. Prentice-Hall, Upper Saddle River (1951). Republished by Dover (1989)
22. Widdows, D., Cohen, T.: Real, complex, and binary semantic vectors. In: Sixth International Symposium on Quantum Interaction, France, Paris (2012)
23. Hannagan, T., Grainger, J.: Protein analysis meets visual word recognition: a case for string kernels in the brain. Cogn. Sci. **36**(4), 575–606 (2012)
24. Yarowsky, D., Wicentowski, R.: Minimally supervised morphological analysis by multimodal alignment. In: Proceedings of the 38th Annual Meeting on Association for Computational Linguistics, ACL '00, PA, USA, pp. 207–216. Association for Computational Linguistics, Stroudsburg (2000)
25. Baroni, M., Matiasek, J., Trost, H.: Unsupervised discovery of morphologically related words based on orthographic and semantic similarity. In: Proceedings of the ACL-02 workshop on Morphological and phonological learning -. MPL '02, PA, USA, vol. 6, pp. 48–57. Association for Computational Linguistics, Stroudsburg (2002)
26. Dennis, S.: Introducing word order within the LSA framework. In: Handbook of Latent Semantic Analysis (2007)
27. Widdows, D., Ferraro, K.: Semantic vectors: a scalable open source package and online technology management application. In: Sixth International Conference on Language Resources and Evaluation (LREC 2008) (2008)
28. Widdows, D., Cohen, T.: The semantic vectors package: New algorithms and public tools for distributional semantics. In: Fourth IEEE International Conference on Semantic Computing (ICSC), pp. 9–15 (2010)

The Sphynx's New Riddle: How to Relate the Canonical Formula of Myth to Quantum Interaction

Sándor Darányi[1](✉), Peter Wittek[1], and Kirsty Kitto[2]

[1] University of Borås, Borås, Sweden
sandor.daranyi@hb.se
[2] Information Systems School, Queensland University of Technology, Brisbane, Australia

Abstract. We introduce Claude Lévi Strauss' canonical formula (CF), an attempt to rigorously formalise the general narrative structure of myth. This formula utilises the Klein group as its basis, but a recent work draws attention to its natural quaternion form, which opens up the possibility that it may require a quantum inspired interpretation. We present the CF in a form that can be understood by a non-anthropological audience, using the formalisation of a key myth (that of Adonis) to draw attention to its mathematical structure. The future potential formalisation of mythological structure within a quantum inspired framework is proposed and discussed, with a probabilistic interpretation further generalising the formula.

1 Introduction

Every society has its myths, and these show many similarities across societies which are themselves markedly different. Thus, a wide range of peoples have a "trickster" character; American Indians have Coyote, the Norse Loki, Africans Anansi, Christians Satan etc. and these characters share universal features despite their very different shapes and backgrounds. They even take similar roles in the mythological cycles that they participate in. Thus, many tricksters are central to creation myths, and equally they participate in the end of the world cycles. Such apparent universalities have led many [1] to wonder if there might be a general pattern to the myths of the world, or even a universal structure or essence [2].

One of the more mathematically oriented attempts to describe such a universal structure was first proposed by Claude Lévi-Strauss [3] in 1955. His *canonical formula* (CF) takes a structural approach to the analysis of myth, utilising mutually opposite value sets encoded in bundles of relations to consider the form that a myth takes as its storyline progresses. Intriguingly, this formula has its roots in group theory [4], which suggests that it might fit within a quantum inspired framework. However, the CF has been dividing anthropologists over the past sixty years and holds a somewhat enigmatic status within that community [5].

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 47–58, 2014.
DOI: 10.1007/978-3-642-54943-4_5,

We believe that this controversy arises from the lack of a consistent interpretative framework from which to understand the CF, which in itself results in no universal understanding of the proper methodology for using the apparatus that Lévi-Strauss created. However, hope lies in the group theoretic formulation of the CF, and this paper is an attempt to propose that the framework of Quantum Interaction (QI) could provide an viable new way forward.

Here, we shall introduce the CF to the QI community, showing that it has strong parallels with many features that can be found in quantum inspired models, and so could provide a new exciting avenue of research. Section 2 gives a brief overview of the CF, and Sect. 2.1 introduces the reader to its usage through the formalisation of a myth (that of Adonis) within the framework. Section 2.1 also explains how to relate the CF to running text by a syntagmatic reading. Section 2.2 discusses the difference between narrative formulae and the CF. Section 2.3 outlines a particular scenario which can generate literally hundreds of narrative formulae, among them 31 equivalents of the CF. Finally, Sect. 3 cites a quaternion interpretation of such formulae with further implications in quantum theory. Section 5 sums up our conclusions.

2 The Canonical Formula

André Weil first wrote the CF as a formula of unfolding, formalizing it by means of group theory [6]. As far as this formula is understood, it describes plot (storyline) development in myths [3] or topic evolution in mythologies [7], encoded as a double transformation of four compound arguments in specific relation to one another:

$$F_x(a) : F_y(b) :: F_x(b) : F_{a^{-1}}(y). \tag{1}$$

Each of these four arguments consist of a term variable (a and b), and a function variable (x and y). The form of this equation requires some explanation, but we caution the reader that (1) has been the subject of ongoing and unresolved debate ever since Lévi-Strauss first proposed it. In what follows we shall closely follow the interpretation proposed by Morava [8], as this mathematically rigorous form will provide the basis for our claim that Eq. (1) can be understood within a quantum-like perspective.

In Morava's rendering, a number of different authors have suggested that (1) describes a *transformation*, which, for a sufficiently large and coherent body of myths, identifies

> *characters a, b and functions x, y, such that the mythical system defines a transformation which sends a to b, y to a^{-1} and b to y, while leaving x invariant.* [8, p.3]

This explanation leaves us with an interesting possibility for generating a mathematical description of myths, that is, the CF describes a *structural relationship* between a set of narrative terms and their transmutative relationships, however, the choice of what concepts these terms and relationships should apply to is left rather open and ill-defined. Intriguingly, at the root of the CF is a Klein group of four elements, e.g. x, $1/x$, $-x$, $-1/x$ [4], applied to one of the two narrative terms a and b or one of the two relations x and y, however Morava makes a convincing argument that the quaternion group of order eight is the correct mathematical structure to adequately represent Lévi-Strauss' conceptualisation, a point that we shall return to in Sect. 3. For now, we shall leave this important point aside, attempting instead to illustrate the key features of the CF with reference to an example.

2.1 Applying the Canonical Formula

The CF (Eq. (1)) describes the relationship between syntagms, i.e. short sentences with condensed content which sum up parts of a myth, leaving one with a considerable amount of freedom when attempting to apply it to a narrative plot. This is a problem that becomes even more extreme when it is acknowledged that many different structural forms of the CF are consistent with the group that it specifies (see Sect. 2.3). This complexity aside, application of the CF to a narrative consists of finding a consistent mapping of the objects and relations according to the structural relationship exemplified by (1), or one of its 32 alternatives (see the discussion in Sect. 2.3, and the further generalization in Sect. 3).

This is no easy task. It requires both the identification of suitable mythological narratives, and then the mapping of their components into a form mandated by (1), practically filling in placeholders in prespecified relationships to one another with fitting syntagmatic content. We shall illustrate this process with reference to an example myth, that of the Ancient Greek story of Adonis, which runs as follows [9, Sects. 14–16][1]:

> *Panyasis says that he was a son of Thias, king of Assyria, who had a daughter Smyrna. In consequence of the wrath of Aphrodite, for she did not honor the goddess, this Smyrna conceived a passion for her father, and with the complicity of her nurse she shared her father's bed without his knowledge for twelve nights. But when he was aware of it, he drew his sword and pursued her, and being overtaken she prayed to the gods that she might be invisible; so the gods in compassion turned her into the tree which they call* smyrna *(myrrh). Ten months afterwards the tree burst and Adonis, as he is called, was born, whom for the sake of his beauty, while he was still an infant, Aphrodite hid in a chest unknown to the gods and entrusted to Persephone. But when Persephone beheld him, she would not give him back. The case being tried before Zeus, the year was divided into three parts, and the god ordained that Adonis should stay by*

[1] The classical texts we used as examples come from the Perseus Digital Library at Tufts University (http://www.perseus.tufts.edu/).

> *himself for one part of the year, with Persephone for one part, and with Aphrodite for the remainder. However Adonis made over to Aphrodite his own share in addition; but afterwards in hunting he was gored and killed by a boar.*

It must be possible to relate each of the terms in the CF consequently to stories such as these. In order to do this it is necessary to identify a set of dichotomies that can be consistently assigned according to the relationships in the CF. The two basic narrative characters, a and b must be identified in a consistent manner, with the added provision that the function y somehow transforms into an inversion of a (i.e. a^{-1}), and b to y, while x remains invariant under the chosen transformation.

Thus, for the above myth, an identification of the character Thias with the label b implies that the action of *killing* should be represented by x.[2] In order to proceed, we could hypothesise a scenario where the representation of the Adonis myth can be started with the following identification:

$F_y(b)$ as Thias "destroys" (in this case he kills Smyrna).

This move starts to limit the available identifications for the other variables in (1). Because the root of the CF in structuralism means that the assignments must be in binary opposition, we require a set of binary opposites for both the terms and the functions in this myth. The following are chosen for our current scenario:

Terms:	Functions:
– Male/female	– Affirm/deny
– Divine/human	– Active voice/passive voice
– Adult/adolescent	– Complete/incomplete

Thus, designating the male human adult Thias as b implies that $-b$ could represent a female human adult, while b^{-1} could be a male human who was adolescent (Adonis in this myth) etc. Essentially this value assignment is open to a certain amount of freedom, yet once one binary value has been designated, its opposite must be interpreted for contrast in some manner. This inversion can be performed in one of the four following ways:

– a is a binary opposite of b
– a is a binary opposite of $-a$

[2] The particular set of values we assigned to variables in the CF for this example was as follows: complete male/female: fertile/adult, incomplete male/female: infertile/adolescent; complete denial active voice: destroy/kill, complete affirmative active voice: procreate/bear, incomplete denial active voice: wound/hurt, incomplete affirmative active voice: heal; passive voice for the above: be destroyed/killed/begotten/born/wound/healed, plus the above being done either to the other or the self.

- a is a binary opposite of a^{-1}.
- $x(a)$ is binary opposite of $a(x)$, here distinguishing between the other and the self.

The CF requires that each of these value assignments be performed consistently across the narrative. Continuing this process for the myth of Adonis, we can represent the full structure of the myth quoted above using the character map depicted in Table 1 and the function map in Table 2.

Table 1. A set of consistent value assignments for the characters in the myth of Adonis.

		complete?	
	male / female	yes	no
divine?	yes	a / $-a$	a^{-1} / $-a^{-1}$
	no	b / $-b$	b^{-1} / $-b^{-1}$

Table 2. A set of consistent function assignments for the myth of Adonis.

		affirm?	
	active voice / passive voice	yes	no (deny)
complete?	yes	x / $-x$	x^{-1} / $-x^{-1}$
	no	y / $-y$	y^{-1} / $-y^{-1}$

This set of mappings allows us to keep assigning variables to the narrative in the myth. Thus, we see a slightly symmetrical relationship between Thias and Adonis start to emerge within this narrative structure, which we can formalise using the item and function variables:

$F_y(b)$ as Thias (a male human adult) destroys someone else (in this case he kills Smyrna),

$F_x(b)$ as Thias creates someone else (i.e. begets Adonis, by sleeping with Smyrna),

$F_{a^{-1}}(y)$ as Adonis (a male adolescent divine) destroys himself (in this case he is killed by a boar but his wounds were obtained during a hunt in which he chose to participate).[3]

Finally, recalling the manner in which Aphrodite was born provides the final missing piece of the formula [11, lines 189–191]:

> *"And so soon as he had cut off the members with flint and cast them from the land into the surging sea, they were swept away over the main a long time: and a white foam spread around them from the immortal flesh, and in it there grew a maiden."*

Which leaves us with an understanding of $F_x(a)$ as the maiden that grew from the white foam that arose in that part of the sea where the genitals of Kronos' father (Uranos) landed:

$F_x(a)$: male divine adult creates someone else (in this case Uranos "creates" Aphrodite when his members were cast into the sea).

As contrasted with syntagms about divine or human adult males procreating and killing others, the crucial difference is the role the adolescent divine male who destroys himself. Thus, we see the final typical step emerge which distinguishes the CF from other possible narrative formulae. Note in particular the manner in which a double inversion of content takes place in the fourth argument: what used to be a term takes a reciprocal value, i.e. a maps to a^{-1}, and a former pair of functions and term values swap roles.

2.2 The Narrative Formula

A formula built from the same term vs. function value distribution but without the characteristic double inversion in the fourth argument is not a CF but something we will call a narrative formula (NF), to distinguish between them. An example would be

$$F_x(a) : F_y(b) :: F_x(b) : F_y(a). \tag{2}$$

This is sometimes called the "weak" variant of the CF, i.e. its existence is acknowledged and explicated [12]. We note that this variant may be used to describe myths with a much more simple narrative structure, in particular, those that do not feature the characteristic double inversion of (1).

[3] "It may be significant, however, that an accident in boar hunting [...] is liable to produce wounds somehow equalling castration; then the boar would be just an exchangeable sign for a deeper meaning" [10, p. 108]. Castration as punishment or a voluntary act is frequent in the cult of a group of minor deities from Asia Minor, to which Adonis also belongs. Strictly speaking it is the boar who mutilates Adonis, not he himself, but as far as we know, on a higher level of abstraction these narrative elements belong together.

While the value sets of the four arguments are not defined but left to guesswork, based on suggested examples, the range of the CF spans from tribal myths [3] to Ancient Greek ones [10,13] which would explain the *canonical* adjective attached to it. In spite of the claimed universal validity, its full potential is unexplored, partly going back to the fact that explanation attempts keep on working top down, i.e. trying to find phenomena which can be characterized by such dynamics.

The NF starts to provide a reason for the ongoing failure of the CF to be generally accepted as *the* mathematicalisation of mythology. Contrary to its name, the CF exists in several variants, partly suggested by Lévi-Strauss himself in different phases of his scholarly career, or by [10,14,15]. This plethora of alternatives already hints at insecurities as to what exactly *the* CF might be, suggesting that perhaps it is just one valid variant among many [8,16,17].

2.3 How Many Narrative vs. Canonical Formulae are There?

It has been long suspected that not one but many forms of the CF exist, all pertinent to myth (and indeed, to several narrative genres). Here we introduce a consistent way to generate, and interpret, families of its variants. A more comprehensive approach to formula generation will have to be dealt with elsewhere. Three observations are pertinent here:

- There are three modifiers of term/function values in the NF and CF: the sign of the argument, the sign of the exponent, and the role swap between term and function values;
- Out of terms a, b and functions x and y plus one of the three modifiers per formula, one can create $4 \times 8 = 32$ "weak" forms of the CF (Table 3, left column). Typical for these is that although they may use one of the modified values, *there is no double inversion with respect to the relational structure of the group in them*;
- Not one but altogether 32 "strong" forms, including the original CF, can be formed by systematic *interaction* between two "weak" forms by exchanging the respective fourth arguments of NF_1 vs. NF_7, NF_2 vs. NF_8, NF_3 vs. NF_5, NF_4 vs. NF_6, NF_5 vs. NF_2, NF_6 vs. NF_1, NF_7 vs. NF_4, and NF_8 vs. NF_3 in the first octet of the collection of "weak" forms, respectively (Table 3). The rules of CF formation are similar for the other "weak" octets as well.

All 31 new forms of the CF, i.e. $\text{CF}_2 - \text{CF}_{32}$ are functionally equivalent with CF_1 but stand for different semantic (conceptual) parameter combinations. In other words the CF as a narrative generation tool performs the same transformations on the plot but under rotation of its group, leading to new actors and actions in new situations. There are also ways to derive more NF variants which can describe increasingly complex mythological situations. A more generic probabilistic approach will be discussed in Sect. 3.

Table 3. By interaction between their respective fourth arguments, 32 CF can be generated from 32 NF.

"Weak" forms	"Strong" forms
Octet A (term +, function +)	Octet E (term +, function +)
$NF_1 = x(a) : y(b) :: x(b) : y(a)$	$CF_1 = x(a) : y(b) :: x(b) : a^{-1}(y)$
$NF_2 = x^{-1}(a) : y^{-1}(b) :: x^{-1}(b) : y^{-1}(a)$	$CF_2 = x^{-1}(a) : y^{-1}(b) :: x^{-1}(b) : a^{-1}(y^{-1})$
$NF_3 = x(a^{-1}) : y(b^{-1}) :: x(b^{-1}) : y(a^{-1})$	$CF_3 = x(a^{-1}) : y(b^{-1}) :: x(b^{-1}) : a(y)$
$NF_4 = x^{-1}(a^{-1}) : y^{-1}(b^{-1}) :: x^{-1}(b^{-1}) : y^{-1}(a^{-1})$	$CF_4 = x^{-1}(a^{-1}) : y^{-1}(b^{-1}) :: x^{-1}(b^{-1}) : a(y^{-1})$
$NF_5 = a(x) : b(y) :: b(x) : a(y)$	$CF_5 = a(x) : b(y) :: b(x) : y^{-1}(a)$
$NF_6 = a(x^{-1}) : b(y^{-1}) :: b(x^{-1}) : a(y^{-1})$	$CF_6 = a(x^{-1}) : b(y^{-1}) :: b(x^{-1}) : y(a)$
$NF_7 = a^{-1}(x) : b^{-1}(y) :: b^{-1}(x) : a^{-1}(y)$	$CF_7 = a^{-1}(x) : b^{-1}(y) :: b^{-1}(x) : y^{-1}(a^{-1})$
$NF_8 = a^{-1}(x^{-1}) : b^{-1}(y^{-1}) :: b^{-1}(x^{-1}) : a^{-1}(y^{-1})$	$CF_8 = a^{-1}(x^{-1}) : b^{-1}(y^{-1}) :: b^{-1}(x^{-1}) : y(a^{-1})$
Octet B (term −, function +)	Octet F (term −, function +)
$NF_9 = x(-a) : y(-b) :: x(-b) : y(-a)$	$CF_9 = x(-a) : y(-b) :: x(-b) : -a^{-1}(y)$
$NF_{10} = x^{-1}(-a) : y^{-1}(-b) :: x^{-1}(-b) : y^{-1}(-a)$	$CF_{10} = x^{-1}(-a) : y^{-1}(-b) :: x^{-1}(-b) : -a^{-1}(y^{-1})$
$NF_{11} = x(-a^{-1}) : y(-b^{-1}) :: x(-b^{-1}) : y(-a^{-1})$	$CF_{11} = x(-a^{-1}) : y(-b^{-1}) :: x(-b^{-1}) : -a(y)$
$NF_{12} = x^{-1}(-a^{-1}) : y^{-1}(-b^{-1}) :: x^{-1}(-b^{-1}) : y^{-1}(-a^{-1})$	$CF_{12} = x^{-1}(-a^{-1}) : y^{-1}(-b^{-1}) :: x^{-1}(-b^{-1}) : -a(y^{-1})$
$NF_{13} = -a(x) : -b(y) :: -b(x) : -a(y)$	$CF_{13} = -a(x) : -b(y) :: -b(x) : y^{-1}(-a)$
$NF_{14} = -a(x^{-1}) : -b(y^{-1}) :: -b(x^{-1}) : -a(y^{-1})$	$CF_{14} = -a(x^{-1}) : -b(y^{-1}) :: -b(x^{-1}) : y(-a)$
$NF_{15} = -a^{-1}(x) : -b^{-1}(y) :: -b^{-1}(x) : -a^{-1}(y)$	$CF_{15} = -a^{-1}(x) : -b^{-1}(y) :: -b^{-1}(x) : y^{-1}(-a^{-1})$
$NF_{16} = -a^{-1}(x^{-1}) : -b^{-1}(y^{-1}) :: -b^{-1}(x^{-1}) : -a^{-1}(y^{-1})$	$CF_{16} = -a^{-1}(x^{-1}) : -b^{-1}(y^{-1}) :: -b^{-1}(x^{-1}) : y(-a^{-1})$
Octet C (term +, function −)	Octet G (term +, function −)
$NF_{17} = -x(a) : -y(b) :: -x(b) : -y(a)$	$CF_{17} = -x(a) : -y(b) :: -x(b) : a^{-1}(-y)$
$NF_{18} = -x^{-1}(a) : -y^{-1}(b) :: -x^{-1}(b) : (-y^{-1}(a)$	$CF_{18} = -x^{-1}(a) : -y^{-1}(b) :: -x^{-1}(b) : a^{-1}(-y^{-1})$
$NF_{19} = -x(a^{-1}) : -y(b^{-1}) :: -x(b^{-1}) : a(-y)$	$CF_{19} = -x(a^{-1}) : -y(b^{-1}) :: -x(b^{-1}) : a(-y)$
$NF_{20} = -x^{-1}(a^{-1}) : -y^{-1}(b^{-1}) :: -x^{-1}(b^{-1}) : -y(a^{-1}$	$CF_{20} = -x^{-1}(a^{-1}) : -y^{-1}(b^{-1}) :: -x^{-1}(b^{-1}) : a(-y^{-1})$
$NF_{21} = a(-x) : b(-y) :: b(-x) : a(-y)$	$CF_{21} = a(-x) : b(-y) :: b(-x) : -y^{-1}(a)$
$NF_{22} = a(-x^{-1}) : b(-y^{-1}) :: b(-x^{-1}) : a(-y^{-1})$	$CF_{22} = a(-x^{-1}) : b(-y^{-1}) :: b(-x^{-1}) : -y(a)$
$NF_{23} = a^{-1}(-x) : b^{-1}(-y) :: b^{-1}(-x) : a^{-1}(-y)$	$CF_{23} = a^{-1}(-x) : b^{-1}(-y) :: b^{-1}(-x) : -y^{-1}(a^{-1})$
$NF_{24} = a^{-1}(-x^{-1}) : b^{-1}(-y^{-1}) :: b^{-1}(-x^{-1}) : a^{-1}(-y^{-1})$	$CF_{24} = a^{-1}(-x^{-1}) : b^{-1}(-y^{-1}) :: b^{-1}(-x^{-1}) : -y(a^{-1})$
Octet D (term −, function −)	Octet H (term −, function −)
$NF_{25} = -x(-a) : -y(-b) :: -x(-b) : -y(-a)$	$CF_{25} = -x(-a) : -y(-b) :: -x(-b) : -a^{-1}(-y)$
$NF_{26} = -x^{-1}(-a) : -y^{-1}(-b) :: -x^{-1}(-b) : -y^{-1}(-a)$	$CF_{26} = -x^{-1}(-a) : -y^{-1}(-b) :: -x^{-1}(-b) : -a^{-1}(-y^{-1})$
$NF_{27} = -x(-a^{-1}) : -y(-b^{-1}) :: -x(-b^{-1}) : -y(-a^{-1})$	$CF_{27} = -x(-a^{-1}) : -y(-b^{-1}) :: -x(-b^{-1}) : -a(-y)$
$NF_{28} = -x^{-1}(-a^{-1}) : -y^{-1}(-b^{-1}) :: -x^{-1}(-b^{-1}) : -y^{-1}(-a^{-1})$	$CF_{28} = -x^{-1}(-a^{-1}) : -y^{-1}(-b^{-1}) :: -x^{-1}(-b^{-1}) : -a(-y^{-1})$
$NF_{29} = -a(-x) : -b(-y) :: -b(-x) : -a(-y)$	$CF_{29} = -a(-x) : -b(-y) :: -b(-x) : -y^{-1}(-a)$
$NF_{30} = -a(-x^{-1}) : -b(-y^{-1}) :: -b(-x^{-1}) : -y(-a)$	$CF_{30} = -a(-x^{-1}) : -b(-y^{-1}) :: -b(-x^{-1}) : -y(-a)$
$NF_{31} = -a^{-1}(-x) : -b^{-1}(-y) :: -b^{-1}(-x) : -a^{-1}(-y)$	$CF_{31} = -a^{-1}(-x) : -b^{-1}(-y) :: -b^{-1}(-x) : -y^{-1}(-a^{-1})$
$NF_{32} = -a^{-1}(-x^{-1}) : -b^{-1}(-y^{-1}) :: -b^{-1}(-x^{-1}) : -b^{-1}(-y^{-1})$	$CF_{32} = -a^{-1}(-x^{-1}) : -b^{-1}(-y^{-1}) :: -b^{-1}(-x^{-1}) : -y(-a^{-1})$

3 The Canonical Formula and Quantum Interaction

The relation between the left hand side and the right hand side of the canonical formula can be treated as a transformation, that is, $F_x(a) : F_y(b) \mapsto F_x(b) : F_{a^{-1}}(y)$.

According to Morava, "Lévi-Strauss is describing a logical system in which truth-values lie in an algebraic system called a noncommutative group" [18, p. 55]. The noncommutative group is identified as the quaternion group of order eight with the elements $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, with the noncommutative product operation defined as $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$, $ii = jj = kk = -1$, and $(-1)^2 = 1$.

Let us define an antiautomorphism $\lambda : Q \mapsto Q$ as $\lambda(i) = k$, $\lambda(j) = -i$, and $\lambda(k) = j$. Assigning $x \mapsto 1$, $a \mapsto i$, $y \mapsto j$, and $b \mapsto k$, this automorphism reproduces the canonical formula [8].

The Pauli matrices are a set of three 2×2 complex matrices which are Hermitian and unitary. They are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

.

Together with the identity matrix I, they form a basis for the real Hilbert space of 2×2 complex Hermitian matrices. Each Pauli matrix is related to an operator that corresponds to an observable describing the spin of a spin-1/2 particle, in each of the corresponding three spatial directions.

The real linear span of $\{I, i\sigma_x, i\sigma_y, i\sigma_z\}$ is isomorphic to the real algebra of quaternions H. The isomorphism from H to this set is given by the following map:

$$1 \mapsto I, \quad i \mapsto -i\sigma_x, \quad j \mapsto -i\sigma_y, \quad k \mapsto -i\sigma_z. \tag{3}$$

Since any 2×2 complex Hermitian matrices can be expressed in terms of the identity matrix and the Pauli matrices, 2×2 mixed states, that is, 2×2 positive semidefinite matrices with trace one, can be represented by the Bloch sphere. This can be seen by simply first writing a Hermitian matrix as a real linear combination of $\{I, \sigma_x, \sigma_y, \sigma_z\}$, then imposing the positive semidefinite and trace one assumptions. Thus a density matrix can be written as $\rho = \frac{1}{2}(I + s\sigma)$, where σ is a vector of the Pauli matrices, and s is called the Bloch vector. For pure states, this provides a one-to-one mapping to the surface of the Bloch sphere, and for mixed states, the Bloch vector lies in the interior of the Bloch ball.

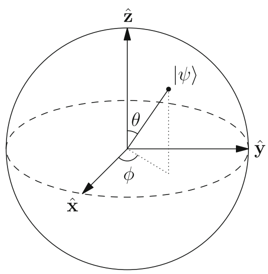
Given the mapping between the canonical formula and the quaternion group of order eight, and in turn, the isomorphism between the real algebra of quaternions and the Pauli basis, we arrive at a probabilistic interpretation of the canonical formula, with a geometry provided by the Bloch sphere. Antiautomorphisms become rotations of pure or mixed states. This far with no apparent upper limit to construct NF, the syntagm occurrences and co-occurrences these match will not be equiprobable, neither will be the 4th arguments following identical tripartite initial strings, which in turn yields the probabilistic raw material the Pauli basis refers to.

Associating elements of the canonical formula with the Pauli matrices has a further advantage. As pointed out in Sects 2.1 and 2.3, it is not necessarily obvious to give a well-cut interpretation of the CF, irrespective of whether we consider the weak or strong variants. Even versions of the same myth might elude interpretation. This is where a Pauli basis helps, where the weight of the basis correspond to probability values of the various components of the CF. Here we take probability values as a degree of belief, and we do not take a frequentist approach, although the latter might prove viable given a proper statistical analysis of myths and CF patterns. In pure states, the probability amplitudes must add to one, leading to a stricter, more formulaic reading of the CF. Mixed states, on the other hand, give full freedom in assigning probability values. In either case, weights might be chosen such that components of the CF are nullified. We believe that this mathematical description of the CF is more general than existing ones, and allows a lenient interpretation with a wider scope that may extend beyond myths.

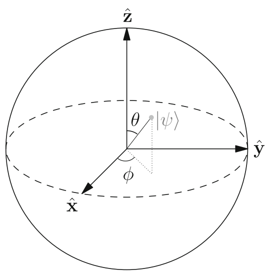
4 Applying the Probabilistic Description

The fundamental difficulty with myths is "belief contamination," also called eclecticism or syncretism, i.e. different concepts belonging to the same category (e.g. the dying deity) can appear in the same plot so that nobody can tell them apart. The other one is the fundamental insecurity of not knowing what factor may be important and how much of its manifestations can be out there. E.g., what is the probability that a text fragment is in state $F_x(a)$, or a whole text as a mix of $F_x(a) : F_y(b) \mapsto F_x(b) : F_{a^{-1}}(y)$ has a given outcome for $F_{a^{-1}}(y)$? A probabilistic tool which, based on scalable text variant scanning, can indicate mixed vs. pure conceptual states and thereby answer such questions is something sorely missed. This is where QI can help.

Given that CF_1-CF_{32} correspond to pure state vectors on the surface of a Bloch sphere (Fig. 1(a)), whereas mixed states of a text appear as vectors pointing inside of the sphere, we tested our working hypothesis on a small corpus



(a) A pure state.



(b) A mixed state.

Fig. 1. A pure state corresponds to a point on the surface of the Bloch sphere, whereas a mixed state is inside the Bloch sphere.

of thirteen texts from ancient Asia Minor, all concerned with Attis, a Phrygian dying god whose cult was imported to Rome as a consort of the Magna Mater, a variant of the Mediterranean Great Goddess. The plot is close to the Adonis myth: a youth either sacrifices his virility to the goddess or is punished by her to the same end. Out of the thirteen variants, in eight, Attis emasculates himself (direct self-mutilation); in one they mutually castrate each other with his partner (indirect self-mutilation); in two, he is either born as an eunuch or is killed by spear through an unspecified wound (indirect not-self mutiliation, i.e. killing by accident or similar); and in another two, it is the goddess who mutilates him (direct not-self mutilation). With $F_x(a)$ as the shift of the origin of the Bloch sphere standing for the beginning of the story, and the axes $F_y(b) = \hat{x}$, $F_x(b) = \hat{z}$, and $F_{a^{-1}}(y) = \hat{y}$, where the latter can have four outcomes as above, Fig. 1(b), shows the mixed state vector weighted by the outcome probabilities and the rest of the story alike.

5 Conclusions

We took a step toward bridging the gap between analytical studies in need of processing methodology vs. processing methodology development in need of raw material, by showing on a concrete example how a topical set of myth variants correspond via their syntagmatic transcripts to narrative formulae, i.e. formulaic expressions of condensed semantic content. The example came from fertility myths concerned with ritual punishment for wrongdoing as compensation underlying codified justice and community welfare regulation. We also demonstrated that there exist families of narrative formulae, some with double inverted values in their arguments, some without, which all share the same group structure with a certain quaternion group of order eight. Such formulae seem to be usable for information filtering. Beyond a group theoretic description, establishing a link to Pauli matrices, a quantum probabilistic framework further generalises the formulae.

References

1. Campbell, J.: The Hero with a Thousand Faces, vol. 17. New World Library, Novato (2008)
2. Klein, S.: The analogical foundations of creativity in language, culture & the arts: The Upper Paleolithic to 2100CE. In: McKevitt, P., Mulvihill, C., O'Nuallain, S., O'Riordan, C. (eds.) Proceedings of CSNLP-8, 8th International Workshop on the Cognitive Science of Natural Language Processing. Galway, Ireland (1999)
3. Lévi-Strauss, C.: The structural study of myth. J. Am. Folklore **68**, 428–444 (1955)
4. Scubla, L.: Hesiod, the three functions, and the canonical formula of myth. In: Maranda, P. (ed.) The Double Twist: From Ethnography to Morphodynamics, pp. 123–155. University of Toronto Press, Toronto (2001)
5. Maranda, P.: The Double Twist: From Ethnography to Morphodynamics, vol. 20. University of Toronto Press, Toronto (2001)

6. Pekonen, O.: The double twist: from ethnography to morphodynamics, edited by Pierre Maranda. the artist and the mathematician: the story of Nicolas Bourbaki, the genius mathematician who never existed by Amir D. Aczel. Math. Intell. **31**, 57–61 (2009)
7. Lévi-Strauss, C.: Mythologiques I-IV. Plon, Paris (1964–1971).
8. Morava, J.: On the canonical formula of C. Lévi-Strauss. arXiv:math/0306174 (2003).
9. Frazer, J.G.: Apollodorus - The Library, vol. 1–2. Harvard University Press, London (1921).
10. Burkert, W.: Structure and History in Greek Mythology and Ritual, vol. 47. University of California Press, Berkeley (1979)
11. Evelyn-White, H.G.: Hesiod, the Homeric Hymns, and Homerica. William Heinemann, London (1914)
12. Mosko, M.S.: The canonic formula of myth and nonmyth. Am. Ethnologist **18**, 126–151 (1991)
13. Leach, E.R.: Claude Lévi-Strauss. Viking Press, New York (1970)
14. Maranda, E.K., Maranda, P.: Structural Models in Folklore and Transformational Essays, vol. 10. Mouton, The Hague (1971)
15. Mosko, M.S., Damon, F.H.: On the Order of Chaos: Social Anthropology and the Science of Chaos. Berghahn Books, New York (2005)
16. Coté, A.: The set of canonical transformations implied in the canonical formula for the analysis of myth. In: Maranda, P. (ed.) The Double Twist: From Ethnography to Morphodynamics, pp. 199–221. University of Toronto Press, Toronto (2001)
17. Darányi, S.: Factor analysis and the canonical formula: where do we go from here? In: Darányi, S. (ed.) HOMO-03, Information Society, Cultural Heritage and Folklore Text Analysis, pp. 55–63. Budapest University of Technology and Economics, Budapest (2003)
18. Morava, J.: From Lévi-Strauss to chaos and complexity. In: Mosko, M., Damon, F.H. (eds.) On the Order of Chaos: Social Anthropology and the Science of Chaos, pp. 47–64. Berghahn Books, New York (2005)

Initial Specifications for the Design of Information Retrieval Systems Based on Quantum Detector Using Kinds

Massimo Melucci(✉)

University of Padua, Padova, Italy
massimo.melucci@unipd.it

Abstract. In our recent work, we investigated how to use quantum detectors (detectors) for improving the effectiveness of Information Retrieval (IR) systems. The implementation of these detectors in IR is still an open problem. In this paper, we give the initial specifications for the design of the systems based on detectors using kinds. In particular, a probabilistic model of the kinds is introduced, and the use of them in ranking by probability of relevance and in relevance feedback are illustrated.

1 Introduction

Information processing often requires the organization and description of data by means of structures; for example, a database is described by a schema and an unstructured document collection is described by an index. In particular, an IR system indexes the document collection and generates a set of index terms associated to lists of document identifiers called postings as illustrated in [1]. Indexing is necessary for making retrieval both efficient and effective. It is a matter of fact that indexing is actually the core methodology of an IR system, since what can be retrieved has necessarily been indexed, and the ranking of the documents retrieved by the system depends on the description of the data stored in the index.

IR systems optimize document ranking given that the data is accurately described and the probabilities of relevance are precisely estimated as possible as dictated by the Probability Ranking Principle (PRP) introduced in [8]. In [6] we wondered whether the ranking can be more effective than that dictated by the PRP – we found that the detectors can in principle provide a more effective ranking given the same data descriptions and probability estimations. However, the detectors are at the current stage only mathematical representations of these descriptors. This immateriality of the detectors is the main motivation for the research reported in this paper.

In this paper we give the initial specifications for the design of systems based on detectors using kinds, which were illustrated in [2] and later inserted into a general framework for IR based on Quantum Mechanics (QM) in [9, Chap. 2].

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 59–70, 2014.
DOI: 10.1007/978-3-642-54943-4_6,

Intuitively, a kind is the twofold representation of a concept – the set of instances of the concept on the one hand and its attributes on the other; for example two kinds might be $K_1 = (\{a, b, c\}, \{x, y\})$ and $K_2 = (\{c, d\}, \{y, z\})$ where a, b, c are all described by both x and y and both y and z describe both c and d. We equip the kinds with a probability space, thus connecting them to the detectors. We also suggest how to implement the kinds as they naturally fit an index generated by an IR system. To this end, Sect. 3 introduces some notions which are relevant to the kinds and to the following sections in addition to the basic definitions summarized in Sect. 2.1. Section 4 briefly illustrates how an IR system might use the kinds. Section 5 illustrates the probability space of the kinds, thus connecting to the detectors introduced in Sect. 2.2. Section 6 describes how the kinds might be used for ordering the documents by probability of relevance and for computing the kinds within a relevance feedback mechanism. Section 7 mentions the directions of the future work. As it will be clear in the remainder of the paper, the topics addressed imply some issues of computational complexity and human computer interaction, which are however out of the scope of this paper.

2 Background

2.1 Kinds

In this section we introduce kinds – further documentation has been given in [2] and [9]. Suppose T_* is a set of traits used for describing the individuals collected in A_*. Consider traits that represent individuals. A kind K is a pair (A, T) where A is a subset of individuals and T is a subset of traits.

Two functions are defined on a kind:

- for every A the function $\mathrm{tr}(A)$ returns the subset of traits that describes every individual of A;
- for every T the function $\mathrm{in}(T)$ returns the subset of individuals described by all the traits of T.

A kind is characterized by the fact that every individual in A instantiates every trait in T and no individual not in A instantiates every trait in T. We write

$$A = \mathrm{in}(T) \qquad T = \mathrm{tr}(A)$$

The kinds are provided with meet (disjunction) and join (conjunction) operations;

- meet is defined as $K_1 \vee K_2 = (\mathrm{in}(T_1 \cap T_2), T_1 \cap T_2)$
- join is defined as $K_1 \wedge K_2 = (A_1 \cap A_2, \mathrm{tr}(A_1 \cap A_2))$

The meet and join of any pair of kinds are kinds. Indeed, $\mathrm{tr}(\mathrm{in}(T_1 \cap T_2)) = T_1 \cap T_2$ and $\mathrm{in}(\mathrm{tr}(A_1 \cap A_2)) = A_1 \cap A_2$.

A partial order relation is defined as $K_1 \leq K_2$ if and only if $K_1 = K_1 \wedge K_2$.

There exists a minimum kind $\mathbf{0}$ and an maximum kind $\mathbf{1}$ such that $\mathbf{1} = (A_*, \emptyset) \geq K$ and $\mathbf{0} = (\emptyset, T_*) \leq K$ for every kind K. Indeed,

$$\begin{aligned} K \wedge \mathbf{0} &= (A \cap \emptyset, \mathrm{tr}(A \cap \emptyset)) = (\emptyset, \mathrm{tr}(\emptyset)) = (\emptyset, T_*) = \mathbf{0} \\ K \wedge \mathbf{1} &= (A \cap A_*, \mathrm{tr}(A \cap A_*)) = (A, \mathrm{tr}(A)) = K \\ K \vee \mathbf{0} &= (\mathrm{in}(T \cap T_*), T \cap T_*) = (\mathrm{in}(T), T) = (A, T) = K \\ K \vee \mathbf{1} &= (\mathrm{in}(T \cap \emptyset), T \cap \emptyset) = (\mathrm{in}(\emptyset), \emptyset) = (A_*, \emptyset) = \mathbf{1} \end{aligned}$$

The distributive law does not hold. Indeed three kinds can be chosen so that $K_1 \wedge (K_2 \vee K_3) \neq \mathbf{0}$ and $(K_1 \wedge K_2) \vee (K_1 \wedge K_3) = \mathbf{0}$. Suppose $A_1 \cap A_2 = \emptyset$ and $A_1 \cap A_3 = \emptyset$. It follows that $\mathrm{tr}(A_1 \cap A_2) = T$ and $\mathrm{tr}(A_1 \cap A_3) = T$ and therefore $K_1 \wedge K_2 = \mathbf{0}$ and $K_1 \wedge K_3 = \mathbf{0}$. It follows that $(K_1 \wedge K_2) \vee (K_1 \wedge K_3) = \mathbf{0}$ since $\mathbf{0} \vee \mathbf{0} = \mathbf{0}$. On the other hand, suppose $T_2 \cap T_3 = T_1$. It follows that $\mathrm{in}(T_2 \cap T_3) = \mathrm{in}(T_1)$ and thus $K_1 = K_2 \vee K_3 = K_1 \wedge (K_2 \vee K_3)$. Therefore, $K_1 \wedge (K_2 \vee K_3) \neq (K_1 \wedge K_2) \vee (K_1 \wedge K_3)$

One can also check that $(K_1 \wedge K_2) \vee (K_1 \wedge K_3) \leq K_1 \leq K_1 \wedge (K_2 \vee K_3)$ for every K_1, K_2, K_3.

2.2 Detection and Information Retrieval

This section summarizes what was in detail illustrated in [6]. Consider an index term occurring in the relevant documents with probability p_1 and in the non-relevant documents with probability p_0. Suppose $|\,1\rangle$ means that the index term occurs in a document and $|\,0\rangle$ does not. If "acceptance" means that the IR system detects a relevant document, the region of acceptance is one of the following subspaces: $\mathbf{0}$, $|\,0\rangle$, $|\,1\rangle, \mathbf{1}$. Given a real threshold $\lambda > 0$ and the mixed density matrices

$$\mu_0 = \begin{pmatrix} p_0 & 0 \\ 0 & 1 - p_0 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} p_1 & 0 \\ 0 & 1 - p_1 \end{pmatrix}$$

the optimal region of acceptance A is such that $\mathrm{trace}(\mu_1 A) > \lambda \mathrm{trace}(\mu_0 A)$ and is given by the positive eigenvectors of $\mu_1 - \lambda\mu_0$. The positive eigenvectors of $\mu_1 - \lambda\mu_0$ actually form the region of acceptance defined by the criterion illustrated in [7]. The calculation of the eigenvectors of $\mu_1 - \lambda\mu_0$ shows that the region of acceptance corresponds to: $|1\rangle\langle 1|$ if and only if $p_1 > \lambda p_0$; $|0\rangle\langle 0|$ if and only if $1 - p_1 > \lambda(1 - p_0)$; $|0\rangle\langle 0| + |1\rangle\langle 1| = \mathbf{1}$ if and only if $0 < \lambda < 1 \wedge \lambda p_0 < p_1 < 1 - \lambda + \lambda p_0$; $\mathbf{0}$ if and only if $\lambda > 1 \wedge 1 - \lambda + \lambda p_0 < p_1 < \lambda p_0$.

Given a threshold λ and the pure density matrices

$$\rho_0 = \begin{pmatrix} \sqrt{p_0} & \sqrt{p_0}\sqrt{1 - p_0} \\ \sqrt{p_0}\sqrt{1 - p_0} & \sqrt{1 - p_0} \end{pmatrix} \qquad \rho_1 = \begin{pmatrix} \sqrt{p_1} & \sqrt{p_1}\sqrt{1 - p_1} \\ \sqrt{p_1}\sqrt{1 - p_1} & \sqrt{1 - p_1} \end{pmatrix}$$

the optimal region of acceptance B is such that $\mathrm{trace}(\rho_1 B) > \lambda \mathrm{trace}(\rho_0 B)$ and is given by the positive eigenvectors of $\rho_1 - \lambda\rho_0$.

Following [3], the probability of correct detection (i.e. recall) $\mathrm{trace}(\rho_1 B)$ is not less than $\mathrm{trace}(\mu_1 A)$ for all α such that the probability of false alarms (i.e. fallout) is $\mathrm{trace}(\mu_0 A) = \mathrm{trace}(\rho_0 B) \leq \alpha$.

The region of acceptance B is called "quantum detector" and it is a one-dimensional subspace. This subspace is a superposition of A and $A^{\perp}$, the latter being the region of non-acceptance when the classical index term occurrence is at the basis of the formation of a region of acceptance. A and B represents incompatible observables and for this reason the distributive law does not hold, that is, $A \wedge (B \vee B^{\perp}) \neq A \wedge B \vee A \wedge B^{\perp}$. Indeed, A, B cannot be compatible because if they were compatible, B could be defined in terms of subspaces of A, thus contradicting the fact that A is optimal when the mixed density matrices are used in detection.

3 Kinds Again

The basic properties of the kinds are illustrated in Sect. 2.1; in this section, we introduce the disjoint kind, the complement kind, and an algebra of kinds useful for the purposes of this paper. Consider the following notions:

- The kind K_1 is disjoint from K_2 when $K_1 \wedge K_2 = \mathbf{0}$.
- The kind K_1 is orthogonal to K_2 when $A_1 \wedge A_2 = \emptyset$ and $T_1 \wedge T_2 = \emptyset$; note that if a kind is orthogonal, it is also disjoint.
- The complement of the kind K is the orthogonal kind $K^{\perp} = (A^{\perp}, T^{\perp})$ such that $K \vee K^{\perp} = \mathbf{1}$, thus implying that

$$A^{\perp} = A_* \setminus A \qquad T^{\perp} = T_* \setminus T$$

In IR there are two notable examples of kind:

- the posting list associated to an index term corresponds to a simple kind $(\mathrm{in}(\{t\}), \{t\})$ where the $\mathrm{in}(\{t\})$ is the set of the individuals indexed by t;
- the terms associated to a document and the document correspond to a simple kind $(\{a\}, \mathrm{tr}(\{a\}))$ where a is the document and $\mathrm{tr}(\{a\})$ is the set of the index terms occurring in the document.

In IR, a collection of documents and an index of terms associated to the documents result in a set of kinds depicted by the co-occurrence matrix (Fig. 1); the submatrices filled with non-null values are the kinds. Figure 2 illustrates the kinds implemented by an index of an IR system.

The set of kinds forms an algebra which is provided with meet, join, complement, minimum kind and maximum kind. This algebra meets the axioms described in [4], that is:

1. If K_1 is disjoint from K_2, then K_2 is disjoint from K_1 and $K_1 \vee K_2 = K_2 \vee K_1$.
2. $K \wedge \mathbf{0} = (A \cap \emptyset, \mathrm{tr}(A \cap \emptyset)) = \mathbf{0}$
3. $K \wedge K^{\perp} = (A \cap (A_* \setminus A), \mathrm{tr}(A \cap (A_* \setminus A))) = (\emptyset, \mathrm{tr}(\emptyset)) = (\emptyset, T_*) = \mathbf{0}$
4. $K \vee K^{\perp} = (\mathrm{in}(T \cap (T_* \setminus T)), T \cap (T_* \setminus T)) = (\mathrm{in}(\emptyset), \emptyset) = (A_*, \emptyset) = \mathbf{1}$
5. If K_1 is disjoint from $K_1^{\perp} \vee K_2$, then K_2 must be the minimum kind. Figure 3(a) illustrates what happens when K_2 is not the minimum kind.

1	1	1	0	0	0	1	1
1	1	1	0	0	0	1	1
1	1	1	0	0	0	1	1
0	1	1	1	0	0	1	1
0	1	1	1	0	0	1	1
0	1	1	1	0	0	1	1
0	1	1	1	0	0	1	1
0	1	1	1	1	1	1	1
0	1	1	1	1	1	1	1
0	0	0	1	1	1	0	0
0	0	0	1	1	1	0	0
0	0	0	1	1	1	0	0

Fig. 1. The index of an IR system is represented as a co-occurrence matrix whose rows correspond to documents (individuals), columns correspond to index terms (traits) and non-null entries ((a, t) where $a \in A$, $t \in T$) correspond to index term occurrence.

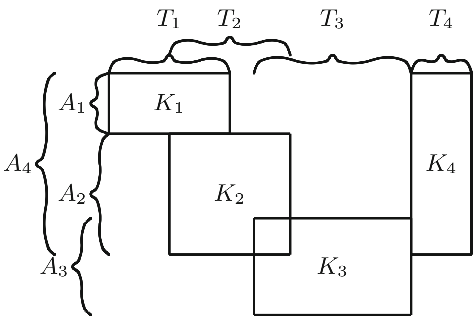


Fig. 2. A kind is a submatrix or block matrix whose entries are all non-null. The kinds may overlap. The simple kinds correspond either to the non-null entries of the column of the index term or the non-null entries of the row of the document.

6. If K_1 is disjoint from $K_1 \vee K_2$, then K_1 must be the minimum kind. Figure 3(b) illustrates what happens when K_1 is not the minimum kind.
7. When K_1, K_2 are mutually disjoint, both K_1 and $K_2^{\perp}$ are disjoint from $(K_1 \vee K_2)^{\perp}$. Figure 3(c) illustrates this property.

From these properties the following additional properties may be derived:

1. $\mathbf{0}^{\perp} = \mathbf{1}$
2. $\mathbf{1}^{\perp} = \mathbf{0}$
3. $K_1 \vee K_2 = K_1 \vee K_3$ only if $K_2 = K_3$
4. $K_1 \vee K_2 = \mathbf{1}$ only if $K_2 = K_1^{\perp}$

4 Kinds and Information Retrieval

When using index terms extracted from texts, retrieval can seamlessly utilize intersection or union of the posting lists, thus implementing classical detection.

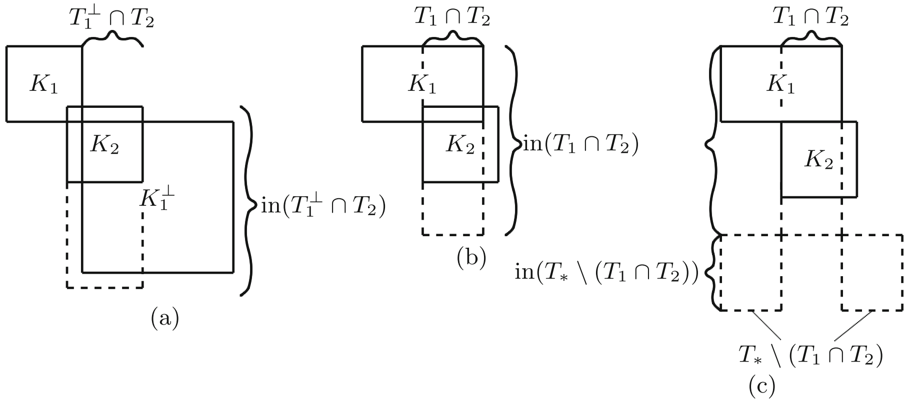


Fig. 3. In Fig. 3(a), if K_1 is disjoint from $K_1^\perp \vee K_2$, then K_2 must be the minimum kind. When K_2 is not the minimum kind, the intersection $T_1^\perp \cap T_2$ determines the set of individuals $\text{in}(T_1^\perp \cap T_2)$ that may include some individuals of A_1, thus making the intersection with A_1 not empty and then the join between K_1 and $K_1^\perp \vee K_2$ not null. In Fig. 3(b), if K_1 is disjoint from $K_1 \vee K_2$, then K_1 must be the minimum kind. When K_1 is not the minimum kind, the intersection $T_1 \cap T_2$ determines the set of individuals $\text{in}(T_1 \cap T_2)$ including some individuals of A_1, thus making the intersection with A_1 not empty and then the join between K_1 and $K_1 \vee K_2$ not null. In Fig. 3(c), when K_1, K_2 are mutually disjoint, both K_1 and $K_2^\perp$ are orthogonal to $(K_1 \vee K_2)^\perp$.

This set-based approach to retrieval and detection fits quite well with textual documents – the index terms are easily recognized and extracted from documents and an index term corresponds to a set of document identifiers stored in a posting list after indexing a document collection.

The main assumption underlying a set-based approach to indexing, retrieval and relevance detection is that an index term has a semantics and its occurrence in a document is meaningful to the end users. When authors are writing their own documents, say, a_1, a_2, a_3, a_4 using, say, four index terms, namely, t_1, t_2, t_3, t_4, they assume that aboutness of documents to index terms can be expressed through the classical logical operators. Suppose, for example, that a_1, a_2, a_3, a_4 are about $\{t_1, t_2\}$, $\{t_2, t_3\}$, $\{t_2, t_3, t_4\}$, $\{t_3, t_4\}$, respectively. According to the set-based approach to IR, the posting lists $t_1 = \{a_1\}$, $t_2 = \{a_1, a_2, a_3\}$, $t_3 = \{a_2, a_3, a_4\}$ and $t_4 = \{a_3, a_4\}$ are obtained. However, the end user can utilize the operators for expressing new concepts not explicitly thought of by the document authors; for example, $t_2 \cap t_3$.

In [9] it was pointed out that the use of a set-based approach to retrieval with non-textual documents is far more complex. When image, video or sound documents are to be indexed, the traits are not conveniently available as index terms and the assumption that the intersection or union of posting lists can express aboutness does not seem as intuitive as it is for text. The reason is that the language of non-textual traits is likely to describe individuals with a logic which will be different from a classical logic.

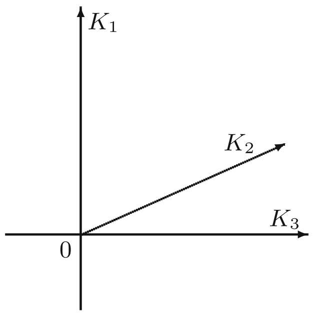


Fig. 4. Kinds and rays

In this paper we also point out that the use of a set-based approach to detection may be inadequate with image, video or sound documents, since the same reasons that make the retrieval of non-textual documents more complex than the retrieval of texts still hold when the regions of acceptance (of the hypothesis of relevance) have to be defined.

The complexity of dealing with non-textual documents and the different language provided by the kinds suggest that a set-based approach to retrieval and detection should be abandoned and a kind-based approach should be preferred.

The kinds exhibit some features similar to those of the detectors, thus suggesting that the kinds may eventually be represented by projectors, in particular:

- The operations intersection and union for sets do not necessarily correspond to the equivalent logical notions for kinds and for projectors. Suppose that a_1, a_2, a_3, a_4 are about $\{t_1, t_2\}$, $\{t_2, t_3\}$, $\{t_2, t_3, t_4\}$, $\{t_3, t_4\}$, respectively. We have the kinds $K_1 = (\{a_1\}, \{t_1, t_2\})$, $K_2 = (\{a_2, a_3\}, \{t_2, t_3\})$, $K_3 = (\{a_3, a_4\}, \{t_3, t_4\})$. The conjunction of index terms by intersection yields other results than the results yielded by the conjunction of kinds by join; for example, $T_1 \cap T_2 = \{a_1\}$ whereas $K_1 \wedge K_2 = \mathbf{0}$.
- The kinds violate the distributive law as the projectors $A, A^\perp$ and $B, B^\perp$ do when the meet and join operations are applied to the kinds. In a two-dimensional space a kind (e.g. K_1) is a ray and two disjoint kinds (e.g. K_2, K_3) are orthogonal rays as depicted by Fig. 4. The plane spanned by K_1 and K_3 is $K_1 \vee K_3$, and the intersection of the ray corresponding to K_1 and the ray corresponding to K_3 is $K_1 \wedge K_3$. The plane corresponding to $K_1 \vee K_3$ will be the entire two-dimensional plane, and thus $K_2 \wedge (K_1 \vee K_3) = K_2$, whereas geometrically $(K_2 \wedge K_1) \vee (K_2 \wedge K_3) = \mathbf{0} \vee (\{a_3\}, \{t_2, t_3, t_4\}) = (\{a_3\}, \{t_2, t_3, t_4\}) \neq K_2$, and the distribution law fails.

5 Probability of Kinds

The features common to kinds and detectors (e.g. the violation of the distributive law) are a necessary yet not sufficient condition that the kinds can be used for implementing the detectors illustrated in Sect. 2.2. As detection is based on probability, a probability measure is also necessary.

A probability function P maps a kind to the real interval $[0, 1]$ and meets the following properties:

- P is function of both A and T in order to exploit all the information provided by K;
- $P(K_1) \leq P(K_2)$ when $K_1 \leq K_2$;
- $P(\mathbf{0}) \leq P(K) \leq P(\mathbf{1})$, since $\mathbf{0} \leq K \leq \mathbf{1}$; therefore, $P(\mathbf{0}) = 0$ and $P(\mathbf{1}) = 1$;
- $\lim_{n\to\infty} P(K_1 \vee \cdots \vee K_n) = 1$.
- $\lim_{n\to\infty} P(K_1 \wedge \cdots \wedge K_n) = 0$.

In IR the probabilities are usually based on the information provided by the posting lists, which store information about both documents (individuals) and index terms (traits). A probability of an index term can be function of the number of documents indexed, while a probability of a document can be function of the index terms stored; indeed, these statistics are exploited by the most effective weighting schemes implemented by the search engines.

If one was induced to consider the area of the rectangles drawn in Fig. 2 as a measure of probability, $\sum_{a\in A}\sum_{t\in T} w(a,t)$ would be used where w is the weight function given to each occurrence of a trait t in an individual a; when $w = 1$ the function is the "volume" or "area" of the kind, that is $|A \times T| = |A||T|$ where $|\cdot|$ is the "volume" such as the cardinality of the set. However, this function does not meet the requirement that $P(\mathbf{1}) > P(\mathbf{0})$. Another probability function is then needed.

Consider a kind $K = (A, T)$ and let $s(T)$ be a natural number function of T. Consider A_* to be an urn of individuals and $P(A)$ to be the probability that an individual is in A. The number x of individuals in A in a sample of size $s(T)$ drawn with replacement from the urn is governed by a binomial probability distribution

$$\binom{s(T)}{x} P(A)^n (1 - P(A))^{s(T)-x}$$

The probability that $x = s(T)$ is exactly

$$P(K) = P(A)^{s(T)} \qquad 0 \leq P(A) \leq 1 \qquad 0 \leq s(T) \in \mathbb{N} \tag{1}$$

K is then the outcome of an experiment governed by a binomial distribution such that each draw is an individual of the kind. The join and the meet of two kinds are then the conjunction and the disjunction of the outcomes of two experiments on the urn of individuals.

This probability function is relevant to our purposes because:

- P is clearly function of both A and T;
- when $K_1 \leq K_2$, $P(A_1 \cap A_2)^{s(\mathrm{tr}(A_1\cap A_2))} \leq P(A_1)^{s(T_1)}$;
- $0 \leq P(K) \leq 1$;
- $P(\mathbf{1}) = P(A_*)^{s(\emptyset)} = P(A_*)^0 = 1$ and $P(\mathbf{0}) = P(\emptyset)^{s(T_*)} = 0^{s(T_*)} = 0$;
- $\lim_{n\to\infty} P(K_1 \vee \cdots \vee K_n) = P(\mathrm{in}(T_1 \cap \cdots \cap T_n))^{s(T_1\cap\cdots\cap T_n)} = 1$, since $T_1 \cap \cdots \cap T_n$ tends to the empty set and $\mathrm{in}(T_1 \cap \cdots \cap T_n)$ tends to A_*;

K_1	K_2	$P(K_1 \vee K_2)$	Interference
(A_1, T_1)	(A_2, T_2)	$P(\text{in}(T_1 \cap T_2))^{s(T_1 \cap T_2)}$	I
(A, T)	$(A^\perp, T^\perp)$	$P(\mathbf{1})$	0
$\mathbf{0}$	(A, T)	$P(K_2)$	0
$\mathbf{1}$	(A, T)	$P(\mathbf{1})$	0

Fig. 5. Probability and interference

– $\lim_{n\to\infty} P(K_1 \wedge \cdots \wedge K_n) = P(A_1 \cap \cdots \cap A_n)^{s(\text{tr}(A_1 \cap \cdots \cap A_n))} = 0$, since $A_1 \cap \cdots \cap A_n$ tends to the empty set and $\text{tr}(A_1 \cap \cdots \cap A_n)$ tends to T_*.

As regards probabilistic IR, the difference between this probability function and the probability function adopted by the Binary Independence Retrieval (BIR) model is that K denotes the individuals described by all traits in T whereas the BIR describes the individuals as both the traits occurring and those not occurring; for example, when there are three individuals and four traits and the following table of 0/1 elements denoting non-occurrence/occurrence of traits,

	t_1	t_2	t_3	t_4
a_1	1	1	1	0
a_2	1	1	0	1
a_3	1	1	0	1

$(\{a_1, a_2, a_3\}, \{t_1, t_2\})$ is a kind whereas $\{a_1\}$ and $\{a_2, a_3\}$ denote two distinct events. The probability of the kind is $P(\{a_1, a_2, a_3\})^{s(\{t_1,t_2\})}$ whereas the probability of the two events are $p_1p_2p_3(1 - p_4)$ and $p_1p_2(1 - p_3)p_4$, respectively.

Following the postulates presented in [5], P should be such that the probability of the conjunction of two orthogonal kinds is the sum of the probabilities of the kinds. Consider K and $K^\perp$, we have that $P(K \vee K^\perp) = P(\mathbf{1}) + P(\mathbf{0}) = 1 + 0 = 1$. However, this property does not hold for any pair of disjoint kinds; indeed, for $K_1 \wedge K_2 = \mathbf{0}$, we have that $P(K_1 \vee K_2) \neq P(K_1) + P(K_2)$.

The difference between $P(K_1 \vee K_2)$ and $P(K_1) + P(K_2)$ is due to the *superposition* of T_1 and T_2 which produces the *interference term*

$$I = P(K_1 \vee K_2) - P(K_1) - P(K_2)$$

which also exists when the kinds are disjoint and not orthogonal. Figure 5 summarizes the probabilities and the cases of null interference.

6 Ranking and Feedback Using Kinds

In this section, we introduce two notable applications of kinds in IR: ranking by probability of relevance (Sect. 6.1) and query expansion through feedback (Sect. 6.2).

6.1 Ranking Kinds According to the PRP

In this section, we briefly describe how to apply the PRP to kinds instead of to documents. Suppose a collection of kinds has been generated from a A_* and T_*; there are plenty of algorithms for computing the complete bipartite graphs obtained when connecting the individuals in A_* to the traits in T_*.

The fundamental lemma illustrated in [7] can be exploited to state that retrieving the kind K is a better decision than $K^\perp$ when

$$K = \arg_{K,K^\perp} \max P_1(K) - \lambda P_0(K) \quad \text{where} \quad P_0(K) < \alpha$$

provided that α is the maximum probability of false alarm. Following the PRP, which is derived from the lemma, the expected recall is maximum when the kinds are listed in order of $P_1(K)$ and a cut-off α is applied when the given probability of false alarm is reached.

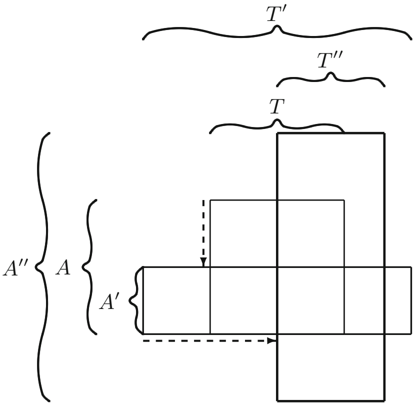
Consider the difference between BIR and kinds mentioned in Sect. 5. According to the PRP and the lemma of [7], the optimal ranking criterion operates upon events in the case of the BIR model, upon kinds in the case of this section, and upon detectors in the case illustrated in [6] and briefly described in Sect. 2.1. Therefore, the ranking criterion is always the same though the objects ranked might change from events, then to kinds, and finally to detectors. The crucial difference is that events commute whereas kinds and detector do not, and this is the difference that in principle make detector and kinds more effective than events in ranking documents by relevance.

When α varies from 0 to 1 at given intervals, and a collection of kinds has to be ranked, the criterion of [7] arranges the kinds in order of expected recall; the kinds $K_1, \dots, K_r$ selected up to rank $r = r(\alpha)$ give the maximum expected recall $P_1(K_1 \vee \cdots \vee K_r)$ provided that $P_0(K_1 \vee \cdots \vee K_r) < \alpha$. Note that the K_i's do not necessarily commute as the events of classical IR do.

From a practical point of view, the collection of kinds ordered by probability of correct detection (i.e. expected recall) should be presented to the end user. However, the user is used to receiving ordered lists of documents, and is not used to receiving kinds; indeed, the current search engines deliver ordered lists of WWW pages. It is convenient presenting the kinds to the end user by presenting the individuals (i.e. the documents) in order of the $P_1(K)$'s. This stratagem would allow the researcher to employ the widely adopted methodologies of laboratory-based experimental studies such as those based on test collections, which require lists of documents in order of measures (e.g. probability) of relevance. However, such a stratagem ignores the presentational issues of the ordered collections of kinds which are likely to require innovative approaches to the general problems encountered within the research in information access and seeking; although important, these issues are out of the scope of this paper.

6.2 Stretching Kinds and Query Expansion Through Feedback

Consider Sect. 2.2 and let $P_i(K)$ be the probability of K in state i where $i = 0, 1$ means non-relevance and relevance, respectively. In particular,



(a)

Require: K
1: converged $\leftarrow$ FALSE
2: **while** not converged **do**
3: $A' \leftarrow \text{distill}(A)$
4: $T' \leftarrow \text{tr}(A')$
5: $q'_0 \leftarrow P_0(A')^{s_0(T')}$
6: $q'_1 \leftarrow P_1(A')^{s_1(T')}$
7: $T'' \leftarrow \text{distill}(T')$
8: $A'' \leftarrow \text{in}(T'')$
9: $q''_0 \leftarrow P_0(A'')^{s_0(T'')}$
10: $q''_1 \leftarrow P_1(A'')^{s_1(T'')}$
11: **if** the criterion of convergence holds **then**
12: converged $\leftarrow$ TRUE
13: **else**
14: $A \leftarrow A''$
15: $T \leftarrow T''$
16: **end if**
17: **end while**
18: $H \leftarrow (A'', T'')$

(b)

Fig. 6. (a) Stretching kinds: A is rst withdrawn to obtain A', which is actually a subset of A; the resulting T' is a stretch of T; T' is then withdrawn to T'', thus stretching A' to obtain A''; at each step, the kinds evolve. (b) Algorithm for stretching kinds.

$P_i(K) = P_i(A)^{s_i(T)}$. The optimal region of acceptance in the case of mixed density matrices can be rewritten as the kind $K = (A, T)$ whereas the optimal region of acceptance in the case of pure density matrices can be rewritten as a new kind H – the problem is to compute H.

An approach to the problem of computing H is to take inspiration from the query expansion techniques used in IR. Suppose K has been computed by an IR system according to the PRP. From K, a series of "stretches" is performed as exemplified by Fig. 6(a) and formalized by the algorithm of Fig. 6(b). A subset A' of individuals is distilled from A (step 3). Document distillation can be implemented by any feedback technique used in IR; for instance, pseudo-relevance feedback techniques have been designed for distilling the candidate documents which store index terms useful for expanding the user's original query. In the case of the kinds, the index terms are given by $T' = \text{tr}(A')$ (step 4). The probabilities of correct detection and of false alarms of the new kind (A', T') are then computed at step 5-6. Similarly to document distillation, term distillation can be implemented by any term selection used when indexing documents (step 7); for example, document parsing, structure and markup processing, link analysis and information extraction techniques have been designed for distilling the candidate index terms useful for retrieving documents. In the case of the kinds, the documents are given by $A'' = \text{tr}(T'')$ (step 8). The probabilities of correct detection and of false alarms of the new kind (A'', T'') are then computed at step 9-10 and compared with the probabilities of (A', T') for testing convergence; in the case of no convergence, steps 3–10 are iterated after replacing K with (A'', T'') at step 14–15. The criterion of convergence is left to future work.

7 Future Work

The directions of future work are the following. The formal definition of the connection between the kinds and the detectors: it is our aim to find the projectors which represent both the kinds and the detectors. The calculation of the kinds requires the use of algorithms finding connected bipartite graphs: although this problem is well studied, a great deal of attention should be paid to the computational issues due to the presence of "big data". A series of experiments aiming at both comparing the retrieval effectiveness of an IR system using kinds with the effectiveness of the systems based on the classical retrieval models, and testing whether stretching the submatrices around the original submatrices would represent a meaningful way of doing query expansion. The presentational issues mentioned in Sect. 6.1: these issues are related both to displaying the kinds to the user and to using of the kinds as a language of interaction between the end user and the IR system.

References

1. Croft, W., Metzler, D., Strohman, T.: Search Engines: Information Retrieval in Practice. Addison Wesley, New York (2009)
2. Hardegree, G.M.: An approach to the logic of natural kinds. Pac. Philos. Q. **63**, 122–132 (1982)
3. Helstrom, C.: Quantum Detection and Estimation Theory. Academic Press, New York (1976)
4. Hughes, R.: The Structure and Interpretation of Quantum Mechanics. Harvard University Press, Cambridge (1989)
5. Kolmogorov, A.: Foundations of the Theory of Probability, II edn. Chelsea Publishing Company, New York (1956)
6. Melucci, M.: Deriving a quantum information retrieval basis. Comput. J. **6**(4–5), 257–405 (2012)
7. Neyman, J., Pearson, E.: On the problem of the most efficient tests of statistical hypotheses. Philos. Trans. R. Soc. Lond. Ser. A **231**, 289–337 (1933)
8. Robertson, S.: The probability ranking principle in information retrieval. J. Doc. **33**(4), 294–304 (1977)
9. van Rijsbergen, K.: The Geometry of Information Retrieval. Cambridge University Press, Cambridge (2004)

Meaning–Focused and Quantum–Inspired Information Retrieval

Diederik Aerts[1], Jan Broekaert[1], Sandro Sozzo[1(✉)], and Tomás Veloz[1,2]

[1] Center Leo Apostel (Clea), Vrije Universiteit Brussel (VUB), Pleinlaan 2, 1050 Brussel, Belgium
{diraerts,jbroekae,ssozzo}@vub.ac.be

[2] Department of Mathematics, University of British Columbia, Kelowna, BC, Canada
tomas.veloz@ubc.ca

Abstract. In recent years, quantum-based methods have promisingly integrated the traditional procedures in information retrieval (IR) and natural language processing (NLP). Inspired by our research on the identification and application of quantum structures in cognition, more specifically our work on the representation of concepts and their combinations, we put forward a 'quantum meaning based' framework for structured query retrieval in text corpora and standardized testing corpora. This scheme for IR rests on considering as basic notions, (i) 'entities of meaning', e.g., concepts and their combinations, and (ii) traces of such entities of meaning, which is how documents are considered in this approach. The meaning content of these 'entities of meaning' is reconstructed by solving an 'inverse problem' in the quantum formalism, consisting of reconstructing the full states of the entities of meaning from their collapsed states identified as traces in relevant documents. The advantages with respect to traditional approaches, such as Latent Semantic Analysis (LSA), are discussed by means of concrete examples.

Keywords: Information Retrieval · Latent Semantic Analysis · Quantum modeling · Concept theory

1 Introduction

Since the appearance of *The Geometry of Information Retrieval* [33], introducing a quantum structure approach to Information Retrieval (IR), Widdows and Peters [34], using a quantum logical negation for a concrete search system, and Aerts and Czachor [7], identifying quantum structure in semantic space theories, such as Latent Semantic Analysis (LSA) [18], the employment of techniques and procedures induced from the mathematical formalisms of quantum physics – Hilbert space, quantum logic and probability, non-commutative algebras, etc. – in fields such as IR and natural language processing (NLP), has produced a number of new and interesting results [21,28,30,35,36]. The latter can be placed within a growing quantum structure research in cognitive

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 71–83, 2014.
DOI: 10.1007/978-3-642-54943-4_7,

domains [8–10, 12–17, 23, 31, 32]. These quantum-based approaches mainly integrate and generalize the standard procedures in IR and NLP. Roughly speaking, one considers 'documents' and 'terms' as basic ingredients, concentrating on the so-called 'document-term matrix' which contains as entries the number of times that a specific term appears in a specific document. Both terms and documents are represented by vectors in a suitable (Euclidean) semantic space, and the scalar product between these vectors is a measure of the similarity of the corresponding documents and terms. This approach has extended to Latent Semantic Analysis (LSA) [18], Hyperspace Analogue to Language (HAL) [22], Probabilistic Latent Semantic Analysis (pLSA) [27], Latent Dirichlet Allocation (LDA) [11]. Several methods in word disambiguation, Information retrieval, question answering, etc., rely on the geometric properties of these linear semantic spaces [19]. Notwithstanding its success, the procedure meets some difficulties, including high computational costs and lack of incremental updates, which limits its applicability. Furthermore, one can claim the 'ad hoc' character of the procedure and, as a consequence, none of the examples mentioned above is immune to criticisms.

Inspired by a two-decade research on the identification and application of quantum structures in disciplines different from the micro-world [1–6, 8–10], we put forward in this paper the first steps leading to a possible conceptually new perspective for IR and NLP. In this approach, we want to replace terms by 'entities of meaning' as primary notions, which can be concepts or combinations of concepts. Such 'entities of meaning' can be in different states and change under the influence of the 'meaning landscape', or 'conceptual landscape', or 'conceptual context'. If we say 'pour out the water', and the meaning landscape is that of a flooded village after a heavy storm, the state of the entity of meaning which is 'pour out the water' is very different, from when the meaning landscape is that of a cafe where we are having some refreshments together. The explicit act of considering entities of meaning in states makes our approach fundamentally 'contextual'. Moreover, documents are not regarded as collection of words, but as traces, i.e. more concrete states, of these entities of meaning, or concepts, or combinations of concepts. This means that a document is considered to be a collapse of full states of different entities of meaning, each entity leaving a trace in the document. The words are only spots of these traces and they are not the main meaning carriers. The technical focus of our approach consists in trying to reconstruct there full states of the different entities of meaning from experiments that can only spot their traces, i.e. that can only look at words in documents. We believe that aspects of our quantum approach to cognition, still in full development, can help in formulating and making technically operational this 'inverse problem', consisting in 'reconstructing the the full states of the different entities of meaning', starting from their collapsed states as traces of word spots in documents.

We introduce in Sect. 2 the basic notions that are needed in our scheme for IR, that is, entities of meaning, which can be concept combinations, documents as traces of such entities of meaning and their technical reconstruction. We point out how our perspective is different from traditional approaches, e.g., LSA. We specify the sense in which the new paradigm is meaning focused. Moreover, the quantum-theoretic formalism we have recently developed to model concept combinations is a possible natural candidate to represent our meaning-based scheme. Indeed, the pair (entities of meaning, documents) is replaced by the pair (concepts/combinations, exemplars) in Sect. 3. Thus, the quantum modeling we have employed in the simple case where entities of meaning are concepts and documents are exemplars can be naturally used also in these more general IR cases. We stress that we have as yet no theory – but specific cases – to solve the functional inverse problem in an IR system, hence we only sketch the first steps for an approach on the theoretical level. Nevertheless, our quantum-inspired scheme is potentially more performant than LSA-based techniques. This is explicitly shown in Sect. 4, where a LSA analysis of Hampton's data on disjunctions of concepts is supplied and compared with our quantum cognition model in Sect. 3. We draw the conclusion that (i) LSA is only partially capable of capturing the effects of overextension and underextension in concept combinations, (ii) LSA supplies only approximate solutions, unlike our quantum modeling approach. This suggests that an application of our quantum and meaning-based approach would be more efficient than classical approaches both in text analysis and in information recovering.

2 Fundamentals of the Meaning-Based Approach

We present here the basics of our meaning-based approach for IR, explaining, in particular, its novelties with respect to the traditional IR and NLP procedures and justifying the use of the quantum-mathematical formalism in Sect. 3 in it.

LSA and its extensions typically use word-counting techniques in which the semantic structure of large bodies of text is incorporated into semantic linear spaces and the 'document-text matrix'. The latter contains as entries the number of times that a given term appears in a given document. If one labels the rows of this matrix by the documents and the columns by the terms, then each row can be viewed as a vector representing the corresponding document and each column as a vector representing the corresponding term. If vectors are normalized, their scalar product is a measure of the 'similarity' of the corresponding documents and terms, hence this data analysis can be used in IR and NLP. In the approach we put forward in this paper, instead, the semantic structure of texts is incorporated directly into concepts and their traces describing documents. This method is closer to the processes concretely working in the human mind, which explains why the quantum modeling in Sect. 3, which faithfully describes human collected data on concept combinations, can be applied to IR in a straightforward way. We stress, however, that the approach we propose is not yet worked out sufficiently to be applied to concrete IR problems, since many of the technical aspects of the inverse problem need to be specified and elaborated in different ways. Notwithstanding this, we will see that some interesting conclusions can already be drawn.

The first fundamental element of our approach is the conceptually new fact that 'terms' are replaced by 'entities of meaning' as basic elements, and specifically such entities of meaning can be 'concepts' or their combinations. This is why we work with 'entities of meaning', usually expressed as concept combinations, rather than with terms. Our procedure takes into account the meaning of the words from the very beginning, and there are valuable reasons to believe that this is how the human mind works. Indeed, whenever we read a text, we understand the 'meaning' of the text, and even have no efficient memory for the 'structure of the terms'. The substitution of terms with entities of meaning also allows us to use the quantum modeling formalism in Sect. 3 for representing concepts and their combinations.

The second basic element is the interpretation of 'pieces of texts' or, better, 'documents', as 'traces of these entities of meaning'. In this perspective, a document is not regarded as a combination of words but, rather, as collapsed states of the considered entities of meaning, or combinations of concepts. And, again, there are valuable reasons to believe that this view is an adequae representation of how the human mind operates. Indeed, if we consider the entity of meaning *The Cat Runs Through The Garden*, which is a combination of concepts, and consider a document telling about the adventures of a cat, a trace of *The Cat Runs Through The Garden* can be identified in this document, depending on the meaning content of the story about the cat. A weight can be identified representing the 'aboutness' of the meaning content of the document with respect to the entity of meaning *The Cat Runs Through The Garden*. And the document itself can be considered as a collapsed state of the full state of the entity of meaning *The Cat Runs Through The Garden*. This is exactly the structure that we can study by means of the quantum modeling formalism, for example what we explained in Sect. 3, namely, a document being a collapsed state of an entity of meaning after a measurement process (a cognitive test on subjects, a query on the web, etc.). The preceding insight has been inspired by what typically occurs in quantum experiments on microscopic particles, where one looks for traces of quantum particles. Whenever an experimental test is performed, a trace, or snapshot, of a quantum particle is left in a suitable apparatus, e.g., a Cloud chamber. A trace of this kind reveals a collapsed process of the quantum particle in the real physical space.

The third element of our approach is developing a technique to reconstruct the full states of the considered entities of meaning starting from weights that these full states contain with respect to their collapsed states, i.e. the documents. This is what in physics is called the 'inverse problem'. We observe that the human mind performs this inverse problem brilliantly, it indeed reconstructs the 'entities of meaning' of a document starting from what is written on the piece of paper, hence starting from – not the words – but the trace of these entities of meaning, or the collapsed states. And this is exactly also what quantum experimentalists and phenomenologists do, they recover the initial state of quantum particles starting from their collapsed states and outcome statistics of repeated experiments.

We stress that the inverse problem above can be technically very complicated, so that we have only given a conceptual description of it here. One could, for example, investigate the methods employed in quantum physics for state reconstruction and tomography, extending them to IR systems. In any case, our quantum cognition approach has already performed a complete reconstruction of the inverse problem in a Hampton's test of typicality [5,6]. Test subjects were asked to choose from a list of 24 exemplars the one that they estimated best represented the concepts *Fruits*, *Vegetables* and their disjunction *Fruits or Vegetables*. We elaborated a 25-dimensional complex Hilbert space which perfectly agreed with empirical data and allowed us to reconstruct and represent the initial states of the concepts. In this quantum model, the given concepts are the 'entities of meaning', while the exemplars, being more concrete states, or traces, of concepts in our approach, play the role of collapsed states of these conceptual entities of meaning. This suggests that a similar Hilbert space scheme can be envisaged where we replace 'exemplars' by 'paragraphs of texts' playing the role of documents, while 'concepts' are replaced by 'entities of meaning'. Of course, a quantum-mechanical model of this kind needs to be specified once a real experiment is performed on human subjects, but it already contains the genuine quantum structures that play a role in an IR process, such as collapse, contextuality, emergence, entanglement, interference and superposition.

3 Effectiveness of a Quantum Cognition Modeling in IR

It is well known that classical logical and probabilistic approaches fail when dealing with conceptual vagueness, the gradation of membership weights and concept combination (see, e.g., [24–26,29]). For this reason, we have recently worked out a quantum-theoretic approach for concept combination [3–6,10]. On the other hand, we have anticipated in Sect. 2 that this quantum cognition formalism is a natural candidate to represent our meaning based approach for IR. The first reason is that both approaches deal with concepts and their states, the second is that the meaning based approach for IR in Sect. 2 faithfully accords with a large collection of experimental data on human subjects on the combination of two concepts [24,25]. But there is a third and even stronger motivation for concretely using our quantum cognition approach in dealing with IR problems: it mathematically follows the same scheme of Sect. 2. Here, the role of 'entities of meaning' is played by the concepts and their disjunctions/conjunctions, while the role of 'documents' is played by the set of 'exemplars'. Indeed, the latter are more concrete states of concepts, hence they can be regarded as traces, or collapsed states, of these conceptual entities of meaning. This means that our quantum modeling scheme would work 'equally well' if we did the experiment with 'concepts, combinations of concepts' and 'real documents', with human subjects performing semantic estimations such as 'aboutness' or'topicality' of certain concepts with respect to a document. It is thus worth focusing on this quantum cognition approach and compare it with traditional approaches.

To model combinations of two concepts we need a Fock space $\mathcal{F}$ which consists of two sectors: 'sector 1' is a Hilbert space $\mathcal{H}$, while 'sector 2' is a tensor product Hilbert space $\mathcal{H} \otimes \mathcal{H}$, so that $\mathcal{F} = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})$. As a general consideration, sector 1 mainly enables modeling of interference connected phenomena, while sector 2 mainly enables modeling of entanglement connected phenomena. Let us consider the membership weights of exemplars of concepts and their conjunctions/disjunctions measured by Hampton [24,25]. He identified systematic deviations from classical (fuzzy set) conjunctions/disjunctions, an effect known as 'overextension' or 'underextension'. We concentrate on disjunctions here, which we will actually compare with LSA in Sect. 4. A completely similar analysis can be done for conjunctions [10]. It can be shown that a large part of Hampton's data cannot be modeled in a classical probability space satisfying Kolmogorov's axioms, due to the following theorem.

Theorem 1. *The membership weights $\mu_x(A), \mu_x(B)$ and $\mu_x(A \text{ or } B)$ of an exemplar x for the concepts A, B and A or B can be represented in a classical probability model if and only if the following two conditions are satisfied.*

$$\Delta_d = \max(\mu_x(A), \mu_x(B)) - \mu_x(A \text{ or } B) \leq 0 \tag{1}$$

$$0 \leq k_d = \mu_x(A) + \mu_x(B) - \mu_x(A \text{ or } B) \tag{2}$$

where Δ_d is the disjunction maximum rule deviation*, and k_d is the* Kolmogorovian disjunction factor*.*

Proof. See (Aerts, 2009), theorem 6.

Equation (1) expresses compatibility with the maximum rule for the conjunction of fuzzy set theory and, more generally, with monotonicity of classical Kolmogorovian probability. A situation with $\Delta_d > 0$ is called 'underextension' [25]. Equation (2) expresses instead compatibility with additivity of classical Kolmogorovian probability. Equations (1) and (2) together provide necessary and sufficient conditions to describe the experimental membership weights $\mu_x(A)$, $\mu_x(B)$ and $\mu_x(A \text{ or } B)$ in a Kolmogorovian probability space $(\Omega, \sigma(\Omega), P)$ $(\sigma(\Omega)$. In this case, indeed, events $P_A, P_B \in \sigma(\Omega)$ exist such that $P(E_A) = \mu_x(A)$, $P(E_B) = \mu_x(B)$, and $P(E_A \cup E_B) = \mu_x(A \text{ or } B)$.

Let us consider a specific example. Hampton estimated the membership weight of *Donkey* with respect to the concepts *Pet*, *Farmyard Animal* and their disjunction *Pet or Farmyard Animal* finding the values $\mu_{Donkey}(Pet) = 0.5$, $\mu_{Donkey}(Farmyard\ Animal) = 0.9$, $\mu_{Donkey}(Pet \text{ or } Farmyard\ Animal) = 0.7$. Thus, the exemplar *Donkey* presents underextension with respect to the disjunction *Pet or Farmyard Animal* of the concepts *Pet* and *Farmyard Animal*. We have in this case $\Delta_d = 0.2 \not\leq 0$, hence no classical probability representation exists for these data, because of Theorem 1. It can instead be proved that a quantum probability model in Fock space exists for these Hampton's data [25].

Theorem 2. *The membership weights $\mu_x(A), \mu_x(B)$ and $\mu_x(A \text{ or } B)$ of an exemplar x for the concepts A, B and A or B can be represented in a quantum probability model where*

$$\mu_x(A \text{ or } B) = m_x^2(\mu_x(A) + \mu_x(B) - \mu_x(A)\mu_x(B)) + n_x^2(\frac{\mu_x(A) + \mu_x(B)}{2} + Int_x(A, B)) \tag{3}$$

where the numbers m_x^2 *and* n_x^2 *are convex coefficients, i.e.* $0 \leq m_x^2, n_x^2 \leq 1$, $m_x^2 + n_x^2 = 1$, *and* θ_x *is the* interference angle *with*

$$Int_x(A,B) = \sqrt{1-\mu_x(A)}\sqrt{1-\mu_x(B)}\cos\theta_x \tag{4}$$

Proof. See Aerts, 2009 [10].

The term $\mu_x(A) + \mu_x(B) - \mu_x(A)\mu_x(B)$ is what one would expect for the disjunction in the case of classical probability. The term $Int_x(A,B))$ is instead the quantum interference term and it is responsible, together with the average $\frac{\mu_x(A)+\mu_x(B)}{2}$, of the deviations from classical expectations. The coefficients m_x^2 and n_x^2 measure the weights of sectors 2 and 1, respectively, of $\mathcal{F}$. For example, in the case of *Donkey* with respect to *Pet*, *Farmyard Animal* and *Pet or Farmyard Animal*, we have that Theorem 2 is satisfied with $m^2_{Donkey} = 0.26$, $n^2_{Donkey} = 0.74$ and $\theta_{Donkey} = 77.34°$.

Theorem 2 and its corresponding theorem for conjunctions – which we do not report, for the sake of brevity – contain the quantum probabilistic expressions allowing the modeling of a large amount of Hampton's data [24,25]. In particular, the quantum modeling above perfectly agrees with the data reported in Sect. 4 and compared with LSA data. We have also proposed an explanation for the fact that a quantum approach of this kind is so successful in modeling the large collection of data by Hampton. We have hypothesized a mechanism in which a genuine quantum effect comes into play, namely 'emergence'. Two processes, a logical one and a conceptual one, occur simultaneously in the human mind, and our quantum approach in Fock space enables both processes to be modeled.

We have seen above the deep reasons why our quantum cognition modeling can be successfully applied to IR. In Sect. 4, we will compare LSA and the quantum model modeling Hampton's data on concept combinations [24,25]. To conclude, we remark that, though our approach is conceptually different form standard IR approaches, such as LSA, it rests on similar basic ideas, that is, 'meaning is expressed in texts by the environment of a term'. We are convinced however that coherence, emergence and contextuality, and their quantum modeling can express meaning in a way that is similar to how meaning is captured by the human mind.

4 A Comparison with LSA

Before analyzing Hampton's data by means of LSA, it is worth dwelling on two aspects that allow one to better grasp the connections between LSA and our quantum cognition modeling in Sect. 3.

(i) LSA typically calculates 'similarity' through a complex technical procedure, which involves a real linear space representation of terms, a document-term matrix, a rank lowering through the reduction to a diagonal matrix by singular value decomposition that drops eigenvalues below a threshold, and thus can be considered as a semantic space construction/reduction technique. In this way, one captures the 'latent nature of similarity' within the studied corpus. The calculation results, though most probably correlated to 'similarity when tested on human subjects', do not express the latter directly. This entails that a LSA analysis of membership weights data, as well as a model for membership weights based on similarity [26], do make sense in this case.

(ii) A LSA process introduces cuts and approximations, so that it models experimental data only approximately. This is usually considered an advantage, at least with respect to an approach, like our quantum-theoretic modeling in Sects. 3 and 2, which can deliver models that 'fit data completely'. Indeed, since one usually maintains that experiments are not perfect, one is led to believe that approaches that model these data approximately have a bigger chance to be close to reality, than approaches that model these data perfectly. There is en error in the above reasoning linked to the difference between 'models that derive from a theory' and 'ad hoc models'. An ad hoc model is specifically made for a situation, and for such a model it would indeed be suspicious if it could fit data correctly. A model that derives from a theory, when fitting data correctly, does not constitute a problem. Indeed when for such a model slightly different data are to be fit, this is also possible by varying some of the parameters. The latter remark expresses a fundamental difference between our quantum modeling and LSA. In our modeling, 'also data that would be slightly different can again be perfectly fitted', which indicates that our models derive from a theory, i.e. quantum theory. This is 'not' true for LSA: there is 'no' corpus of texts that would fit, e.g., Hampton's data, as we see in what follows, which indicates that LSA is closer to an ad hoc way of model building.

We considered Hampton's membership weight data for the disjunction of eight pairs of concepts and 24 exemplars for each pair [10,25]. We computed the one-to-many similarity between the exemplars of the concepts and the concept, for each concept and their disjunction, considering the corpus *General reading up to 1st college (300 factors)* of the LSA Colorado website.[1] The LSA similarity website could not compute the similarity of one exemplar in three pairs of concepts, and of two exemplars on one pair of concepts, because these exemplars were not present in the corpus. Therefore, we computed the similarity of 187 exemplars within the eight pairs of concepts in total.

The aim of this analysis is twofold. Firstly, we test whether the LSA similarity between the exemplar of a concept and the term denoting the concept can be used to estimate Hampton's membership data. To this end, we compare the LSA similarity with Hampton's membership data, and we also verify whether or not a membership model based on similarity, the 'threshold model' [26], improves the LSA estimations. The threshold model is a simple model which assigns membership weight zero to the exemplars that are below a similarity threshold s_l, assigns membership weight one to the exemplars that are above a similarity threshold s_h, and using a parameter s_t builds a quadratic function to assign membership in the range $[s_l, s_h]$. Secondly, we identify the type of data that the LSA similarity (and the threshold model) delivers. To this aim, we compare the average number

[1] See the link http://lsa.colorado.edu/.

of exemplars that verify (or do not verify) Eqs. (1) and (2) in Hampton's data, in the LSA data, and in the threshold model. Whenever Eqs. (1) and (2) are satisfied, we call the exemplar of a 'classical' type. If Eq. (1) is not satisfied, we call the exemplar of a Δ_d type, and if Eq. (2) is not satisfied, we call the exemplar of a k_d type. Note that both inequalities cannot be violated simultaneously, and when any of the two inequalities is violated, the data cannot be modeled by a Kolmogorovian probability model (see Theorem 1).

Note that the LSA-similarity function can be negative. However, the membership function is assumed to be non-negative. We therefore set the negative similarities to be equal to zero. Indeed, this is not a significant modification to our data set because only 10 of the 187 tested instances deliver negative similarities, and none of these values are lower than −0.1.

For reasons of space, we cannot illustrate the performance of LSA and the threshold model in fitting Hampton's data for each concept. However, we illustrate that neither approach performs well. In Fig. 1, we compare the LSA and threshold models, using $s_l = 0.1, s_t = 0.5$, and $s_h = 0.9$, to the Hampton's data. The top row shows from left to right Hampton's data, the LSA similarity, and the threshold model data for the concepts *Home Furnishing* in black, *Furniture* in grey, and *Home Furnishing or Furniture* in dashed black. The exemplars of the concepts are on the x-axis, and the membership weights on the y-axis. It is clear that the range of values that both LSA and threshold model deliver are not close to the actual membership weights measured by Hampton. The second row shows the Pearson correlation between Hampton's data and the LSA model in the center, and the threshold model on the right. The x-axis identifies the concept pair, and the y-axis identifies the correlation found for each concept, and their combination, with respect to Hampton's data. The coloring of the curves is the same as in the first row: the first concept of the pair is plotted in black, the second concept in grey and their combination in dashed black. We see that there is not significant correlation for any concept in both models. We therefore conclude that LSA and the threshold model using LSA data deliver weak estimations of Hampton's data, in their values and co-variations (correlations).

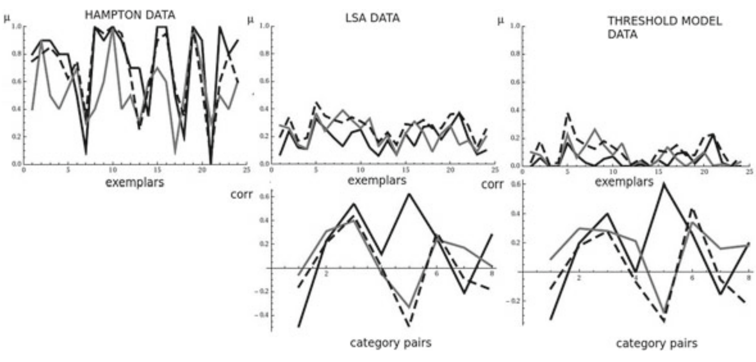


Fig. 1. Contrasting Hampton's data to LSA and Threshold models.

To compare the different types of Hampton's data with the LSA and threshold models, we compute a graph as follows: we define three nodes $\{C, D, K\}$, referring to the types 'classical', Δ_d, and k_d, respectively, and we build an edge $x \to y$ for each exemplar of each pair of concepts. The edge $x \to y$ indicates that the exemplar is of the type x for Hampton's data, while it is of the type y for the model we consider. For example, in the LSA data graph, the link $D \to C$ counts the number of exemplars that are of type Δ_d for Hampton's data and of 'classical' type for the LSA data. Self-loops indicate no type difference between the two data sets, and so on. We draw these graphs in Fig. 2. The LSA data graph is shown on the left, the threshold model data, with parameters $s_l = 0.1, s_t = 0.5$ and $s_h = 0.9$, is shown in the center, and the threshold model with parameters $s_l = 0.3, s_t = 0.5$ and $s_h = 0.7$, is shown on the right. Note that we consider all the exemplars of all concept pairs in this graph. In this sense we plot the average behavior of the models within the concepts. We do not show the detailed analysis for each pair of concepts, for reasons of space, but we mention that the tendencies we observe in the average case are also strong in the majority of the concept pairs. For the three models, the tendency of having the same type for a given exemplar follows the order C, D, K. Where C and D are much larger than K. Particularly for the threshold model, increasing the threshold region, i.e. increasing s_l and decreasing s_h, we observe that C becomes even larger than D, and K decreases to zero. In addition, there is a strong tendency for D exemplars in Hampton's data to become C elements in three models we consider. Moreover there is not a slight preference for the transition $D \to K$ and $K \to D$, or for the transition $C \to K$ and $K \to C$, except for the center plot, where we observe a significantly larger amount of transitions going from D to K than from K to D, and a significantly larger amount of transitions from C to K than from K to C. We infer that LSA and the threshold model using LSA similarity have a weak capacity to identify instances of the Δ_d type, and that they cannot discern properly among instances of the classical and k_d. However, they have a non-neglectible capacity to identify classical data.

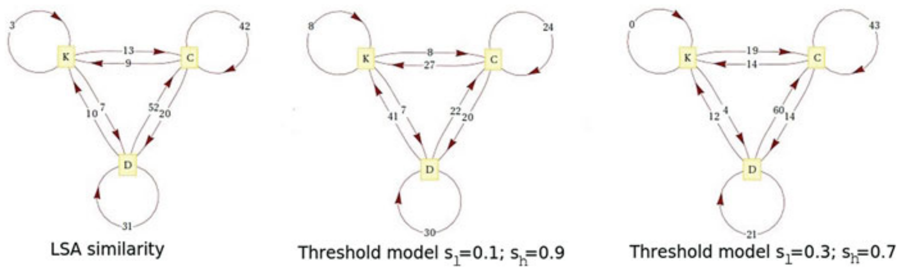


Fig. 2. Contrasting Hampton's data-type to LSA and Threshold models.

We can draw some general conclusions from the above analysis, as follows: (i) LSA and the threshold model feeded with LSA data does not agree to what

the human mind does in evaluating concept combinations. It might be that the specific of categories we considered were problematic, or that the corpus was not particularly suited for this task [20]. However we conclude that the 'bag of words' way of functioning of LSA, where terms and documents are not 'entities of meaning' and 'collapses of these entities of meaning' is not a well suited model to evaluate concept combinations.
(ii) LSA captures quite some of the non-classical aspects of underextension and overextension. This can be technically understood from the point of view of our Fock space modeling in Sect. 3, as follows.

(ii.a) Word vectors are summed, which implies that LSA works in sector 1 of our Fock space model. Actually, semantic spaces are real rather than complex linear spaces, but this is enough to introduce the possibility of interference. On the other hand, since 'only' sector 1 of Fock space is taken into account in LSA, the bag of words problem is present: no difference can be made between 'John hits Mary' and 'Mary hits John'. This ordering problem is avoided in our Fock space approach by also taking into account sector 2, which is a tensor product.

(ii.b) The exclusion in LSA of the smallest eigenvalues after diagonalization introduces 'latent values', i.e. weights become different from zero between terms and documents. By means of this simplification technique, LSA manages to grasp something that is closer to the 'states of the entities of meaning'. But it does so in a completely not understood way, as a by-product of a technique. This suggests that it should be possible to find more sophisticated techniques that directly and openly work toward the construction of 'states of the entities of meaning'. This is exactly what the meaning-based and quantum-inspired approach we put forward in the present paper aims at.

References

1. Aerts, D.: A possible explanation for the probabilities of quantum mechanics. J. Math. Phys. **27**, 202–210 (1986)
2. Aerts, D.: Foundations of quantum physics: a general realistic and operational approach. Int. J. Theor. Phys. **38**, 289–358 (1999)
3. Aerts, D., Sozzo, S.: Quantum structure in cognition: why and how concepts are entangled. In: Song, D., Melucci, M., Frommholz, I., Zhang, P., Wang, L., Arafat, S. (eds.) QI 2011. LNCS, vol. 7052, pp. 116–127. Springer, Heidelberg (2011)
4. Aerts, D., Broekaert, J., Gabora, L., Veloz, T.: The guppy effect as interference. In: Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.) QI 2012. LNCS, vol. 7620, pp. 36–47. Springer, Heidelberg (2012)
5. Aerts, D., Gabora, L., S. Sozzo, S.: Concepts and their dynamics: a quantum-theoretic modeling of human thought. Top. Cogn. Sci. (in print). ArXiv: 1206.1069 [cs.AI]
6. Aerts, D., Broekaert, J., Gabora, L., Sozzo, S.: Quantum structure and human thought. Behav. Bra. Sci. (in print)
7. Aerts, D., Czachor, M.: Quantum aspects of semantic analysis and symbolic artificial intelligence. J. Phys. A **37**, L123–L132 (2004)
8. Aerts, D., Aerts, S.: Applications of quantum statistics in psychological studies of decision processes. Found. Sci. **1**, 85–97 (1995)

9. Aerts, D., Gabora, L.: A theory of concepts and their combinations I & II. Kybernetes **34**, 167–191, 192–221 (2005)
10. Aerts, D.: Quantum structure in cognition. J. Math. Psychol. **53**, 314–348 (2009)
11. Blei, D.M., Ng, A.Y., Jordan, M.I.: Latent dirichlet allocation. J. Mach. Learn. Res. **3**, 993–1022 (2003)
12. Bruza, P.D., Lawless, W., van Rijsbergen, C.J., Sofge, D. (eds.): Proceedings of the AAAI Spring Symposium on Quantum Interaction. Stanford University, Stanford, 27–29 March 2007
13. Bruza, P.D., Lawless, W., van Rijsbergen, C.J., Sofge, D. (eds.): Quantum Interaction: Proceedings of the Second Quantum Interaction Symposium. College Publications, London (2008)
14. Bruza, P.D., Sofge, D., Lawless, W., van Rijsbergen, K., Klusch, M. (eds.): QI 2009. LNCS, vol. 5494. Springer, Heidelberg (2009)
15. Busemeyer, J.R., Pothos, E., Franco, R., Trueblood, J.S.: A quantum theoretical explanation for probability judgment 'errors'. Psychol. Rev. **118**, 193–218 (2011)
16. Busemeyer, J.R., Bruza, P.D.: Quantum Models of Cognition and Decision. Cambridge University Press, Cambridge (2012)
17. Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.): QI 2012. LNCS, vol. 7620. Springer, Heidelberg (2012)
18. Deerwester, S., Dumais, S.T., Furnas, G.W., Landauer, T.K., Harshman, R.: Indexing by latent semantic analysis. J. Am. Soc. Inf. Sci. **41**, 391–407 (1990)
19. Dumais, S.: Latent semantic analysis. Ann. Rev. Inform. Sci. Technol. **38**(1), 188–230 (2005)
20. Laham, D.: Latent semantic analysis approaches to categorization. In: Proceedings of the 19th Annual Conference of the Cognitive Science Society, p. 979 (1997)
21. Li, Y., Cunningham, H.: Geometric and quantum methods for information retrieval. ACM SIGIR Forum **42**, 22–32 (2008)
22. Lund, K., Burgess, C.: Producing high-dimensional semantic spaces from lexical co-occurrence. Behav. Res. Meth. Instr. Comp. **28**, 203–208 (1996)
23. Khrennikov, A.Y.: Ubiquitous Quantum Structure. Springer, Berlin (2010)
24. Hampton, J.A.: Overextension of conjunctive concepts: evidence for a unitary model for concept typicality and class inclusion. J. Exp. Psychol: Learn. Mem. Cogn. **14**, 12–32 (1988a)
25. Hampton, J.A.: Disjunction of natural concepts. Mem. Cogn. **16**, 579–591 (1988b)
26. Hampton, J.A.: Typicality, graded membership and vagueness. Cogn. Sci. **31**, 355–384 (2007)
27. Hofmann, T.: Probabilistic latent semantic indexing. In: Proceedings of the 22nd Annual International SIGIR Conference on Research and Development in Information Retrieval (1999)
28. Melucci, M.: A basis for information retrieval in context. ACM Trans. Inf. Sys. **26**, 1–41 (2008)
29. Osherson, D., Smith, E.: On the adequacy of prototype theory as a theory of concepts. Cognition **9**, 35–58 (1981)
30. Piwowarski, B., Frommholz, I., Lalmas, M., Van Rijsbergen, K.: What Can Quantum Theory Bring to IR. In: Huang, J., Koudas, N., Jones, G., Wu, X., Collins-Thompson, K., An, A. (eds.) Proceedings of the 19th ACM Conference on Information and Knowledge Management (2010)
31. Pothos, E.M., Busemeyer, J.R.: A quantum probability model explanation for violations of 'rational' decision theory. Proc. Roy. Soc. B **276**, 2171–2178 (2009)
32. Song, D., Melucci, M., Frommholz, I., Zhang, P., Wang, L., Arafat, S. (eds.): QI 2011. LNCS, vol. 7052. Springer, Heidelberg (2011)

33. Van Rijsbergen, K.: The Geometry of Information Retrieval. Cambridge University Press, Cambridge (2004)
34. Widdows, D., Peters, S.: Word Vectors and Quantum Logic: Experiments with Negation and Disjunction. In: Mathematics of Language 8, pp. 141–154. Bloomington, Indiana (2003)
35. Widdows, D.: Geometry and Meaning. CSLI Publications, University of Chicago Press, Illinois (2006)
36. Zuccon, G., Azzopardi, L.: Using the quantum probability ranking principle to rank interdependent documents. In: Gurrin, C., He, Y., Kazai, G., Kruschwitz, U., Little, S., Roelleke, T., Rüger, S., van Rijsbergen, K. (eds.) ECIR 2010. LNCS, vol. 5993, pp. 357–369. Springer, Heidelberg (2010)

Entanglement Zoo I: Foundational and Structural Aspects

Diederik Aerts and Sandro Sozzo(✉)

Center Leo Apostel (CLEA), Vrije Universiteit Brussel (VUB), Pleinlaan 2, 1050 Brussels, Belgium
{diraerts,ssozzo}@vub.ac.be

Abstract. We put forward a general classification for a structural description of the entanglement present in compound entities experimentally violating Bell's inequalities, making use of a new entanglement scheme that we developed in [1]. Our scheme, although different from the traditional one, is completely compatible with standard quantum theory, and enables quantum modeling in complex Hilbert space for different types of situations. Namely, situations where entangled states and product measurements appear ('customary quantum modeling'), and situations where states and measurements and evolutions between measurements are entangled ('nonlocal box modeling', 'nonlocal non-marginal box modeling'). The role played by Tsirelson's bound and marginal distribution law is emphasized. Specific quantum models are worked out in detail in complex Hilbert space within this new entanglement scheme.

Keywords: Quantum modeling · Bell's inequalities · Entanglement · Nonlocal boxes

1 Introduction

Entanglement is one of the most intriguing aspects of quantum physics. It is the feature that most neatly marked the departure from ordinary intuition and common sense, on which classical physics rest. The structural and conceptual novelties brought in by quantum entanglement were originally put forward by John Bell in 1964. He proved that, if one introduces 'reasonable assumptions for physical theories', one derives an inequality for the expectation values of coincidence measurements performed on compound entities ('Bell's inequality') which does not hold in quantum theory [2]. In quantum physics, entanglement is responsible for the violation of this inequality, which entails that quantum particles share statistical correlations that cannot be described in a single classical Kolmogorovian probability framework [3–5]. Another amazing observation was that entanglement, together with a number of other quantum features, such as 'contextuality', 'emergence', 'interference' and 'superposition', also appears outside the microscopic domain of quantum theory. These findings constituted the beginning of a systematic and promising search for quantum structures and

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 84–96, 2014.
DOI: 10.1007/978-3-642-54943-4_8,

the employment of quantum-based models in domains where classical structures prove to be problematical [6–18].

As for our own research, many years ago we already identified situations in macroscopic physics which violate Bell's inequalities [19–23]. One of these macroscopic examples, the 'connected vessels of water', exhibits even a maximal possible violation of Bell's inequalities, i.e. more than the typical entangled spin example in quantum physics. More recently, we performed a cognitive experiment showing that a specific combination of concepts, *The Animal Acts*, violates Bell's inequalities [24–26]. These two situations present deep structural and conceptual analogies which we analyze systematically in Ref. [1,27].

In the present paper, we put forward a classification that enables us to represent experimental situations of compound entities which violate Bell's inequalities identifying the quantum-theoretic modeling involved in these violations. We show that a complete quantum-mechanical representation can be worked out, and we prove that quantum entanglement not only appears on the level of the states, but also on the level of the measurements. Indeed, we show that the empirical data we collected on *The Animal Acts* (Sect. 2), as well as the situation of the 'connected vessels of water' [27], can be modeled only when both states and measurements are entangled. The existence of a quantum model for the 'connected vessels of water' was not a priori expected and constitutes an original result. Our modeling scheme, although completely compatible with standard quantum theory, is more general than the traditional one, because within this traditional scheme certain ways in which subentities can be part of compound entities have been overlooked. It is when the marginal probability law is violated that this shortcoming of the traditional entanglement scheme comes on the surface, and hence some of the entanglement situations that we consider in the present paper would not be possible to be modeled within the traditional entanglement scheme.

Our classification gives rise to the following different types of situations and entities: **Type 1**: Situations where Bell's inequalities are violated within 'Tsirelson's bound' [28] and the marginal distribution law holds ('customary quantum modeling'), (Sect. 3); **Type 2**: Situations where Bell's inequalities are violated within Tsirelson's bound and the marginal distribution law is violated ('nonlocal non-marginal box modeling 1'), (Sect. 3). We recall that situations of type 2 seem to be present in 'real quantum spin experiments'. A reference to the 'experimental anomaly' that, in our opinion, indicates the presence of entangled measurements, occurs already in Alain Aspects PhD thesis [29,30]. Our framework accommodates these situations too; **Type 3**: Situations where Bell's inequalities are violated beyond Tsirelson's bound and the marginal distribution law is violated ('nonlocal non-marginal box modeling 2'), (Sect. 3); **Type 4**: Situations where Bell's inequalities are violated beyond Tsirelson's bound and the marginal distribution law holds ('nonlocal box modeling'), (Sect. 4).

Additionally to introducing the framework, we analyze in this paper the hypothesis that 'satisfying the marginal distribution law' is merely a consequence of extra symmetry being present in situations that contain full-type entanglement, e.g., situations of types 2 and 3. Whenever enough symmetry is present, such that all the entanglement of the situation can be pushed into the state, allowing a model with product measurements, and product unitary transformations, the marginal law is satisfied. We give two examples, a cognitive 'gedanken experiment' violating Bell's inequalities, which is a 'variation adding more symmetry' to an example that was introduced in Ref. [23], and in this variation the marginal law is satisfied. We introduce in a similar way extra symmetry in our 'vessels of water' example, to come to a variation where the marginal law is satisfied. Both examples are isomorphic and realizations of the so-called 'nonlocal box', which is studied as a purely theoretical construct – no physical realizations were found prior to the ones we present here – in the foundations of quantum theory [31].

Let us state clearly, to avoid misunderstandings, that we use the naming 'entanglement' referring explicitly to the structure within the theory of quantum physics that a modeling of experimental data take, if (i) these data are represented, following carefully the rules of standard quantum theory, in a complex Hilbert space, and hence states, measurements, and evolutions, are presented respectively by vectors (or density operators), self-adjoint operators, and unitary operators in this Hilbert space; (ii) a situation of coincidence joint measurement on a compound entity is considered, and the subentities are identified following the tensor product rule of 'compound entity description in quantum theory' (iii) within this tensor product description of the compound entity entanglement is identified, as 'not being product', whether it is for states (non- product vectors), measurements (non-product self-adjoint operators), or transformations (non-product unitary transformations).

2 Technical Aspects of Modeling Quantum Entanglement

To develop our new scheme for the study of entanglement, we will first introduce some basic notions and results, which we developed in [1] in detail. The more general nature of our scheme, as compared to the standard one, is that we carefully analyse the different ways in which two entities can be subentities of a compound entity. Indeed, entanglement depends crucially on these different possible way of 'being a subentity', and this has not been recognised sufficiently in the standard scheme. Let us also remark explicitly that, although our scheme is more general than the standard one, it is completely compatible with standard quantum theory. Hence, the limitations and simplifications as compared to our scheme of the standard one are only linked to an overlooking of the more subtle ways in which subentities can place themselves within a compound entity in situations described by quantum theory.

First we introduce the notions of 'product state', 'product measurement' and 'product dynamical evolution' as we will use it in our new entanglement scheme. For this we consider the general form of an isomorphism $I : \mathbb{C}^4 \rightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$, by linking the elements of an ON basis $\{|x_1\rangle, |x_2\rangle, |x_3\rangle, |x_4\rangle\}$ of $\mathbb{C}^4$ to the elements $\{|c_1\rangle \otimes |d_1\rangle, |c_1\rangle \otimes |d_2\rangle, |c_2\rangle \otimes |d_1\rangle, |c_2\rangle \otimes |d_2\rangle\}$ of the type of ON basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$

where $\{|c_1\rangle, |c_2\rangle\}$ and $\{|d_1\rangle, |d_2\rangle\}$ are ON bases of $\mathbb{C}^2$ each

$$I|x_1\rangle = |c_1\rangle \otimes |d_1\rangle,\ I|x_2\rangle = |c_1\rangle \otimes |d_2\rangle,\ I|x_3\rangle = |c_2\rangle \otimes |d_1\rangle,\ I|x_4\rangle = |c_2\rangle \otimes |d_2\rangle \quad (1)$$

Definition 1. *A state p represented by the unit vector $|p\rangle \in \mathbb{C}^4$ is a 'product state', with respect to I, if there exists two states p_a and p_b, represented by the unit vectors $|p_a\rangle \in \mathbb{C}^2$ and $|p_b\rangle \in \mathbb{C}^2$, respectively, such that $I|p\rangle = |p_a\rangle \otimes |p_b\rangle$. Otherwise, p is an 'entangled state' with respect to I.*

Definition 2. *A measurement e represented by a self-adjoint operator $\mathcal{E}$ in $\mathbb{C}^4$ is a 'product measurement', with respect to I, if there exists measurements e_a and e_b, represented by the self-adjoint operators $\mathcal{E}_a$ and $\mathcal{E}_b$, respectively, in $\mathbb{C}^2$ such that $I\mathcal{E}I^{-1} = \mathcal{E}_a \otimes \mathcal{E}_b$. Otherwise, e is an 'entangled measurement' with respect to I.*

Definition 3. *A dynamical evolution u represented by a unitary operator $\mathcal{U}$ in $\mathbb{C}^4$ is a 'product evolution', with respect to I, if there exists dynamical evolutions u_a and u_b, represented by the unitary operators $\mathcal{U}_a$ and $\mathcal{U}_b$, respectively, in $\mathbb{C}^2$ such that $I\mathcal{U}I^{-1} = \mathcal{U}_a \otimes \mathcal{U}_b$. Otherwise, u is an 'entangled evolution' with respect to I.*

Remark that the notion of product states, measurements and evolutions, are defined with respect to the considered isomorphism between $I : \mathbb{C}^4 \rightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$, which expresses already the new aspect of our entanglement scheme, making entanglement depending on 'how sub entities are part of the compound entity'. The following theorems can then be proved.

Theorem 1. *The spectral family of a self-adjoint operator $\mathcal{E} = I^{-1}\mathcal{E}_a \otimes \mathcal{E}_b I$ representing a product measurement with respect to I, has the form $\{I^{-1}|a_1\rangle\langle a_1| \otimes |b_1\rangle\langle b_1|I,\ I^{-1}|a_1\rangle\langle a_1| \otimes |b_2\rangle\langle b_2|I,\ I^{-1}|a_2\rangle\langle a_2| \otimes |b_1\rangle\langle b_1|I, I^{-1}|a_2\rangle\langle a_2| \otimes |b_2\rangle\langle b_2|I\}$, where $\{|a_1\rangle\langle a_1|, |a_2\rangle\langle a_2|\}$ is a spectral family of $\mathcal{E}_a$ and $\{|b_1\rangle\langle b_1|, |b_2\rangle\langle b_2|\}$ is a spectral family of $\mathcal{E}_b$.*

Theorem 1 shows that the spectral family of a product measurement is made up of product orthogonal projection operators.

Theorem 2. *Let p be a product state represented by the vector $|p\rangle = I^{-1}|p_a\rangle \otimes |p_b\rangle$ with respect to the isomorphism I, and e a product measurement represented by the self-adjoint operator $\mathcal{E} = I\mathcal{E}_a \otimes \mathcal{E}_b I^{-1}$ with respect to the same I. Let $\{|y_1\rangle, |y_2\rangle, |y_3\rangle, |y_4\rangle\}$ be the ON basis of eigenvectors of $\mathcal{E}$, and $\{|a_1\rangle, |a_2\rangle\}$ and $\{|b_1\rangle, |b_2\rangle\}$ the ON bases of eigenvectors of $\mathcal{E}_a$ and $\mathcal{E}_b$ respectively. Then, we have $p(A_1) + p(A_2) = p(B_1) + p(B_2) = 1$, and $p(Y_1) = p(A_1)p(B_1)$, $p(Y_2) = p(A_1)p(B_2)$, $p(Y_3) = p(A_2)p(B_1)$ and $p(Y_4) = p(A_2)p(B_2)$, where $\{p(Y_1), p(Y_2), p(Y_3), p(Y_4)\}$ are the probabilities to collapse to states $\{|y_1\rangle, |y_2\rangle, |y_3\rangle, |y_4\rangle\}$, and $\{p(A_1), p(A_2)\}$ and $\{p(B_1), p(B_2)\}$ are the probabilities to collapse to states $\{|a_1\rangle, |a_2\rangle\}$ and $\{|b_1\rangle, |b_2\rangle\}$ respectively.*

With this theorem we prove that if there exists an isomorphism I between $\mathbb{C}^4$ and $\mathbb{C}^2 \otimes \mathbb{C}^2$ such that state and measurement are both product with respect to this isomorphism, then the probabilities factorize. A consequence is that in case the probabilities do not factorize the theorem is not satisfied. This means that there does not exist an isomorphism between $\mathbb{C}^4$ and $\mathbb{C}^2 \otimes \mathbb{C}^2$ such that both state and measurement are product with respect to this isomorphism, and there is genuine entanglement. The above theorem however does not yet prove where this entanglement is located, and how it is structured. The next theorems tell us more about this.

We consider now the coincidence measurements AB, AB', $A'B$ and $A'B'$ from a typical Bell-type experimental setting. For each measurement we consider the ON bases of its eigenvectors in $\mathbb{C}^4$. For the measurement AB this gives rise to the unit vectors $\{|ab_{11}\rangle, |ab_{12}\rangle, |ab_{21}\rangle, |ab_{22}\rangle\}$, for AB' to the vectors $\{|ab'_{11}\rangle, |ab'_{12}\rangle, |ab'_{21}\rangle, |ab'_{22}\rangle\}$, for $A'B$ to the unit vectors $\{|a'b_{11}\rangle, |a'b_{12}\rangle, |a'b_{21}\rangle, |a'b_{22}\rangle\}$ and for $A'B'$ to the vectors $\{|a'b'_{11}\rangle, |a'b'_{12}\rangle, |a'b'_{21}\rangle, |a'b'_{22}\rangle\}$. We introduce the dynamical evolutions $u_{AB'AB}, \ldots$, represented by the unitary operators $\mathcal{U}_{AB'AB}, \ldots$, connecting the different coincidence experiments for any combination of them, i.e. $\mathcal{U}_{AB'AB} : \mathbb{C}^4 \rightarrow \mathbb{C}^4$, such that

$$|ab_{11}\rangle \mapsto |ab'_{11}\rangle, |ab_{12}\rangle \mapsto |ab'_{12}\rangle, |ab_{21}\rangle \mapsto |ab'_{21}\rangle, |ab_{22}\rangle \mapsto |ab'_{22}\rangle \quad (2)$$

Theorem 3. *There exists a isomorphism between* $\mathbb{C}^4$ *and* $\mathbb{C}^2 \otimes \mathbb{C}^2$ *with respect to which both measurements* AB *and* AB' *are product measurements iff there exists an isomorphism between* $\mathbb{C}^4$ *and* $\mathbb{C}^2 \otimes \mathbb{C}^2$ *with respect to which the dynamical evolution* $u_{AB'AB}$ *is a product evolution and one of the measurements is a product measurement. In this case the marginal law is satisfied for the probabilities connected to these measurements, i.e.* $p(A_1, B_1) + p(A_1, B_2) = p(A_1, B'_1) + p(A_1, B'_2)$.

The above theorem introduces an essential deviation of the customary entanglement scheme, which we had to consider as a consequence of our experimental data on the concept combination *The Animal Acts*. Indeed, considering our description of the situation in Sect. 3, we have $P(A_1, B_1) + p(A_1, B_2) = 0.679 \neq 0.618 = p(A_1, B'_1) + p(A_1, B'_2)$, which shows that the marginal law is not satisfied for our data. Hence, for the our experimental data on *The Animal Acts* there does not exist an isomorphism between $\mathbb{C}^4$ and $\mathbb{C}^2 \otimes \mathbb{C}^2$, such that with respect to this isomorphism all measurements that we performed in our experiment can be considered to be product measurements. It right away shows that we will not able to model our data within the customary entanglement scheme. We could have expected this, since indeed, in this customary scheme all considered measurements are product measurements, and entanglement only appears in the state of the compound entity. We refer to Ref. [1] for proof of Theorems. 1–3.

Let us summarise the structural situation. Entanglement is a property attributed to states, measurements, or unitary transformations, when looked at the tensor product identification (isomorphism) with the Hilbert space describing the compound entity. The 'physics' of the compound entity is expressed in this one Hilbert space describing directly the compound entity, which makes entanglement itself dependent on 'the way in which subentities of the compound entity are attempted to be identified'. For one state and one compound measurements, the identification between tensor product and compound entity Hilbert space can always be chosen such that the measurement appears as a product, and all the entanglement is pushed in the state. Theorem 3 shows that, whenever the marginal distribution law is violated, this can no longer be achieved, and entanglement is also present in measurements and the dynamical transformations connecting these measurements. In [1] we show that in case different isomorphisms of identification are considered, an entanglement scheme with again product measurements and product dynamical transformations is possible. But the price to pay is that the entangled state cannot be presented any longer in a unique way within the tensor product space, i.e. a different representation is needed for each coincidence experiment context. All this of course related to the marginal law for the probabilities connected to these different coincidence measurements not being satisfied. A direct consequence of the above is that, if a set of experimental data violate both Bell's inequalities and the marginal distribution law, it is impossible to work out a quantum-mechanical representation in a fixed Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ which satisfies the data and where only the initial state is entangled while all measurements are products. We will make this more explicit in the next sections.

3 Examples of Systems Entailing Entanglement

The first example we shortly present is that of a macroscopic entity violating Bell's inequalities in exactly the same way as a pair of spin-1/2 quantum particles in the singlet spin state when faraway spin measurements are performed [22]. We only sketch this example here to make our zoo collection as complete as possible, and refer to [22,23] for a detailed presentation.

This mechanical entity simulates the singlet spin state of a pair of spin-1/2 quantum particles by means of two point particles P_1 and P_2 initially located in the centers C_1 and C_2 of two separate unit spheres B_1 and B_2, respectively. The centers C_1 and C_2 remain connected by a rigid but extendable rod, which introduces correlations. We denote this state of the overall entity by p_s. A measurement $A(a)$ is performed on P_1 which consists in installing a piece of elastic of 2 units of length between the diametrically opposite points $-a$ and $+a$ of B_1. At one point, the elastic breaks somewhere and P_1 is drawn toward either $+a$ (outcome $\lambda_{A_1} = +1$) or $-a$ (outcome $\lambda_{A_2} = -1$). Due to the connection, P_2 is drawn toward the opposite side of B_2 as compared to P_1. Now, an analogous measurement $B(b)$ is performed on P_2 which consists in installing a piece of elastic of 2 units of length between the two diametrically opposite points $-b$ and $+b$ of B_2. The particle P_2 falls onto the elastic following the orthogonal path and sticks there. Next the elastic breaks somewhere and drags P_2 toward either $+b$ (outcome $\lambda_{B_1} = +1$) or $-b$ (outcome $\lambda_{B_2} = -1$). To calculate the transition probabilities, we assume there is a uniform probability of breaking on the elastics. The single and coincidence probabilities coincide with the standard probabilities for spin-1/2 quantum particles in the singlet spin state when spin measurements are performed along directions a and b. In particular, the probabilities for the coincidence counts $\lambda_{A_1B_1} = \lambda_{A_2B_2} = +1$ and $\lambda_{A_1B_2} = \lambda_{A_2B_1} = -1$ of the joint measurement $AB(a,b)$ in the state p_s are given by

$$p(p_s, AB(a,b), \lambda_{A_1B_1}) = p(p_s, AB(a,b), \lambda_{A_2B_2}) = \frac{1}{2}\sin^2\frac{\gamma}{2} \quad (3)$$

$$p(p_s, AB(a,b), \lambda_{A_1B_2}) = p(p_s, AB(a,b), \lambda_{A_2B_1}) = \frac{1}{2}\cos^2\frac{\gamma}{2} \quad (4)$$

respectively, where γ is the angle between a and b, in exact accordance with the quantum-mechanical predictions. Furthermore, this model leads to the same violation of Bell's inequalities as standard quantum theory. Hence, the 'connected spheres model' is structurally isomorphic to a standard quantum entity. This means that it can be represented in the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ in such a way that its initial state is the singlet spin, i.e. a maximally entangled state, and the measurements are products. Furthermore, the marginal distribution law holds and Bell's inequalities are violated within the Tsirelson's bound $2\sqrt{2}$, hence the connected spheres model is an example of a 'customary identified standard quantum modeling' in our theoretical framework.

The presence of entanglement in concept combination has recently also been identified in a cognitive test [24–26] and subsequently improved by elaborating a quantum Hilbert space modeling of it [1,27]. We analyze it in the light of our new entanglement scheme exposed in Sect. 2. For a detailed description of the conceptual entity, and the measurements considered, we refer to [27], Sec. 2.1, or [1]. We consider the typical Bell inequality situation of four coincidence measurements AB, AB', $A'B$ and $A'B'$, performed on the sentence *The Animal Acts* as a conceptual combination of the concepts *Animal* and *Acts*. Measurements consists of asking participants in the experiment to answer the question whether a given exemplar 'is a good example' of the considered concept or conceptual combination.

We had 81 subjects participating in our experiment. If we denote by $p(A_1, B_1)$, $p(A_1, B_2)$, $p(A_2, B_1)$, $p(A_2, B_2)$ the probabilities for the different Bell-type situation choices, we find $p(A_1, B_1) = 0.049$, $p(A_1, B_2) = 0.630$, $p(A_2, B_1) = 0.259$, $p(A_2, B_2) = 0.062$, $p(A_1, B'_1) = 0.593$, $p(A_1, B'_2) = 0.025$, $p(A_2, B'_1) = 0.296$, $p(A_2, B'_2) = 0.086$, $p(A'_1, B_1) = 0.778$, $p(A'_1, B_2) = 0.086$, $p(A'_2, B_1) = 0.086$, $p(A'_2, B_2) = 0.049$, $p(A'_1, B'_1) = 0.148$, $p(A'_1, B'_2) = 0.086$, $p(A'_2, B'_1) = 0.099$, $p(A'_2, B'_2) = 0.667$, and the expectation values are $E(A, B) = p(A_1, B_1) - p(A_1, B_2) - p(A_2, B_1) + p(A_2, B_2) = -0.7778$, $E(A, B') = p(A_1, B'_1) - p(A_1, B'_2) - p(A_2, B'_1) + p(A_2, B'_2) = 0.3580$, $E(A', B) = p(A'_1, B_1) - p(A'_1, B_2) - p(A'_2, B_1) + p(A'_2, B_2) = 0.6543$, $E(A', B') = p(A'_1, B'_1) - p(A'_1, B'_2) - p(A'_2, B'_1) + p(A'_2, B'_2) = 0.6296$. Inserting them into the Clauser-Horne-Shimony-Holt (CHSH) version of Bell's inequality [32]

$$-2 \le E(A', B') + E(A', B) + E(A, B') - E(A, B) \le 2. \quad (5)$$

we find $E(A', B') + E(A', B) + E(A, B') - E(A, B) = 2.4197$. This violation proves the presence of entanglement in the conceptual situation considered.

The probabilities corresponding to the coincidence measurements cannot be factorized, which means that a result stronger than the one in Theorem 2 holds. For example, for the measurement AB, there do not exist real numbers $a_1, a_2, b_1, b_2 \in [0, 1]$, $a_1 + a_2 = 1$, $b_1 + b_2 = 1$, such that $a_1 b_1 = 0.05$, $a_2 b_1 = 0.63$, $a_1 b_2 = 0.26$ and $a_2 b_2 = 0.06$. Indeed, supposing that such numbers do exist, from $a_2 b_1 = 0.63$ follows that $(1 - a_1) b_1 = 0.63$, and hence $a_1 b_1 = 1 - 0.63 = 0.37$. This is in contradiction with $a_1 b_1 = 0.05$. It is also easy to verify that marginal law is not satisfied, for example $p(A'_1, B_1) + p(A'_1, B_2) = 0.864 \neq 0.234 = p(A'_1, B'_1) + p(A'_1, B'_2)$. Following Theorem 3 a quantum representation where only the state is entangled, while all measurements are products, does not exist. But a representation which entails entangled measurements can be elaborated [1,27]. In the quantum modeling we worked out, the state of *The Animal Acts* is represented by a non-maximally entangled state, while all coincidence measurements are entangled. Since the violation of the CHSH inequality we found satisfies Tsirelson's bound, this quantum modeling for the concept combination *The Animal Acts* is an example of a 'nonlocal non-marginal box modeling 1'.

Next we consider the 'vessels of water' example [19–21]. Two vessels V_A and V_B are interconnected by a tube T, vessels and tube containing 20 l of transparent water. The measurements A and B consist in siphons S_A and S_B pouring out water from vessels V_A and V_B, respectively, and collecting the water in reference vessels R_A and R_B, where the volume of collected water is measured. If more than 10 l are collected for A or B, we put $\lambda_{A_1} = +1$ or $\lambda_{B_1} = +1$, respectively, and if fewer than 10 l are collected for A or B, we put $\lambda_{A_2} = -1$ or $\lambda_{B_2} = -1$, respectively. Measurements A' and B' consist in taking a small spoonful of water out of the left vessel and the right vessel, respectively, and verifying whether the water is transparent. We have $\lambda_{A'_1} = +1$ or $\lambda_{A'_2} = -1$, depending on whether the water in the left vessel turns out to be transparent or not, and $\lambda_{B'_1} = +1$ or $\lambda_{B'_2} = -1$, depending on whether the water in the right vessel turns out to be transparent or not. We put $\lambda_{A_1 B_1} = \lambda_{A_2 B_2} = +1$ if $\lambda_{A_1} = +1$ and $\lambda_{B_1} = +1$ or $\lambda_{A_2} = -1$ and $\lambda_{B_2} = -1$, and $\lambda_{A_1 B_2} = \lambda_{A_2 B_1} = -1$ if $\lambda_{A_1} = +1$ and $\lambda_{B_2} = -1$ or $\lambda_{A_2} = -1$ and $\lambda_{B_1} = +1$, if the coincidence measurement AB is performed. We proceed analogously for the outcomes of the measurements AB', $A'B$ and $A'B'$. We can then define the expectation values $E(A, B)$, $E(A, B')$, $E(A', B)$ and $E(A', B')$ associated with these coincidence measurements. Since each vessel contains 10 l of transparent water, we find that $E(A, B) = -1$, $E(A', B) = +1$, $E(A, B') = +1$ and $E(A', B') = +1$, which gives $E(A', B') + E(A', B) + E(A, B') - E(A, B) = +4$. This is the maximal violation of the CHSH inequality and it obviously exceeds Tsirelson's bound. We further have $0.5 = p(\lambda_{A_1 B_1}) + p(\lambda_{A_1 B_2}) \neq p(\lambda_{A_1 B'_1}) + p(\lambda_{A_1 B'_2}) = 1$, which shows that the marginal distribution law is violated. In [27] we constructed a quantum model in complex Hilbert space for the vessels of water situation, where the state p with transparent water and the state q with non-transparent water are entangled, and the measurement AB, since it has product states in its spectral decomposition, is a product measurement (Theorem 1). Compatible with Theorem 3 we can see that AB', $A'B$ and $A'B'$ are entangled measurements. Summarizing, we can say that the 'vessels of water' situation is an example of a 'nonlocal non-marginal box modeling 2'.

4 Nonlocal Boxes

We conclude this paper by giving two examples, the one physical and the other cognitive, which maximally violate Bell's inequalities, i.e. with value 4, but satisfy the marginal distribution law. These examples are also inspired by the macroscopic non-local box example worked out already in 2005 by Sven Aerts, using a breakable elastic and well defined experiments on this elastic [33]. In physics, a system that behaves in this way is called a 'nonlocal box' [31].

For the first example, we again consider the vessels of water and two measurements for each side A and B. The first consists in using the siphon and checking the water. If there are more than 10 l and the water is transparent ($\lambda_{A_1B_1}$) or if there are fewer than 10 l and the water is not transparent ($\lambda_{A_2B_2}$), the outcome of the first measurement is $+1$. In case there are fewer than 10 l and the water is transparent $\lambda_{A_2B_1}$, or if there are more than 10 l and the water is not transparent $\lambda_{A_1B_2}$, the outcome is -1. The second measurement consists in taking out some water with a little spoon to see if it is transparent or not; if it is transparent, the outcome is $\lambda_{A_1B'_1} = \lambda_{A_2B'_2} = +1$, and if it is not transparent, the outcome is $\lambda_{A_2B'_1} = \lambda_{A_1B'_2} = -1$. The water is prepared in a mixed state m of the states p (transparent water) and q (not transparent water) with equal weights. Thus, m is represented by the density operator $\rho = 0.5|p\rangle\langle p| + 0.5|q\rangle\langle q|$, where $|p\rangle = |0, \sqrt{0.5}e^{i\alpha}, 0.5e^{i\beta}, 0\rangle$ and $|q\rangle = |0, \sqrt{0.5}e^{i\alpha}, -0.5e^{i\beta}, 0\rangle$ [27].

The coincidence measurement AB is represented by the ON set $|r_{A_1B_1}\rangle = |1, 0, 0, 0\rangle$, $|r_{A_1B_2}\rangle = |0, 1, 0, 0\rangle$, $|r_{A_2B_1}\rangle = |0, 0, 1, 0\rangle$, $|r_{A_2B_2}\rangle = |0, 0, 0, 1\rangle$, which gives rise to a self-adjoint operator

$$\mathcal{E}_{AB} = \begin{pmatrix} \lambda_{A_1B_1} & 0 & 0 & 0 \\ 0 & \lambda_{A_1B_2} & 0 & 0 \\ 0 & 0 & \lambda_{A_2B_1} & 0 \\ 0 & 0 & 0 & \lambda_{A_2B_2} \end{pmatrix} \tag{6}$$

Applying Lüders' rule, we calculate the density operator representing the state after AB. This gives

$$\rho_{AB} = \sum_{i,j=1}^{2} |r_{A_iB_j}\rangle\langle r_{A_iB_j}|\rho|r_{A_iB_j}\rangle\langle r_{A_iB_j}| = \rho \tag{7}$$

as one can easily verify. This means that the nonselective measurement AB leaves the state m unchanged or, equivalently, the marginal distribution law holds.

Measurement AB' is represented by the ON set $|r_{A_1B'_1}\rangle = |0, \sqrt{0.5}e^{i\alpha}, \sqrt{0.5}e^{i\beta}, 0\rangle$, $|r_{A_1B'_2}\rangle = |1, 0, 0, 0\rangle$, $|r_{A_2B'_1}\rangle = |0, 0, 0, 1\rangle$, $|r_{A_2B'_2}\rangle = |0, \sqrt{0.5}e^{i\alpha}, -\sqrt{0.5}e^{i\beta}, 0\rangle$, which gives rise to a self-adjoint operator

$$\mathcal{E}_{AB'} = \begin{pmatrix} \lambda_{A_1B'_2} & 0 & 0 & 0 \\ 0 & 0.5(\lambda_{A_1B'_1} + \lambda_{A_2B'_2}) & 0.5e^{i(\alpha-\beta)}(\lambda_{A_1B'_1} - \lambda_{A_2B'_2}) & 0 \\ 0 & 0.5e^{-i(\alpha-\beta)}(\lambda_{A_1B'_1} - \lambda_{A_2B'_2}) & 0.5(\lambda_{A_1B'_1} + \lambda_{A_2B'_2}) & 0 \\ 0 & 0 & 0 & \lambda_{A_2B'_1} \end{pmatrix} \tag{8}$$

Applying Lüders' rule, we calculate the density operator representing the state oafter AB', which gives

$$\rho_{AB'} = \sum_{i,j=1}^{2} |r_{A_iB'_j}\rangle\langle r_{A_iB'_j}|\rho|r_{A_iB'_j}\rangle\langle r_{A_iB'_j}| = \rho \tag{9}$$

Also in this case, the nonselective measurement AB' leaves the state m unchanged. The measurements $A'B$ and $A'B'$ are analogous to AB', hence the marginal distribution law is always satisfied.

We now calculate the expectation values corresponding to the four measurements above in the mixed state m and insert them into the CHSH inequality. This gives

$$\mathcal{E}_{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{E}_{AB'} = \mathcal{E}_{A'B} = \mathcal{E}_{A'B'} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{10}$$

$$B = \mathcal{E}_{A'B'} + \mathcal{E}_{A'B} + \mathcal{E}_{AB'} - \mathcal{E}_{AB} = \begin{pmatrix} -4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} \tag{11}$$

Hence, the CHSH inequality $tr\rho B = 4$, which shows that Bell inequalities are maximally violated in the mixed state m. This construction of a Hilbert space modeling for the 'connected vessels of water' is new and was not expected when the original example was conceived.

Next we look at the cognitive example. We consider the concept *Cat* and two concrete exemplars of it, called *Glimmer* and *Inkling*, the names of two brother cats that lived in our research center [23]. The concept *Cat* is abstractly described by the state p. The experiments we consider are realizing physical contexts that influence the collapse of the concept *Cat* to one of its exemplars, or states, *Glimmer* or *Inkling*, inside the mind of a person being confronted with the physical contexts. It is a 'gedanken experiment', in the sense that we put forward plausible outcomes for it, taking into account the nature of the physical contexts, and Liane, the owner of both cats, playing the role of the person. We also suppose that Liane sometimes puts a collar with a little bell around the necks of both cats, the probability of this happening being equal to 1/2. We also suppose that if she does, she always puts them around the necks of both cats.

The measurement A consists in '*Glimmer* appearing in front of Liane as a physical context'. We consider outcome λ_{A_1} to occur if Liane thinks of *Glimmer* and there is a bell, or if she thinks of *Inkling* and there is no bell, while outcome λ_{A_2} occurs if Liane thinks of *Inkling* and there is a bell, or if she thinks of *Glimmer* and there is no bell. The measurement B consists in '*Inkling* appearing in front of Liane as a physical context'. We consider outcome λ_{B_1} to occur if Liane thinks of *Inkling* and there is a bell, or if she thinks of *Glimmer* and there is no bell, while outcome λ_{B_2} occurs if Liane thinks of *Glimmer* and there is a bell, or if she thinks of *Inkling* and there is no bell. Experiment A' consists in '*Inkling* appearing in front of Liane as a physical context', and outcome $\lambda_{A'_1}$ occurs if Inkling wears a bell, and outcome $\lambda_{A'_2}$, if Inkling does not. Experiment B' consists in '*Glimmer* appearing in front of Liane as a physical context', and outcome $\lambda_{B'_1}$ occurs if Glimmer wears a bell, outcome $\lambda_{B'_1}$, if Glimmer does not.

The measurement AB consists in both cats showing up as physical context. Because of the symmetry of the situation, it is plausible to suppose probability 1/2 that Liane thinks of *Glimmer* and probability 1/2 that she thinks of *Inkling*, however, they are mutually exclusive. Also, since both cats either wear bells or do not wear bells, AB produces strict anti-correlation, probability 1/2 for outcome $\lambda_{A_1B_2}$ and probability 1/2 for outcome $\lambda_{A_2B_1}$. Hence $p(\lambda_{A_1B_2}) = p(\lambda_{A_2B_1}) = 1/2$ and $p(\lambda_{A_1B_1}) = p(\lambda_{A_2B_2}) = 0$, which gives $E(A, B) = -1$. The measurement AB' consists in *Glimmer* showing up as a physical context. This gives rise to a perfect correlation, outcome $\lambda_{A_1B'_1}$ or outcome $\lambda_{A_2B'_2}$, depending on whether *Glimmer* wears a bell or not, hence both with probability 1/2. As a consequence, we have $p(\lambda_{A_1B'_1}) = p(\lambda_{A_2B'_2}) = 1/2$ and $p(\lambda_{A_1B'_2}) = p(\lambda_{A_2B'_1}) = 0$, and $E(A, B') = +1$. The measurement $A'B$ consists in *Inkling* showing up as a physical context, again giving rise to a perfect correlation, outcome $\lambda_{A'_1B_1}$ or outcome $\lambda_{A'_2B_2}$, depending on whether *Inkling* wears a bell or not, hence both with probability 1/2. This gives $p(\lambda_{A'_1B_1}) = p(\lambda_{A'_2B_2}) = 1/2$ and $p(\lambda_{A'_1B_2}) = p(\lambda_{A'_2B_1}) = 0$ and $E(A', B) = +1$. The measurement $A'B'$ consists in both cats showing up as physical context, giving rise to a perfect correlation, outcome $\lambda_{A'_1B'_1}$ or outcome $\lambda_{A'_2B'_2}$, depending on whether both wear bells or not, hence both with probability 1/2. This gives $p(\lambda_{A'_1B'_1}) = p(\lambda_{A'_2B'_2}) = 1/2$ and $p(\lambda_{A'_1B'_2}) = p(\lambda_{A'_2B'_1}) = 0$ and $E(A', B') = +1$.

We find $E(A', B') + E(A', B) + E(A, B') - E(A, B) = 4$ in the CHSH inequalitiy. The marginal distribution law is satisfied here, because, e.g., $p(\lambda_{A_1B_1}) + p(\lambda_{A_1B_2}) = p(\lambda_{A_1B'_1}) + p(\lambda_{A_1B'_2}) = 1/2$. It is easy to check that the marginal distribution law globally holds in this case.

The two examples above are structurally isomorphic, i.e. one can provide the same quantum Hilbert space model for both of them. Moreover, they are realizations of what quantum foundations physicists call a 'nonlocal box', that is, systems obeying the marginal distribution law but violating Bell's inequalities maximally [31]. Following our classification, we call this modeling a 'nonlocal box modeling'. The above examples show that it is possible to realise nonlocal boxes in nature and elaborate a Hilbert space modeling for them, contrary to what is usually believed in quantum foundation circles.

References

1. Aerts, D., Sozzo, S: Quantum Entanglement in Concept Combinations. Int. J. Theor. Phys. ArXiv:1302.3831 [cs.AI] (2013)
2. Bell, J.S.: On the einstein-podolsky-rosen paradox. Physics **1**, 195–200 (1964)
3. Accardi, L., Fedullo, A.: On the statistical meaning of complex numbers in quantum theory. Lett. Nuovo Cim. **34**, 161–172 (1982)
4. Aerts, D.: A possible explanation for the probabilities of quantum mechanics. J. Math. Phys. **27**, 202–210 (1986)

5. Pitowsky, I.: Quantum Probability, Quantum Logic. Lecture Notes in Physics, vol. 321. Springer, Berlin (1989)
6. Aerts, D., Aerts, S.: Applications of quantum statistics in psychological studies of decision processes. Found. Sci. **1**, 85–97 (1995)
7. Aerts, D., Gabora, L.: A theory of concepts and their combinations I: the structure of the sets of contexts and properties. Kybernetes **34**, 167–191 (2005)
8. Aerts, D., Gabora, L.: A theory of concepts and their combinations II: a hilbert space representation. Kybernetes **34**, 192–221 (2005)
9. Bruza, P.D., Lawless, W., van Rijsbergen, C.J., Sofge, D. (eds.): Proceedings of the AAAI Spring Symposium on Quantum Interaction. Stanford University, Stanford, 27–29 March 2007
10. Bruza, P.D., Lawless, W., van Rijsbergen, C.J., Sofge, D. (eds.): Quantum Interaction: Proceedings of the Second Quantum Interaction Symposium. College Publications, London (2008)
11. Aerts, D.: Quantum structure in cognition. J. Math. Psychol. **53**, 314–348 (2009)
12. Bruza, P., Sofge, D., Lawless, W., van Rijsbergen, K., Klusch, M. (eds.): QI 2009. LNCS(LNAI), vol. 5494. Springer, Heidelberg (2009)
13. Pothos, E.M., Busemeyer, J.R.: A quantum probability model explanation for violations of 'rational' decision theory. Proc. Roy. Soc. B **276**, 2171–2178 (2009)
14. Khrennikov, A.Y.: Ubiquitous Quantum Structure. Springer, Berlin (2010)
15. Song, D., Melucci, M., Frommholz, I., Zhang, P., Wang, L., Arafat, S. (eds.): QI 2011. LNCS, vol. 7052. Springer, Heidelberg (2011)
16. Busemeyer, J.R., Pothos, E., Franco, R., Trueblood, J.S.: A quantum theoretical explanation for probability judgment 'errors'. Psychol. Rev. **118**, 193–218 (2011)
17. Busemeyer, J.R., Bruza, P.D.: Quantum Models of Cognition and Decision. Cambridge University Press, Cambridge (2012)
18. Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.): QI 2012. LNCS, vol. 7620. Springer, Heidelberg (2012)
19. Aerts, D.: Example of a macroscopical situation that violates bell inequalities. Lett. Nuovo Cim. **34**, 107–111 (1982)
20. Aerts, D.: The physical origin of the EPR Paradox and How to violate bell inequalities by macroscopical systems. In: Lathi, P., Mittelstaedt, P. (eds.) Symposium on the Foundations of Modern Physics: 50 Years of the Einstein-Podolsky-Rosen Gedankenexperiment, pp. 305–320. World Scientific, Singapore (1985)
21. Aerts, D.: A possible explanation for the probabilities of quantum mechanics and a macroscopical situation that violates bell inequalities. In: Mittelstaedt, P., Stachow, E.W. (eds.) Recent Developments in Quantum Logic, pp. 235–251. Bibliographisches Institut, mannheim (1985)
22. Aerts, D.: A Mechanistic Classical Laboratory Situation Violating the Bell Inequalities with $2\sqrt{2}$, Exactly 'in the Same Way' as its Violations by the EPR Experiments. Helv. Phys. Acta **64**, 1–23 (1991)
23. Aerts, D., Aerts, S., Broekaert, J., Gabora, L.: The violation of bell inequalities in the macroworld. Found. Phys. **30**, 1387–1414 (2000)
24. Aerts, D., Sozzo, S.: Quantum structure in cognition: why and how concepts are entangled. In: Song, D., Melucci, M., Frommholz, I., Zhang, P., Wang, L., Arafat, S. (eds.) QI 2011. LNCS, vol. 7052, pp. 116–127. Springer, Heidelberg (2011)
25. Aerts, D., Gabora, L., S. Sozzo, S.: concepts and their dynamics: a quantum-theoretic modeling of human thought. Top. Cogn. Sci. (in print). ArXiv: 1206.1069 [cs.AI]
26. Aerts, D., Broekaert, J., Gabora, L., Sozzo, S.: Quantum structure and human thought. Behav. Bra. Sci. **36**, 274–276 (2013)

27. Aerts, D., Sozzo, S.: Entanglement Zoo II. Applications in Physics and Cognition, accepted in LNCS (2013)
28. Tsirelson, B.S.: Quantum generalizations of bell's inequality. Lett. Math. Phys. **4**, 93–100 (1980)
29. Aspect, A.: Bell's inequality test: more ideal than ever. Nature **398**, 189–190 (1982)
30. Adenier, G., Khrennikov, A.Y.: Is the fair sampling assumption supported by EPR experiments? J. Phys. A **40**, 131–141 (2007)
31. Popescu, S., Rohrlich, D.: Nonlocality as an axiom. Found. Phys. **24**, 379–385 (1994)
32. Clauser, J.F., Horne, M.A., Shimony, A., Holt, R.A.: Proposed experiment to test local hidden-variable theories. Phys. Rev. Lett. **23**, 880–884 (1969)
33. Aerts, S.: A realistic device that simulates the non-local PR box without communication. http://arxiv.org/abs/quant-ph/0504171 (2005)

Entanglement Zoo II: Examples in Physics and Cognition

Diederik Aerts and Sandro Sozzo(✉)

Center Leo Apostel (CLEA), Vrije Universiteit Brussel (VUB),
Pleinlaan 2, 1050 Brussels, Belgium
{diraerts,ssozzo}@vub.ac.be

Abstract. We have recently presented a general scheme enabling quantum modeling of different types of situations that violate Bell's inequalities [1]. In this paper, we specify this scheme for a combination of two concepts. We work out a quantum Hilbert space model where 'entangled measurements' occur in addition to the expected 'entanglement between the component concepts', or 'state entanglement'. We extend this result to a macroscopic physical entity, the 'connected vessels of water', which maximally violates Bell's inequalities. We enlighten the structural and conceptual analogies between the cognitive and physical situations which are both examples of a nonlocal non-marginal box modeling in our classification.

Keywords: Quantum cognition · Vessels of water · Bell's inequalities · Entanglement

1 Introduction

The presence of entanglement in microscopic quantum particles is typically revealed by a violation of Bell-type inequalities [2,3]. Such a violation also implies that the corresponding coincidence measurements contain correlations that cannot be modeled in a classical Kolmogorovian probability structure [4–6], which led to the widespread belief that such correlations only appear in the micro-world. Many years ago, we already showed that Bell's inequalities can be violated by macroscopic physical entities, e.g., two connected vessels of water [7–11]. Little attention was paid to this result at that time, however, because most quantum foundations physicists were convinced that it was impossible to violate Bell's inequalities in situations pertaining to domains different from the micro-world. Now that quantum interaction research is flourishing [12–26], it is valuable to reconsider some of these examples, also because we have recently found that it is possible to build Hilbert space models for them, something we did not look into at the time. The fact that it is possible to explicitly construct complex Hilbert space models for these situations became clear to us when we were struggling

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 97–109, 2014.
DOI: 10.1007/978-3-642-54943-4_9,

to quantum-model the experimental correlation experiments we performed on a conceptual combination *The Animal Acts*. The Hilbert space modeling of our cognitive correlation data produced a range of new insights. After we had verified that the given concept combination violated Bell's inequalities [25–27], the elaboration of a Hilbert space representation showed the presence of 'conceptual entanglement' and proved that this entanglement is only partly due to the component concepts, or 'state entanglement', because it is also caused by 'entangled measurements' and 'entangled dynamical evolutions between measurements' [1]. This discovery of the presence of entanglement on the level of measurements and evolutions shed unexpectedly new light on traditional in-depth studies of aspects of entanglement, such as the possible violation of the marginal distribution law.

We have since developed a general quantum modeling scheme for the structural description of the entanglement present in different types of situations violating Bell's inequalities [28]. In this perspective, situations are possible in which only states are entangled and measurements are products ('customary entanglement'), but also situations in which entanglement appears on the level of the measurements, in the form of the presence of entangled measurements and the presence of entangled evolutions ('nonlocal box situation', 'nonlocal non-marginal box situation'). In the present paper, after briefly resuming our empirical results on *The Animal Acts* (Sect. 2.1), we provide a synthetized version of our quantum-theoretic modeling for this conceptual entity (Sect. 2.2). We then build a quantum model in complex Hilbert space for the entity 'vessels of water' (Sects. 3 and 4). This modeling in Hilbert space was not expected when this example was originally conceived, and hence constitutes a new result. We also study the conceptual and structural connections between these two situations in the light of our classification in Ref. [28]. The two cases we consider here are paradigmatic of 'nonlocal non-marginal box situations', that is, experimental situations in which (i) joint probabilities do not factorize, (ii) Bell's inequalities are violated, and (iii) the marginal distribution law does not hold. Whenever these conditions are simultaneously satisfied, a form of entanglement appears which is stronger than the 'customarily identified quantum entanglement in the states of microscopic entities'. In these cases, it is not possible to work out a quantum-mechanical representation in a fixed $\mathbb{C}^2 \otimes \mathbb{C}^2$ space which satisfies empirical data and where only the initial state is entangled while the measurements are products. It follows that entanglement is a more complex property than usually thought, a situation we investigate in depth in [1]. Shortly, if a single measurement is at play, one can distribute the entanglement between state and measurement, but if more measurements are considered, the marginal distribution law imposes drastic limits on the ways to model the presence of the entanglement [28]. This is explicitly shown by constructing an alternative $\mathbb{C}^4$ modeling for the original vessels of water example (Sect. 5).

Let us remark that we use the naming 'entanglement' referring explicitly to the structure within the theory of quantum physics that a modeling of experimental data takes, if (i) these data are represented, following carefully the rules of standard quantum theory, in a complex Hilbert space, and hence states, measurements, and evolutions, are presented respectively by vectors (or density operators), self-adjoint operators, and unitary operators in this Hilbert space; (ii) a situation of coincidence joint measurement on a compound entity is considered, and the subentities are identified following the tensor product rule of 'compound entity description in quantum theory' (iii) within this tensor product description of the compound entity entanglement is identified, as 'not being product', whether it is for states (non-product vectors), measurements (non-product self-adjoint operators), or evolutions (non-product unitary transformations).

Let us also remark that the research we present in this paper frames within the general emergence of 'quantum interaction research'. Indeed, there is increasing evidence that quantum structures are systematically present in domains other than the micro-world described by quantum physics. Cognitive science, economics, biology, computer science - they all entail situations that can be modeled more faithfully by elements of quantum theory than by approaches rooted in classical theories such as classical probability [12–26]. Our inspiration to identify quantum structures in domains different from the micro-world originally arose when we were investigating the structures of classical and quantum probability, more specifically, when we were analyzing the question of whether classical probability can reproduce the predictions of quantum theory [4, 29]. Understanding the structural difference between classical and quantum probability led us firstly to identify situations in the macroscopic world entailing aspects that are usually attributed only to microscopic quantum entities, such as 'contextuality', 'emergence', 'entanglement, 'interference' and 'superposition' [7–11]. Later, we extended our search to the realm of human cognition, the structure of human decision processes and the way in which the human mind handles concepts, their dynamics and combinations [12–14, 17, 27, 30].

2 The Animal Acts and Its Quantum Representation

We have recently performed a cognitive test on the combination of concepts *The Animal Acts* [25–27] which violated Bell's inequalities. We have also worked out a quantum representation which fits the collected data and reveals entanglement between the component concepts *Animal* and *Acts*. And, more, it shows a 'stronger form of entanglement' involving not only entangled states but also entangled measurements and entangled evolutions [1]. In the following, we present these results in the light of the classification elaborated in Ref. [28].

2.1 Description of the Cognitive Test

We consider the sentence *The Animal Acts* as a conceptual combination of the concepts *Animal* and *Acts*. Measurements consists of asking participants in the experiment to answer the question whether a given exemplar 'is a good example' of the considered concept or conceptual combination. The measurement A, respectively A', considers the exemplars *Horse* and *Bear*, respectively *Tiger* and

Cat, of the concept *Animal*, the measurement B, respectively B', considers the exemplars *Growls* and *Whinnies*, respectively *Snorts* and *Meows*, of the concept *Acts*. For the coincidence experiments, for AB, participants choose among the four possibilities (1) *The Horse Growls*, (2) *The Bear Whinnies* – and if one of these is chosen we put $\lambda_{A_1B_1} = \lambda_{A_2B_2} = +1$ – and (3) *The Horse Whinnies*, (4) *The Bear Growls* – and if one of these is chosen we put $\lambda_{A_1B_2} = \lambda_{A_2B_1} = -1$. For the measurement AB', they choose among (1) *The Horse Snorts*, (2) *The Bear Meows* – and in case one of these is chosen we put $\lambda_{A_1B'_1} = \lambda_{A_2B'_2} = +1$ – and (3) *The Horse Meows*, (4) *The Bear Snorts* – and in case one of these is chosen we put $\lambda_{A_1B'_2} = \lambda_{A_2B'_1} = -1$. For the measurement $A'B$, they choose among (1) *The Tiger Growls*, (2) *The Cat Whinnies* – and in case one of these is chosen we put $\lambda_{A'_1B_1} = \lambda_{A'_2B_2} = +1$ – and (3) *The Tiger Whinnies*, (4) *The Cat Growls* – and in case one of these is chosen we put $\lambda_{A'_1B_2} = \lambda_{A'_2B_1} = -1$. For the measurement $A'B'$ participants choose among (1) *The Tiger Snorts*, (2) *The Cat Meows* – and in case one of these is chosen we put $\lambda_{A'_1B'_1} = \lambda_{A'_2B'_2} = +1$ – and (3) *The Tiger Meows*, (4) *The Cat Snorts* – and in case one of these is chosen we put $\lambda_{A'_1B'_2} = \lambda_{A'_2B'_1} = -1$.

We now evaluate the expectation values $E(A', B')$, $E(A', B)$, $E(A, B')$ and $E(A, B)$ associated with the coincidence experiments $A'B'$, $A'B$, AB' and AB, respectively, and substitute these values into the Clauser-Horne-Shimony-Holt (CHSH) version of Bell's inequality [3]

$$-2 \leq E(A', B') + E(A', B) + E(A, B') - E(A, B) \leq 2 \tag{1}$$

One typically says that, if Eq. (1) is violated, a classical Kolmogorovian probabilistic description of the data is not possible [4–6]. In analogy with the quantum violation of the CHSH inequality, we call the phenomenon 'cognitive entanglement' between the given concepts, and remark that it is the necessary appearance of non-product structures in the explicit quantum-theoretic model in Sect. 2.2 that in our approach justifies this naming.

We actually performed a test involving 81 subjects who were presented with a form to be filled out in which they were asked to choose among the above alternatives in experiments AB, $A'B$, AB' and $A'B'$. If we denote by $p(A_1, B_1)$, $p(A_1, B_2)$, $p(A_2, B_1)$, $p(A_2, B_2)$, the probability that *The Horse Growls*, *The Bear Whinnies*, *The Horse Whinnies*, *The Bear Growls*, respectively, is chosen in the coincidence experiment AB, and so on in the other experiments, these probabilities are $p(A_1, B_1) = 0.049$, $p(A_1, B_2) = 0.630$, $p(A_2, B_1) = 0.259$, $p(A_2, B_2) = 0.062$, in experiment AB, $p(A_1, B'_1) = 0.593$, $p(A_1, B'_2) = 0.025$, $p(A_2, B'_1) = 0.296$, $p(A_2, B'_2) = 0.086$, in experiment AB', $p(A'_1, B_1) = 0.778$, $p(A'_1, B_2) = 0.086$, $p(A'_2, B_1) = 0.086$, $p(A'_2, B_2) = 0.049$, in experiment $A'B$, $p(A'_1, B'_1) = 0.148$, $p(A'_1, B'_2) = 0.086$, $p(A'_2, B'_1) = 0.099$, $p(A'_2, B'_2) = 0.667$, in experiment $A'B'$. Therefore, the expectation values are $E(A, B) = p(A_1, B_1) - p(A_1, B_2) - p(A_2, B_1) + p(A_2, B_2) = -0.7778$, $E(A, B') = p(A_1, B'_1) - p(A_1, B'_2) - p(A_2, B'_1) + p(A_2, B'_2) = 0.3580$, $E(A', B) = p(A'_1, B_1) - p(A'_1, B_2) - p(A'_2, B_1) + p(A'_2, B_2) = 0.6543$, $E(A', B') = p(A'_1, B'_1) - p(A'_1, B'_2) - p(A'_2, B'_1) + p(A'_2, B'_2) = 0.6296$.

Hence, Eq. (1) gives $E(A',B')+E(A',B)+E(A,B')-E(A,B)=2.4197$, which is significantly greater than 2. This violation is close to Tsirelson's bound [31], the maximal quantum violation of $2\sqrt{2}$, in case only product measurements are considered, so that it does reveal the presence of genuine entanglement in the situation considered with *The Animal Acts*, as we will see in the next section.

2.2 A Quantum Representation in Complex Hilbert Space

Let us now construct a quantum representation in complex Hilbert space for the collected data by starting from an operational description of the conceptual entity *The Animal Acts*. The entity *The Animal Acts* is abstractly described by an initial state p. Measurement AB has four outcomes $\lambda_{A_1B_1}$, $\lambda_{A_1B_2}$, $\lambda_{A_2B_1}$ and $\lambda_{A_2B_2}$, and four final states $p_{A_1B_1}$, $p_{A_1B_2}$, $p_{A_2B_1}$ and $p_{A_2B_2}$. Measurement AB' has four outcomes $\lambda_{A_1B'_1}$, $\lambda_{A_1B'_2}$, $\lambda_{A_2B'_1}$ and $\lambda_{A_2B'_2}$, and four final states $p_{A_1B'_1}$, $p_{A_1B'_2}$, $p_{A_2B'_1}$ and $p_{A_2B'_2}$. Measurement $A'B$ has four outcomes $\lambda_{A'_1B_1}$, $\lambda_{A'_2B_1}$, $\lambda_{A'_1B_2}$ and $\lambda_{A'_2B_2}$, and four final states $p_{A'_1B_1}$, $p_{A'_1B_2}$, $p_{A'_2B_1}$ and $p_{A'_2B_2}$. Measurement $A'B'$ has four outcomes $\lambda_{A'_1B'_1}$, $\lambda_{A'_2B'_1}$, $\lambda_{A'_1B'_2}$ and $\lambda_{A'_2B'_2}$, and four final states $p_{A'_1B'_1}$, $p_{A'_1B'_2}$, $p_{A'_2B'_1}$ and $p_{A'_2B'_2}$. Then, we consider the Hilbert space $\mathbb{C}^4$ as the state space of *The Animal Acts* and represent the state p by the unit vector $|p\rangle \in \mathbb{C}^4$. We assume that $\{|p_{A_1B_1}\rangle, |p_{A_1B_2}\rangle, |p_{A_2B_1}\rangle, |p_{A_2B_2}\rangle\}$, $\{|p_{A_1B'_1}\rangle, |p_{A_1B'_2}\rangle, |p_{A_2B'_1}\rangle, |p_{A_2B'_2}\rangle\}$, $\{|p_{A'_1B_1}\rangle, |p_{A'_1B_2}\rangle, |p_{A'_2B_1}\rangle, |p_{A'_2B_2}\rangle\}$, $\{|p_{A'_1B'_1}\rangle, |p_{A'_1B'_2}\rangle, |p_{A'_2B'_1}\rangle, |p_{A'_2B'_2}\rangle\}$ are orthonormal (ON) bases of $\mathbb{C}^4$. Therefore, $|\langle p_{A_1B_1}|\psi\rangle|^2 = p(A_1B_1)$, $|\langle p_{A_1B_2}|\psi\rangle|^2 = p(A_1B_2)$, $|\langle p_{A_2B_1}|\psi\rangle|^2 = p(A_2B_1)$, $|\langle p_{A_2B_2}|\psi\rangle|^2 = p(A_2B_2)$, in the measurement AB. We proceed analogously for the other probabilities. Hence, the self-adjoint operators

$$\begin{aligned}
\mathcal{E}_{AB} &= \lambda_{A_1B_1}|p_{A_1B_1}\rangle\langle p_{A_1B_2}| + \lambda_{A_1B_2}|p_{A_1B_2}\rangle\langle p_{A_1B_1}| \\
&\quad + \lambda_{A_2B_1}|p_{A_2B_1}\rangle\langle p_{A_2B_1}| + \lambda_{A_2B_2}|p_{A_2B_2}\rangle\langle p_{A_2B_2}| \\
\mathcal{E}_{AB'} &= \lambda_{A_1B'_1}|p_{A_1B'_1}\rangle\langle p_{A_1B'_2}| + \lambda_{A_1B'_2}|p_{A_1B'_2}\rangle\langle p_{A_1B'_1}| \\
&\quad + \lambda_{A_2B'_1}|p_{A_2B'_1}\rangle\langle p_{A_2B'_1}| + \lambda_{A_2B'_2}|p_{A_2B'_2}\rangle\langle p_{A_2B'_2}| \\
\mathcal{E}_{A'B} &= \lambda_{A'_1B_1}|p_{A'_1B_1}\rangle\langle p_{A'_1B_2}| + \lambda_{A'_1B_2}|p_{A'_1B_2}\rangle\langle p_{A'_1B_1}| \\
&\quad + \lambda_{A'_2B_1}|p_{A'_2B_1}\rangle\langle p_{A'_2B_1}| + \lambda_{A'_2B_2}|p_{A'_2B_2}\rangle\langle p_{A'_2B_2}| \\
\mathcal{E}_{A'B'} &= \lambda_{A'_1B'_1}|p_{A'_1B'_1}\rangle\langle p_{A'_1B'_2}| + \lambda_{A'_1B'_2}|p_{A'_1B'_2}\rangle\langle p_{A'_1B'_1}| \\
&\quad + \lambda_{A'_2B'_1}|p_{A'_2B'_1}\rangle\langle p_{A'_2B'_1}| + \lambda_{A'_2B'_2}|p_{A'_2B'_2}\rangle\langle p_{A'_2B'_2}|
\end{aligned} \tag{2}$$

represent the measurements AB, AB', $A'B$ and $A'B'$ in $\mathbb{C}^4$, respectively.

Let now the state p of *The Animal Acts* be the entangled state represented by the unit vector $|p\rangle = |0.23e^{i13.93^\circ}, 0.62e^{i16.72^\circ}, 0.75e^{i9.69^\circ}, 0e^{i194.15^\circ}\rangle$ in the canonical basis of $\mathbb{C}^4$. This choice is not arbitrary, but deliberately 'as close as possible to a situation of only product measurements', as we explain in [1,28]. Moreover, we choose the outcomes $\lambda_{A_1B_1}, \ldots, \lambda_{A'_2B'_2}$ to be ± 1, as in Sect. 2.1. In this case, we have proved in Ref. [1] that

$$\mathcal{E}_{AB} = \begin{pmatrix} 0.952 & -0.207-0.030i & 0.224+0.007i & 0.003-0.006i \\ -0.207+0.030i & -0.930 & 0.028-0.001i & -0.163+0.251i \\ 0.224-0.007i & 0.028+0.001i & -0.916 & -0.193+0.266i \\ 0.003+0.006i & -0.163-0.251i & -0.193-0.266i & 0.895 \end{pmatrix} \quad (3)$$

$$\mathcal{E}_{AB'} = \begin{pmatrix} -0.001 & 0.587+0.397i & 0.555+0.434i & 0.035+0.0259i \\ 0.587-0.397i & -0.489 & 0.497+0.0341i & -0.106-0.005i \\ 0.555-0.434i & 0.497-0.0341i & -0.503 & 0.045-0.001i \\ 0.035-0.0259i & -0.106+0.005i & 0.045+0.001i & 0.992 \end{pmatrix} \quad (4)$$

$$\mathcal{E}_{A'B} = \begin{pmatrix} -0.587 & 0.568+0.353i & 0.274+0.365i & 0.002+0.004i \\ 0.568-0.353i & 0.090 & 0.681+0.263i & -0,110-0.007i \\ 0.274-0.365i & 0.681-0.263i & -0.484 & 0.150-0.050i \\ 0,002-0.004i & -0,110+0.007i & 0.150+0.050i & 0.981 \end{pmatrix} \quad (5)$$

$$\mathcal{E}_{A'B'} = \begin{pmatrix} 0.854 & 0.385+0.243i & -0.035-0.164i & -0.115-0.146i \\ 0.385-0.243i & -0.700 & 0.483+0.132i & -0.086+0.212i \\ -0.035+0.164i & 0.483-0.132i & 0.542 & 0.093+0.647i \\ -0.115+0.146i & -0.086-0.212i & 0.093-0.647i & -0.697 \end{pmatrix} \quad (6)$$

Our quantum-theoretic modeling in the Hilbert space $\mathbb{C}^4$ of our cognitive test is completed. By recalling the following canonical isomorphisms, $\mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ and $L(\mathbb{C}^4) \cong L(\mathbb{C}^2) \otimes L(\mathbb{C}^2)$, and the definitions of entangled states and measurements in Refs. [1,28], it can be proved that all measurements AB, AB', $A'B$ and $A'B'$ are entangled with this choice of the entangled state. Moreover, the marginal distribution law is violated by all measurements, e.g., $p(A_1B_1)+p(A_1B_2) \neq p(A_1B'_1)+p(A_1B'_2)$. Since we are under Tsirelson's bound, this modeling is an example of a 'nonlocal non-marginal box modeling 1', following the classification we have proposed in Ref. [28].

3 The Vessels of Water Entity

We have seen in Sect. 2 that there are unexpected connections between how a sentient human being connects conceptual entities through meaning in cognitive tests and how microscopic quantum entities are connected in entangled states in space-like separated spin experiments. Both kinds of entities violate Bell's inequalities and present entanglement. In this section, we consider a macroscopic entity, namely, 'two vessels of water connected by a tube', which behaves in an analogous way. We believe that the 'connected vessels of water example' still is a very good example because it provides an intuitive insight into 'what entanglement is about', i.e. what conditions are necessary and sufficient for entanglement to manifest itself in reality, irrespective of whether it is physical or cognitive reality. We came upon this example many years ago, when we were demonstrating how Bell's inequalities can be violated by ordinary macroscopic material entities by different examples [7–11], and we will discuss it in some detail here.

We consider two vessels V_A and V_B connected by a tube T, containing a total of 20 l of transparent water. Coincidence experiments A and B consist in siphons

S_A and S_B pouring out water from vessels V_A and V_B, respectively, and collecting the water in reference vessels R_A and R_B, where the volume of collected water is measured. If more than 10 l are collected for experiments A or B we put $E(A) = +1$ or $E(B) = +1$, respectively, and if fewer than 10 l are collected for experiments A or B, we put $E(A) = -1$ or $E(B) = -1$, respectively. We define experiments A' and B', which consist in taking a small spoonful of water out of the left vessel and the right vessel, respectively, and verifying whether the water is transparent. We have $E(A') = +1$ or $E(A') = -1$, depending on whether the water in the left vessel turns out to be transparent or not, and $E(B') = +1$ or $E(B') = -1$, depending on whether the water in the right vessel turns out to be transparent or not. We put $E(AB) = +1$ if $E(A) = +1$ and $E(B) = +1$ or $E(A) = -1$ and $E(B) = -1$, and $E(AB) = -1$ if $E(A) = +1$ and $E(B) = -1$ or $E(A) = -1$ and $E(B) = +1$, if the coincidence experiment AB is performed. We can thus define the expectation value $E(A, B)$ for the coincidence experiment AB in a traditional way. Similarly, we put $E(A'B) = +1$ if $E(A') = +1$ and $E(B) = +1$ or $E(A') = -1$ and $E(B) = -1$ and the coincidence experiment $A'B$ is performed. And we have $E(AB') = +1$ if $E(A) = +1$ and $E(B') = +1$ or $E(A) = -1$ and $E(B') = -1$ and the coincidence experiment AB' is performed, and further $E(A'B') = +1$ if $E(A') = +1$ and $E(B') = +1$ or $E(A') = -1$ and $E(B') = -1$ and the coincidence experiment $A'B'$ is performed. Hence, we can define the expectation values $E(A', B)$, $E(A, B')$ and $E(A', B')$ corresponding to the coincidence experiments $A'B$, AB' and $A'B'$, respectively. Now, since each vessel contains 10 l of transparent water, we find that these expectation values are $E(A, B) = -1$, $E(A', B) = +1$, $E(A, B') = +1$ and $E(A', B') = +1$, which gives $E(A', B') + E(A', B) + E(A, B') - E(A, B) = +4$. This is the maximum possible violation of the CHSH form of Bell's inequalities.

There are deep structural and conceptual connections between the cognitive and physical situations violating Bell's inequalities. The main reason why these interconnected water vessels can violate Bell's inequalities is because the water in the vessels has not yet been subdivided into two volumes before the measurement starts. The water in the vessels is only 'potentially' subdivided into volumes whose sum is 20 l. It is not until the measurement is actually carried out that one of these potential subdivisions actualizes, i.e. one part of the 20 l is collected in reference vessel R_A and the other part is collected in reference vessel R_B. This is very similar to the combination of concepts *The Animal Acts* in Sect. 2 not collapsing into one of the four possibilities *The Horse Growls*, *The Bear Whinnies*, *The Bear Growls* or *The Bear Whinnies* before the coincidence measurement AB starts. It is the coincidence measurement itself which makes the combination *The Animal Acts* collapse into one of these four possibilities. The same holds for the interconnected water vessels. The coincidence experiment AB with the siphons is what causes the total volume of 20 l of water to be split into two volumes, and it is this which creates the correlation for AB giving rise to $E(A, B) = -1$. It can easily be calculated that if we take away the tube and suppose that, before the measurement, the water is already subdivided over the two vessels, which are now no longer interconnected, although still an anti-correlation would be measured for the coincidence experiments between A and B, the perfect correlations between A and B', and between A' and B no longer hold, one of them changing into an anti-correlation. This makes that Bell's inequality is no longer violated, i.e. $E(A',B')+E(A',B)+E(A,B')-E(A,B)=+2$ – by the way, this is also true in general when the initial state of the vessels of water is a mixture of product states. This proves that the tube, provoking the 'potentiality of the anti-correlation for A and B', is essential for Bell's inequality to be violated. In *The Animal Acts*, it is the presence of 'meaning' in the mind of the choosing person that is necessary to provoke a violation of Bell's inequalities. In the vessels of water, it is the presence of 'water' in the two connected vessels which is necessary to provoke a violation of Bell's inequalities.

4 A Quantum Representation of the Vessels of Water

In this section, we elaborate a Hilbert space representation for the vessels of water situation: this result is new and was not investigated neither expected when this example was originally conceived.

Let us provide a preliminary description of the experiments with the vessels of water, as in Sect. 2.2. The entity 'vessels of water' is abstractly described each time by a state p. Measurement AB has four outcomes, $\lambda_{A_1B_1}$, $\lambda_{A_1B_2}$, $\lambda_{A_2B_1}$ and $\lambda_{A_2B_2}$, and four final states, $p_{A_1B_1}$, $p_{A_1B_2}$, $p_{A_2B_1}$ and $p_{A_2B_2}$. Measurement AB' has four outcomes, $\lambda_{A_1B'_1}$, $\lambda_{A_1B'_2}$, $\lambda_{A_2B'_1}$ and $\lambda_{A_2B'_2}$, and four final states, $p_{A_1B'_1}$, $p_{A_1B'_2}$, $p_{A_2B'_1}$ and $p_{A_2B'_2}$. Measurement $A'B$ has four outcomes $\lambda_{A'_1B_1}$, $\lambda_{A'_1B_2}$, $\lambda_{A'_2B_1}$ and $\lambda_{A'_2B_2}$, and four final states $p_{A'_1B_1}$, $p_{A'_1B_2}$, $p_{A'_2B_1}$ and $p_{A'_2B_2}$. Measurement $A'B'$ has four outcomes $\lambda_{A'_1B'_1}$, $\lambda_{A'_1B'_2}$, $\lambda_{A'_2B'_1}$ and $\lambda_{A'_2B'_2}$, and four final states $p_{A'_1B'_1}$, $p_{A'_1B'_2}$, $p_{A'_2B'_1}$ and $p_{A'_2B'_2}$.

To work out a quantum-mechanical model in the Hilbert space $\mathbb{C}^4$ for the vessels of water situation, we consider the entangled state p represented by the unit vector $|p\rangle = |0,\sqrt{0.5}e^{i\alpha},\sqrt{0.5}e^{i\beta},0\rangle$ as describing the vessels of water situation with transparent water, and represent the measurement AB by the ON (canonical) basis $|p_{A_1B_1}\rangle = |1,0,0,0\rangle$, $|p_{A_1B_2}\rangle = |0,1,0,0\rangle$, $|p_{A_2B_1}\rangle = |0,0,1,0\rangle$, $|p_{A_2B_2}\rangle = |0,0,0,1\rangle$. This gives indeed the correct probabilities in the state p, that is, $p(\lambda_{A_1B_1}) = |\langle p_{A_1B_1}|p\rangle|^2 = 0$, $p(\lambda_{A_1B_2}) = |\langle p_{A_1B_2}|p\rangle|^2 = 0.5$, $p(\lambda_{A_2B_1}) = |\langle p_{A_2B_1}|p\rangle|^2 = 0.5$, $p(\lambda_{A_2B_2}) = |\langle p_{A_2B_2}|p\rangle|^2 = 0$. In the coincidence measurement AB', we take the ON basis $|p_{A_1B'_1}\rangle = |0,\sqrt{0.5}e^{i\alpha},\sqrt{0.5}e^{i\beta},0\rangle$, $|p_{A_1B'_2}\rangle = |0,\sqrt{0.5}e^{i\alpha},-\sqrt{0.5}e^{i\beta},0\rangle$, $|p_{A_2B'_1}\rangle = |1,0,0,0\rangle$, $|p_{A_2B'_2}\rangle = |0,0,0,1\rangle$. We have the correct probabilities in the state p, that is, $p(\lambda_{A_1B'_1}) = |\langle p_{A_1B'_1}|p\rangle|^2 = 1$, $p(\lambda_{A_1B'_2}) = |\langle p_{A_1B'_2}|p\rangle|^2 = 0$, $p(\lambda_{A_2B'_1}) = |\langle p_{A_2B'_1}|p\rangle|^2 = 0$, $p(\lambda_{A_2B'_2}) = |\langle p_{A_2B'_2}|p\rangle|^2 = 0$. In the coincidence measurement $A'B$, we choose the ON basis $|p_{A'_1B_1}\rangle = |0,\sqrt{0.5}e^{i\alpha},\sqrt{0.5}e^{i\beta},0\rangle$, $|p_{A'_1B_2}\rangle = |1,0,0,0\rangle$, $|p_{A'_2B_1}\rangle = |0,\sqrt{0.5}e^{i\alpha},-\sqrt{0.5}e^{i\beta},0\rangle$, $|p_{A'_2B_2}\rangle = |0,0,0,1\rangle$. As expected, we get probability 1 for the outcome $\lambda_{A'_1B_1}$ in the state p. In the coincidence measurement $A'B'$, we take the ON basis $|p_{A'_1B'_1}\rangle = |0,\sqrt{0.5}e^{i\alpha},\sqrt{0.5}e^{i\beta},0\rangle$, $|p_{A'_1B'_2}\rangle = |1,0,0,0\rangle$, $|p_{A'_2B'_1}\rangle = |0,0,0,1\rangle$, $|p_{A'_2B'_2}\rangle = |0,\sqrt{0.5}e^{i\alpha},-\sqrt{0.5}e^{i\beta},0\rangle$. As expected, we get probability 1 for the outcome $\lambda_{A'_1B'_1}$ in the state p.

Let us now explicitly construct the self-adjoint operators representing the measurements AB, AB', $A'B$ and $A'B'$. They are respectively given by

$$\mathcal{E}_{AB} = \sum_{i,j=1}^{2} \lambda_{A_iB_j} |p_{A_iB_j}\rangle\langle p_{A_iB_j}| = \begin{pmatrix} \lambda_{A_1B_1} & 0 & 0 & 0 \\ 0 & \lambda_{A_1B_2} & 0 & 0 \\ 0 & 0 & \lambda_{A_2B_1} & 0 \\ 0 & 0 & 0 & \lambda_{A_2B_2} \end{pmatrix} \quad (7)$$

$$\mathcal{E}_{AB'} = \sum_{i,j=1}^{2} \lambda_{A_iB'_j} |p_{A_iB'_j}\rangle\langle p_{A_iB'_j}|$$
$$= \begin{pmatrix} \lambda_{A_2B'_1} & 0 & 0 & 0 \\ 0 & 0.5(\lambda_{A_1B'_1} + \lambda_{A_1B'_2}) & 0.5e^{i(\alpha-\beta)}(\lambda_{A_1B'_1} - \lambda_{A_1B'_2}) & 0 \\ 0 & 0.5e^{-i(\alpha-\beta)}(\lambda_{A_1B'_1} - \lambda_{A_1B'_2}) & 0.5(\lambda_{A_1B'_1} + \lambda_{A_1B'_2}) & 0 \\ 0 & 0 & 0 & \lambda_{A_2B'_2} \end{pmatrix} \quad (8)$$

$$\mathcal{E}_{A'B} = \sum_{i,j=1}^{2} \lambda_{A'_iB_j} |p_{A'_iB_j}\rangle\langle p_{A'_iB_j}|$$
$$= \begin{pmatrix} \lambda_{A'_1B_2} & 0 & 0 & 0 \\ 0 & 0.5(\lambda_{A'_1B_1} + \lambda_{A'_2B_1}) & 0.5e^{i(\alpha-\beta)}(\lambda_{A'_1B_1} - \lambda_{A'_2B_1}) & 0 \\ 0 & 0.5e^{-i(\alpha-\beta)}(\lambda_{A'_1B_1} - \lambda_{A'_2B_1}) & 0.5(\lambda_{A'_1B_1} + \lambda_{A'_2B_1}) & 0 \\ 0 & 0 & 0 & \lambda_{A'_2B_2} \end{pmatrix} \quad (9)$$

$$\mathcal{E}_{A'B'} = \sum_{i,j=1}^{2} \lambda_{A'_iB'_j} |p_{A'_iB'_j}\rangle\langle p_{A'_iB'_j}|$$
$$= \begin{pmatrix} \lambda_{A'_1B'_2} & 0 & 0 & 0 \\ 0 & 0.5(\lambda_{A'_1B'_1} + \lambda_{A'_2B'_2}) & 0.5e^{i(\alpha-\beta)}(\lambda_{A'_1B'_1} - \lambda_{A'_2B'_2}) & 0 \\ 0 & 0.5e^{-i(\alpha-\beta)}(\lambda_{A'_1B'_1} - \lambda_{A'_2B'_2}) & 0.5(\lambda_{A'_1B'_1} + \lambda_{A'_2B'_2}) & 0 \\ 0 & 0 & 0 & \lambda_{A'_2B'_1} \end{pmatrix} \quad (10)$$

The self-adjoint operators corresponding to measuring the expectation values are instead obtained by putting $\lambda_{A_iB_i} = \lambda_{A_iB'_i} = \lambda_{A'_iB_i} = \lambda_{A'_iB'_i} = +1$, $i = 1, 2$ and $\lambda_{A_iB_j} = \lambda_{A_iB'_j} = \lambda_{A'_iB_j} = \lambda_{A'_iB'_j} = -1$, $i, j = 1, 2; i \neq j$, as in our experiment in Sect. 3. If we now insert these values into Eqs. (7)–(10) and define the 'Bell operator' as

$$B = \mathcal{E}_{A'B'} + \mathcal{E}_{A'B} + \mathcal{E}_{AB'} - \mathcal{E}_{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2e^{i(\alpha-\beta)} & 0 \\ 0 & 2e^{-i(\alpha-\beta)} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (11)$$

its expectation value in the entangled state p is

$$\langle p|B|p\rangle = \begin{pmatrix} 0 & \sqrt{0.5}e^{-i\alpha} & \sqrt{0.5}e^{-i\beta} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2e^{i(\alpha-\beta)} & 0 \\ 0 & 2e^{-i(\alpha-\beta)} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{0.5}e^{i\alpha} \\ \sqrt{0.5}e^{i\beta} \\ 0 \end{pmatrix} = 4 \quad (12)$$

which gives the same value in the CHSH inequality as in Sect. 3. A completely analogous construction can be performed if the entangled state q represented by the unit vector $|q\rangle = |0, \sqrt{0.5}e^{i\alpha}, -\sqrt{0.5}e^{i\beta}, 0\rangle$ is chosen to describe the vessels with non-transparent water.

We add some conclusive remarks that are discussed in detail in Ref. [28]. The measurement AB is a product measurement, since it has the product states represented by the vectors in the canonical basis of $\mathbb{C}^4$ as final states. Indeed, AB 'divides' the water into two separated volumes of water, thus 'destroying' entanglement, to arrive at a situation of a product state. The measurements AB' and $A'B$ are instead entangled measurements, since they have the entangled states represented by $|0, \sqrt{0.5}e^{i\alpha}, \sqrt{0.5}e^{i\beta}, 0\rangle$ and $|0, \sqrt{0.5}e^{i\alpha}, -\sqrt{0.5}e^{i\beta}, 0\rangle$ as final states. Indeed, since all the water is poured out of the two vessels, the water has not been divided, and inside the reference vessel it keeps being a whole, i.e. entangled. If another two siphons are put in the reference vessel where all the water has been collected, the same experiment can be performed, violating Bell's inequalities. The measurement $A'B'$ has also entangled states as possible final states, hence it is entangled. This measurement leaves the vessels of water unchanged, hence it is naturally an entangled measurement. We finally observe that the marginal ditribution law is violated in the case of the vessels of water. Indeed, we have, e.g., $0.5 = p(\lambda_{A_1B_1}) + p(\lambda_{A_1B_2}) \neq p(\lambda_{A_1B'_1}) + p(\lambda_{A_1B'_2}) = 1$. Since this vessels of water model violates Bell's inequalities beyond Tsirelson's bound, we can say that our $\mathbb{C}^4$ representation is an example of a 'nonlocal non-marginal box modeling 2', if we follow the classification in Ref. [28].

5 An Alternative Model for the Vessels of Water

In this section, we provide an alternative model in $\mathbb{C}^4$ for the vessels of water situation. This model is interesting, in our opinion, because it serves to show that where entanglement is located, in the state, or on the level of the measurement, depends on the way in which the tensor product isomorphism with the compound entity Hilbert space is chosen.

In the representation in Sect. 4, we have given preference to the first coincidence measurement AB which we have chosen as a product, i.e. we have represented it by the canonical basis of $\mathbb{C}^4$. This means that the entanglement of this 'state-measurement' situation has been completely put into the state. Let us identify this entanglement on the level of the probabilities by using Th. 2 in Ref. [28]. We have $p(\lambda_{A_1B_1}) = p(\lambda_{A_2B_2}) = 0$, $p(\lambda_{A_1B_1}) = p(\lambda_{A_2B_2}) = 0.5$ in both states p and q. Suppose that we search numbers $a, b, a', b' \in [0,1]$ such that $p(\lambda_{A_1B_1}) = a \cdot b$, $p(\lambda_{A_1B_2}) = a \cdot b'$, $p(\lambda_{A_2B_1}) = a' \cdot b$ and $p(\lambda_{A_2B_2}) = a' \cdot b'$. Then, we get that $a = 0$ or $b = 0$. Since $a \cdot b' = 0.5$, we cannot have that $a = 0$, and hence $b = 0$. But then $a' \cdot b$ cannot be equal to 0.5. This entails that the probabilities do not compose into a product, hence there is entanglement in the considered 'state-measurement' situation, this entanglement being 'a joint property of state and measurement', and not of one apart.

Let us now consider the probabilities of the measurement AB'. We have $p(\lambda_{A_1B'_1}) = 1$, $p(\lambda_{A_1B'_2}) = p(\lambda_{A_2B'_1}) = p(\lambda_{A_2B'_2}) = 0$ in the state p. We can again look for numbers $a, b, a', b' \in [0,1]$ such that $p(\lambda_{A_1B'_1}) = a \cdot b$, $p(\lambda_{A_1B'_2}) = a \cdot b'$, $p(\lambda_{A_2B'_1}) = a' \cdot b$ and $p(\lambda_{A_2B'_1}) = a' \cdot b'$. We find the solution $a' = b' = 0$, and $a = b = 1$, which is unique. Indeed, from $a \cdot b = 1$ follows that $a \neq 0$ and $b \neq 0$, and hence from $a \cdot b' = 0$ and $a' \cdot b = 0$ follows then $a' = b' = 0$. This implies that we could model this 'state-measurement' situation by a product state and a product measurement. Let us do this explicitly in $\mathbb{C}^4$. If this time we represent the state p' with transparent water by the unit vector $|p'\rangle = |1,0,0,0\rangle$, and the measurement AB' in the canonical basis, we get the wanted result. This also implies that we have single probabilities $p(\lambda_{A_1})$, $p(\lambda_{A_2})$, $p(\lambda_{B'_1})$ and $p(\lambda_{B'_2})$ such that $p(\lambda_{A_1}) = p(\lambda_{A_2}) = 1$, $p(\lambda_{B'_1}) = p(\lambda_{B'_2}) = 0$. We can construct also the second and third 'state measurement' in the same space, and with the same state. It gives $p(\lambda_{A'_1}) = p(\lambda_{B_1}) = 1$, $p\lambda_{A'_2}) = p(\lambda_{B_2}) = 0$.

Proceeding in this way, we can propose an alternative quantum model where we use the product state p', represented by $|1,0,0,0\rangle$, and the product measurements AB', $A'B$ and $A'B'$, all represented by the canonical ON basis in $\mathbb{C}^4$. Only AB is entangled in this construction and corresponds to the ON set $|p'_1\rangle = |0,1,0,0\rangle$, $|p'_2\rangle = |\sqrt{0.5}e^{i\alpha}, 0, 0, \sqrt{0.5}e^{i\beta}\rangle$, $|p'_3\rangle = |\sqrt{0.5}e^{i\alpha}, 0, 0, -\sqrt{0.5}e^{i\beta}\rangle$, $|p'_4\rangle = |0,0,1,0\rangle$, as one can verify at once. This gives rise to the self-adjoint operators

$$
\begin{aligned}
\mathcal{E}'_{AB} &= \lambda_{A_1B_1}|p_1\rangle\langle p_1| + \lambda_{A_1B_2}|p_2\rangle\langle p_2| + \lambda_{A_2B_1}|p_3\rangle\langle p_3| + \lambda_{A_2B_2}|p_4\rangle\langle p_4| \\
&= \begin{pmatrix} 0.5(\lambda_{A_1B_1} + \lambda_{A_2B_1}) & 0 & 0 & 0.5e^{i(\alpha-\beta)}(\lambda_{A_1B_2} - \lambda_{A_2B_1}) \\ 0 & \lambda_{A_1B_1} & 0 & 0 \\ 0 & 0 & \lambda_{A_2B_2} & 0 \\ 0.5e^{-i(\alpha-\beta)}(\lambda_{A_1B_2} - \lambda_{A_2B_1}) & 0 & 0 & 0.5(\lambda_{A_1B_2} + \lambda_{A_2B_1}) \end{pmatrix}
\end{aligned} \quad (13)
$$

$$
\mathcal{E}'_{AB'} = \begin{pmatrix} \lambda_{A_1B'_1} & 0 & 0 & 0 \\ 0 & \lambda_{A_1B'_2} & 0 & 0 \\ 0 & 0 & \lambda_{A_2B'_1} & 0 \\ 0 & 0 & 0 & \lambda_{A_2B'_2} \end{pmatrix} \quad \mathcal{E}'_{A'B} = \begin{pmatrix} \lambda_{A'_1B_1} & 0 & 0 & 0 \\ 0 & \lambda_{A'_1B_2} & 0 & 0 \\ 0 & 0 & \lambda_{A'_2B_1} & 0 \\ 0 & 0 & 0 & \lambda_{A'_2B_2} \end{pmatrix} \quad (14)
$$

$$
\mathcal{E}'_{A'B'} = \begin{pmatrix} \lambda_{A'_1B'_1} & 0 & 0 & 0 \\ 0 & \lambda_{A'_1B'_2} & 0 & 0 \\ 0 & 0 & \lambda_{A'_2B'_1} & 0 \\ 0 & 0 & 0 & \lambda_{A'_2B'_2} \end{pmatrix} \quad (15)
$$

As usual, if we measure expectation values, i.e. the outcomes are all either +1 or −1, and insert them into Eqs. (13)–(15), we can directly calculate the expectation values in the state p'. We find $\langle p'|\mathcal{E}'_{A'B'} + \mathcal{E}'_{A'B} + \mathcal{E}'_{AB'} - \mathcal{E}'_{AB}|p'\rangle = 4$, as expected. An analogous construction can be worked out for the state of the vessels with non-transparent water.

References

1. Aerts, D., Sozzo, S: Quantum entanglement in concept combinations. Int. J. Theor. Phys. ArXiv:1302.3831 [cs.AI] (2013)
2. Bell, J.S.: On the einstein-podolsky-rosen paradox. Physics **1**, 195–200 (1964)

3. Clauser, J.F., Horne, M.A., Shimony, A., Holt, R.A.: Proposed experiment to test local hidden-variable theories. Phys. Rev. Lett. **23**, 880–884 (1969)
4. Aerts, D.: A possible explanation for the probabilities of quantum mechanics. J. Math. Phys. **27**, 202–210 (1986)
5. Accardi, L., Fedullo, A.: On the statistical meaning of complex numbers in quantum theory. Lett. Nuovo Cim. **34**, 161–172 (1982)
6. Pitowsky, I.: Quantum Probability, Quantum Logic. Lecture Notes in Physics, vol. 321. Springer, Berlin (1989)
7. Aerts, D.: Example of a macroscopical situation that violates bell inequalities. Lett. Nuovo Cim. **34**, 107–111 (1982)
8. Aerts, D.: The physical origin of the EPR paradox and how to violate bell inequalities by macroscopical systems. In: Lathi, P., Mittelstaedt, P. (eds.) Symposium on the Foundations of Modern Physics: 50 Years of the Einstein-Podolsky-Rosen Gedankenexperiment, pp. 305–320. World Scientific, Singapore (1985)
9. Aerts, D.: A possible explanation for the probabilities of quantum mechanics and a macroscopical situation that violates bell inequalities. In: Mittelstaedt, P., Stachow, E.W. (eds.) Recent Developments in Quantum Logic, pp. 235–251. Bibliographisches Institut, Mannheim (1985)
10. Aerts, D.: A mechanistic classical laboratory situation violating the bell inequalities with $2\sqrt{2}$, exactly 'in the same way' as its violations by the epr experiments. Helv. Phys. Acta **64**, 1–23 (1991)
11. Aerts, D., Aerts, S., Broekaert, J., Gabora, L.: The violation of bell inequalities in the macroworld. Found. Phys. **30**, 1387–1414 (2000)
12. Aerts, D., Aerts, S.: Applications of quantum statistics in psychological studies of decision processes. Found. Sci. **1**, 85–97 (1995)
13. Aerts, D., Gabora, L.: A theory of concepts and their combinations i: the structure of the sets of contexts and properties. Kybernetes **34**, 167–191 (2005)
14. Aerts, D., Gabora, L.: A theory of concepts and their combinations ii: a hilbert space representation. Kybernetes **34**, 192–221 (2005)
15. Bruza, P.D., Lawless, W., van Rijsbergen, C.J., Sofge, D. (eds.): Proceedings of the AAAI Spring Symposium on Quantum Interaction. Stanford University, Stanford, 27–29 March 2007
16. Bruza, P.D., Lawless, W., van Rijsbergen, C.J., Sofge, D. (eds.): Quantum Interaction: Proceedings of the Second Quantum Interaction Symposium. College Publications, London (2008)
17. Aerts, D.: Quantum structure in cognition. J. Math. Psychol. **53**, 314–348 (2009)
18. Bruza, P., Sofge, D., Lawless, W., van Rijsbergen, K., Klusch, M. (eds.): QI 2009. LNCS (LNAI), vol. 5494. Springer, Heidelberg (2009)
19. Pothos, E.M., Busemeyer, J.R.: A quantum probability model explanation for violations of 'rational' decision theory. Proc. Roy. Soc. B **276**, 2171–2178 (2009)
20. Khrennikov, A.Y.: Ubiquitous Quantum Structure. Springer, Berlin (2010)
21. Song, D., Melucci, M., Frommholz, I., Zhang, P., Wang, L., Arafat, S. (eds.): QI 2011. LNCS, vol. 7052. Springer, Heidelberg (2011)
22. Busemeyer, J.R., Pothos, E., Franco, R., Trueblood, J.S.: A quantum theoretical explanation for probability judgment 'errors'. Psychol. Rev. **118**, 193–218 (2011)
23. Busemeyer, J.R., Bruza, P.D.: Quantum Models of Cognition and Decision. Cambridge University Press, Cambridge (2012)
24. Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.): QI 2012. LNCS, vol. 7620. Springer, Heidelberg (2012)
25. Aerts, D., Gabora, L., Sozzo, S.: Concepts and their dynamics: a quantum-theoretic modeling of human thought. Top. Cogn. Sci. (in print). ArXiv: 1206.1069 [cs.AI].

26. Aerts, D., Broekaert, J., Gabora, L., Sozzo, S.: Quantum structure and human thought. Behav. Bra. Sci. **36**, 274–276 (2013)
27. Aerts, D., Sozzo, S.: Quantum structure in cognition: why and how concepts are entangled. In: Song, D., Melucci, M., Frommholz, I., Zhang, P., Wang, L., Arafat, S. (eds.) QI 2011. LNCS, vol. 7052, pp. 116–127. Springer, Heidelberg (2011)
28. Aerts, D, Sozzo, S.: The Entanglement Zoo I. Foundational and Structural Aspects, accepted in LNCS (2013)
29. Aerts, D.: Foundations of quantum physics: a general realistic and operational approach. Int. J. Theor. Phys. **38**, 289–358 (1999)
30. Aerts, D.: Quantum particles as conceptual entities: a possible explanatory framework for quantum theory. Found. Sci. **14**, 361–411 (2010)
31. Tsirelson, B.S.: Quantum generalizations of bell's inequality. Lett. Math. Phys. **4**, 93–100 (1980)

Contextual Query Using Bell Tests

João Barros, Zeno Toffano(✉), Youssef Meguebli, and Bich-Liên Doan

SUPELEC, 3 Rue Joliot Curie, 91192 Gif-sur-Yvette, France
{joaocarlos.pintobarros,zeno.toffano,youssef.meguebli,
bich-lien.doan}@supelec.fr

Abstract. Tests are essential in Information Retrieval and Data Mining in order to evaluate the effectiveness of a query. An automatic measure tool intended to exhibit the meaning of words in context has been developed and linked with Quantum Theory, particularly entanglement. "Quantum like" experiments were undertaken on semantic space based on the Hyperspace Analogue Language (HAL) method. A quantum HAL model was implemented using state vectors issued from the HAL matrix and query observables, testing a wide range of window sizes. The Bell parameter S, associating measures on two words in a document, was derived showing peaks for specific window sizes. The peaks show maximum quantum violation of the Bell inequalities and are document dependent. This new correlation measure inspired by Quantum Theory could be promising for measuring query relevance.

Keywords: Bell inequality · Entanglement · Information retrieval · Cooccurrence · HAL · Tests · Context · IR algorithms · Quantum Theory

1 Introduction

In this work we present original "quantum-like" tests that could be useful in the domain of Information Retrieval (IR) and Data Mining.

Context is used to disambiguate terms. Melucci [1] showed that a query or a document can be generalized, in different contexts, as vectors, where the likelihood of context of a set of documents can be considered. Quantum Mechanics has been invoked to enrich the search capabilities in IR by Rijsbergen [2] by using the mathematical formalism of the Hilbert vector space.

Analogies between concepts derived from Quantum Theory with Information Retrieval tools have been made by several authors. Widdows [3] uses the quantum formalism for experiments with negation and disjunction and Arafat [4] shows that user needs can be represented by a state vector. Other analogies have been stated by Li and Cunningham [5] such as: "state vector"/"objects" in a collection; "observable"/"query", "eigenvalues"/"relevance or not for one object"; "probability of getting one eigenvalue"/"relevance degree of object to a query".

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 110–121, 2014.
DOI: 10.1007/978-3-642-54943-4_10,

Bruza and Cole explicitly calculated eigenvectors associated to a word [6] and in the field of concept representation an explicit Bell inequality violation was found [7].

2 Bell Inequality and Bell Parameters for Binary Outcomes

Entanglement, which can be made manifest through Bell inequality violations [8] (commonly presented in the form of the CHSH inequality [9]) has become a very important research trend in Physics. Several experiments have proved the existence of entangled particles [10,11] and this fact is now widely accepted. The field has fascinated many scientists throughout the last decades also leading to much parallel scientific and pseudoscientific research as is well described by Keiser in a recent book [12]. Part of the attraction arises because of the concept of "nonlocality" of the quantum world suggesting "spooky action at distance" (a discussion can be found in [13]). Even though in general the violation of Bell inequalities demands entanglement, higher violations of the inequalities do not necessarily mean more entanglement.

Quantum Information has emerged bridging physics and information science. Though initially discovered in the context of foundations of Quantum Mechanics, the violations of Bell inequalities referred to above are nowadays a key point in a wide range of branches of Quantum Information. Entanglement is at the heart of this field because it is seen as a potential "resource" for new applications such as coding or computing [14]. New theorems of the kind of Bell's, named no-go theorems (for example the Kochen-Specker theorem [15]), have been proposed.

In practice most experiments have used polarized photons as in the famous experiment in 1982 by Aspect et al. [11]. More sophisticated set-ups are constantly being proposed and discussed [16,17] very often to rule out local hidden variable models.

Some macroscopic tests have been proposed in the form of thought experiments or combined with yes-no questions showing also Bell inequality violations, for example by Aerts [18].

The CHSH-Bell parameter S_{Bell} for tests with two binary outcomes, $+1$ or -1, can be defined as follows:

$$S_{Bell} = |E(A,B) - E(A,C)| + |E(B,D) + E(C,D)| \tag{1}$$

where A, B, C and D are tests and $E(X,Y)$ stands for the expectation value of the outcome of mutual tests X and Y.

We briefly recall some important facts about the Bell parameter. It is easily verified that S_{Bell} can never exceed 4. More specifically in the so called classical, local and separable situation S_{Bell} lies between 0 and 2. In this case, for example, we could write $E(X,Y) = E(X)E(Y)$.

The case $2 \leq S_{Bell} \leq 2\sqrt{2}$ can be achieved with quantum entangled states obtained experimentally with photons. Less underlined is the case where

$S_{Bell} > 2\sqrt{2}$, also known as the Tsirelson's bound [16,19]. This zone between $2\sqrt{2}$ and 4 is called the "no-signaling" region. The maximum value $S_{Bell} = 4$ can be attained with logical probabilistic constructions often named PR boxes [20].

3 Bell Tests in Semantic Space Using HAL

Our approach presented here can be perceived as an experiment done on objects outside the domain of physics. The objects are words within texts. We study the relationships between words within a document; these relationships can be formed by creating a "semantic space" using the Hyperspace Analogue Language (HAL) method [21].

The HAL algorithm does not require any explicit human a-priori judgment. In this procedure a HAL lexical co-occurence matrix is built with a "window" representing a span of words passed over the corpus being analyzed. The width of this window can be varied. Words within the window are recorded as co-occurring with strength inversely proportional to the number of other words separating them within the window.

The point of the co-occurrence matrix is that the rows effectively constitute vectors in a high-dimensional space, so that that the elements of the normalized vectors are frequency counts (probabilities), and the dimensionality of the space is determined by the number of columns in the matrix (context vectors).

The HAL method has already been used as a tool for a physical analogy between semantic space and Quantum Theory, where at each word was associated a given spectrum (in analogy with spectral emission lines of atoms) [22].

Our method uses the HAL matrices for calculating quantum mechanical mean values of query observables and combining them in order to calculate a Bell parameter S_{query}. We carried out our tests in a symmetric matrix obtained by the sum of the HAL matrix and its transpose (equivalent to running HAL backwards). This is due to the fact that we did not consider the order in which words appear in a text.

The tests were carried out on documents in English. The programming scheme of the algorithm implementation is represented in Fig. 3 (Appendix).

4 Quantum Model for Bell Tests Using HAL

In this section we intend to define operators, in analogy with Quantum Theory, that will give a new possible approach to document queries. We make the following definitions.

4.1 Document Vector States

In the N dimensional HAL space each document will have an associated vector. The vector state of the document is the sum of all the word vectors $|w_i\rangle$ it contains. Each word vector state is extracted from the lines of the symmetric HAL matrix. The document vector state is defined as:

$$|\Psi\rangle = \sum_{i}^{N} |w_i\rangle \tag{2}$$

We are now interested in analyzing how two words are connected within a document, namely word A and word B. The two word vectors $|w_A\rangle$ and $|w_B\rangle$ define a plane on the semantic space. We will not consider the part of the document corresponding to the orthogonal projection with the two chosen words. The resulting normalized state vector $|\psi\rangle$ from now on will be the document vector state.

To obtain $|\psi\rangle$ we take the vectors $|w_A\rangle$ and $|w_B\rangle$ and normalize them obtaining two new vectors: $|u_A\rangle$ and $|u_B\rangle$. Now we apply the Gram-Schmidt orthogonalization process to the non-orthogonal basis $\{|u_A\rangle, |u_B\rangle\}$, and doing so, we can obtain two new bases that describe the plane formed by the original vectors $|w_A\rangle$ and $|w_B\rangle$: the bases $\{|u_A\rangle, |u_{A\perp}\rangle\}$ and $\{|u_B\rangle, |u_{B\perp}\rangle\}$. In this way we can parameterize the plane in two ways: in the first we make explicit the parallel component of a vector with respect to $|w_A\rangle$ and in the second we make explicit the parallel component of a vector with respect to $|w_B\rangle$. By projecting the vector $|\Psi\rangle$ on one of these basis, we obtain its projection onto this plane. Taking this vector and normalizing it gives us the desired vector $|\psi\rangle$. Explicitly what we get is:

$$|\psi\rangle = \alpha |u_A\rangle + \alpha_\perp |u_{A\perp}\rangle = \beta |u_B\rangle + \beta_\perp |u_{B\perp}\rangle \tag{3}$$

The coefficients α, $\alpha_\perp$, β and $\beta_\perp$ are obtained by projecting the state $|\Psi\rangle$ on both basis vectors and then normalizing to unity. For example for α we have:

$$\alpha = \frac{\langle u_A |\Psi\rangle}{\sqrt{\langle u_A |\Psi\rangle^2 + \langle u_{A\perp} |\Psi\rangle^2}} \tag{4}$$

4.2 Query Operators

We want now to define query operators. The purpose of these operators is to quantify a query within our formalism. The query operators $\hat{A}$ and $\hat{B}$ are defined in a way that they attribute the value +1 to the component of the state that corresponds to the word meaning we are interested in, and −1 in the orthogonal direction. More precisely we will use operators acting as the spin Pauli matrix $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on their respective decomposition basis. These operators are associated with observables because they are Hermitian. Explicitly:

$$\hat{A}|\psi\rangle = \alpha |u_A\rangle - \alpha_\perp |u_{A\perp}\rangle, \ \hat{B}|\psi\rangle = \beta |u_B\rangle - \beta_\perp |u_{B\perp}\rangle \tag{5}$$

The expectation values of these operators are calculated in the same way as in quantum mechanics using the Born rule; for example, the mean value in the context of document associated to $|\psi\rangle$ for the query about A is written as usual in quantum mechanics:

$$\langle A\rangle_{\psi} = \langle\psi|\,\hat{A}\,|\psi\rangle = \alpha^2 - \alpha_{\perp}^2 = 2\alpha^2 - 1 \tag{6}$$

From this example we see that we can obtain a score for a search related to word A. This corresponds to something reasonable for the query score since it increases with α which is equal to the scalar product between the document vector and the word vector as shown in Eq. 4. Score values range from +1 to −1. +1 is obtained when the document vector is parallel to the query vector, and −1 when it is orthogonal. Following this line of thought other operators can be defined using, for example, the Pauli matrix $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

For the choice of the query operator $\hat{A}_x = \hat{\sigma}_x$ in the $\{|u_A\rangle, |u_{A\perp}\rangle\}$ basis we have:

$$\hat{A}_x\left(\alpha\,|u_A\rangle + \alpha_{\perp}\,|u_{A\perp}\rangle\right) = \alpha_{\perp}\,|u_A\rangle + \alpha\,|u_{A\perp}\rangle \tag{7}$$

We see that this operator switches the components of the vector state. This can be interpreted as a measure of different meaning in the document with respect to the original direction corresponding to word A.

We do not consider the expectation values for the spin Pauli matrix $\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ due to the fact that here the components of the vector state issued from the HAL matrix are always real. Possible future generalizations may include a way of obtaining vector components on the complex plane.

4.3 Combining Operators and Expectation Values of Two Queries

For technical reasons we choose the basis associated to the word A given above in Eq. 5 and write the operators with respect to this basis. We can write the transformation matrix $\hat{M}$ from the A basis to the B basis. It is easy to see that:

$$\hat{M} = \begin{pmatrix} \langle u_B|\,u_A\rangle & \langle u_B|\,u_{A\perp}\rangle \\ \langle u_{B\perp}|\,u_A\rangle & \langle u_{B\perp}|\,u_{A\perp}\rangle \end{pmatrix} = \begin{pmatrix} p & \sqrt{1-p^2} \\ -\sqrt{1-p^2} & p \end{pmatrix} \tag{8}$$

By construction, this matrix can be simply expressed by the scalar product $p = \langle u_B|\,u_A\rangle$ which in our case is always positive and smaller than 1, unless there is a perfect alignment between the two words; then it is 1.

Any operator expressed in its matrix form on the basis associated to the word B can be written in the basis associated to the word A using the transformation matrix $\hat{M}$. From our previous definition of $\hat{B}$, its matrix form in the basis associated to word A becomes:

$$\hat{B} = \begin{pmatrix} 2p^2 - 1 & 2p\sqrt{1-p^2} \\ 2p\sqrt{1-p^2} & 1 - 2p^2 \end{pmatrix} \tag{9}$$

With the two operators expressed in the common basis we can now combine two query operators and calculate quantum mechanical mean values using the Born rule. For example for the query of the combination of words A and B in the context of the document represented by Ψ we will write:

$$\left\langle \hat{A}\hat{B} \right\rangle_{\psi} = \langle\psi| \hat{A}\hat{B} |\psi\rangle \tag{10}$$

4.4 Bell Parameter Calculation

Bell tests are usually a proof of non-separability of the combination of two different systems. Here we make a connection to this scenario in physics using words and their meanings within the document.

For purposes of document analysis we have chosen to take an approach leading to the calculation of a Bell parameter as defined in Eq. 1. Concretely we calculate quantum means defined in Eq. 10, using different query operators which can be considered as measuring devices, and then define the Bell query parameter:

$$S_{query} = \left| \left\langle \hat{A}\hat{B}_+ \right\rangle + \left\langle \hat{A}_x\hat{B}_+ \right\rangle \right| + \left| \left\langle \hat{A}\hat{B}_- \right\rangle - \left\langle \hat{A}_x\hat{B}_- \right\rangle \right| \tag{11}$$

using the following operators

$$\hat{A};\ \hat{A}_x;\ \hat{B}_+ = -\frac{\hat{B} + \hat{B}_x}{\sqrt{2}};\ \hat{B}_+ = \frac{\hat{B} - \hat{B}_x}{\sqrt{2}} \tag{12}$$

Our particular operator choice was inspired from the usual example that maximizes the violation of the Bell inequalities. All operators have the property of being their own inverse, that is, their square is the identity (property of the Pauli matrices) which means that their eigenvalues are $+1$ and -1. With this we can calculate the corresponding parameters considering different queries among different documents. Two examples are presented in the next section.

5 Results and Discussion

With the formalism presented before we are in a position to apply it to different documents. We calculated the Bell parameter defined in Eq. 11 using the algorithm presented in the Appendix. We will discuss the obtained results in a relevance perspective. In the following examples all the documents were taken from Wikipedia (see following section).

5.1 Test on Documents: "Reagan" and "Iran"

As a first application we considered an example originally introduced by Bruza and Cole [6], which is the query for the word "Reagan" in the context of "Iran". If we talk about Reagan alone one usually associates this with the fact that he was President, but if we include Iran it will be more likely that we are interested in the Iran-contra scandal. Four documents were considered which are close to the query: "Reagan administration scandals"[1], "Reagan"[2], "Iran-Contra affair"[3] and "Iran"[4]

We plot the parameters S_{query} defined above in Eq. 11 as a function of the HAL window length for the query of the words "Reagan" and "Iran". The results are shown in Fig. 1.

The considered HAL window starts at the beginning of the document, in the first word, and will run through all the words until the end of the text. When finished we start the same process again. with a new window length (increasing the length by one unit). This is done for all window lengths we desire to analyze.

There is clearly a common behavior for the three queries in the documents (with just one exception): the parameter starts from zero and increases until it reaches a maximum, never crossing the Tsirelson's bound $2\sqrt{2}$, but getting very close to it, and then drops again. This suggests that each document, given a two word query, has an optimal HAL window size that maximizes the parameter S_{query}.

For the query of "Iran - Reagan", among the four documents, it is predictable that the document that is more closely related to this query is the "Iran-Contra affair", followed by the documents: "Reagan administration scandals" and "Reagan", with an expected greater relevance for the first. The least related document should be "Iran".

At first sight it may appear that since we are looking for "Reagan - Iran", the documents "Reagan" and "Iran" should appear on the same level in the search. However in general, the meaning "Reagan" has less importance in the context "Iran" (because the common concept "Iran" includes its history, culture, geographical situation, etc.) than "Iran" in the context of "Reagan". In Fig. 1 we also observe that this is basically the order in which the peaks appear.

The document regarding "Iran" always gives a constant value of S_{query} equal to 2. This fact is easily explained in the framework of our model. In fact it is not hard to see that when we do a two word query in which one of these words is not present in the document, the result for S_{query} is always 2. Besides if neither of the two words is present the result is always zero. Let us now consider the other three documents.

The first peak appears using a window length around $l = 30$. The S_{query} curve peaks before this value in the document of "Iran-Contra affair". The other two documents cannot be clearly distinguished. This corresponds to our previous prediction. In fact, it makes sense that the "sooner" a peak appears the less interaction, in the sense of window length, we have to consider to get higher correlation between the two words. Bearing this in mind, the document "Iran-Contra affair" is clearly the one selected by the model. The other two documents

[1] http://en.wikipedia.org/wiki/Reagan_administration_scandals (accessed 12/04/2013).
[2] http://en.wikipedia.org/wiki/Reagan (accessed 12/04/2013).
[3] http://en.wikipedia.org/wiki/Iran-Contra_affair (accessed 12/04/2013).
[4] http://en.wikipedia.org/wiki/Iran (accessed 12/04/2013).

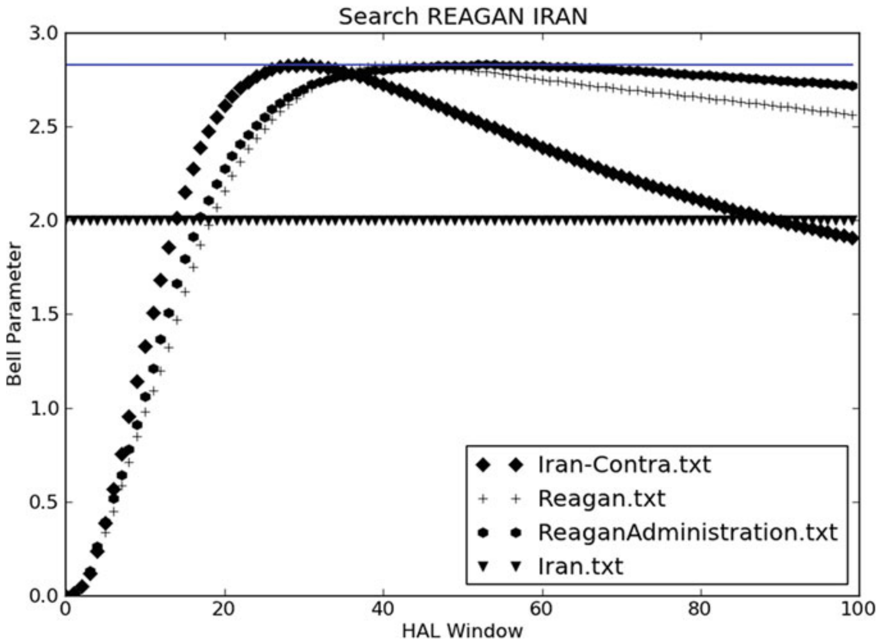


Fig. 1. Bell parameter for the query of words "Reagan - Iran" in four documents: "Reagan administration scandals", "Reagan", "Iran-Contra affair" and "Iran".

("Reagan" and "Reagan administration scandals") are not clearly distinguishable. On one side the peak of the "Reagan" document appears first, but the curve for "Reagan administration scandals" has a bigger extension close to the Tsirelson's bound $2\sqrt{2}$, meaning high correlation for several window sizes, which can also be a clue for some strong correlation between words.

5.2 Test on Documents About "Orange"

The second case considered concerns the polysemy of the word "orange" and associated concepts. In this example we are interested in the ambiguity between the meanings color and fruit. We also associate the concept of juice. The documents considered were: Orange (Colour)[5], Orange (Fruit)[6], Orange Juice[7] and Juice[8] Two queries are considered: "Orange Fruit" and "Orange Juice". The results are presented in Fig. 2.

The query "Orange - Fruit" presents the first peak around $l = 22$ for the document "Orange color", the second in $l = 29$ for the document "Orange fruit", then for $l = 40$ the document "Orange juice" and very far away the document

[5] http://en.wikipedia.org/wiki/Orange_(colour) (accessed 12/04/2013).
[6] http://en.wikipedia.org/wiki/Orange_(fruit) (accessed 12/04/2013).
[7] http://en.wikipedia.org/wiki/Orange_juice (accessed 12/04/2013).
[8] http://en.wikipedia.org/wiki/Juice (accessed 12/04/2013).

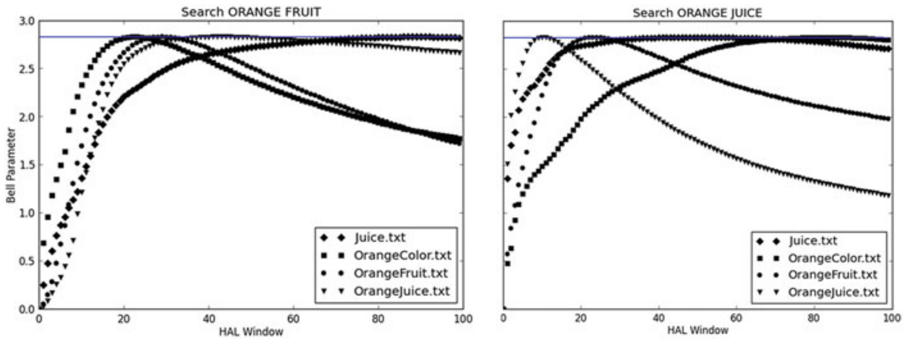


Fig. 2. Bell parameter for the query on the words "Orange - Fruit" and "Orange - Juice" in four different documents: "Juice", "Orange (Color)", "Orange (Fruit)" and "Orange Juice".

"Juice". It is interesting to note that, even though the peaks are close, the peak of the curve "Orange - Color" appears before the one for "Orange - Fruit". This may be the suggestion of a strong correlation between the origin of the name of the color and the name of the fruit. The poor correlation of the general term "Juice" with the specific query "Orange - Fruit" is very clear on the graph according to this criterion.

Finally, the last query was "Orange - Juice". Again, here, we recover precisely the order that we would expect for the documents: "Orange juice", "Orange Fruit", "Juice" and "Orange Color". It is worth noticing that in the latter case the peak corresponds to a window length range considered to be optimal for implementation of HAL ($l = 10$) which may indicate an even a stronger correlation between the words.

6 Conclusion and Perspectives

In this work, we presented a novel search experiment based on the Bell parameter extraction in semantic space using the HAL method.

The semantic vectors in HAL are representations that are essentially measures of context. The HAL method has already been used for the analogies with Quantum Theory by Bruza and Woods [23] for activating associations of concepts and by Wittek and Darány [22] for extracting spectral content from the semantic space. HAL shows high potentiality because it is a simple way to build a semantic space with a measure that is independent of user judgment.

The main feature of Quantum Theory explored in this work is the violation of the Bell inequalities which can be related to entanglement and non-locality, impossible at a classical level. The results show Bell inequality violation up to the maximal value of $S_{Bell} = 2\sqrt{2}$, (the Tsirelson's bound).

In our model each document is associated to a two dimensional Hilbert space (dependent on the search we are interested in), and queries are observables acting on it. A Bell parameter is then defined.

We found that the Bell parameter is strongly dependent on the HAL window size. From our results it is suggested that for this kind of model there is an optimal window size that maximizes the Bell parameter. This is reminiscent of what was also noticed by Bruza and Woods [23]: if the window size is set too large spurious co-occurrence associations are represented in the matrix and, if the window size is too small, relevant associations may be missed. In this model we see that too large windows may also dilute connections between associated words. Only one document, the one that did not present one of the words of the query, did not violate the Bell inequality. In general, a pattern of "early" appearance of the peak (smallest window sizes) seems to be related to the relevance of the document for the search.

In a near future other measures of quantum properties (as proper measures of entanglement) will also serve to make a better comparison between the results derived from the standard information retrieval methods.

It is not clear how to interpret the Bell inequality violation here and what is the meaning of the optimal length that maximizes the Bell parameter. Can correlation and entanglement give a measure of query relevance? Experiments and systematic comparisons with other methods used in IR, such as Latent Semantic Analysis (LSA) and the ranking method Okapi BM25, could give further indications.

An important technical point is that we introduced a new tool which has connections with the Quantum Theory: query observables. Here we made a practical choice similar to the spin Pauli matrices, but we think that it should be possible, after much experimentation on different documents, to introduce new families of query observables adapted to different purposes and contexts. In the domain of IR many concepts are introduced to define, for example, opinion-like queries in social networks [24]. Efforts are also being made in order to diversify query results of ambiguous queries; for example concepts such as sentiment diversification to identify positive, negative and neutral sentiments about the search topic can be used [25].

Acknowledgements. We would like to thank the students of SUPELEC Fabien d'ANGELO and Sixte BOISSÉ for helping on the implementation and test the HAL algorithm.

Appendix

The algorithm was implemented using the Python programming language along with the string module and pylab. All words are considered and simple plurals (constructed by adding an "s") are treated as if singular words. Lower and upper case letters are not distinguished, which means that if two words differ from each other by changes on the case, they are considered equivalent. Even if words have the same origin, they are treated differently (for example "battle" and "battling" are distinct).

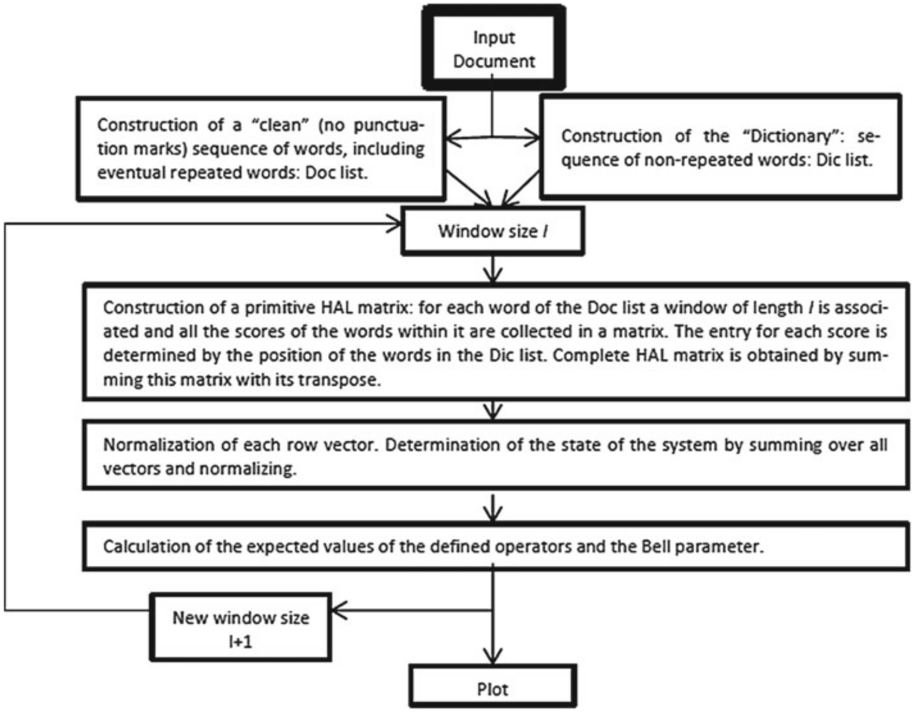


Fig. 3. Flow diagram of the Quantum HAL algorithm described.

References

1. Melucci, M.: Context modeling and discovery using vector space bases. In: Proceedings of the ACM Conference on Information and Knowledge Management, pp. 808–815 (2005)
2. van Rijsbergen, C.J.: The Geometry of Information Retrieval. Cambridge University Press, Cambridge (2004)
3. Widdows, D.: Semantic vector products: some initial investigations. In: Proceedings of the Second AAAI Symposium on Quantum Interaction, vol. 26 (2008)
4. Arafat, S.: Foundations research in information retrieval inspired by quantum theory. Ph.D. thesis, University of Glasgow (2008)
5. Li, Y., Cunningham, H.: Geometric and quantum methods for information retrieval. SIGIR Forum **42**, 22–32 (2008)
6. Bruza, P.D., Cole, R.J.: Quantum logic of semantic space: an exploratory investigation of context effects in practical reasoning. In: Artemov, S., Barringer, H., d'Avila Garcez, A.S., Lamb, L.C., Woods, J. (eds.) We Will Show Them: Essays in Honour of Dov Gabbay, pp. 339–361. College Publications, London (2005)
7. Kitto, K., Bruza, P.D.: Tests and models of non-compositional concepts. In: Proceedings of the 34th Annual Conference of the Cognitive Science Society, Sapporo, Japan (2012)
8. Bell, J.S.: On the Einstein Podolsky Rosen paradox. Physics **1**, 195 (1964)

9. Clauser, J.F., Horne, M.A., Shimony, A., Holt, R.A.: Proposed experiment to test local hidden-variable theories. Phys. Rev. Lett. **23**(15), 880–884 (1969)
10. Freedman, S.J., Clauser, J.F.: Experimental test of local hidden-variable theories. Phys. Rev. Lett. **28**, 938 (1972)
11. Aspect, A., Grangier, P., Roger, G.: Experimental realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: a new violation of Bell's inequalities. Phys. Rev. Lett. **49**, 91 (1982)
12. Kaiser, D.: How the Hippies Saved Physics. W.W. Norton & Company, New York (2012)
13. Khrennikov, A.: Bell argument: locality or realism? time to make the choice. In: AIP Conference Proceedings, vol. 1424, pp. 160–175 (2011)
14. Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information. Cambridge University Press, New York (2000)
15. Kochen, S., Specker, E.P.: The problem of hidden variables in quantum mechanics. J. Math. Mech. **17**, 59–87 (1967)
16. Cabello, A.: Violating Bell's inequality beyond Cirel'son's bound. Phys. Rev. Lett. **88**, 060403 (2002)
17. Rössler, O.E., Parisi, J., Fröhlich, D., Toffano, Z.: Direction-correlated correlated photons cannot self-interfere: a prediction. Z. Naturforsch **61a**, 418 (2006)
18. Aerts, D.: Quantum structure in cognition. J. Math. Psychol. **53**, 314–348 (2009)
19. Cirel'son (Tsirelson), B.S.: Quantum generalizations of Bell's inequality. Lett. Math. Phys. **4**(2), 93–100 (1980)
20. Popescu, S., Rohrlich, D.: Quantum nonlocality as an axiom. Found. Phys. **24**, 379–385 (1994)
21. Lund, K., Burgess, C.: Producing high-dimensional semantic spaces from lexical co-occurrence. Behav. Res. Meth. Instrum. Comput. **28**, 203–208 (1996)
22. Wittek, P., Darányi, S.: Spectral composition of semantic spaces. In: Song, D., Melucci, M., Frommholz, I., Zhang, P., Wang, L., Arafat, S. (eds.) QI 2011. LNCS, vol. 7052, pp. 60–70. Springer, Heidelberg (2011)
23. Bruza, P.D., Woods, J.: Quantum collapse in semantic space: interpreting natural language argumentation. In: Second Quantum Interaction Symposium (2008)
24. Meguebli, Y., Popineau, F., Doan, B.L.: A novel architecture for a smart information retrieval system based on opinions engineering. In: Proceedings of the Symposium on Human-Computer Interaction and Information Retrieval, Cambridge, Massachusetts (2012)
25. Kacimi, M., Gamper, J.: Diversifying search results of controversial queries. In: 20th ACM International Conference on Information and Knowledge Management, pp. 93–98 (2011)

Select the Appropriate Map Depending on Context in a Hilbert Space Model (SCOP)

Juergen Hahn(✉) and Andrew U. Frank

Department of Geodesy and Geoinformation, Vienna University of Technology,
Gusshausstrasse 27-29, 1040 Vienna, Austria
{hahn,frank}@geoinfo.tuwien.ac.at
http://www.geoinfo.tuwien.ac.at

Abstract. Human use of categories exhibits a prototype effect; concepts become more defined through a conversation. Modelling these gradual clarifications of what a word signifies is equally important in human - computer interactions, for example in interactions about geographic concepts and the information that is needed in a given situation. We address here the simplified, but essentially realistic, question of what is meant by "map" and how the concept is refined. We apply the methods Aerts, Gabora and Rosch have described and explore how they can be integrated into practical systems.

In this paper we explore the optimal selection of a map through a conversation with the client to elucidate their intentions. The example case contains effects which are similar to the "guppy effect" that is known from the literature and is a key reason to apply quantum mechanical formalism. The results are promising, and we sketch the extension to the construction of "custom made" maps from layers. This will provide users with maps that optimally reflect what map elements should be visible for use in a given context.

Keywords: Map prediction · Geographic concepts · Hilbert space model · SCOP

1 Introduction

Rosch has demonstrated prototype effects in the use of concepts by humans; the same word may have different meanings depending on the context. Montello [18] lists the influence of context as one of the most important problems for GIScience research and asks for the incorporation of models of human categories in Geographic Information Systems.

Aerts, Gabora, and Rosch have described a computational theory of prototypes based on quantum mechanical formalizations [12]. Their survey of previous efforts to compute categories showing prototype effects led them to conclude that a formalization has to deal with contextuality, concept combination, similarity, compatibility, and correlation [12]. It can be achieved with a formalism based on quantum mechanics.

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 122–133, 2014.
DOI: 10.1007/978-3-642-54943-4_11,

The interactions between humans or between humans and computerized information systems is based on the exchange of words (or graphical tokens on maps) which are interpreted in the *context* of the conversation. The words used may originally have a broad meaning (comparable to the "pet" in Aerst, Gabora, and Rosch [12]); through conversation the context becomes more precise, and the categories obtain more specific values (e.g. "goldfish" or "snake").

Understanding the meaning of natural language words is important in information retrieval and database access; the use of quantum mechanics as a formalization has been discussed [23], and compared there with other fuzzy and probabilistic methods for data retrieval. The focus of the current paper is more comparable to the work by Aerts and Gabora [3]; it intends to apply their insight into the field of geographic information science [20] where, as mentioned before, context is assumed to be one of the major research challenges today. The proof of concept we present here is meant to understand the context of the user, and to determine the best response, without relying on the user to select among technical terms that assume technical knowledge on the user side. This is somewhat similar to a recommender system [17] for which others have suggested methods based on quantum mechanics. Relations can be drawn to Formal Concept Analysis [13].

It should be possible for a user to describe their situation from their point of view - as presented by other statements made - and for the system to then guess the most likely optimal response to the user request. Through the additional information, the produced contexts transform the concept initially invoked in the base state into a more specific one.

The paper is intended as a "proof of concept" of the applicability of the method in regards to a practical problem in Geographic Information Systems [16]. For a proof of concept we restrict the selection to the selection of a number of predetermined map types (e.g. street map, political map, map for hiking, ski routes). The goal is a computational model which can be incorporated into GIS software. The input for user preferences is produced by the authors by introspection; a real-use system would need data from a user group. Aerts and Gabora [11] have shown how such contextual frequencies can be obtained by questionnaires. It is likely that methods of Volunteered Geographic Information [14] could be used to obtain valid data for different user groups.

The paper is structured as follows: the next section outlines a brief survey of prototype effects, methods to deal with them, and the computational model Aerts, Gabora and Rosch propose; it concludes with an overview of the SCOP model used here. That section is mostly intended to establish the terminology used, and to make the paper self-contained. The following section discusses categories of maps, the map production process, and how maps are used to set the stage for the production of "customized maps" in a given context. Section 4 introduces the example proof of concept case and the context-dependent selection probabilities. Section 5 connects two contexts in an entangled state. The concluding section lists further application opportunities of quantum mechanics in geographic information processing, and discusses research issues necessary to overcome possible impediments to their widespread use.

2 Review of Theories and Models for Categories

Rosch and Mervis [21] studied the internal structure of categories. They hypothesised that family resemblance correlates with the prototypicality of items, and used polls to confirm their hypothesis. They concluded that categorized elements have some attributes in common with a prototype. This prototype can be seen as a reference point for a concept [22]: The instances of a concept are more or less prototypical and are ranked in a graded structure around the prototype.

They used fuzzy set theory [27] that is able to handle objects with graded boundaries. Smith and Osherson [24] demonstrated that fuzzy sets cannot completely model how humans use concepts. They asked people to rate the typicality of instances for the concepts: pet, fish, and pet-fish. It was found that a guppy is more typical for the combination pet-fish than for the constituent concepts (pet, fish).

Gabora et al. found that none of the then known theories formalize the effects of: (1) contextuality, (2) concept combination, and (3) similarity, compatibility, and correlation [12]. These three effects were analysed by Gabora [10]. She illustrates the contextual effect for a concept as shown in Fig. 1. Starting with no contextual influence at time t0, concept p can collapse into all possible states p1(t1), p2(t1), p3(t1), and p4(t1). Influencing the concept by a particular context e3 the concept realizes state p3(t1). In state p3(t1) the concept can also collapse into all states p1(t2), .., p4(t2). Where context e7 influences the concept which collapses in state p7(t2).

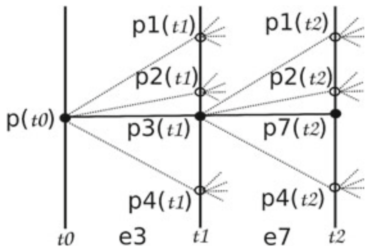


Fig. 1. Influence from contexts to concept states, see Fig. 11.1 [10]

As a result of this analysis, Gabora et al. [3] presented a different formalization for concepts. They called their approach state-context-property (SCOP) formalism based on quantum mechanics. They mapped elements taken from operational foundations of quantum mechanics like states, measurements, and observables to concepts, contexts, and properties of human cognition, and they chose Hilbert spaces as the foundation for the model. In Hilbert spaces all possible states for a concept can be included. An example with one concept in three states and eight contexts is shown above in Fig. 1.

The model is defined as a formal model [1] $(\Sigma, \mathcal{M}, \mathcal{L}, \mu, \nu)$. The sets are:

- $\Sigma = \{p1, p2, ...\}$ representing the *states* a concept can assume
- $\mathcal{M} = \{e1, e2, ..., f1, f2, ...\}$ including *contexts* for a concept
- $\mathcal{L} = \{a1, a2, ...\}$ containing *properties* or features for a concept

The functions are:

- $\mu(q, e, p)$ calculates the transition probability from one state q to another state p under the influence of context e
- $\nu(p, a)$ weights the importance of one property a in a particular state p

If no context is applied to a concept, the state is called ground state $x_{\hat{p}}$ [4].

(1) The entanglement that is typically found in microscopic quantum system can model combined concepts [11]. Combining two concepts with two distinct probability values into one concept creates a new probability value that cannot be split again into two probability values.

(2) The guppy effect described before can be formalized by the interference effect [7]. The Liar [2], Ellsberg and Machina [6] paradoxes can also be formalized with SCOP.

3 Characteristics of Geographic Maps

The American Heritage Dictionary provides four definitions of maps. The most suitable in the scope of this paper is as follows: "A map is a representation, usually on a plane surface, of a region of the earth ...". As a representation, certain features or aspects of geographic entities are not taken into account, whereas others are emphasized [8]: A roadmap includes no isolines, but highlights the highway.

Smith and Mark [25] listed properties of geographical concepts (geographers use the term "feature") and found: (1) Geographic objects are tied intrinsically to geographic space and inherit many properties from it, such as topology and geometry. (2) The scale used to categorize geographic objects influences the concept used, for example: pond, lake, sea, and ocean. (3) The boundaries of several geographic objects are indeterminate, e.g. beach, mountain, and dune.

Maps cannot possibly show all geographic features found on the surface of the earth. The cartographer produces a map for a set of potential users. In response to their expected needs, the cartographer selects and highlights features which are deemed important for the intended class of potential map users and, correspondingly, omits other features. In mapmaking, these processes are subsumed under the term "cartographic generalization" [19]. In practice, maps are categorized often with respect to their potential use as street map, road map, ski route map etc.

4 Prediction of an Appropriate Map with a Hilbert Space Model Within SCOP

This section applies SCOP to predict an answer for the question: "Which map is appropriate for a given context?", where the intended activities are used as contexts. The usage of the model is illustrated in Fig. 2. A concept and a context serve as input parameters. The model calculates the collapsed state and returns it. In this collapsed state probability values for exemplars of the concept can be calculated.

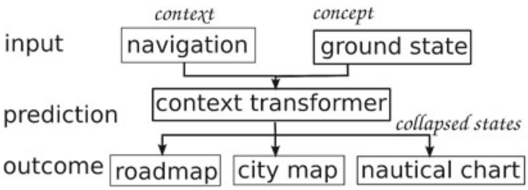


Fig. 2. Model for the prediction

The following example conversation may justify the example. A person states to another "Yesterday, I bought a map." What kind of map is meant remains undefined for the second person; the concept "map" is in ground state, where all maps have some non-zero probability to be meant. The first person continues: "I plan to go on a bicycling trip." Now the second person is influenced by the context, and the state of the concept "map" collapses into a bicycling map. The conversation may continue to indicate the region where the trip is planned, thus further restricting the map (beyond what is modelled here).

Using SCOP, a computational model for his conversation is possible. We implemented the relevant formulae [3] in the functional programming language Haskell [15], using the available matrix calculation packages (eventually using the standard implementations of GSL, BLAS or LAPACK).

To predict probability values, we define possible exemplars in Table 1 for the set Σ. The set Σ also includes the ground state $\widehat{p}$. For the term exemplar, SCOP also uses the term state of the concept.

Table 1. States of the concept map.

States of the concept, set Σ	Kind of the map
$\widehat{p}$	Map
p1	Roadmap
p2	Hiking map
p3	City map
p4	Nautical chart
p5	Ski runway map
p6	Bicycling map

Table 2 inherits properties for all elements giving set $\mathcal{L}$. In Table 3 the weights of the properties for each context are listed. These values are based on our own experience and appear realistic; the values are sufficient for a proof of concept, but are not the result of a representative experiment. Function ν uses this table.

The meaning of the concept "map" depends on the intended activity; the mapmaker creates the map for the intended purpose and typically labels it accordingly. We posit that users have, from experience, similar sub-categories for "map". Table 4 lists the intended activities the map should serve. These activities are the contexts included into set $\mathcal{M}$.

The input parameters for SCOP are the frequency values included in Tables 5 and 3.

At the start of the conversation the concept map is in ground state $x_{\widehat{p}}$. In this state none of the possible states is preferred. This results in a probability for each state as found in the ground state, i.e. (1). The value 1800 consists of the sum of the states without any context $(342 + 252 + .. + 252)$. The variable $|u\rangle$ indicates vectors of the Hilbert space. The sum identifies the selected vectors. In the ground state all vectors available in the set $\mathcal{M}$ are chosen.

$$|x_{\widehat{p}}\rangle = \sum_{u \in M} \frac{1}{\sqrt{1800}} |u\rangle \tag{1}$$

Table 2. Properties of the concept map.

Properties for the concepts, set $\mathcal{L}$	Layers of the map
a1	Road
a2	Lake
a3	Buildings
a4	Mountains
a5	Ski runs
a6	Bicycling lanes
a7	Hiking path
a8	Contour lines

Table 3. Weights of the properties by context

Weights for properties		e1	e2	e3	e4	e5	e6
a1	road	0.9	0.5	0.9	0.3	0.4	0.8
a2	lake	0.6	0.8	0.2	0.8	0.7	0.3
a3	buildings	0.7	0.5	0.99	0.5	0.5	0.01
a4	mountains	0.1	0.9	0.4	0.8	0.7	0.7
a5	ski runway	0.01	0.6	0.01	0.01	0.99	0.01
a6	bicycling lanes	0.4	0.6	0.5	0.1	0.01	0.99
a7	hiking path	0.01	0.99	0.4	0.01	0.4	0.6
a8	contour lines	0.1	0.99	0.1	0.6	0.7	0.5

Table 4. Activities used as contexts for maps

Contexts, set $\mathcal{M}$	Activities for maps
1	I choose a map
e1	I choose a map for navigation
e2	I choose a map for hiking
e3	I choose a map for sight seeing
e4	I choose a map for sailing
e5	I choose a map for skiing
e6	I choose a map for bicycling

Table 5. Frequency values in percentages and Hilbert States

Exemplars	**e1**		**e2**		**e3**		**e4**		**e5**		**e6**		**1**	
	Freq.	States	Freq.	States	Freq.	States	Freq.	States	Freq.	States	Freq.	States	Freq.	States
roadmap	54	216	9	27	21	105	13	39	5	15	7	7	19	342
hiking map	0	0	77	231	2	10	0	0	2	6	0	0	14	252
city map	4	16	3	9	67	380	18	54	0	0	5	5	22	396
nautical chart	7	28	0	0	0	0	69	207	0	0	0	0	15	270
ski runway map	0	0	0	0	0	0	0	0	93	279	0	0	16	288
bicycle map	36	144	10	30	2	10	0	0	0	0	88	88	14	252
sum		**404**		**297**		**505**		**300**		**300**		**100**	**100**	**1800**

By influencing the ground state with the context e6 "I plan to go on a bicycling trip" the state collapses into state x_{p_6}, where 100 states are present.

$$|x_{p_6}\rangle = \frac{P_{e_6}|x_{\widehat{p}}\rangle}{\sqrt{\langle x_{\widehat{p}}|P_{e_6}|x_{\widehat{p}}\rangle}} = \sum_{u \in e6} \frac{1}{\sqrt{100}} |u\rangle \tag{2}$$

With the function μ the weight of each map type (p1.. p7) can be checked. This equation yields a value between zero and one. A value closer to one identifies a highly appropriate exemplar, a value close to zero an inappropriate one.

For example, to check if the nautical chart is an appropriate map in state x_{p_6}, the projector P_{e_4} for nautical charts is used.

$$\mu\left(p_4, e4, x_{p_6}\right) = \langle x_{p_6}|P_{e4}|x_{p_6}\rangle = 0 \tag{3}$$

For state x_{p_6}, the probability for nautical charts equals zero, whereas the probability for the bicycle map equals 0.88. This is calculated as:

$$\mu\left(p_6, e6, x_{p_6}\right) = \langle x_{p_6}|P_{e6}|x_{p_6}\rangle = 0.88 \tag{4}$$

In this state the weight of the properties can also be calculated with Eq. (5). To calculate the weight of a road in the state x_{p_6} the equation is:

$$\nu\left(x_{p_6}, a_1\right) = 0.8 \tag{5}$$

The property of a map to show roads has a weight of 0.8 in state x_{p_6} and is therefore an important property, in contrast to the property ski-run (0.01). This values indicates whether the map should include this layer or not. This example will therefore include roads and will exclude ski-slopes. The calculated relevance of a property could be used to produce maps on demand for particular activities (currently, maps on demand are produced when user select the layers explicitly, which is usually too demanding for non-technical users and introduces confusing jargon; who knows what "bathymetry" is and when it is used? - but it should be used on maps used for sailing and boating!)

If we take the above conversation to be between a potential map user and a map producing service, then SCOP could be included in the program and calculate the probability for given maps, given the known contexts. If a map type receives a clear preference, it can be produced for the user. If not, additional questions can be asked to obtain more context from the user. These contexts can be processed partially, as suggested by Weiser and Frank [26].

5 Prediction of an Appropriate Map Combining Cycling and Buying of a Map

In this section we combine the concept of "map" from the previous section with a concept of "buying things"; in this situation effects like the known "guppy-effect" can occur, and can be handled through the formalism of "entanglement" from quantum mechanics [7].

In both concepts "buying a bicycle map" occurs and connects the two. The frequencies from Tables 6 and 5 declare the input values. Table 6 includes two contexts influencing the concept "I buy things", modelled as Hilbert space $\mathcal{H}^{buy}$. Context f1 includes the context cycling. As a further step, context f2 appends the context map to f1. Context f2 will result in an entangled state with the second Hilbert space.

Table 6. Frequency values in percentages and Hilbert States for $\mathcal{H}^{buy}$

Context	f1		f2		1	
Exemplar	I buy things for cycling		I buy a map for cycling		I buy things	
	Freq	States	Freq	States	Freq	States
Bread	1	4	0	0	30	900
Milk	3	12	0	0	28	840
Rain jacket	13	52	0	0	14	420
First aid kit	32	128	0	0	13	390
Bicycle chain	21	84	0	0	8	240
Cycle helmet	19	76	0	0	3	90
Road map	4	16	13	13	1	30
Bicycle map	7	28	87	87	3	90

The second Hilbert space $\mathcal{H}^{map}$ models the values from Table 5. The context e6 is selected for entanglement. Context e6 and f2 declare the same context shown by essentially equivalent statements. This is the foundation for entanglement, which brings different received informations into a single context. The following Eq. (6) describes this mathematically.

$$e_6, f_2 \in M^{map,buy} \tag{6}$$

SCOP uses the tensor product to describe combined systems; the tensor product combines all possible combinations of the basic states [4]. The entanglement set $M^{map,buy}$ is defined including the states from f_2, e_6. To create this state the two concepts f_2 and e_6 are combined by the Cartesian product, where each element from f_2 is combined with each element from e_6. The entangled state is formulated by Eq. (7).

$$|s\rangle = \sum_{u \in E^{map,buy}} \frac{1}{\sqrt{100}} |u\rangle \otimes |u\rangle \tag{7}$$

Projectors can be applied to this state, to predict an answer for: "If one buys a map for cycling, is this a bicycle map?". The answer predicted by the model is a probability value for the projected exemplar. For the exemplar bicycle map the following projector is used:

$$P_{F_2}^{map} \otimes 1^{buy} = \sum_{u \in F_2^{map}} |u\rangle\langle u| \otimes 1 \tag{8}$$

Applying this projector to the state s and reducing this state will result in the following equation:

$$|s'\rangle\langle s'|^{buy} = \sum_{u \in E_{f_2^{map}} \cap E_{f_2^{map}}} \frac{1}{100} |u\rangle\langle u| \tag{9}$$

By applying function μ to the reduced state, the probability for the bicycle map can be determined, which equals 0.88; much higher than the probability for roadmap, with a value of 0.07. The value is also higher than the probability in both independent contexts; entanglement connects the information gained in one and reinforces it in the other.

6 Conclusions and Future Research

Geographic concepts often exhibit prototype effects: the prototypical mountain to a Swiss person is not what a mountain in the Netherlands looks like. There are a great many similar effects - indeed it is hard to find a geographic concept which does not exhibit a prototype effect. The understanding of context effects is considered a major impediment for GIScience [18].

A computational model to deal with prototype effects is urgently needed. The selection of layers for maps is just one example of context effects: a prototypical map (say, a road map) serves many purposes, but by far not all. Berendt et al. described how to build maps depending on aspects of uses [8]. The experiments reported here show that with SCOP a computational solution to maps constructed from individual layers for particular purposes becomes possible.

SCOP is an appropriate model for formalizing concepts influenced by contexts, including the combination of concepts. Aerts [5] presented a further model using Fock spaces [9] to treat the disjunction of concepts.

The experiment reported here shows that the SCOP formalism of computing with contexts and combinations of contexts can be applied to geographic concepts. It promises (1) to help with the selection of maps for particular uses and (2) to contribute to the construction of maps on demand for a particular use without asking the user to construct the maps from individual layers.

Interesting and challenging research questions remain:

- Collect for several meaningful communities the data describing how they use the concept of map, following the example of Aerts and Gabora [3].
- Extend the example from before to include the selection of maps depending on the intended location of an activity. This increases the number of ground states for the concept "map", namely by regions (e.g. road map for Italy, France...)
- Users do not desire specific map types, they request maps with certain informations which are relevant for their planned activities. Instead of using SCOP to identify the map type suitable for an intended activity, one could identify the map layers which contain relevant information for the planned activity directly, and produce customized maps for different activities. Selecting the layers to be shown seems possible with SCOP, leaving the issues of graphical interactions between the map layers; the customized map must not only include the desired layers, but present them in a form which allows reading the presented information!
- Apply the SCOP computational model to other geographic concepts, e.g. town, village, mountain, forest, and observe how this affects statistical data collected across communities with different conceptualizations of e.g. forest. What is the correct answer to the question of the total forest area of Europe, if one exist considering the differences in the concept "forest"?

For inclusion in a practical system, the SCOP formalism could be further developed into an incremental algorithm; in particular, give computational solutions to adding one additional ground state or an additional context (the first seems difficult, the second trivial).

References

1. Aerts, D., Broekaert, J., Gabora, L.: Intrinsic contextuality as the crux of consciousness. In: Yasne, K., Jibu, M., Della Senta, T. (eds.) No Matter, Never Mind: Proceedings of Toward a Science of Consciousness: Fundamental Approaches, pp. 173–182. John Benjamins Publishing Company, Philadelphia (1999)
2. Aerts, D., Broekaert, J., Smets, S.: A quantum structure description of the liar paradox. Int. J. Theor. Phys. **38**, 3231–3239 (1999)
3. Aerts, D., Gabora, L.: A theory of concepts and their combinations I: the structure of the sets of contexts and properties. Kybernetes **34**, 167–191 (2005)
4. Aerts, D., Gabora, L.: A theory of concepts and their combinations II: a Hilbert space representation. Kybernetes **34**, 192–221 (2005)
5. Aerts, D.: General Quantum Modeling of Combining Concepts: A Quantum Field Model in Fock Space. arXiv, preprint arXiv:0705.1740. 51 (2007)
6. Aerts, D., Sozzo, S., Tapia, J.: A quantum model for the ellsberg and machina paradoxes. In: Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.) QI 2012. LNCS, vol. 7620, pp. 48–59. Springer, Heidelberg (2012)
7. Aerts, D., Broekaert, J., Gabora, L., Veloz, T.: The Guppy effect as interference. In: Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.) QI 2012. LNCS, vol. 7620, pp. 36–47. Springer, Heidelberg (2012)
8. Berendt, B., Barkowsky, T., Freksa, C., Kelter, S.: Spatial representation with aspect maps. In: Freksa, C., Habel, C., Wender, K.F. (eds.) Spatial Cognition 1998. LNCS (LNAI), vol. 1404, pp. 313–336. Springer, Heidelberg (1998)
9. Fock, V.: Konfigurationsraum und zweite quantelung. Zeitschrift für Physik **75**, 622–647 (1932)
10. Gabora, L.: cognitive mechanisms underlying the origin and evolution of culture (2001)
11. Gabora, L., Aerts, D.: Contextualizing concepts using a mathematical generalization of the quantum formalism. J. Exp. Theor. Artif. Intell. **14**, 327–358 (2002)
12. Gabora, L., Rosch, E., Aerts, D.: Toward an ecological theory of concepts. Ecol. Psychol. **20**, 84–116 (2008)
13. Ganter, B., Stumme, G., Wille, R. (eds.): Formal Concept Analysis. Springer, Heidelberg (2005)
14. Goodchild, M.: Citizens as sensors: the world of volunteered geography. GeoJournal **69**, 211–221 (2007)
15. Jones, S., Hughes, J., Augustsson, L., Barton, D.: Report on the programming language Haskell 98 (1999)
16. Longley, P.A., Goodchild, M.F., Maguire, D.J., Rhind, D.W.: Geographic Information Systems and Science. Wiley, New York (2005)
17. Lü, L., Medo, M., Yeung, C.H., Zhang, Y.-C., Zhang, Z.-K., Zhou, T.: Recommender systems. Phys. Rep. **519**, 1–49 (2012)
18. Montello, D.R., Freundschuh, S.: Cognition of geographic information. In: McMaster, R.B., Usery, E.L. (eds.) A Research Agenda for Geographic Information Science, pp. 61–91. CRC Press, Boca Raton (2005)
19. Muller, J.J.-C., Lagrange, J.J.-P., Weibel, R.: GIS and Generalization: Methodology and Practice. Taylor & Francis Group, London (1995)
20. Raper, J.: Geographic information science. Annu. Rev. Inf. Sci. Technol. **43**, 1–117 (2009)
21. Rosch, E., Mervis, C.B.: Family resemblances: studies in the internal structure of categories. Cogn. Psychol. **7**, 573–605 (1975)

22. Rosch, E.: Cognitive reference points. Cogn. Psychol. **7**, 532–547 (1975)
23. Schmitt, I.: QQL: a DB&IR query language. VLDB J. **17**, 39–56 (2007)
24. Smith, E., Osherson, D.: Conceptual combination with prototype concepts. Cogn. Sci. **8**, 337–361 (1984)
25. Smith, B., Mark, D.: Ontology with human subjects testing. Am. J. Econ. Sociol. **58**, 245–312 (1998)
26. Weiser, P., Frank, A., Abdalla, A: Process composition and process reasoning over multiple levels of detail. In: 7th International Conference, GIScience 2012, Columbus, Ohio (2012)
27. Zadeh, L.A.: Fuzzy sets. Inform. Control **8**, 338–353 (1965)

Measuring Conceptual Entanglement in Collections of Documents

Tomás Veloz[1,3(✉)], Xiaozhao Zhao[2], and Diederik Aerts[3]

[1] Department of Mathematics, University of British Columbia, Kelowna, BC, Canada
tomas.veloz@ubc.ca
[2] School of Computer Science, Tianjin University, Tianjin, China
0.25eye@gmail.com
[3] Center Leo Apostel (Clea) and Department of Mathematics, Brussels Free University (VUB), Pleinlaan 2, 1050 Brussel, Belgium
diraerts@vub.ac.be

Abstract. Conceptual entanglement is a crucial phenomenon in quantum cognition because it implies that classical probabilities cannot model non-compositional conceptual phenomena. While several psychological experiments have been developed to test conceptual entanglement, this has not been explored in the context of Natural Language Processing. In this paper, we apply the hypothesis that words of a document are traces of the concepts that a person has in mind when writing the document. Therefore, if these concepts are entangled, we should be able to observe traces of their entanglement in the documents. In particular, we test conceptual entanglement by contrasting language simulations with results obtained from a text corpus. Our analysis indicates that conceptual entanglement is strongly linked to the way in which language is structured. We discuss the implications of this finding in the context of conceptual modeling and of Natural Language Processing.

Keywords: Quantum cognition · Conceptual entanglement · Quantum interaction · Information retrieval · Semantic modeling · Theory of concepts

1 Introduction

The extraction of relevant information from the web has been a fundamental area of research and development during the last decades [13]. In particular, several quantum-inspired approaches have been elaborated to perform standard tasks in Information Retrieval (see [28,29,31,38,40,41], and references therein). In order to extract information from the web it is necessary to establish a representation model. The most efficient techniques represent documents and terms as vectors in high dimensional spaces. These vectors are built based on statistical analyses of large corpora of web pages and documents. Examples of such methods include Latent Semantic Analysis [19], Hyperspace Analogue to Language [26], and Latent Dirichlet Allocation [15]. While these approaches have been shown

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 134–146, 2014.
DOI: 10.1007/978-3-642-54943-4_12,

to be extremely useful for information retrieval (IR) tasks, they rely on that counting term occurrences is **sufficient** to represent the meaning of a document in a corpus [37].

Increasing evidence shows that a term-based analysis of a corpus of text is not sufficient to perform information extraction in an optimal way [11,12]. As an alternative to this term-based methodology, semantic-based models which emphasize the conceptual information that is represented by the terms in the document have been proposed [22,23]. This means that the statistical information that we can obtain from the distribution of terms in a document can be complemented with conceptual information that we have in advance [25].

The concept-based methods applied in Natural Language Processing (NLP) and IR are based mainly on the use of Ontologies [30], and Word-net like structures [21]. These approaches have been useful to improve syntactic approaches to NLP and IR. However, they cannot account for elements that have been identified as of primary interest in concept theory. Namely, the context-dependence of concepts' meaning [34], and the non-compositional meaning of concept combinations [35]. While Word-net and some ontology-based approaches consider context, it has been shown that it is computationally unfeasible to handle context-sensitivity in general [27,36]. Moreover, none of both approaches considers the non-compositionality of concept combination.

To overcome these limitations, quantum-inspired models have been proposed [6,7]. In the quantum approach to concepts, the conceptual entity is assumed to be contextual. This means that its meaning emerges from its interaction with a certain context, analogous to how quantum states become actual after interacting with a measurement apparatus. Moreover, the mathematical formalism of Fock space allows to model the emergence of non-compositional meaning in conceptual combinations [2,3].

Conceptual entanglement has been investigated to detect quantum behaviour in the phenomenology of concepts [8,9,17,18]. The discovery of conceptual entanglement has been crucial because it implies that a classical probability framework is not sufficient to describe the non-compositionality of conceptual phenomena [16]. Particularly when a concept combination is entangled, a strong dependence between the terms that form instances of this concept combination is found. Such dependence is revealed by statistical tests, the so-called Bell-like inequalities [5].

The aim of the present paper is to investigate whether conceptual entanglement is of significant importance on written texts. In particular, we view a piece of text as a trace of the concepts the subject who write had in mind at the while writing (for a complete elaboration of this perspective see [10]). From here, we assume that it is possible to measure conceptual entanglement by observing statistical properties of these conceptual traces. A previous attempt to investigate this aspect is [4]. However, they analyze one concept combination only. We want to go one step further and investigate the extent to which entanglement can be found by automatic methods. If conceptual entanglement is significant on the web, then non-classical probabilities should become the *standard framework* for modeling in IR and NLP.

In Sect. 2 we explain how to measure entanglement using a co-occurrence corpus of text; in Sect. 3 we explain the methodology we used to analyze the corpus; and Sects. 4 and 5 illustrate the results and discuss their relevance.

2 Conceptual Entanglement

2.1 Detecting Conceptual Entanglement

Following [5], if we want to test whether two abstract entities $\mathcal{A}$ and $\mathcal{B}$ are entangled, we need to set two observables for each entity, each observable having two possible outcomes. Hence, we will denote these observables and their outcomes by $A = \{A_1, A_2\}, A' = \{A'_1, A'_2\}$ for entity $\mathcal{A}$, and $B = \{B_1, B_2\}$, and $B' = \{B'_1, B'_2\}$ for entity $\mathcal{B}$. Next, we assume that each measurement $X \in \{A, A', B, B'\}$ corresponds the value 1 if X_1 is observed and -1 if X_2 is observed. From here, we can define a composed experiment $XY \in \{AB, A'B, AB', A'B'\}$, corresponding to 1 in case X_1Y_1 or X_2Y_2 is observed, and to -1 in case X_1Y_2 or X_2Y_1 is observed.

If we perform each experiment XY a large number of times, we can estimate the expected value $E(XY)$ of each composed experiment. The Clauser-Horn-Shimony-Holt (CHSH) inequality states that if

$$-2 \leq E(AB) + E(A'B) + E(AB') - E(A'B') \leq 2, \tag{1}$$

is not hold, then the entities are entangled. The violation of (1) implies the non-existence of one Kolmogorovian probability model for the considered joint experiments [1].

It is important to mention that the entities $\mathcal{A}$ and $\mathcal{B}$ need not be physical entities. The CHSH inequality is a statistical test that verifies whether or not it is possible to model a set of data in a classical probability setting.

For example in [8], the entities $\mathcal{A}$ and $\mathcal{B}$ refer to the concepts *Animal* and *Acts*, respectively. The possible collapse states, i.e. the observables, of these entities were defined as A = {'Horse','Bear'}, A' ={'Tiger','Cat'}, and B ={'Growls', 'Whinnies'}, and B' ={'Snorts','Meows'}. A psychological experiment where participants chose the combination that best represented the combination *The Animal Acts*, considering elements from AB, AB', $A'B$, and $A'B'$ was performed, and the expected values of these joint observables were found to violate (1) with value 2.4197. Hence, it is concluded that the concepts $\mathcal{A}$ and $\mathcal{B}$ are entangled.

2.2 Concepts and Meaning of a Document

Someone who is writing a piece of text usually does not have the exact wording in mind, but only a particular idea of the intended meaning. If an idea is not concrete enough, it is hard to express it properly. Usually in this case, the idea is broken into a set of interconnected concrete ideas. These concrete ideas are easier to put into sentences, which in turn form the paragraphs of the document.

Moreover, it is only at the time of writing of a document that the words that express these concrete ideas are elicited. Hence, in this process we can identify two steps: an abstract idea is converted into several concrete ideas, and these concrete ideas are converted into sentences. Following this reflection, we propose a cognitive interpretation of the notion of meaning for pieces of text. More in particular, we assume that the process of converting an abstract idea into concrete ideas corresponds to the formation of entities of meaning, and that these entities of meaning collapse to states which are represented by text, in the form of words or sentences. The meaning of a piece of text can thus be understood as the solution of an inverse problem, i.e. the piece of text plays the role of the collapsed state of an entity of meaning, a so-called conceptual trace, and the meaning of the piece of text is obtained by identifying the entity of meaning that collapsed to this piece of text.

In this work we do not focus on the inverse problem formulation as a general framework for meaning in documents. For an in-depth elaboration of this idea we refer to [10]. Instead, we assume the existence of concepts underlying the meaning of a document, and test whether or not these conceptual traces exhibit entanglement in text corpora.

2.3 Measuring Conceptual Entanglement in Text Corpora

Let T be a corpus of text containing a set of terms E. Note that each term $t \in E$ can be considered an instance of one or many concepts. For example, the term *Dog* can be considered an instance of concepts *Pet*, *Animal*, *Mammal*, etc. Let $C_1, C_2 \subseteq E$ be sets of exemplars of concepts $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Let W be a positive integer. We say that concepts $\mathcal{C}_1$ and $\mathcal{C}_2$ W-co-occur if there exists a sequence s of W consecutive terms in the corpus T such that one term in C_1 and one term in C_2 co-occur in s. We call s a window of size W. In general, we can compute the W-co-occurrence frequency $F_W(\mathcal{C}_1, \mathcal{C}_2) : C_1 \times C_2 \to \mathbb{N}$ of exemplars of concepts $\mathcal{C}_1$ and $\mathcal{C}_2$ within windows of size W in the corpus.

Suppose we choose C_1 and C_2 to have 4 terms each, and we partition $C_1 = \{A, A'\}$ and $C_2 = \{B, B'\}$ such that each set in each partition has two terms. We then have that A and A' (B and B') are pairs of exemplars referring to concept C_1 (C_2). Hence, the co-occurrence measurements $AB, A'B, AB'$ and $A'B'$ can be used to measure the entanglement of concepts C_1 and C_2 in the corpus T. This is analogous to what has been done in previous psychological studies in conceptual entanglement [4,8,17,18]. However, in this work, instead of performing a psychological experiment where participants are requested to choose co-occurrent terms from a list, we extract these co-occurrences from a corpus of text.

For each pair $XY \in \{AB, A'B, AB', A'B'\}$, we compute their term co-occurrence

$$F_W(X_iY_j) = \sum_{s \in T} N(X_iY_j, s), \tag{2}$$

where $i, j \in \{1, 2\}$ and $N(X_i, Y_j, s)$ is equal to one if the pair X_iY_j co-occurs in the window s of the corpus T, and zero if not.

From here we can estimate the expected values of each co-occurrence experiment

$$E(XY) = \frac{F(X_1Y_1) + F(X_2Y_2) - F(X_1Y_2) - F(X_2Y_1)}{F(X_1Y_1) + F(X_2Y_2) + F(X_1Y_2) + F(X_2Y_1)}. \quad (3)$$

Note that to measure conceptual entanglement in a corpus of text, we must first identify a set of exemplars for each concept. However, a multitude of concepts can be built upon a list of words, so that it is not possible to know in advance how to group words representing instances concepts. Indeed, psychological experiments where participants have to build categories choosing groups of words from a list show that, although some words fall regularly into similar kinds of categories, every word set can be potentially considered as a category [32].

Instead of proposing a methodology to identify concepts, we assume that relevant words in a document correspond to relevant traces of the concepts that entail the document's meaning. Hence, without knowing exactly which concepts we are measuring entanglement for, we are able to represent them by sets of terms C_1 and C_2 from the relevant words of a document, following a given relevance criterion. Therefore, we propose the following *brute force* algorithm to measure entanglement:

1. Select two sets C_1 and C_2 of relevant terms from the corpus, having k words each.
2. Verify the CHSH inequality for all the elements $(c_1, c_2) \in \mathcal{P}(C_1, 4) \times \mathcal{P}(C_2, 4)$, where $\mathcal{P}(C, n)$ is the set of all the possible subsets of length n of the set C.

In order to perform the second step of the algorithm, we verify if there is a way of partitioning c_1 and c_2 such that (1) is violated. To do so, we consider all the possible row/column permutations of the 4×4 matrix $F_W(c_1, c_2)$, and apply the expected co-occurrence formula (3) to estimate (1). Note that we have 12 different partitions for each c_1 and c_2, leading to 144 partitions for (c_1, c_2). Hence, we say that C_1 and C_2 are entangled if there exists $(c_1, c_2) \in \mathcal{P}(C_1, 4) \times \mathcal{P}(C_2, 4)$ such that (at least) one of the 144 co-occurrence matrices generated from (c_1, c_2) violates the CHSH inequalities. Our aim is to estimate the likelihood of finding instances (c_1, c_2) of concepts C_1 and C_2, such that they can be partitioned in a way that violates the CHSH inequalities.

2.4 Statistical Considerations

Consider the following question: Assume we know in advance the term co-occurrence frequency distribution of the corpus T, i.e. we know the probability $\rho_T(n)$ that two terms co-occur n times in T for each value of n; what is the likelihood of building concepts $(c_1, c_2) \in \mathcal{P}(T, 4) \times \mathcal{P}(T, 4)$ that violate the CHSH inequality in the corpus T?

In order to answer this question, we need to compare the kinds of co-occurrence matrix that $\rho_T(n)$ delivers, to the kinds of frequency matrix that violate the CHSH inequality.

Note that inequality (1) is likely to be violated if the three first elements have the same signs and relatively large values, and the fourth term has the

Table 1. Co-occurrence table that violates the CHSH inequality. The leftmost column and top row indicate terms of concepts C_1 and C_2, respectively. The interior 4×4 table indicates the types of co-occurrence. L stands for 'Large' and S stands for 'Small'. If we switch columns B_2/B_1 and B_2'/B_1', the inequality is violated with a negative value.

$F_W()$	B_1	B_2	B_1'	B_2'
A_1	L	S	L	S
A_2	S	L	S	L
A_1'	L	S	S	L
A_2'	S	L	L	S

opposite sign and a relatively large value. Indeed, in such case each row leads to an expected value near to 1 (or -1), except for the fourth row, which leads to an expected value near to -1 (or 1). Table 1 shows an example of frequency data that violates the CHSH inequality with a positive value near 4.

From Table 1 we infer that if $\rho_T(n)$ does not have co-occurrences that are relatively larger than others, then the likelihood of violating the CHSH inequality is low. Therefore, a corpus T will be able to violate the CHSH inequality if $\rho_T(n)$ is significant for small values of n, and also for large values of n. This argument is qualitative and intuitive, but it can be better explained with an example. Suppose the corpus T has a maximal co-occurrence equal to 100, and the co-occurrences are divided into three groups of values: small values $G_S = \{1, ..., 5\}$, intermediate values $G_I = \{10, ..., 20\}$, and large values $G_L = \{50, ..., 100\}$. Note that, in case that all the co-occurrence frequencies belong to a single group, the expected value $E(XY)$ given by Eq. (3) is likely to be small. However, if one of the values belong to G_L and all the other values belong to G_S, then $E(XY)$ will be dominated by the value in G_L. A list of other possible cases is shown in Table 2.

From Table 2 we can see that intermediate co-occurrence values tend to diminish the likelihood if violating the CHSH inequality. Note that the question we analyze in this section assumes we can choose the partition of c_1 and c_2 that maximizes the evaluation of (1). Indeed, Table 2 considers the partition that maximizes the evaluation of (1) for each case.

Table 2. Co-occurrence frequencies and their expected values determined from Eq. (3). Intermediate frequencies tend to decrease the likelihood of obtaining large expected values.

	Frequencies	Expected values		Frequencies	Expected values		Frequencies	Expected values
1	SSSS	S	5	IIII	S	9	LIII	L
2	ISSS	S	6	LSSS	L	10	LSSL	L
3	ISSI	I	7	LSSI	L	11	LSIL	L
4	ISII	S	8	LSII	L	12	LIIL	L

3 Method

3.1 Statistical Language Analysis

We first computed the probability $p_B(\lambda)$ that, given a distribution $\rho(n)$ of co-occurrences, there exists a partition for two random sets, having 4 terms each, that violates the CHSH inequality. The distribution of co-occurrences $\rho(n)$ is modeled with a B-bounded Zipfian distribution $Z_B(\lambda, n)$ of parameter λ. We have chosen a Zipfian distribution because it has been shown to model the statistics of term co-occurrence in English language [24], and we bounded the distribution by a parameter B to model the fact that the term co-occurrence frequency is limited by the corpus size. We also computed the likelihood of finding conceptual entanglement using homogeneous and Poisson distributions. These distributions do not exhibit significant entanglement so we do not present their results. However, we want to remark that this is consistent with the reasoning of Sect. 2.4 because neither homogenous nor Poisson distributions assign large probabilities to both small and large values of n.

3.2 The Collection of Documents

We evaluate the aforementioned entanglement test on TREC collection: WSJ8792. Lemur 4.12[1] is used for indexing. The collection is pre-processed by removing stop terms and applying the Porter stemmer. Among TREC topics 151–200, 32 topics with more than 70 truly relevant documents (judged by users) are selected. For certain topics, all truly relevant documents are segmented into windows of W consecutive terms ($W = 20, 10, 5$, in our experiments). We considered two alternative ways to automatically generate concepts C_1 and C_2:

- ☐ Top 20 frequent terms $\{t_1^i, t_2^i, \ldots, t_{20}^i\}$ of each topic $i = 1, ..., 32$, are extracted from the set of truly relevant documents. Concept C_1^i consists of top 10 frequent terms, i.e., $\{t_1^i, t_2^i, \ldots, t_{10}^i\}$, and the rest of terms forms concept C_2^i.
- ☐ Top 20 terms with largest tf-idf are extracted from the set of truly relevant documents, denoted as $\{{t'_1}^i, {t'_2}^i, \ldots, {t'_{20}}^i\}$ (ranked by tf-idf values). Concept C_1^i consists of top 10 terms, i.e., $\{{t'_1}^i, {t'_2}^i, \ldots, {t'_{10}}^i\}$, and the rest of terms form concept C_2^i.

Then the windows are used to count the co-occurrence frequencies between exemplars from concepts C_1^i and C_2^i for each $i = 1, ..., 32$.

4 Results

4.1 Statistical Language Analysis

Figure 1 shows the probability $p_B(\lambda)$ described in Sect. 3.1, for $0 < \lambda \leq 2$, and $B = 10, 50, 100, 500$, on the left. Note that for $\lambda \sim 0.3$, the proportion

[1] Lemur is an open source project that develops search engines and text analysis tools for research and development of information retrieval and text mining softwares.

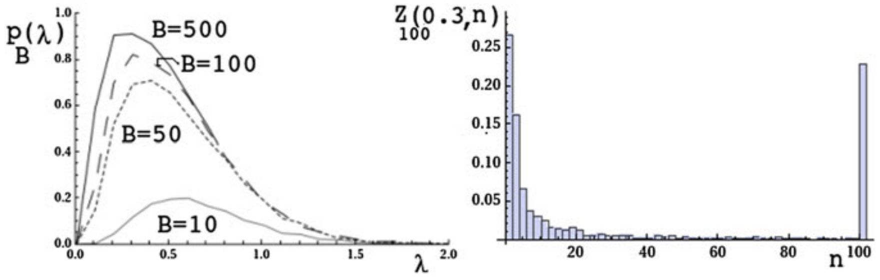


Fig. 1. Entanglement probability for random co-occurrence matrix on the left, and two examples of Zipfian distributions at the center and right, respectively.

of entangled concepts is maximized for all values of B. Note that English co-occurrence distribution is estimated to be a Zipfian distribution of parameter $0.66 \leq \lambda \leq 0.8$ [24]. This range exhibits around 50 % of entanglement when $B \geq 50$. This indicates that when the size of the corpus is statistically significant, and hence co-occurrences are allowed to have large values, the likelihood of finding entangled concepts is non-neglectable. Therefore, we predict that entanglement is not a rare effect, and hence it should be observable from the *brute force* procedure proposed in Sect. 2.3.

The middle and right plots in Fig. 1 show the Zipfian co-occurrence distributions $Z_B(\lambda, n)$ for $B = 100$, using $\lambda = 0.3$ and $\lambda = 0.7$, respectively. These distributions confirm that when co-occurrence probability mass is concentrated on small and large values, conceptual entanglement is more likely to be detected.

4.2 Corpus Analysis

Figure 2 shows the proportion $p_W(T)$ of entangled subsets of terms for the different topics in the corpus. The black curve corresponds to $W = 20$, the gray curve corresponds to $W = 10$ and the black dashed curve corresponds to $W = 5$. The left plot is based on the data obtained using the frequency relevance method, and the right plot is based on the tf-idf method. In both cases, $p_W(T)$ is strictly decreasing with respect to W. Moreover, we observe that $p_W(T)$ does not have uniform variation with respect to W. For example in the tf-idf case (right plot in Fig. 2), $p_W(7)$ has a very close value for $W = 5, 10$, and 20. Analogously, $p_W(T)$ does not exhibit major variations for topics $1, 10, 16, 19$, and topics 24 to 32 for $W = 20$ and 10. However, $p_W(T)$ exhibits a major change when W changes from 10 to 5 for most topics. Therefore, given two topics T_1, T_2, and two window sizes W_1, W_2, we cannot ensure that $p_{W_1}(T_1) < p_{W_1}(T_2)$ implies that $p_{W_2}(T_1) < p_{W_2}(T_2)$. Hence, there is not a topic-sorting such that $p_W(T)$ is a strictly decreasing function for $W = 5, 10$, and 20 simultaneously. Analogously, for the term-relevance method, while topics have a different set of relevant terms, there is not a unique sorting of the topics set that leads to $p_W(T)$ decreasing for $W = 20, 10$ and 5 simultaneously.

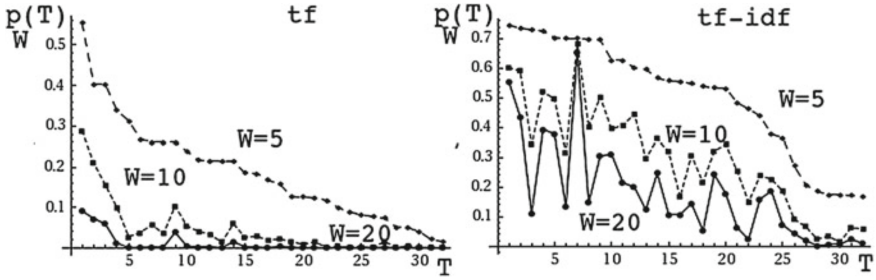


Fig. 2. Entanglement probability for the 20 most frequent terms. The left plot corresponds to the co-occurrence data for most frequent terms, and the right plot corresponds to the co-occurrence data for the td-idf highest ranked terms.

In both methods, topics were sorted such that $p_5(T)$ is decreasing. By doing so, we avoid that curves cross each other because $p_W(T)$ is strictly decreasing with respect to W. Note that $p_W(T)$ reaches significantly larger values for the tf-idf method. This is consistent with the fact that tf-idf is a better indicator of term relevance than the term frequency.

4.3 Distribution of Co-occurrences

We computed the histogram $\rho_T(n)$ for each topic T in the corpus, and for each window size. In Fig. 3, we plot examples considering window sizes $W = 20, 10$

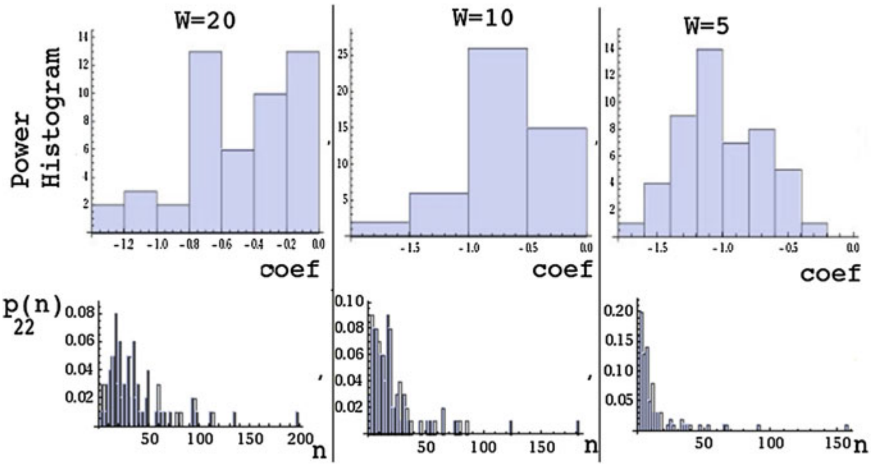


Fig. 3. Co-occurrence distribution examples extracted from the tf-idf method. The first row corresponds to the third topic, and the second row corresponds to topic number 7. The left plot of each row corresponds to $W = 22$, the center plot to $W = 10$ and the right plot to $W = 5$.

and 5, for topics 3 and 7 of the co-occurrences data obtained from the tf-idf method.

We observe that when the window size decreases, $\rho_T(n)$ tends to decrease the number of co-occurrences on intermediate values. However, large co-occurrence values remain probable. From here we infer that when the window size decreases, the co-occurrence distribution becomes similar to a bounded Zipfian distribution. Observe for example the first row of Fig. 3; we can see qualitative changes on the probability distribution for the different window sizes. This is consistent with the strong changes observed in $p_W(T = 3)$ of Fig. 2 for the tf-idf method. Analogously, the second row does not exhibit major changes in its co-occurrence distribution. This is again reflected in $p_W(T = 3)$ of Fig. 2, where no major changes are also observed. This is consistent with the fact that the $p_W(T)$ is strictly decreasing with respect to W.

5 Conclusion

We performed a qualitative analysis to corroborate that conceptual entanglement is a significant effect in written texts. The analysis is based on the hypothesis that pieces of text are traces of concepts, and that such concepts entail the meaning of documents [10]. This corroboration is supported by an analysis of statistical properties of the English language, and of 32 topic-structured text corpora. Our language analysis indicates that the statistical co-occurrence distribution of the English language has a significant tendency to build conceptual entities that are entangled. Particularly, term co-occurrence matrices built from a distribution that models co-occurrence of words in English, violate the CHSH inequality in around 50 % of the cases. It should be noted that this analysis is based only on the statistical distribution of co-occurrences of English, and does not consider any semantic or linguistic aspect of the English language.

We further analyzed a corpus separated by 32 topics, and found results consistent with the language analysis. For each topic, we built a matrix computing the term co-occurrence of the 10 most relevant terms with respect to the next 10 most relevant terms, considering window sizes $W = 5, 10$, and 20 to measure co-occurrence. From here, we computed the proportion of 4×4 sub-matrices of the co-occurrence matrix that violate the CHSH inequality. We observe that the tf-idf relevance delivers more entanglement than the term frequency relevance measure. This is consistent with the fact that td-idf is a better relevance measure than term frequency. Although some topics exhibit more entanglement than others, we identify a strong tendency to find conceptual entanglement for most topics. Moreover, conceptual entanglement decreases with respect to the window size. This is consistent with the fact that word correlations are noisy for large window sizes [24]. For short window sizes, we have more chance to keep only meaningful correlations, and hence entanglement is observed with more clarity. In addition, we found that for shorter window sizes, the distribution of co-occurrences becomes more similar to a bounded Zipfian distribution. Indeed, in Fig. 2, we can see that when the distributions look more like a Zipfian distribution, i.e. when $W = 5$, the average conceptual entanglement for words selected by the tf-idf relevance measure is 50 % of entanglement.

A fundamental and novel element of this work is that we do not build in advance the concepts for which we will measure entanglements. Instead, we assume that relevant words of a topic-corpus are relevant traces of the concepts that entail the meaning of the topic [10]. Hence, if the co-occurrence of these traces violates the CHSH inequality, we conclude that the concepts that entail the meaning of the document are entangled. Therefore, we do not focus on the generation of categories or taxonomies. However, we also suggest that one interesting extension of this work would consist in testing conceptual entanglement from categories that are automatically built as in [33,39]. We also propose to study why some topics exhibit more entanglement than others, count term co-occurrence in structured sentences, e.g. Frame-Net [14], rather than in windows of text, and evaluate other statistical conditions that provide a more precise classification of entanglement [20].

References

1. Accardi, L., Fedullo, A.: On the statistical meaning of complex numbers in quantum mechanics. Lettere Al Nuovo Cimento (1971–1985) **34**(7), 161–172 (1982)
2. Aerts, D.: General quantum modeling of combining concepts: a quantum field model in Fock space. arXiv preprint arXiv:0705.1740 (2007)
3. Aerts, D.: Quantum structure in cognition. J. Math. Psychol. **53**(5), 314–348 (2009)
4. Aerts, D.: Interpreting quantum particles as conceptual entities. Int. J. Theor. Phys. **49**(12), 2950–2970 (2010)
5. Aerts, D., Aerts, S., Broekaert, J., Gabora, L.: The violation of bell inequalities in the macroworld. Found. Phys. **30**(9), 1387–1414 (2000)
6. Aerts, D., Gabora, L.: A theory of concepts and their combinations i: the structure of the sets of contexts and properties. Kybernetes **34**(1/2), 167–191 (2005)
7. Aerts, D., Gabora, L.: A theory of concepts and their combinations ii: a Hilbert space representation. Kybernetes **34**(1/2), 192–221 (2005)
8. Aerts, D., Sozzo, S.: Quantum structure in cognition: why and how concepts are entangled. In: Song, D., Melucci, M., Frommholz, I., Zhang, P., Wang, L., Arafat, S. (eds.) QI 2011. LNCS, vol. 7052, pp. 116–127. Springer, Heidelberg (2011)
9. Aerts, D., Sozzo, S.: Quantum entanglement in concept combinations. arXiv, preprint arXiv:1302.3831 (2013)
10. Aerts, D., Sozzo, S., Broekaert, J., Veloz, T.: Meaning-focused and quantum-inspired information retrieval. In: Atmanspacher, H., Haven, E., Kitto, K., Raine, D. (eds.) QI 2013, LNCS, vol. 8369, pp. 71–83. Springer, Heidelberg (2014)
11. Baeza-Yates, R., Broder, A.Z., Maarek, Y.: The new frontier of web search technology: seven challenges. In: Ceri, S., Brambilla, M. (eds.) Search Computing II. LNCS, vol. 6585, pp. 3–9. Springer, Heidelberg (2011)
12. Baeza-Yates, R., Raghavan, P.: Chapter 2: next generation web search. In: Ceri, S., Brambilla, M. (eds.) Search Computing. LNCS, vol. 5950, pp. 11–23. Springer, Heidelberg (2010)
13. Baeza-Yates, R., Ribeiro-Neto, B., et al.: Modern Information Retrieval, vol. 463. ACM Press, New York (1999)

14. Baker, C.F., Fillmore, C.J., Lowe, J.B.: The Berkeley Framenet project. In: Proceedings of the 36th Annual Meeting of the Association for Computational Linguistics and 17th International Conference on Computational Linguistics, vol. 1, pp. 86–90. Association for Computational Linguistics (1998)
15. Blei, D.M., Ng, A.Y., Jordan, M.I.: Latent Dirichlet allocation. J. Mach. Learn. Res. **3**, 993–1022 (2003)
16. Bruza, P.D., Kitto, K., Ramm, B., Sitbon, L., Song, D., Blomberg, S.: Quantum-like non-separability of concept combinations, emergent associates and abduction. Logic J. IGPL **20**(2), 445–457 (2012)
17. Bruza, P., Kitto, K., Nelson, D., McEvoy, C.: Extracting spooky-activation-at-a-distance from considerations of entanglement. In: Bruza, P., Sofge, D., Lawless, W., van Rijsbergen, K., Klusch, M. (eds.) QI 2009. LNCS, vol. 5494, pp. 71–83. Springer, Heidelberg (2009)
18. Bruza, P.D., Kitto, K., McEvoy, D., McEvoy, C.: Entangling Words and Meaning. Oxford University Press, Oxford (2008)
19. Dumais, S.T.: Latent semantic analysis. Annu. Rev. Inf. Sci. Technol. **38**(1), 188–230 (2005)
20. Dzhafarov E., Kujala, J.: On selective influences, marginal selectivity, and Bell/CHSH inequalities. Topics in Cognitive Science (in press, 2013)
21. Fellbaum, C.: Wordnet. In: Poli, R., Healy, M., Kameas, A. (eds.) Theory and Applications of Ontology: Computer Applications, pp. 231–243. Springer, Heidelberg (2010)
22. Fernández, M., Cantador, I., López, V., Vallet, D., Castells, P., Motta, E.: Semantically enhanced information retrieval: an ontology-based approach. Web Semant.: Sci., Serv. Agents World Wide Web **9**(4), 434–452 (2011)
23. Giunchiglia, F., Kharkevich, U., Zaihrayeu, I.: Concept search. In: Aroyo, L., Traverso, P., Ciravegna, F., Cimiano, P., Heath, T., Hyvönen, E., Mizoguchi, R., Oren, E., Sabou, M., Simperl, E. (eds.) ESWC 2009. LNCS, vol. 5554, pp. 429–444. Springer, Heidelberg (2009)
24. Ferrer i Cancho, F., Solé, R.V.: The small world of human language. Proc. R. Soc. London. Series B: Biol. Sci. **268**(1482), 2261–2265 (2001)
25. Kharkevich, U.: Concept search: semantics enabled information retrieval. Ph.D. thesis, University of Trento (2010)
26. Lund, K., Burgess, C.: Producing high-dimensional semantic spaces from lexical co-occurrence. Behav. Res. Methods **28**(2), 203–208 (1996)
27. Mandala, R., Takenobu, T., Hozumi, T.: The use of WordNet in information retrieval. In: Use of WordNet in Natural Language Processing Systems: Proceedings of the Conference, pp. 31–37 (1998)
28. Melucci, M.: An investigation of quantum interference in information retrieval. In: Cunningham, H., Hanbury, A., Rüger, S. (eds.) IRFC 2010. LNCS, vol. 6107, pp. 136–151. Springer, Heidelberg (2010)
29. Melucci, M., Rijsbergen, K.: Quantum mechanics and information retrieval. Adv. Top. Inf. Retrieval **33**, 125–155 (2011)
30. Paralic, J., Kostial, I.: Ontology-based information retrieval. In: Proceedings of the 14th International Conference on Information and Intelligent systems (IIS 2003), Varazdin, Croatia, pp. 23–28 (2003)
31. Piwowarski, B., Lalmas, M.: A quantum-based model for interactive information retrieval. In: Azzopardi, L., Kazai, G., Robertson, S., Rüger, S., Shokouhi, M., Song, D., Yilmaz, E. (eds.) ICTIR 2009. LNCS, vol. 5766, pp. 224–231. Springer, Heidelberg (2009)

32. Ranjan, A., Srinivasan, N.: Dissimilarity in creative categorization. J. Creative Behav. **44**(2), 71–83 (2010)
33. Roark, B., Charniak, E.: Noun-phrase co-occurrence statistics for semiautomatic semantic lexicon construction. In: Proceedings of the 17th International Conference on Computational Linguistics, vol. 2, pp. 1110–1116. Association for Computational Linguistics (1998)
34. Rosch, E.: Principles of categorization. In: Margolis, E., Laurence, S. (eds.) Concepts: Core Readings, pp. 189–206. MIT Press, Cambridge (1999)
35. Smith, E.E., Osherson, D.N.: Conceptual combination with prototype concepts. Cogn. Sci. **8**(4), 337–361 (1984)
36. Strang, T., Linnhoff-Popien, C.: A context modeling survey. In: Workshop Proceedings (2004)
37. Wallach, H.M.: Topic modeling: beyond bag-of-words. In: Proceedings of the 23rd International Conference on Machine Learning, pp. 977–984. ACM (2006)
38. Wang, J., Song, D., Kaliciak, L.: Tensor product of correlated text and visual features: A quantum theory inspired image retrieval framework (2010)
39. Widdows, D., Dorow, B.: A graph model for unsupervised lexical acquisition. In: Proceedings of the 19th International Conference on Computational Linguistics, vol. 1, pp. 1–7. Association for Computational Linguistics (2002)
40. Widdows, D.: Geometry and Meaning. CSLI Publications, Stanford (2004)
41. Zuccon, G., Piwowarski, B., Azzopardi, L.: On the use of complex numbers in quantum models for information retrieval. In: Amati, G., Crestani, F. (eds.) ICTIR 2011. LNCS, vol. 6931, pp. 346–350. Springer, Heidelberg (2011)

Looking at Vector Space and Language Models for IR Using Density Matrices

Alessandro Sordoni(✉) and Jian-Yun Nie(✉)

DIRO, Université de Montréal, Montréal, QC H3C 3J7, Canada
{sordonia,nie}@iro.umontreal.ca

Abstract. In this work, we conduct a joint analysis of both Vector Space and Language Models for IR using the mathematical framework of Quantum Theory. We shed light on how both models allocate the space of density matrices. A density matrix is shown to be a general representational tool capable of leveraging capabilities of both VSM and LM representations thus paving the way for a new generation of retrieval models. We analyze the possible implications suggested by our findings.

1 Introduction

Information Retrieval (IR) has nowadays become the focus of a multidisciplinary research, combining mathematics, statistics, philosophy of language and of the mind and cognitive sciences. In addition to these, it has been recently argued that IR researchers should be looking into particular concepts borrowed from physics. Particularly, it was first evoked in 2004 in Van Rijsbergen's pioneering manuscript "The Geometry of Information Retrieval" [15] that Quantum Theory principles could be beneficial to IR.

Despite Quantum Theory (QT) being an extremely successful theory in a number of fields, the idea of giving a quantum look to Information Retrieval could be at first classified as unjustified euphoria. However, the main motivation for this big leap should be found in the powerful mathematical framework embraced by the theory which offers a generalized view of probability measures defined on vector spaces. Events correspond to subspaces and generalized probability measures are parametrized by a special matrix, usually called *density matrix* or *density operator*. From an IR point of view, it is extremely attractive to deal with a formalism which embraces probability and geometry, those being two amongst the pillars of modern retrieval models. Even if we believe that an unification of retrieval approaches would be out-of-reach due to the intrinsic complexity of modern models, the framework of QT could give interesting overviews and change of perspective thus fostering the design of new models. The opening lines of Van Rijsbergen manuscript perfectly reflect this interpretation: "It is about a way of looking, and it is about a formal language that can be used to describe the objects and processes in Information Retrieval" [15]. To this end, the last chapter of Van Rijsbergen's book is mainly dedicated to a preliminary analysis

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 147–159, 2014.
DOI: 10.1007/978-3-642-54943-4_13,

of IR models and tasks by means of the language of QT. Amongst others, the author deals with coordinate level matching and pseudo-relevance feedback.

Since then, the methods that stemmed from Van Rijsbergen's initial intuition provided only limited evidence about the real usefulness and effectiveness of the framework for IR tasks [13,20,26,29]. Several proposed approaches took inspiration from the key notions of the theory such as superposition, interference or entanglement. In [28], the authors use interference effects in order to model document dependence thus relaxing the strong assumption imposed by the probability ranking principle (PRP). An alternative solution to this problem has been proposed in [26], in which a novel reranking approach is proposed using a probabilistic model inspired by the notion of quantum measurement. In [13], the authors represent documents as subspaces and queries as density matrices. However, both documents and queries are estimated through passage-retrieval like heuristics, i.e. a document is divided into passages and is associated to a subspace spanned by the vectors corresponding to document passages. Different representations for the query density matrix are tested but none of them led to good retrieval performance. In [20], the authors work out an explicit interference formula in a topic model setting. Although marginal improvements are obtained over the baseline model, the ad-hoc application of the interference formula does not provide solid evidence towards the usefulness of the theory itself.

In order to give a stronger theoretical status to QT as a necessary or more general theory for IR, some authors step back into more theoretical considerations exposing potential improvements achievable over state-of-the-art models [9,10,14,23]. In [10], the author shows how detection theory in QT offers a generalization of the Neyman-Pearson Lemma (NPL), which is shown to be strictly linked to the PRP. Dramatic potential improvements could be obtained by switching to such more general framework. Widdows [23] observed that the Vector Space Model (VSM) lacked a logic like the Boolean model. Through the formalism for quantum logic illustrated by Birkoff and Von Neumann [1], Widdows defines a geometry of word meaning by expressing word negation based on the notion of orthogonality. Recently, the work by Melucci and Van Risjbergen [11] and Song et al. [18] offered a comprehensive review of QT methods for IR along with some insightful thoughts about possible reinterpretations of general IR methods (such as LSI [3]) from a quantum point of view. This paper shares the main purpose of the latter works.

In the ending section of his book, Van Rijsbergen calls for a reinterpretation of the Language Modeling (LM) approach for IR by means of the quantum framework. To our knowledge, such an interpretation has not been presented yet in the literature and this work can be considered as a first attempt to fill this gap. We provide a theoretical analysis of both LM and the VSM approach from a quantum point of view. In both models, documents and queries can be represented by means of density matrices. A density matrix is shown to be a general representational tool capable of leveraging capabilities of both VSM and LM representations thus paving the way for a new generation of retrieval models. As a conclusion, we analyze the possible implications suggested by our findings.

2 Quantum Probability and Density Matrices

In QT, the probabilistic space is naturally encapsulated in a complex vector space, specifically a Hilbert space, noted $\mathbb{H}^n$. We adopt the notation $|e_1\rangle, \ldots, |e_n\rangle$[1] to denote the standard basis vectors in $\mathbb{H}^n$. In QT, events are no more defined as subsets but as subspaces, more specifically as projectors onto subspaces. Given a ket $|u\rangle$, the projector $|u\rangle\langle u|$ onto $|u\rangle$ is an elementary event of the quantum probability space, also called *dyad*. A dyad is always a projector onto a 1-dimensional space. Generally, a unit vector $|v\rangle = \sum_i \upsilon_i |u_i\rangle$, $\upsilon_i \in \mathbb{H}$, $\sum_i |\upsilon_i|^2 = 1$, is called a *superposition* of the $|u_i\rangle$ where $|u_1\rangle, \ldots, |u_n\rangle$ form an orthonormal basis for $\mathbb{H}^n$.

A density matrix ρ is a symmetric positive semi-definite matrix of trace one. In QT, a density matrix defines the state of a system (a particle or an ensemble of particles) under consideration. Gleason's famous theorem [5] ensures that a density matrix is the unique way of defining quantum probability measures through the mapping $\mu_\rho(|u\rangle\langle u|) = \mathrm{tr}(\rho|u\rangle\langle u|)$. The measure μ ensures that $\forall |u\rangle, \mu(|u\rangle\langle u|) \geq 0$. This is because, $\mu_\rho(|u\rangle\langle u|) = \langle u|\rho|u\rangle \geq 0$ because ρ is positive semi-definite. Moreover, if $|u_1\rangle, \ldots, |u_n\rangle$ form an orthonormal system for $\mathbb{H}^n$, the probabilities for the dyads $|u_i\rangle\langle u_i|$ sum to one, i.e. they can be understood as disjoints events of a classical sample space. Given that $\sum_i |u_i\rangle\langle u_i| = I_n$, the identity matrix, we have $\sum_i \mathrm{tr}(\rho|u_i\rangle\langle u_i|) = \mathrm{tr}(\rho \sum_i |u_i\rangle\langle u_i|) = \mathrm{tr}(\rho) = 1$. Therefore, for orthogonal decompositions of the vector space[2], a quantum probability measure μ reduces to a classical probability measure.

Any classical discrete probability distribution can be seen as a mixture over n elementary points, i.e. a parameter $\vec{\theta} = (\theta_1, \ldots, \theta_n)$, $\theta_i \geq 0$, $\sum_i \theta_i = 1$. The density matrix is the straightforward generalization of this idea by considering a mixture over orthogonal dyads[3], i.e. $\rho = \sum_i \upsilon_i |u_i\rangle\langle u_i|$, $\upsilon_i \geq 0$, $\sum_i \upsilon_i = 1$. Given a density matrix ρ, one can find the components dyads by taking its eigendecomposition and building a dyad for each eigenvector. We note such decomposition by $\rho = R\Lambda R^\dagger = \sum_{i=1}^n \lambda_i |r_i\rangle\langle r_i|$, where $|r\rangle_i$ are the eigenvectors and λ_i their corresponding eigenvalues. This decomposition always exists for density matrices [12]. Note that the vector of eigenvalues $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ belongs to the simplex of classical discrete distributions over n points. If the distribution $\vec{\lambda}$ lies

[1] The Dirac notation establishes that $|u\rangle$ denotes a unit norm vector in $\mathbb{H}^n$ and $\langle u|$ its conjugate transpose.

[2] In a more general formulation of the theory, a quantum probability measure reduces to a classical probability measure for any set $\mathcal{M} = \{M_i\}$ of positive operators M_i such that $\sum_i M_i = I_n$. The set $\mathcal{M}$ is called Positive-Operator Valued Measure (POVM) [12]. Therefore, the properties reported in this paper which apply to a complete set of mutually orthogonal projectors equally hold for a general POVM.

[3] In general, the dyads in the mixture don't need to be orthogonal. However, in this case, the coefficients υ_i cannot be easily interpreted as the probabilities assigned by the density matrix to each dyad.

at a corner of the multinomial simplex, i.e. $\lambda_i = 1$ for some i, then the resulting density matrix consists of a single dyad and is called *pure state*. In the other cases, the density is called *mixed state*.

Conventional probability distributions can be represented by diagonal density matrices. In this case, a classical sample space of n points corresponds to the set of projectors onto the standard basis $\{|e_1\rangle\langle e_1|, \ldots, |e_n\rangle\langle e_n|\}$. Hence, the density matrix corresponding to the multinomial parameter $\vec{\theta}$ above can be represented as a mixture, $\rho_\theta = \mathrm{diag}(\vec{\theta}) = \sum_i \theta_i |e_i\rangle\langle e_i|$. As an example, the density matrix ρ_θ below corresponds to a classical probability distribution with $n = 2$, σ is a pure state and ρ is a general quantum density, a mixed state:

$$\rho_\theta = \frac{1}{2}|e_a\rangle\langle e_a| + \frac{1}{2}|e_b\rangle\langle e_b| = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}, \quad \rho = \begin{pmatrix} 0.5 & 0.25 \\ 0.25 & 0.5 \end{pmatrix}.$$

3 Looking at Language Models

In the Language Modeling approach to IR, each document is usually assigned a unigram language model $\vec{\theta}_d = (\theta_{d1}, \ldots, \theta_{dn})$, i.e. a categorical distribution over the vocabulary sample space $\mathcal{V}$ (of size n), $w \in \mathcal{V}$, $p_{\theta_d}(w) = \theta_{dw}$ [25]. A query is represented as a sequence of terms $\{q_1, \ldots, q_m\}$, sampled i.i.d. (independent and identically distributed) from the document model. The score for a document is obtained by computing the likelihood for the query to be generated by the corresponding document model:

$$\mathcal{L}(\{q_1, \ldots, q_m\}|\vec{\theta}_d) = \prod_{i=1}^{m} p_{\theta_d}(q_i).$$

This scoring function is generally called Query Likelihood (QL). On the other hand, Kullback-Leibler (KL) divergence models can be seen as a generalization of QL models introduced in order to facilitate the use of feedback information in Language Modeling framework [25]. In KL-divergence models, both documents and queries are assigned to unigram language models. The score for a document is calculated as the negative query to document KL-divergence:

$$-\mathcal{KL}(\vec{\theta}_q \| \vec{\theta}_d) = -\sum_w \theta_{qw} \log \frac{\theta_{qw}}{\theta_{dw}}.$$

3.1 Query Likelihood View

As presented in Sect. 2, conventional probability distributions can be seen as diagonal density matrices. A straightforward quantum interpretation of the QL scoring function can be obtained by associating a diagonal density matrix to each document and consider a query as a sequence of dyads. Formally, we associate the vocabulary sample space to the orthogonal set of projectors on the standard basis, $\mathcal{E} = \{|e_1\rangle\langle e_1|, \ldots, |e_n\rangle\langle e_n|\}$. The density matrix ρ for a document is a mixture over $\mathcal{E}$ whose vector of weights corresponds to the parameter $\vec{\theta}_d$. Therefore,

$\rho = \text{diag}(\vec{\theta}_d) = \sum_i \theta_{di}|e_i\rangle\langle e_i|$. It is straightforward to show that restricted to $\mathcal{E}$, μ_ρ generates the same statistics as $p_{\theta_d}(\cdot)$, i.e. $\forall w \in \mathcal{V}$:

$$\mu_\rho(|e_w\rangle\langle e_w|) = \text{tr}(\rho|e_w\rangle\langle e_w|) = \sum_i \theta_{di}\,\text{tr}(|e_i\rangle\langle e_i||e_w\rangle\langle e_w|) = \theta_{dw} = p_{\theta_d}(w).$$

In the query likelihood view, the query is represented as an i.i.d. sample of word events. As word events correspond to projectors onto the standard basis, we represent a query as a sequence of i.i.d.[4] quantum events belonging to $\mathcal{E}$, $\{|e_{q_1}\rangle\langle e_{q_1}|, \ldots, |e_{q_m}\rangle\langle e_{q_m}|\}$. Therefore, the score for a document is computed by the following product:

$$\mathcal{L}(\{|e_{q_1}\rangle\langle e_{q_1}|, \ldots, |e_{q_m}\rangle\langle e_{q_m}|\}|\rho) = \prod_{i=1}^{m} \mu_\rho(|e_{q_i}\rangle\langle e_{q_i}|) = \prod_{i=1}^{m} p_{\theta_d}(q_i), \qquad (1)$$

which indeed corresponds to the classical QL scoring function. However, we shall stress out an important point about the equation above. If the projectors included in the query sequence are mutually orthogonal (as above), the calculation above behaves as a proper classical likelihood, i.e. the sum of the likelihoods of all possible samples of length m is one. On the contrary, the product cannot be considered as a classical likelihood because quantum probabilities for arbitrary events does not need to sum to one. Further considerations on these issues will be made in Sect. 6.

3.2 Divergence View

The KL scoring function computes a divergence between a query language model $\vec{\theta}_q$ and document language model $\vec{\theta}_d$. In QT, the KL-divergence is a special case of a more general divergence function acting on density matrices called Von-Neumann (VN) Divergence. Note $\rho = \sum_i \lambda_i|r_i\rangle\langle r_i|$, and $\sigma = \sum_i \zeta_i|s_i\rangle\langle s_i|$ the eigendecompositions of two arbitrary density matrices. In the following, the log function applied to a matrix refers to the matrix logarithm, i.e. the natural logarithm applied to the matrix eigenvalues, $\log\rho = \sum_i \log\lambda_i|r_i\rangle\langle r_i|$. The VN divergence writes as:

$$\mathcal{VN}(\rho\|\sigma) = \text{tr}(\rho(\log\rho - \log\sigma)) = \sum_i \lambda_i\log\lambda_i - \sum_{i,j}\lambda_i\log\zeta_j|\langle r_i|s_j\rangle|^2.$$

This divergence quantifies the difference in the eigenvalues as well as in the eigenvectors of the two density matrices [21].

In order to see how the classical KL retrieval framework is recovered, we assign a density matrix to the query very similarly to what has been done for

[4] In quantum physics, the meaning of i.i.d. can be associated to the physical notion of measurement. If a density matrix ρ represents the state of a system, an i.i.d. set of m quantum events is obtained by performing a measurement on m different copies of ρ and by recording the outcomes.

a document. Precisely, ρ_q and ρ_d are diagonal density matrices such that $\rho_d = \sum_i \theta_{qi}|e_i\rangle\langle e_i|$ and $\rho_d = \sum_i \theta_{di}|e_i\rangle\langle e_i|$. As ρ_q (ρ_d) is diagonal in the standard basis, its eigenvalues correspond to $\vec{\theta}_q$ ($\vec{\theta}_d$), thus:

$$\mathcal{VN}(\rho_q \| \rho_d) = \sum_i \theta_{qi} \log \theta_{qi} - \sum_{i,j} \theta_{qi} \log \theta_{dj} |\langle e_i|e_j\rangle|^2 = \sum_i \theta_{qi} \log \frac{\theta_{qi}}{\theta_{di}}, \quad (2)$$

which corresponds to the KL divergence. As conventional probability distributions correspond to diagonal density matrices, their eigensystem is fixed to be the identity matrix. Intuitively, KL divergence captures the dissimilarities in the way they distribute the probability mass on that eigensystem, i.e. by their eigenvalues.

4 Looking at the Vector Space Model

In this section, we are attempting to look at the VSM [17] in a new way. In its original formulation, no probabilistic interpretation could be given because of the lack of an explicit link between vector spaces and probability theory [24]. In the model, documents and queries are represented in the non-negative part of the vector space $\mathbb{R}^n_+$, where n is the number of terms in the collection vocabulary. In VSM, each term corresponds to a standard basis vector. The location of each object in the term space is defined by term weights (i.e. *tf*, *idf*, *tf-idf*) on each dimension. Similarity between documents and queries are computed through a vector similarity score $\vec{q}^\top \vec{d}$, where $\vec{q}, \vec{d}$ are the vector representations of the query and the document. In [17], the authors show that normalizing document vectors is important to reduce bias introduced by variance on document lengths. By normalizing both document vector and query vector, the similarity score reduces to the cosine similarity between the two vectors, which is an effective similarity measure in the model [27]. From now on, we consider $|q\rangle, |d\rangle \in \mathbb{R}^n_+$, the normalized ($\|\cdot\|_2$) query vectors. Documents can thus be safely ranked by decreasing cosine $\langle q|d\rangle \in [0, 1]$, which cannot be negative because the ambient space is $\mathbb{R}^n_+$.[5]

4.1 Query Likelihood View

In this interpretation of the VSM, each document is associated to a probabilistic "model" in the same spirit of the Language Modeling approach. We define a

[5] In this paper, we do not explicitly take into account situations in which the vectors could contain negative entries. For example, this could easily happen after the application of Rocchio's algorithm [16] in feedback situations or by reducing the dimensionality of the vector space by LSI [3]. Besides the historically encountered difficulties in the interpretation of such negative entries [6], in these particular cases, the rank equivalence situations discussed here could not hold. However, we argue that ignoring these situations causes no harm to the generality of our conclusions on the need of an enlarged representation space.

density matrix ρ for the document as $\rho_d = |d\rangle\langle d|$, which is a pure state, i.e. its mixture weights are concentrated onto the projector $|d\rangle\langle d|$. Note that this density matrix does not have a statistical meaning. It has been determined by merely normalizing heuristic weighing schemes and it cannot be related to a statistical estimators such as Maximum Likelihood (MLE).

A query can be represented as the quantum event corresponding to the subspace spanned by $|q\rangle$. This subspace naturally corresponds to the dyad $|q\rangle\langle q|$. Hence, a query can be seen as the sequence of quantum events of length one $\{|q\rangle\langle q|\}$. In this setting, the likelihood given the document model is calculated by:

$$\mathcal{L}(\{|q\rangle\langle q|\}|\rho_d) = \mu_{\rho_d}(|q\rangle\langle q|) = \text{tr}(\rho_d|q\rangle\langle q|) = \text{tr}(\langle q|d\rangle\langle d|q\rangle) = |\langle q|d\rangle|^2, \quad (3)$$

The above calculation shows that the quantum "likelihood" assigned to the event $|q\rangle\langle q|$ by the density ρ_d is the square of the cosine similarity between the query and the document. When restricted to the non-negative domain, the square function is a monotonic, increasing transformation. This means that $\mu_{\rho_d}(|q\rangle\langle q|) \stackrel{rank}{=} \langle q|d\rangle$, i.e. the two formulations lead to the same document ranking.

4.2 Divergence View

According to the original VSM, queries and documents should share the same representation and the scoring function should be a distance measure between these representations. In the previous formalization, this initial paradigm seems apparently lost. The following alternative quantum interpretation of the VSM is perhaps closer to the original vision of the model. We associate a density matrix both to the document and to the query. Specifically, those density matrices would be pure states, projectors onto the corresponding vectors, i.e. $\rho_d = |d\rangle\langle d|$, $\rho_q = |q\rangle\langle q|$. It turns out that computing the Fidelity measure [12] between density matrices produces a ranking function equivalent to cosine similarity:

$$\mathcal{F}(\rho_q, \rho_d) = \text{tr}\left(\sqrt{\sqrt{\rho_q}\rho_d\sqrt{\rho_q}}\right) = \text{tr}\left(\sqrt{|q\rangle\langle q|d\rangle\langle d|q\rangle\langle q|}\right) = |\langle q|d\rangle|\text{tr}(\rho_q) = |\langle q|d\rangle|, \quad (4)$$

obtained by noting that ρ_q is a projector thus $\sqrt{\rho_q} = \rho_q$, and $\text{tr}(\rho_q) = 1$. As $|q\rangle, |d\rangle \in \mathbb{R}^n_+$, ranking by Fidelity measure is equivalent to ranking by cosine similarity, thus $\mathcal{F}(\rho_q, \rho_d) \stackrel{rank}{=} \langle q|d\rangle$.

5 A Joint Analysis

In this section, we will try to summarize the commonalities and the differences arising from the quantum formalizations of the two models given in the preceding sections. The following analysis is succinctly reported in Table 1. As a starting point, we shall note that the ambient space for both models is the Hilbert space $\mathbb{H}^n$, where n is the size of the collection vocabulary. Each standard basis vector $\mathcal{E} = \{|e_1\rangle, \ldots, |e_n\rangle\}$ is associated to a word event. Therefore, the vocabulary sample space corresponds to the set of projectors onto the standard basis vectors $\{|e_i\rangle\langle e_i|\}_{i=1}^n$.

Table 1. Summary of the representations for documents and queries and the scoring functions of the two studied methods.

Query likelihood view			
	Query	Document	Scoring
VSM	$\{\|q\rangle\langle q\|\}$	$\rho_d = \|d\rangle\langle d\|$	$\mu_\rho(\|q\rangle\langle q\|)$
LM	$\{\|e_{q_1}\rangle\langle e_{q_1}\|, \dots, \|e_{q_m}\rangle\langle e_{q_m}\|\}$	$\rho_d = \sum_w \theta_{dw}\|e_w\rangle\langle e_w\|$	$\prod_i \mu_{\rho_q}(\|e_{q_i}\rangle\langle e_{q_i}\|)$
Divergence view			
VSM	$\rho_q = \|q\rangle\langle q\|$	$\rho_d = \|d\rangle\langle d\|$	$\mathcal{F}(\rho_q, \rho_d)$
LM	$\rho_q = \sum_w \theta_{qw}\|e_w\rangle\langle e_w\|$	$\rho_d = \sum_w \theta_{dw}\|e_w\rangle\langle e_w\|$	$-\mathcal{VN}(\rho_q \\| \rho_d)$

5.1 Query Likelihood View

In query likelihood interpretations, the query is represented as a sequence of i.i.d. dyads. In the VSM, the sequence contains one dyad corresponding to the projector onto the query vector $\{|q\rangle\langle q|\}$. On the contrary, in the LM approach the sequence contains a dyad for each classical word event, i.e. $\{|e_{q_1}\rangle\langle e_{q_i}|\}_{i=1}^{m}$.

Besides the number of dyads included in the sequence, a major difference distinguishes the two formalizations. Contrary to probabilistic retrieval models such as LM, a query is not considered as a sequence of independent classical word events but as a single event and a particular kind thereof. The query event is a *superposition* of word events. This can be seen because the vector $|q\rangle$ can be expressed, up to normalization, as $|q\rangle = \sum_w f(w)\,|e_w\rangle$ where $f(w)$ is the weight for term w in the query vector. This kind of event cannot be expressed using set theoretic operations neither it has a clear classical probabilistic interpretation: it does not belong to $\mathcal{E}$ thus it can only be justified in the quantum probabilistic space. Arguing further, we would say that, in the case of VSM, term weighting methods aim at estimating the "best" query event, i.e. the event which is the most representative for the information need of the user. Intuitively, if a single choice would be given to us on what to observe, we would rather be observing in the "direction" of important words in the query.

It follows from the considerations above that VSM creates query representations by accessing the whole projective space through appropriate choices of $f(w)$. On the contrary, LM "sees", and consequently can handle, only events from the classical sample space $\mathcal{E}$. However, the principled probabilistic foundations of the model give the flexibility of adding an arbitrary number of such events in the sequence, thus refining query representation[6]. In the next section, this kind of duality between VSM and LM approaches will be strengthened by analyzing the properties of the density matrices used in the two models.

Before continuing, we shall make one last consideration about the "likelihood" written in Eq. 1. This equation and its corresponding maximization algorithm have already been proposed by Lvovsky et al. [7] in Quantum Tomography applications in order to achieve a Maximum Likelihood Estimation (MLE) of a

[6] This is indeed the practice of Query Expansion (QE), see for example [2].

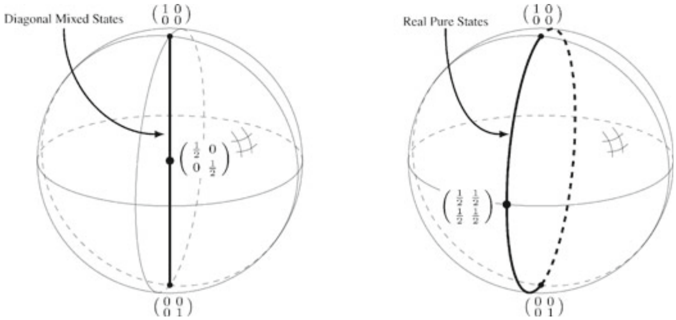


Fig. 1. The set $\mathcal{D}^2$ visualized using the Bloch sphere parametrization [12]. Highlighted in black are the region of $\mathcal{D}^2$ used by LM (to the left) and VSM (to the right).

density matrix. As we have already pointed out, $\mathcal{L}$ reduces to a classical likelihood if and only if the projectors in the sequence are picked from the same eigensystem. Therefore, the product in its general form cannot be understood as a proper likelihood. We believe that it would be interesting to focus future research in finding a proper likelihood formulation in the quantum case that would enable principled statistical estimation and Bayesian inference (see [22] for a recent attempt in formulating a Bayesian calculus for density matrices).

5.2 Divergence View

In the divergence view, a density matrix is associated both to the document and to the query and the scoring function is a divergence defined on the set $\mathcal{D}^n$ of $n \times n$ density matrices. Valuable insights can be provided by noting that the models gain access to different regions within $\mathcal{D}^n$. As an example, in Fig. 1, we plot the set $\mathcal{D}^2$ using the well known Bloch parametrization [12]. Highlighted in black are the regions of the space used by LM (to the left) and VSM (to the right). Distinct regions are likely to denote different representational capabilities.

In the case of LM, density matrices are restricted to be diagonal, i.e. mixtures over the identity eigensystem. For two density matrices to be different, one has to modify the distribution of the eigenvalues. Therefore, LM ranks based upon differences in the eigenvalues between density matrices. The picture of the VSM approach appears as the perfect dual of the preceding situation. Query and documents are represented by *pure states*, i.e. dyads. Whatever the dimensionality of the Hilbert space, the mixture weights of these density matrices are concentrated onto a single projector. In order to be different, density matrices must be defined over different eigensystems. Therefore, VSM ranks based on the difference in the eigensystem between query and document density matrices.

The set of diagonal density matrices is represented in Fig. 1 (left). Any two antipodal points on the surface of the sphere correspond to a particular eigensystem. Diagonal density matrices are restricted to the identity eigensystem. However, they can delve inside the sphere by spreading the probability mass across their eigenvalues. The black circle in Fig. 1 (right) highlights pure states with real positive entries. These naturally lie on the surface of the Bloch sphere.

In summary, the VSM restriction to pure states leaves free choice on the eigensystem while fixing the eigenvalues. Conversely, by restricting density matrices to be diagonal, i.e. classical probability distributions, LM leaves free choice on the eigenvalues while fixing the eigensystem. Leveraging both degrees of freedom by employing the machinery of density matrices seems to be a natural step in order to achieve more precise representation for documents and queries. VSM and LM also differ in the choice of scoring functions. The former uses the Fidelity measure which is a metric on $\mathcal{D}^n$. The latter uses an asymmetric divergence on $\mathcal{D}^n$. More insights into these differences are given in the next section, where we try to contextualize our considerations by referring to common IR issues and concepts.

6 A Joint Interpretation and Perspectives

In [25], the author presents KL divergence models as "essentially similar to the vector-space model except that text representation is based on probability distributions rather than heuristically weighted term vectors". The analysis done in the previous section extends this remark and highlights how VSM and LM leverage very different degrees of freedom by allocating different regions in $\mathcal{D}^n$. However, no clue is given about what should be the meaning of the eigensystems and the eigenvalues from an IR point of view, nor why controlling both could be useful for IR. We will try to give some perspective for the potential usefulness of the enlarged representation space.

In basic bag-of-words retrieval models such as LM or VSM, terms are assumed to be unrelated, in the sense that each term is considered to be an atomic unit of information. To enforce this view, LM associates to each term a sample point and the VSM a dimension in a vector space. Our analysis showed that sample points correspond to dimensions in a vector space. The heritage left by LSI [3] suggests that a natural interpretation for such dimensions is to consider them as *concepts*. In this work, we interpret projectors onto directions as concepts. Because terms are considered as unrelated, the projectors onto the standard basis $|e_1\rangle\langle e_1|, \ldots, |e_n\rangle\langle e_n|$ in $\mathbb{H}^n$ form a *conceptual basis* in which each term labels its own underlying concept.[7]

From this point of view, LM builds representations of queries and documents by expressing uncertainty on which concept chosen from the standard basis represents the information need. On the contrary, VSM does not have the flexibility of spreading probability weights. However, it can represent documents and queries by a unique but arbitrary concept. In VSM, the similarity score is

[7] In [8], *each* basis of a vector space is considered as describing a *contextual property* and the vectors in the basis as *contextual factors*. We prefer not to adopt such interpretation for two reasons: (1) in this paper, classical sample spaces are exclusively associated to orthonormal basis and (2) we believe that referring to concepts leads to a more general formulation, better tailored to our needs.

computed by comparing how similar the query concept is to the document concept. In this picture, the cosine similarity reveals to be a measure of relatedness between concepts. In LM, the score is not at all computed on concept similarity, but by considering how the query and the document spread uncertainty on the same conceptual basis.

In order to see how this all could be instantiated, let us suppose that compound phrases such as *"computer architecture"* express a different concept than *"computer"* and *"architecture"* taken separately. Modeling interactions between terms has been a longstanding problem in IR (for example, see [4]). We conjecture that a very natural way to handle such cases stems from our analysis. Assume that both *"computer"* and *"architecture"* are associated to their corresponding single term concepts, i.e. $|e_c\rangle\langle e_c|$, $|e_a\rangle\langle e_a|$. The concept expressed by the compound could be associated to a superposition event $|k_{ca}\rangle\langle k_{ca}|$ where $|k_{ca}\rangle = f(c)|e_c\rangle + f(a)|e_a\rangle$ and f is a weight function (assuming normalization) expressing how compound and single term concepts are related. In this setting, the enlarged representation space turns out to be the perfect fit in order to express uncertainty on this set of concepts. One could build a density matrix associated both to a query and to a document assigning uncertainty to both single term concepts $|e_c\rangle\langle e_c|, |e_a\rangle\langle e_a|$ and compound concepts $|k_{ca}\rangle\langle k_{ca}|$. This could be done, for example, by leveraging quantum estimation methods such as described in [7]. As we have pointed out before, the VN divergence could be the right scoring function in order to take into account both divergences in uncertainty distribution and concept similarities. The practical instantiation of such general framework can be found in [19]. Experiments on several TREC collections show that the model leads to higher retrieval effectiveness than the existing models (in particular, LM).

As a last remark, we shall point out that the considerations made until now do not necessitate of the whole machinery of complex vector spaces. We do not have a practical justification for the usefulness of vector spaces defined over the complex fields (see [29] for a discussion on these issues). However, we speculate that these could bring improved representational power and thus remains an interesting direction to explore.

7 Conclusion

In this work, we showed how VSM and LM can be considered dual in how they allocate the representation space of density matrices and in the nature of their scoring functions. In our interpretation, VSM adopt a symmetric scoring function which measures the concept similarity. LM fixes the standard conceptual basis and scores documents against queries based on how they spread the probability mass on such basis. We argued that leveraging both degrees of freedom could lend a more precise representations of documents and queries and could be especially effective in modelling compound concepts arising from phrasal structures. The soundness and usefulness of this general setting has been confirmed experimentally in [19].

References

1. Birkhoff, G., Von Neumann, J.: The logic of quantum mechanics. Ann. Math. **37**(4), 823 (1936)
2. Carpineto, C., Romano, G.: A survey of automatic query expansion in information retrieval. ACM Comput. Surv. **44**(1), 1:1–1:50 (2012)
3. Deerwester, S., Dumais, S.T., Furnas, G.W., Landauer, T.K., Harshman, R.: Indexing by latent semantic analysis. J. Am. Soc. Inf. Sci. **41**, 391–407 (1990)
4. Gao, J., Nie, J.Y., Wu, G., Cao, G.: Dependence language model for information retrieval. In: Proceedings of SIGIR, pp. 170–177 (2004)
5. Gleason, A.: Measures on the closed subspaces of a hilbert space. J. Math. Mech. **6**, 885–893 (1957)
6. Hofmann, T.: Unsupervised learning by probabilistic latent semantic analysis. Mach. Learn. **42**(1), 177–196 (2001)
7. Lvovsky, A.I.: Iterative maximum-likelihood reconstruction in quantum homodyne tomography. J. Opt. B: Quant. Semiclassical Opt. **6**, S556–S559 (2004)
8. Melucci, M.: A basis for information retrieval in context. ACM Trans. Inf. Syst. **26**, 14:1–14:41 (2008)
9. Melucci, M.: An investigation of quantum interference in information retrieval. In: Cunningham, H., Hanbury, A., Rüger, S. (eds.) IRFC 2010. LNCS, vol. 6107, pp. 136–151. Springer, Heidelberg (2010)
10. Melucci, M.: Deriving a quantum information retrieval basis. The Computer Journal **56**(11), 1279–1291 (2013). doi:10.1093/comjnl/bxs095
11. Melucci, M., Rijsbergen, K.: Quantum mechanics and information retrieval. Adv. Top. Inf. Retrieval **33**, 125–155 (2011). (Springer, Berlin)
12. Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2004)
13. Piwowarski, B., Frommholz, I., Lalmas, M., van Rijsbergen, K.: What can quantum theory bring to information retrieval. In: Proceedings of CIKM, pp. 59–68 (2010)
14. Piwowarski, B., Amini, M.R., Lalmas, M.: On using a quantum physics formalism for multidocument summarization. JASIST **63**(5), 865–888 (2012)
15. van Rijsbergen, K.: The Geometry of Information Retrieval. Cambridge University Press, Cambridge (2004)
16. Rocchio, J.: Relevance feedback in information retrieval. In: Salton, G. (ed.) The SMART Retrieval System, pp. 313–323. Prentice-Hall, Englewood Cliffs (1971)
17. Salton, G., Buckley, C.: Term-weighting approaches in automatic text retrieval. Inf. Process. Manage. **24**(5), 513–523 (1988)
18. Song, D., Lalmas, M., van Rijsbergen, K., Frommholz, I., Piwowarski, B., Wang, J., Zhang, P., Zuccon, G., Bruza, P., Arafat, S., Azzopardi, L., Huertas-Rosero, A., Hou, Y., Melucci, M., Rüger, S.: How quantum theory is developing the field of information retrieval. In: Proceedings of QI, pp. 105–108 (2010)
19. Sordoni, A., Nie, J.-Y., Bengio, Y.: Modeling term dependencies with quantum language models for IR. In: Proceedings of SIGIR, pp. 653–662 (2013)
20. Sordoni, A., He, J., Nie, J.-Y.: Modeling latent topic interactions with quantum interference for IR. To appear in Proceedings of CIKM (2013)
21. Tsuda, K., Ratsch, G., Warmuth, M.K.: Matrix exponentiated gradient updates for on-line learning and bregman projection. J. Mach. Learn. Res. **6**(1), 995 (2006)
22. Warmuth, M.K., Kuzmin, D.: Bayesian generalized probability calculus for density matrices. Mach. Learn. **78**(1–2), 63–101 (2009)

23. Widdows, D., Peters, S.: Quantum logic of word meanings: Concept lattices in vector space models (2003)
24. Wong, S.K.M., Yao, Y.Y.: On modeling information retrieval with probabilistic inference. ACM Trans. Inf. Syst. **13**(1), 38–68 (1995)
25. Zhai, ChX: Statistical language models for information retrieval a critical review. Found. Trends Inf. Retr. **2**(3), 137–213 (2007)
26. Zhao, X., Zhang, P., Song, D., Hou, Y.: A novel re-ranking approach inspired by quantum measurement. In: Clough, P., Foley, C., Gurrin, C., Jones, G.J.F., Kraaij, W., Lee, H., Mudoch, V. (eds.) ECIR 2011. LNCS, vol. 6611, pp. 721–724. Springer, Heidelberg (2011)
27. Zobel, J., Moffat, A.: Exploring the similarity space. SIGIR Forum **32**(1), 18–34 (1998)
28. Zuccon, G., Azzopardi, L.: Using the quantum probability ranking principle to rank interdependent documents. In: Gurrin, C., He, Y., Kazai, G., Kruschwitz, U., Little, S., Roelleke, T., Rüger, S., van Rijsbergen, K. (eds.) ECIR 2010. LNCS, vol. 5993, pp. 357–369. Springer, Heidelberg (2010)
29. Zuccon, G., Piwowarski, B., Azzopardi, L.: On the use of complex numbers in quantum models for information retrieval. In: Amati, G., Crestani, F. (eds.) ICTIR 2011. LNCS, vol. 6931, pp. 346–350. Springer, Heidelberg (2011)

Combining Word Semantics within Complex Hilbert Space for Information Retrieval

Peter Wittek[1(✉)], Bevan Koopman[2,3], Guido Zuccon[2], and Sándor Darányi[1]

[1] University of Borås, Borås, Sweden
peterwittek@acm.org
[2] Australian e-Health Research Centre, CSIRO, Brisbane, Australia
[3] Queensland University of Technology, Brisbane, Australia

Abstract. Complex numbers are a fundamental aspect of the mathematical formalism of quantum physics. Quantum-like models developed outside physics often overlooked the role of complex numbers. Specifically, previous models in Information Retrieval (IR) ignored complex numbers. We argue that to advance the use of quantum models of IR, one has to lift the constraint of real-valued representations of the information space, and package more information within the representation by means of complex numbers. As a first attempt, we propose a complex-valued representation for IR, which explicitly uses complex valued Hilbert spaces, and thus where terms, documents and queries are represented as complex-valued vectors. The proposal consists of integrating distributional semantics evidence within the real component of a term vector; whereas, ontological information is encoded in the imaginary component. Our proposal has the merit of lifting the role of complex numbers from a computational byproduct of the model to the very mathematical texture that unifies different levels of semantic information. An empirical instantiation of our proposal is tested in the TREC Medical Record task of retrieving cohorts for clinical studies.

1 Introduction

In quantum theory, states are represented by vectors defined on a complex-valued Hilbert space. Complex numbers are a fundamental aspect in the mathematical formalism of quantum physics. For example, mathematically, the quantum interference term in the law of total probability for disjoint events arises because the probability amplitudes of events are modelled by complex numbers. Quantum-like formalisms were proposed to model systems outside of physics, for example in cognitive science, decision making, economy, etc. In information retrieval (IR), the pioneering work by van Rijsbergen [1] showed that the quantum formalism encompasses many state-of-the-art retrieval models; subsequent works proposed many quantum-like models for IR [2]. Common to all these proposals is the assumption that information objects (queries, documents, etc.) are represented in real-valued Hilbert spaces, even when the key modelling aspect is the quantum interference phenomenon [3]. Zuccon and Piwowarski argued that this

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 160–171, 2014.
DOI: 10.1007/978-3-642-54943-4_14,

assumption is not imposed by the models themselves, which, being grounded on the mathematics of quantum theory, allow for complex valued representations. Instead it is rooted in the difficulties of understanding how complex numbers could be obtained from term counts in documents [4].

We derive a complex-valued representation of information by encoding semantics by complex numbers. The proposal helps to increase the "semantic power" of traditional semantic space models for IR, by combining different sources of semantic evidence. The intuition underlying our proposal stems from the observation that different models of semantic spaces for IR and text categorisation [5–8] apparently share a joint limitation: they are minimally semantic only, inasmuch as all they utilise is the distributional meaning of words based on their co-occurrence in context. Although not measured explicitly, this limitation probably constrains their performance as well. This long-standing convention has been relying on the use of real numbers only. Our proposal of creating a complex-valued information object grounded in semantic information allows to go beyond the use of mere distributional semantics and includes, e.g., paradigmatic vs. syntagmatic elements of word meaning [9,10] to merge term co-occurrence statistics with ontological information, e.g., from WordNet [11].

Using an example from medical IR and departing from an earlier study on the use of semantics in such a task [12], we empirically demonstrate how a more sophisticated model of word semantics is implemented in Hilbert space by means of complex numbers. While the retrieval performance is better, we must point out that storage requirements double.

The paper is structured as follows. In Sect. 2, we briefly discuss the major types of bi- and tripartite theories of word meaning which were consulted to engineer mathematical objects with a higher than usual capacity for information representation. Section 3 inspects random indexing used for model building. Section 4 discusses experiment design. Section 4.2 interprets the results and Sect. 5 concludes the paper.

2 From Signs to Meaning: Engineering Sign Spaces

Next, we provide a brief account of the theories of meaning that were consulted for engineering the complex-valued representation of word meaning proposed.

In semiotics, a sign (e.g., a word, as well as non-linguistic symbols or clues) is defined as a unity of form and content. Signs are characterised by many important characteristics, which are captured by the following typology:

- Signs are located in some concrete or abstract space vs. having a temporal, e.g., causal nature;
- They are bi- or tripartite. Bipartite theories go back to Aristotle (*form* vs. *substance*) and St. Augustine (*content* vs. *expression*); they found their way into 20th century general linguistics thanks to Ferdinand de Saussure whose sign as word form vs. word meaning was contrasted by American semiotics, itself relying on a tripartite sign concept, adding pragmatics as a third component by the question *for whom* is a sign meaningful;

– Wherever we happen onto a meaningful word or sentence, it is always an instance of a sign, which lends importance to designing such spaces for computer processing.

When focusing on linguistic signs, an alternative way to distinguish between kinds of word meaning is to juxtapose "meaning is use" (i.e., the distributional hypothesis proposed by Wittgenstein [13], Harris [14] and Firth [15]), with "meaning is change" (stimulus-response theories of Bloomfield [16], Morris [17] and Uexkull [18]), and "meaning is equivalence" (referential theories, e.g., by Peirce [19] and Frege [20]). It is of great importance that because none of these theories are able to account for word semantics alone, one has to regard word meaning as being composite. This in turn leads to the insight that unless this compound nature of word meaning is encapsulated in mathematical objects, less progress beyond today's IR models can be realistically expected. The proposal put forward aims to address this observation.

Metric spaces are often used to represent signs by assigning word meaning as substance to a certain location expressed by its coordinates as form. What lends importance to such "charged" locations goes back to another aspect of word semantics specified by Harris' distributional hypothesis [14], stating that words able to replace one another in the same context have highly similar meaning. This is the cornerstone upon which the meaningfulness of semantic spaces rests. Our approach here will be to expand on this practice by unifying language use with its conceptual underpinnings, and merge them as the form and content side of signs, the building blocks of using complex vector space for information representation.

3 Bringing Different Semantics Together: A Complex-Valued Representation of Information

Inspired by the observation made in Sect. 2, we hypothesise that a combination of different sources or types of semantic information is achieved using complex numbers for the representation of information objects in IR models. Such a proposal would provide a means for generating a representation of information based on complex numbers, which could form the basis for more advanced quantum-like models for IR.

In this section, we describe how our proposal is instantiated, i.e., how a complex-valued representation of terms, documents and queries are generated that brings together different types of word semantics. To do so, we combined previous techniques for random indexing and concept-based document indexing, which are briefly outlined in the next paragraphs.

3.1 Random Indexing

Random indexing does not rely on the use of computationally intensive matrix decomposition algorithms like singular value decomposition (SVD) to achieve a fairly low-dimensional representation of a document or term space. This makes random indexing a much more scalable technique in practice as it builds an incremental word space model in a two-step process as follows [7,21]:

- First, every context (e.g., each document or each word) in the data is assigned a unique and randomly generated representation called an index vector. These index vectors are sparse, high-dimensional, and ternary, that is, their dimensionality (k) is in the order of thousands, and they consist of a small number of randomly distributed +1s and -1s, with the rest of the elements of the vectors set to 0;
- Then, context vectors are produced by scanning through the text, and every time a word occurs in a context (e.g., in a document, or within a sliding window), that context's k-dimensional index vector is added to the context vector for the word in question. Words are thus represented by k-dimensional context vectors that are effectively the sum of the words' contexts.

Document vectors are simply the sums of their constituent word vectors, hence the document space is also k-dimensional. The number of dimensions is defined by k, and random indexing does not provide an explicit way of computing it, being a parameter of the model. These dimensions are not topics, in contrast to other low-dimensional embeddings such as latent semantic indexing [5]. Efficient and extendable open source implementations of random indexing exist [22].

3.2 Concept-Based Document Indexing

In concept-based indexing, documents are represented by concepts rather than terms, as is instead the case for traditional term-based representations. Concepts are usually defined by an ontology or are knowledge-based, and different strings of text are represented by the same concept, indicating that these have identical meaning. For example, in the medical domain, the expressions "heart attack" and "myocardial infarction" have the same meaning and are usually mapped to the same underlying concept. The simpler form of concept-based indexing consists of extracting concepts from the textual content of documents and then representing documents as a bag-of-concepts (BOC) vector, as opposed to the traditional bag-of-words (BOW) approach. More advanced forms of concept-based indexing have been proposed; for example, in concept-based indexing for medical IR, Koopman et al., 2012, and Zuccon et al., 2012 capture the relations (implicit or formal, respectively) between concepts encoded in the ontology of reference [23,24]. These approaches are, however, beyond our scope.

We consider a simple BOC representation, where documents correspond to vectors of concept identifiers, and we thus assume that a mapping between strings of texts and concepts exist. In the experiments of Sect. 4, we use the procedure outlined in Koopman et al. [12], which involves converting both queries and documents to concepts[1] using the medical natural language processing system called MetaMap [26]. Both documents and queries are thus represented not as

[1] Specifically, concepts from the SNOMED-CT medical ontology [25].

Table 1. Term and concept statistics for the TREC Medical Records Track collection.

Unique terms	218,574	Unique concepts	36,467
Total terms	40,212,729	Total concepts	67,183,177

their original terms but as concept identifiers from the SNOMED-CT ontology. A standard IR indexing and retrieval process can then be applied to the concept documents and queries.

A concept may correspond to an n-gram of text, e.g., concept 165664003 refers to "Entire articular process of cervical vertebra"; likewise, an n-gram (or just a single word) may stand for several concepts. As such, the term statistical behaviour observed in language does not always apply to concept representations. In particular, concept representations does not obey Zipf's law, i.e., do not exhibit the typical long tail distribution of very infrequent concepts that instead a term representation commonly exhibits [27]. Similarly, the number of concepts used to represent a corpus of documents differ greatly from the size of the term vocabulary for that same corpus. To exemplify this, we anticipate the statistics obtained from the representations of the document corpus used in the experiments of Sect. 4. Table 1 summarises the term and concept statistics obtained when indexing the TREC Medical Records Track collection; more details about the indexing procedure are given later. The table shows that while the corpus contains many more concepts than terms, the vocabulary size for concepts is one order of magnitude smaller than that for terms.

3.3 Combining Word Semantics: Documents in Complex Space

Our proposal revolves around using complex numbers to combine different forms of semantic information. In the following, we consider two instances of semantic information, namely distributional and referential. Specifically, we draw distributional semantic information using the random indexing technique described in Sect. 3.1; whereas, referential semantic information is drawn from the concept-based indexing procedure outlined in Sect. 3.2. The concept index vectors were assigned to SNOMED-CT concepts, and corresponding document vectors were derived from these by superposition. The position of an index term with a bipartite sign nature in the complex vector space is composed from term frequency and other statistics in a term-document matrix, and represented as the real component of the resulting complex vector, with the representation of concepts from the concept-based indexing constituting its imaginary component. Thereby in any complex term weight, the real component encoded distributional semantics whereas the imaginary component hosted referential semantics. Such weights are then used to build complex term, document and query vectors for retrieval.

Because of the difference in vocabulary sizes between the term and concept representations, the dimensionality of the term space and that of the concept space does not match. The use of random indexing provides a solution to address this issue, where the dimension of the random vectors is used to force a common dimensionality among the two sources of information, as detailed in the following. Assume a document-term matrix of dimension $m \times n$ is built from the corpus of documents, where m is the number of documents and n is the size of the term vocabulary. Similarly, assume that the corresponding document-concept matrix for that corpus is of dimension $m \times p$, with p being the number of concepts in the concept-vocabulary. By applying random indexing to both matrices maintaining the number of dimensions of the random vectors to the same k, not only distributional semantic information is extracted from the respective original matrices, but compatible representations of the two spaces are also obtained. That is, the random indexing representation for of the original document-term matrix will have a dimensionality of $m \times k$; similarly, the random indexing representation for concepts will have a dimensionality of $m \times k$, eliminating issues of dimensionality mismatches between vectors from term or concept representations. In the representation proposed here, document similarity calculated as the inner product between complex-valued vectors reflects both the distributional and the ontological facets of document content. Similarly, the comparison of the vectors associated to the real part of the representation with those associated with the imaginary part would provide the similarity between the statistical term space representation of a document and its ontological concept space representation.

Merging the two in a complex space is a trivial exercise, and it allows for measurements of phase between a statistical term space vs. an ontological concept space (see Fig. 1), with the inner product of document similarity reflecting both the distributional and the ontological facets of content.

The method proposed here is similar to that suggested by van Rijsbergen, where different functions of term statistics (specifically, term frequency and inverse document frequency) are assigned to the real and imaginary components of complex numbers [1]. That approach provided poor retrieval effectiveness, and the power of representation was questioned [4]. However, differently from van Rijsbergen's proposal, we suggest to encode in the components of the complex numbers semantic information (purely distributional information and higher-level conceptual information) rather than functions of term statistics. Another Hilbert space-based representation embedded the two kinds of semantics by a

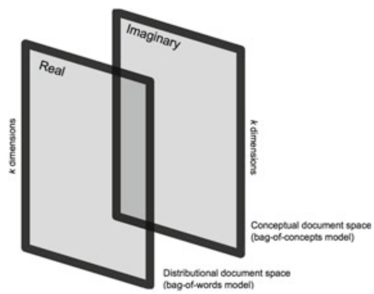


Fig. 1. Schematic representation of a complex document space.

seriation of the feature space and using the L_2 space of square-integrable functions [28]. Compared to that approach, the only preprocessing we need is mapping the terms to concepts, which is faster than seriation.

To aid the understanding of our method, we present a simple example. Consider the following documents: D_1 = "`kidney stones`", D_2 = "`kidney`" and D_3 = "`renal calculi`", and the query Q = "`kidney stones`". A term-based retrieval system processing Q would return only the documents D_1, D_2 (in this order). However, `renal calculi` in D_3 is actually a synonym of the query `kidney stones`, while D_2 is actually not relevant. Therefore, D_3 should be ranked higher than D_2. Using our method, when a concept representation is included, the phrases `kidney stones` and `renal calculi` both map to the same SNOMED CT concept `155868000`. Thus, our ranking approach would retrieve D_1 in the first rank position because of the contribution of both term and concept weighting; and D_3 above D_2 because of the inclusion of the concept weighting.

4 Empirical Investigation

Next, we outline the experiment we devised to test our proposal. It describes an initial effort to evaluate the merit of the complex-value representation; further validation will be subject of future work.

4.1 Experiment Settings

To benchmark the efficiency of the proposed representation, we evaluated the method in the IR task provided in the TREC Medical Records Track, which consists in retrieving medical records of patients that satisfy clinical and demographic criteria specified as queries. We followed previous work in this area for combining medical records belonging to a single patient into a unique document, called a patient visit document [23,24]. We followed the procedure outlined by Koopman et al. [12] to obtain a concept representation. Consistently with previous work, SNOMED-CT was chosen as ontology of reference for the concept representation. In total, the collection consisted of 17,198 patient visit documents and 81 queries; statistics for both term and concept representations are outlined in Table 1. Retrieval effectiveness was measured by mean average precision (MAP) and precision of the top 10 ranked documents (P@10), as well as a precision-recall analysis.

We used Lucene 3.6.2 to index and retrieve documents, while the SemanticVectors package, version 3.8 [22,29], was used to construct a random indexing representation for terms and concepts bag of words. The context window of a word or a concept was the full document. The use of the respective random indexes for retrieval formed the two baselines, we contrasted our method against in this evaluation, namely term-based (BOW) random indexing and concept-based (BOC) random indexing. To implement our method, we constructed a complex space by using the random indices of the BOW and BOC models as described in Sect. 3.3. Queries were represented in the complex space in the same way as the documents. The number k of random dimensions was set to 200 for all methods; experimentation with other dimensions is left for future work, but initial results showed effectiveness not to vary considerably when changing k.

For the standard BOW and BOC models, the distance function used to judge the similarity between documents and queries was the inner product of the corresponding real space (as formed by the random index). The counterpart of the inner product in the complex space is the Hermitian product, but this yields a complex number. Since complex numbers do not have a natural ordering, this product cannot be used for ranking. We adapted the inner product to measure the overlap between the real and complex part in the same way as in a real space, yielding a real number. This approach is different from the similarity measure defined for complex spaces in SemanticVectors. None of the spaces were normalised, as the inner product is insensitive to the norm of vectors.

4.2 Results

The retrieval effectiveness of the methods is reported in Table 2. The mean average precision of the method that exploits the complex-valued representation is found to be 15 % higher than that of the concept-based approach, and 40 % higher compared to the term-based random indexing approach. Similar findings are obtained when considering P@10. The analysis of the 11-point precision-recall interpolation across all queries is showed in Fig. 2. The results suggest that precision is markedly higher at lower recall levels; that is, the proposed complex-valued representation retrieves more relevant documents in the top results than baselines methods considered here.

The results obtained by all methods considered here are generally lower than those reported by state-of-the-art IR method on the same task [30]. This suggests that random indexing alone is not an effective method for document retrieval in the medical domain. Similar findings were obtained when using random indexing for query expansion in this task [31], although that method delivered higher effectiveness than the results presented here. Nevertheless, in our evaluation, we are interested in understanding the value the complex-valued representation adds to the baseline methods, rather than the actual absolute effectiveness of the instantiation investigated here. More effective instantiations of our proposal may in fact consider distributional semantic techniques other than semantic indexing, boosting effectiveness.

We further analysed the empirical results obtained in our experiments by examining the values of the angles formed by the real and imaginary components

Table 2. MAP and P@10 for term retrieval, concept retrieval, and our method (labeled “complex” retrieval).

Measure	Term-based	Concept-based	Complex
MAP	0.0886	0.1084	0.1245
P@10	0.1593	0.1963	0.2235

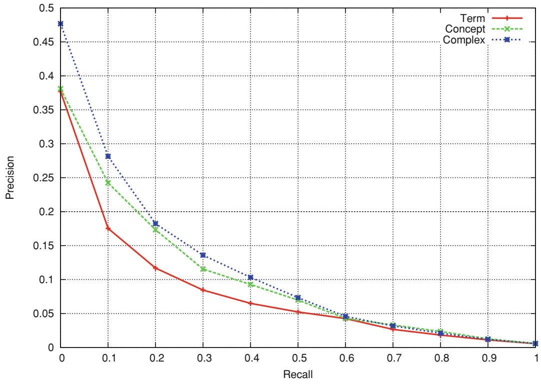


Fig. 2. 11 points precision-recall graph.

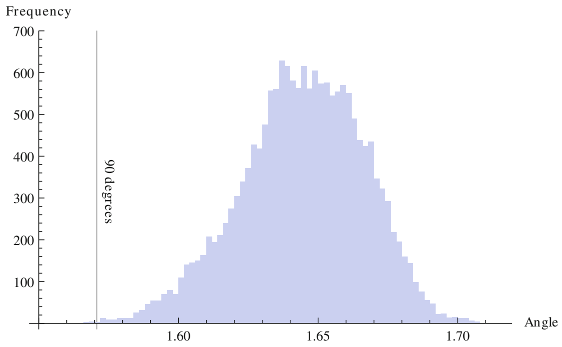


Fig. 3. Distribution of angles between real and imaginary components of complex vectors representing documents from the TREC Medical Records Track collection. Angles were expressed in radians and were normalised in the range $[0, 2\pi]$.

Table 3. Mean value and extrema of phase

Mean absolute phase	Lowest phase	Highest phase
6.9520	-39.9163	28.6791

of the complex vectors representing documents. Figure 3 shows the distribution of these angles in radians normalised in the range $[0, 2\pi[$, while Table 3 reports the mean, maximum and minimum values of the (unnormalised) angles.

We observed that documents for which the angles assumed the lowest phase value in radians across the collection (< -30) generally contained a large amount of numerical laboratory test results, as well as long lists of medication brand names and dosages. This sort of content is often not mapped to concepts by MetaMap, which led to large amounts of information found in the term representation missing in the concept representation.

Vice versa, we observed that documents characterised by large angles (> 20 radians) are often short in length (due to the small number of terms), but when mapped to concepts, their concept representation was considerably longer than the term-based one (i.e., a document is represented by more concepts than terms). This is the case when MetaMap assigns multiple concepts to the same (or overlapping) strings of text. This finding suggests that in such cases, the concept representation provides a more accurate description of the content of short documents than the original term representation.

5 Conclusions

Complex numbers have a key role in the mathematical framework of quantum theory; however they have been overlooked when using quantum-like formalisms for modelling systems outside of physics: this is the case especially in IR. We departed from previous approaches that considered complex numbers as a simple computational byproduct of the models, and we proposed a novel representation of information based on complex numbers which brings together different sources of semantics. Specifically, in the empirical instantiation of our proposal, we mixed distributional (contextual) and referential (ontological) semantics, although the proposed approach is not limited to the particular techniques used to derive semantics. The empirical evaluation of the proposed complex-valued representation has lead to encouraging results, although further evaluation is required. We must point out that the storage requirement complex-valued representation the sum of the a term-based and concept-based model, which means it is nearly double than the space required by either. Given the sparse data structures, this trade-off is hardly an issue.

The proposed model allows for the measurement of quantitative distances between the distributional form of a word or a phrase and its referential representation in a concept space. Technically this distance corresponds to the angle between the real and imaginary components of information objects (terms, documents, etc.). The representation invites the study of the compositionality of meaning in noun compounds from a new perspective, a field of intensive study within Quantum Interaction [32,33]. In addition, user relevance is an indirect but inherent component in our framework as the sources of referential semantics within the representation (e.g., SNOMED-CT in the medical domain) are constructed with domain-specific relevance in mind. A query effectively interacts with both the distributional pattern of terms and the underlying concept space where relevance judgements are implicitly encoded.

References

1. van Rijsbergen, C.J.: The Geometry of Information Retrieval. Cambridge University Press, New York (2004)

2. Song, D., Lalmas, M., van Rijsbergen, K., Frommholz, I., Piwowarski, B., Wang, J., Zhang, P., Zuccon, G., Bruza, P.D., Arafat, S., et al.: How quantum theory is developing the field of information retrieval. In: Proceedings of QI, Arlington, VA, USA, pp. 105–108, November 2010
3. Zuccon, G., Azzopardi, L.: Using the quantum probability ranking principle to rank interdependent documents. In: Gurrin, C., He, Y., Kazai, G., Kruschwitz, U., Little, S., Roelleke, T., Rüger, S., van Rijsbergen, K. (eds.) ECIR 2010. LNCS, vol. 5993, pp. 357–369. Springer, Heidelberg (2010)
4. Zuccon, G., Piwowarski, B., Azzopardi, L.: On the use of complex numbers in quantum models for information retrieval. In: Amati, G., Crestani, F. (eds.) ICTIR 2011. LNCS, vol. 6931, pp. 346–350. Springer, Heidelberg (2011)
5. Deerwester, S., Dumais, S., Furnas, G., Landauer, T., Harshman, R.: Indexing by latent semantic analysis. JASIST **41**(6), 391–407 (1990)
6. Lund, K., Burgess, C.: Producing high-dimensional semantic spaces from lexical co-occurrence. Behav. Res. Meth. Instrum. Comput. **28**, 203–208 (1996)
7. Kanerva, P., Kristofersson, J., Holst, A.: Random indexing of text samples for latent semantic analysis. In: Proceedings of CogSci., vol. 1036, Philadelphia, PA, USA (2000)
8. Karlgren, J., Sahlgren, M.: From words to understanding. In: Uesaka, Y., Kanerva, P., Asoh, H. (eds.) Foundations of Real-World Intelligence, pp. 294–308. CSLI Publications, Stanford (2001)
9. Sahlgren, M.: The Word-Space Model: using distributional analysis to represent syntagmatic and paradigmatic relations between words in high-dimensional vector spaces. Ph.D. thesis, Institutionen för lingvistik. Department of Linguistics, Stockholm University (2006)
10. Symonds, M., Bruza, P., Sitbon, L., Turner, I.: Modelling word meaning using efficient tensor representations. In: Proceedings of PacLic., pp. 313–322, November 2011
11. Fellbaum, C.: WordNet: An Electronic Lexical Database. MIT Press, Cambridge (1998)
12. Koopman, B., Bruza, P., Sitbon, L., Lawley, M.: Towards semantic search and inference in electronic medical records: an approach using concept-based information retrieval. AMJ **9**, 482–488 (2012)
13. Wittgenstein, L.: Philosophical Investigations. Blackwell Publishing, Oxford (1967)
14. Harris, Z.: Distributional structure. In: Harris, Z. (ed.) Papers in Structural and Transformational Linguistics. Formal Linguistics, pp. 775–794. Humanities Press, New York (1970)
15. Firth, J.R.: Papers in Linguistics 1934–1951. Oxford University Press, London (1957)
16. Bloomfield, L.: Language. Holt, Reinhart and Winston, New York (1933)
17. Morris Charles, W.: Signs, Language and Behavior. Prentice Hall, New York (1946)
18. von Uexküll, J.: The theory of meaning. Semiotica **42**(1), 25–82 (1982)
19. Peirce, C.: Logic as semiotic: the theory of signs. In: Peirce, C., Buchler, J. (eds.) Philosophical Writings of Peirce, pp. 98–119. Dover Publications, New York (1955)
20. Frege, G.: Sense and reference. Philos. Rev. **57**(3), 209–230 (1948)
21. Sahlgren, M.: An introduction to random indexing. In: Proceedings of TKE, Copenhagen, Denmark (2005)
22. Widdows, D., Ferraro, K.: Semantic vectors: a scalable open source package and online technology management application. In: Proceedings LREC, Marrakech, Morocco, May 2008

23. Koopman, B., Zuccon, G., Bruza, P., Sitbon, L., Lawley, M.: Graph-based concept weighting for medical information retrieval. In: Proceedings of ADCS, Dunedin, New Zealand, pp. 80–87, December 2012
24. Zuccon, G., Koopman, B., Nguyen, A., Vickers, D., Butt, L.: Exploiting medical hierarchies for concept-based information retrieval. In: Proceedings of ADCS, Dunedin, New Zealand, pp. 111–114, December 2012
25. Spackman, K.: SNOMED Clinical Terms Basics. International Health Terminology Standards Development Organisation Technical report, August 2008
26. Aronson, A.R., Lang, F.M.: An overview of MetaMap: historical perspective and recent advances. JAMIA **17**(3), 229–236 (2010)
27. Wu, S.T., Liu, H., Li, D., Tao, C., Musen, M.A., Chute, C.G., Shah, N.H.: Unified medical language system term occurrences in clinical notes: a large-scale corpus analysis. JAMIA **19**(e1), e149–e156 (2012)
28. Wittek, P., Tan, C.L.: Compactly supported basis functions as support vector kernels for classification. IEEE Trans. Pattern Anal. Mach. Intell. **33**(10), 2039–2050 (2011)
29. Widdows, D., Cohen, T.: Real, complex, and binary semantic vectors. In: Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.) QI 2012. LNCS, vol. 7620, pp. 24–35. Springer, Heidelberg (2012)
30. Voorhees, E., Tong, R.: Overview of the TREC Medical Records Track. In: Proceedings of TREC, Gaithersburg, MD, USA, November 2011
31. Wu, S., Masanz, J., Ravikumar, K., Liu, H.: Three questions about clinical information retrieval. In: Proceedings of TREC, Gaithersburg, MD, USA, November 2012
32. Aerts, D., Czachor, M.: Quantum aspects of semantic analysis and symbolic artificial intelligence. J. Phys. A: Math. Gen. **37**, L123–L132 (2004)
33. Bruza, P., Kitto, K., Ramm, B., Sitbon, L., Song, D., Blomberg, S.: Quantum-like non-separability of concept combinations, emergent associates and abduction. Logic J. IGPL **20**(2), 445–457 (2012)

Polynomial Time Quantum Algorithm for Search Problem and Its Application

Satoshi Iriyama(✉) and Masanori Ohya

Tokyo University of Science, 2641 Yamazaki, Noda City, Chiba, Japan
iriyama@is.noda.tus.ac.jp

Abstract. The well known quantum algorithm for search problem is Grover's one. However, its computational complexity is not a polynomial in the input. In this study, we propose a polynomial time quantum algorithm for it based on quantum binary search and an amplification process. This process can be written as a quantum Turing machine form, a so called generalized quantum Turing machine (GQTM). We introduce the definition of GQTM and its language classes.

1 Introduction

Let X and Y be two finite sets and a function $f : X \to Y$. A search problem is to find $x \in X$ such that $f(x) = y$ for a given $y \in Y$ [1,2]. There are two different cases for the search problem: (S1) one is the case that we know there exists at least one solution x of $f(x) = y$ in X [3]. (S2) The other is the case that we do not know the existence of such a solution [5]. The second one is more difficult than the first one. S1 belongs to a class NP, however S2 does to a class NP-hard.

Since S2 contains S1 as a special case, we will discuss S2 only here. A search problem is defined by the following.

Problem 1. (S2) For a given f and $y \in Y$, we ask whether there exists $x \in X$ such that $f(x) = y$.

Without loss of generality for discrete cases, we take $X = \{0, 1, \cdots, 2^n - 1\}$ and $Y = \{0, 1\}$. Let $M_{f,X,Y}$ be a Turing machine calculating $f(x)$ and checking whether $f(x) = y$ with $x \in X$ and $y \in Y$. It outputs 1 when $f(x) = y$, 0 otherwise. To solve this problem, one can construct a Turing machine M_f running as follows:

Step1: Set a counter $i = 0$.

Step2: If $i > 2^n - 1$, then M_f outputs "reject", else calls $M_{f,X,Y}$ with the inputs $x = i$ and y, so that M_f obtains the result.

Step3: If the result of Step 2 is 1, then it outputs x.

Step4: If the result is 0, then it goes back to Step2 with the counter $i + 1$.

In the worst case, M_f must call $M_{f,X,Y}$ for all x to check whether $f(x) = y$ or not, so that the computational complexity of the searching algorithm is the cardinal number of X.

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 172–183, 2014.
DOI: 10.1007/978-3-642-54943-4_15,

In the sequel sections, we construct a quantum algorithm to solve the problem S2, and discuss on the computational complexity of it.

In the paper [11], we developed a new quantum algorithm for the search problem, and showed that the computational complexity of it is polynomial of n. Moreover, we applied this quantum algorithm into prime factorization [12].

2 Quantum Algorithm

A quantum algorithm is constructed by the following steps:

1. Prepare a Hilbert space $\mathcal{H} = \mathbb{C}^{\otimes n}$.
2. Construct an initial state $|\psi_{in}\rangle \in \mathcal{H}$.
3. Construct unitary operators U to solve the problem.
4. Apply them for the initial state and obtain a result state $|\psi_{out}\rangle = U |\psi_{in}\rangle$.
5. If necessary, amplify the probability of the correct result.
6. Measure an observable with the result state.

In the first step, we define the Hilbert space depending on the problem. Let $\mathbb{C}^2$ be a Hilbert space spanned by $|0\rangle = (1,0)^t$ and $|1\rangle = (0,1)^t$. A normalized vector $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ on this space is called a qubit. Since we can use a superposition of $|0\rangle$ and $|1\rangle$ as an initial state vector, the quantum algorithm is more effective than the classical one.

One can apply the Hadamard transformation

$$U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

to create a superposition. The Hadamard transformation has a very important role in a quantum algorithm. Applying $(U_H)^{\otimes n}$ to the vector $|\psi\rangle = |0\rangle \otimes \cdots \otimes |0\rangle \in \left(\mathbb{C}^2\right)^{\otimes n}$, we have

$$(U_H)^{\otimes n} |\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |e_i\rangle ,$$

where $\{|e_i\rangle\}$ is a complete orthnormal system of $\left(\mathbb{C}^2\right)^{\otimes n}$ defined as

$$|e_0\rangle = |0\rangle \otimes \cdots \otimes |0\rangle , \cdots , |e_{2^n-1}\rangle = |1\rangle \otimes \cdots \otimes |1\rangle .$$

One can represent any positive integer less than 2^n by $|e_i\rangle$.

Here we introduce unitary gates, which are NOT gate, C-NOT gate and CC-NOT gate. We call these gates fundamental gates. We can construct AND and OR gates by the product of fundamental gates [8,9]. The NOT gate U_{NOT} is defined on a Hilbert space $\mathbb{C}^2$ as

$$U_{NOT} = |1\rangle \langle 0| + |0\rangle \langle 1| .$$

C-NOT U_{CN} gate and CC-NOT U_{CCN} are given on two and three qubit Hilbert space as

$$U_{CN} = |0\rangle \langle 0| \otimes I + |1\rangle \langle 1| \otimes U_{NOT}$$

$$U_{CCN} = |0\rangle \langle 0| \otimes I \otimes I + |1\rangle \langle 1| \otimes |0\rangle \langle 0| \otimes I + |1\rangle \langle 1| \otimes |1\rangle \langle 1| \otimes U_{NOT},$$

respectively.

2.1 Computational Complexity of a Quantum Algorithm

In order to discuss the computational complexity of a quantum algorithm, we introduced fundamental gates above. Here we define the computational complexity of a quantum algorithm as the number of fundamental gates in it. For example, we say the computational complexity is n if the unitary operator U is constructed by a product of n fundamental gates.

Moreover, we define a generalized quantum Turing machine (GQTM) using a quantum channel and density operator. Based on GQTM we defined some language classes. BGQPP is the language class which is recognized by a GQTM in polynomial time with a halting probability 1/2. We proved that the class NP in included by BGQPP [9,10].

3 Generalized Quantum Turing Machine

In this section, we discuss the generalized quantum Turing machine (GQTM) [9,10] which contains the unitary quantum Turing machine as a special case.

3.1 Generalized Quantum Turing Machine

Let Q be a set of states including an initial state q_0 and a set of final states $\{q_F\}$ and Σ a set of alphabets including a blank symbol #. We denote a set of all infinite sequences of alphabets in Σ as

$$\Sigma^* = \Sigma \times \Sigma \times \cdots = \Sigma^\infty.$$

Let $A \in \Sigma^*$ be a sequence of alphabets representing the tape state. Each tape alphabet has a unique index. We write the k-th tape alphabet a_k as $A(k)$.

The GQTM M_{gq} is defined by the quadruplet $(Q, \Sigma, \mathcal{H}, \Lambda_\delta)$, where Λ_δ is a quantum transition function from $\mathcal{C} = Q \times \Sigma \times \mathbb{Z}$ to $\mathcal{C}$. Q and Σ are represented by a density operator on Hilbert spaces $\mathcal{H}_Q$ and $\mathcal{H}_\Sigma$, which are spanned by a canonical basis $\{|q\rangle\,; q \in Q\}$ and $\{|a\rangle\,; a \in \Sigma\}$, respectively. A tape configuration A is a sequence of elements of Σ represented by a density operator on the Hilbert space $\mathcal{H}_\Sigma$ spanned by a canonical basis $\{|A\rangle\,; A \in \Sigma^*\}$. A position of the tape head is represented by a density operator on the Hilbert space $\mathcal{H}_Z$ spanned by a canonical basis $\{|i\rangle\,; i \in \mathbb{Z}\}$. Then a configuration of GQTM M_{gq} is described by a density operator ρ in $\mathcal{H} \equiv \mathcal{H}_Q \otimes \mathcal{H}_\Sigma \otimes \mathcal{H}_Z$. Let $\mathfrak{S}(\mathcal{H})$ be the set of all density operators in the Hilbert space $\mathcal{H}$.

Here, we define the transition function

$$\delta_1 : \mathbb{R} \times Q \times \Sigma \times Q \times \Sigma \times Q \times \Sigma \times \{0, \pm 1\} \times Q \times \Sigma \times \{0, \pm 1\} \to \mathbb{C}.$$

A quantum transition function is given by a quantum channel

$$\Lambda_{\delta_1} : \mathfrak{S}(\mathcal{H}) \to \mathfrak{S}(\mathcal{H}),$$

satisfying the following condition.

Definition 1. *Λ_{δ_1} is called a quantum transition channel if there exists a transition function δ_1 such that for any quantum configuration $\rho = \sum_k \lambda_k |\psi_k\rangle\langle\psi_k|$, $|\psi_k\rangle = \sum_l \alpha_{k,l} |q_{k,l}, A_{k,l}, i_{k,l}\rangle$, $\sum_k \lambda_k = 1, \forall\lambda_k \geq 0$, $\sum_l |\alpha_{k,l}|^2 = 1, \forall\alpha_{k,l} \in \mathbb{C}$ it holds*

$$\Lambda_{\delta_1}(\rho) \equiv \sum_{k,l,m,n,p,b,d,p',b',d'} \delta_1\left(\lambda_k, q_{k,l}, A_{k,l}\left(i_{k,l}\right), q_{m,n}, A_{m,n}\left(i_{m,n}\right), p, b, d, p', b', d'\right) \times |p, B, i_{k,l} + d\rangle\langle p', B', i_{m,n} + d'|$$

$$B(j) = \begin{cases} b & j = i_{k,l} \\ A_{k,l}(j) & otherwise \end{cases}$$

$$B'(j) = \begin{cases} b' & j = i_{m,n} \\ A_{m,n}(j) & otherwise \end{cases}$$

so that the RHS is a state.

Definition 2. *$M_{gq} = (Q, \Sigma, \mathcal{H}, \Lambda_\delta)$ is called a LQTM (Linear Quantum Turing Machine) if there exists a transition function*

$$\delta_2 : Q \times \Sigma \times Q \times \Sigma \times Q \times \Sigma \times \{0, \pm 1\} \times Q \times \Sigma \times \{0, \pm 1\} \to \mathbb{C}$$

such that for any quantum configuration ρ_k, Λ_{δ_2} is written as

$$\Lambda_{\delta_2}(\rho_k) \equiv \sum_{k,l,m,n,p,b,d,p',b',d'} \delta_2\left(q_{k,l}, A_{k,l}\left(i_{k,l}\right), q_{m,n}, A_{m,n}\left(i_{m,n}\right), p, b, d, p', b', d'\right) \times |p, B, i_{k,l} + d\rangle\langle p', B', i_{m,n} + d'|$$

so that the RHS is a state. For any quantum configuration $\rho = \sum_k \lambda_k \rho_k$, Λ_{δ_2} is affine;

$$\Lambda_{\delta_2}\left(\sum_k \lambda_k \rho_k\right) = \sum_k \lambda_k \Lambda_{\delta_2}(\rho_k)$$

Definition 3. *A GQTM M_{gq} is called a unitary QTM (UQTM), if the quantum transition channel Λ_{δ_3} is a unitary channel, implemented as $\Lambda_{\delta_3} \cdot = U_{\delta_3} \cdot U^*_{\delta_3}$, where U_{δ_3} is given by, for any $|\psi\rangle = |q, A, i\rangle$,*

$$\begin{aligned} U_{\delta_3}|\psi\rangle &= U_{\delta_3}|q, A, i\rangle \\ &= \sum_{p,b,d} \delta_3\left(q, A(i), p, b, d\right)|p, B, i + d\rangle \end{aligned}$$

where

$$\delta_3 : Q \times \Sigma \times Q \times \Sigma \times \{0, 1\} \to \mathbb{C}$$

is a transition function and it satisfies for any $q \in Q, a \in \Sigma$, $q'(\neq q) \in Q, a'(\neq a) \in \Sigma$,

$$\sum_{p,b,d} |\delta_3 (q, a, p, b, d)|^2 = 1.$$

$$\sum_{p,b,d,d'} \delta_3 (q', a', p, b, d')^* \delta_3 (q, a, p, b, d) = 0.$$

Remark 1. A classical Turing Machine (CTM) is represented as a LQTM where the transition channel has only a diagonal part. Moreover, for any $q, p \in Q, a, b \in \Sigma, d \in \{0, \pm 1\}$, put $\delta_3 (q, a, p, b, d) = 0$ or 1 then UQTM is a reversal CTM.

3.2 Computation Process of GQTM

Let $M = (Q, \Sigma, \mathcal{H}, \Lambda_\delta)$ and $\rho_0 = |\psi_0\rangle \langle\psi_0|$ where $|\psi_0\rangle = |q_0, A, 0\rangle$; we call this state an initial state and A an input of M. The computation of a GQTM proceeds, applying Λ_δ to ρ_0, till the processor state becomes $q_f \in \{q_F\}$, then it halts. This process is described by the products of Λ_δ as$\Lambda_\delta \circ \cdots \circ \Lambda_\delta (\rho_0) = \rho_f$

Let $\mathfrak{S}_f (\mathcal{H}) = \{\rho | \rho \restriction \mathfrak{S} (\mathcal{H}_Q) \in \{q_f\}\}$ be the set of final configurations, where $\restriction$ means the restriction of a state on $\mathfrak{S} (\mathcal{H}_Q)$. ρ is called a final state if ρ is in the form

$$\rho = \sum_k \lambda_k \rho_k + \sum_l \mu_l \sigma_l$$

$$\sum_k \lambda_k + \sum_l \mu_l = 1, \quad \forall \lambda_k, \mu_l \geq 0$$

where $\sigma_l \in \mathfrak{S}_f (\mathcal{H})$. We call $p = \sum_l \mu_l$ the halting probability.

3.3 Language Classes Defined by GQTM

Let M be a GQTM and L a set of alphabet sequences; we say M *recognizes* L if M halts with any input $x \in L$ and does not halt for $x \notin L$.

We call L a *language* if there exists GQTM M which recognizes L.

Definition 4. *If languages are accepted by non-deterministic classical Turing machine in polynomial time of input size with a certain probability p, then this class of languages is called a bounded probability polynomial time (BPP).*

If there is a polynomial time algorithm to solve a NP-complete problem in the above sense, then P = NP. The existence of such an algorithm is demonstrated in [6,7] in an extended quantum domain, as is reviewed in the next section. We will show that the OV SAT algorithm can be written by the GQTM.

Let us define the *recognition* of a GQTM and some classes of languages.

Definition 5. *Given a GQTM M_{gq} and a language L, if there exists N steps when M_{gq} recognizes L with the probability p, we say that the GQTM M_{gq} recognizes L with probability p and its computational complexity is N.*

Definition 6. *A language L is bounded quantum probability polynomial time GQTM (BGQPP) if there is a polynomial time GQTM M_{gq} which accepts L with probability $p \geq \frac{1}{2}$.*

Similarly, we can define the class of languages BUQPP (=BQPP) and BLQPP corresponding to UQTM and LQTM, respectively.

In the paper [9,10], it is pointed out that LQTM includes classical TM, which implies

$$BPP \subseteq BLQPPL \subseteq BGQPP.$$

Moreover, if NLQTM accepts the SAT OV algorithm in polynomial time with probability $p \geq \frac{1}{2}$, then we have the inclusion [9]

$$NP \subseteq BGQPP$$

4 Quantum Searching Algorithm

From this section, we use a discrete function f. Let n be a positive number, and f a function from $X = \{0, 1, \cdots, 2^n - 1\}$ to $Y = \{0, 1\}$.

We show a quantum algorithm to solve the problem S2. To solve this problem, we denote x by the following binary expression

$$x = \sum_{k=1}^{n} 2^{k-1} \varepsilon_k,$$

where $\varepsilon_1, \cdots, \varepsilon_n \in \{0, 1\}$.

We divide the problem S2 into several problems as below. Here we start the following problem:

Problem 2. Does there exist an x such that $f(x) = 1$ with $\varepsilon_1 = 0$?

If the answer is "yes", namely $\varepsilon_1 = 0$, then there exists at least one $x = 0\varepsilon_2 \cdots \varepsilon_n$ such that $f(x) = 1$. If the $\varepsilon_1 \neq 0$, then one considers two cases: that $\varepsilon_1 = 1$, or there does not exist any x such that $f(x) = 1$.

We go to the next problem with the result of the above problem:

Problem 3. Does there exist an x such that $f(x) = 1$ with $\varepsilon_2 = 0$ for the obtained ε_1?

After solving this problem, we know the value of ε_2; for example, when $\varepsilon_2 = 0$, x is written by $00\varepsilon_3 \cdots \varepsilon_n$ or $10\varepsilon_3 \cdots \varepsilon_n$.

Furthermore, we check the ε_i, $i = 3, \cdots, n$ by the same way as above using the information of the bits from ε_1 to ε_{i-1}. We run the algorithm from ε_1 to ε_n, and we look for one x satisfying $f(x) = 1$. Finally in the case that the result of the algorithm is $x = 1 \cdots 1$, we calculate $f(1 \cdots 1)$ and check whether

$f(1\cdots1) = 1$ or not. We conclude that (1) if it becomes 1, $x = 1\cdots1$ is a solution of the search problem, and (2) otherwise, there does not exist an x such that $f(x) = 1$.

Let m be a positive integer which can be written by a polynomial in n. Let $\mathcal{H} = \left(\mathbb{C}^2\right)^{\otimes n+m+1}$ be a Hilbert space. The m qubits are used for the computation of f, and the dust qubits are produced by this computation. When f is given, we can fix m. We will show in the next section that this algorithm can be done in a polynomial time.

We construct the following quantum algorithm $M_Q^{(1)}$ to solve problem 2. Let $\left|\psi_{in}^{(1)}\right\rangle = |0^n\rangle \otimes |0^m\rangle \otimes |0\rangle \in \mathcal{H}$ be an initial vector for $M_Q^{(1)}$, where the upper index (1) comes from the quantum algorithm checking the bit ε_1. The last qubit of $\left|\psi_{in}^{(1)}\right\rangle$ is for the answer to it, namely "yes" or "no". If the answer is "yes", then the last qubit becomes $|1\rangle$, otherwise $|0\rangle$.

The quantum algorithm $M_Q^{(1)}$ is given by the following steps. We start $M_Q^{(1)}$ with $\varepsilon_1 = 0$.

Step1: Apply Hadamard gates from the 2nd qubit to the n-th qubit.

$$\begin{aligned} I \otimes U_H^{\otimes n-1} \otimes I^{m+1} \left|\psi_{in}^{(1)}\right\rangle &= \frac{1}{\sqrt{2^{n-1}}} \left|\varepsilon_1 (= 0)\right\rangle \otimes \left(\sum_{i=0}^{2^{n-1}-1} |e_i\rangle \right) \otimes |0^m\rangle \otimes |0\rangle \\ &= \left|\psi_1^{(1)}\right\rangle \end{aligned}$$

where $|e_i\rangle$ are

$$\begin{aligned} |e_0\rangle &= |0\cdots0\rangle \\ |e_1\rangle &= |1\cdots0\rangle \\ &\vdots \\ |e_{2^{n-1}-1}\rangle &= |1\cdots1\rangle \end{aligned}$$

Let U_f be the unitary operator on $\mathcal{H} = \left(\mathbb{C}^2\right)^{\otimes n+m+1}$ to compute f, defined by

$$U_f |x\rangle \otimes |0^m\rangle \otimes |0\rangle = |x\rangle \otimes |z_x\rangle \otimes |f(x)\rangle$$

where z_x is the dust qubit produced by the computation.

Step2: Apply the unitary operator U_f to the state made in Step1, and store the result in the last qubit.

$$\begin{aligned} U_f \left|\psi_1^{(1)}\right\rangle &= \frac{1}{\sqrt{2^{n-1}}} |0\rangle \otimes \left(\sum_{i=0}^{2^{n-1}-1} |e_i\rangle \otimes |z_i\rangle \otimes |f(0e_i)\rangle \right) \\ &= \left|\psi_2^{(1)}\right\rangle \end{aligned}$$

where z_i is the dust qubits depending on e_i.

Step3: We take the last qubit by the projection from the final state $\left|\psi_2^{(1)}\right\rangle$ such that

$$(1-p)\left|0\right\rangle\left\langle 0\right| + p\left|1\right\rangle\left\langle 1\right| = proj.\left|\psi_2^{(1)}\right\rangle\left\langle\psi_2^{(1)}\right|$$

where $p = card\left\{x|f(x)=1, x=0\varepsilon_2\cdots\varepsilon_n\right\}/2^{n-1}$.

Step4: After the above formula, the state is a pure state or a mixed state. If the state is mixed and $p\neq 0$ however very small, then apply the Chaos Amplifier given in the Appendix to check whether the last qubit is in the state $\left|1\right\rangle\left\langle 1\right|$. If we find that the last qubit is in the state $\left|1\right\rangle\left\langle 1\right|$, then $p\neq 0$, which implies that there exists at least one solution of $f(x)=1$ for $\varepsilon_1=0$. If we do not find that the last qubit is in the state $\left|1\right\rangle\left\langle 1\right|$, namely $p=0$, then there are two possibilities that are $\varepsilon_1=1$ or no solutions $x\in X$ of $f(x)=1$.

After this algorithm, we know that if $\varepsilon_1=0$ or 1, then the last qubit is 1 or 0, respectively. We write this process as $M_Q^{(1)}(0^n)=\varepsilon_1$ where 0^n means the initial vector.

Next we modify Step1 of the algorithm $M_Q^{(1)}$ as:

Step1: Apply Hadamard gates from 3rd qubit to n-th qubit.

And we call this algorithm $M_Q^{(2)}$. The index (2) means that the algorithm check ε_2. We start $M_Q^{(2)}$ with the initial vector $\left|\psi_{in}^{(2)}\right\rangle = \left|\varepsilon_1, 0^{n-1}\right\rangle\otimes\left|0^m\right\rangle\otimes\left|0\right\rangle$ instead of $\left|\psi_{in}^{(1)}\right\rangle$.

So forth we obtain the bit ε_2, and write as

$$M_Q^{(2)}\left(\varepsilon_1, 0^{n-1}\right) = M_Q^{(2)}\left(M_Q^{(1)}(0^n), 0^{n-1}\right) = \varepsilon_2$$

.

In generally, we write the algorithm $M_Q^{(i)}\left(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{i-1},0^{n-i+1}\right)$ for an initial vector $\left|\psi_{in}^{(i)}\right\rangle = \left|\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{i-1},0^{n-i+1}\right\rangle\otimes\left|0^m\right\rangle\otimes\left|0\right\rangle$ as the following:

Step1: Apply Hadamard gates from $i+1$-th to n-th qubits.

$$\begin{aligned} I^{\otimes i}\otimes U_H^{\otimes n-i}\otimes I^{m+1}\left|\psi_{in}^{(i)}\right\rangle &= \frac{1}{\sqrt{2^{n-i}}}\left|\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{i-1}\right\rangle\otimes\left(\sum_{k=0}^{2^{n-i}-1}\left|e_k\right\rangle\right) \\ &\otimes\left|0^m\right\rangle\otimes\left|0\right\rangle \\ &= \left|\psi_1^{(i)}\right\rangle \end{aligned}$$

Step2: Apply the unitary gate to compute f for the superposition made in Step1, and store the result in $n+m+1$-th qubit.

$$\begin{aligned} U_f\left|\psi_1^{(i)}\right\rangle &= \frac{1}{\sqrt{2^{n-i}}}\left|\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{i-1}\right\rangle \\ &\otimes\left(\sum_{k=0}^{2^{n-1}-1}\left|e_k\right\rangle\otimes\left|z_k\right\rangle\otimes\left|f\left(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{i-1},e_k\right)\right\rangle\right) \\ &= \left|\psi_2^{(i)}\right\rangle \end{aligned}$$

Step3: Take the last qubit by the projection from the final state $\left|\psi_2^{(i)}\right\rangle$ such that

$$(1-p)\left|0\right\rangle\left\langle 0\right| + p\left|1\right\rangle\left\langle 1\right| = proj.\left|\psi_2^{(i)}\right\rangle\left\langle\psi_2^{(i)}\right|$$

Step4: Apply the Chaos Amplifier, which is explained below, to find that the last qubit is $\left|1\right\rangle\left\langle 1\right|$.

After this algorithm $M_Q^{(i)}\left(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{i-1}, 0^{n-i+1}\right)$, we know the bit ε_i such that $f(x) = 1$. Each $M_Q^{(i)}$, $i \geq 2$ use the result of all $M_Q^{(j)}$ $(j < i)$ as an initial vector.

5 Computational Complexity of the Quantum Searching Algorithm

Here, we calculate the computational complexity of the quantum searching algorithm. The computational complexity is the number of the total unitary gates and amplification channels in our search algorithm.

In the above section, the quantum algorithm for binary search is given by the products of unitary gates denoted by U_i below. Let $\left|\psi_{in}^{(i)}\right\rangle$ be an initial vector for the algorithm $M_Q^{(i)}$ as

$$\left|\psi_{in}^{(i)}\right\rangle = \left|\varepsilon_1 \cdots \varepsilon_{i-1}, 0^{n-i}\right\rangle \otimes \left|0^m\right\rangle \otimes \left|0\right\rangle,$$

and it goes to the final vector

$$\begin{aligned} U_i\left|\psi_{in}^{(i)}\right\rangle &= \frac{1}{\sqrt{2^{n-i}}}\left|\varepsilon_1, \cdots, \varepsilon_{i-1}\right\rangle \otimes \left(\sum_{k=0}^{2^{n-1}-1}\left|e_k\right\rangle \otimes \left|z_k\right\rangle \otimes \left|f\left(\varepsilon_1, \cdots, \varepsilon_{i-1}, e_k\right)\right\rangle\right) \\ &= \left|\psi_2^{(i)}\right\rangle \end{aligned}$$

where $f\left(\varepsilon_1, \cdots, \varepsilon_{i-1}, e_k\right)$ is the result of the objective function for searching. The above unitary gate U_i for the algorithm M_i is defined by

$$U_i = U_f\left(U_H\right)^{\otimes n-i}\prod_{\{x_k|x_k=1\}} U_{NOT}(k)$$

where $U_{NOT}(k)$ is to apply a NOT gate for the k-th qubit only when the result of stage k is 1, $(k = 1, 2, \cdots i-1)$.

The computational complexity T of the quantum binary search algorithm $T\left(U_n\right)$ is given by the total number of unitary gates and quantum channels for the amplification. We obtain the following theorem [11].

Theorem 1. *We have*

$$T = \frac{13}{8}n^2 - \frac{9}{4}n + nT\left(U_f\right)$$

where $T\left(U_f\right)$ is a given complexity associated to the function f.

6 Chaos Amplifier

In this section, let us review the Chaos Amplifier along the papers [6,7] and the book [8].

Consider the so called logistic map which is given by the equation

$$x_{n+1} = ax_n(1 - x_n) \equiv g_a(x), \quad x_n \in [0,1] .$$

The properties of the map depend on the parameter a. If we take, for example, $a = 3.71$, then the Lyapunov exponent is positive, the trajectory is very sensitive to the initial value and one has chaotic behavior. It is important to notice that if the initial value $x_0 = 0$, then $x_n = 0$ for all n.

In the previous section we took the last qubit by the projection from the final state $\left|\psi_2^{(i)}\right\rangle$ such that

$$proj. \left|\psi_2^{(i)}\right\rangle \left\langle\psi_2^{(i)}\right| = (1-p)\left|0\right\rangle\left\langle 0\right| + p\left|1\right\rangle\left\langle 1\right|$$

One has to notice that $\left|0\right\rangle\left\langle 0\right|$ and $\left|1\right\rangle\left\langle 1\right|$ generate an Abelian algebra which can be considered as a classical system. Here we put

$$\bar{\rho} = (1-p)\left|0\right\rangle\left\langle 0\right| + p\left|1\right\rangle\left\langle 1\right|$$

Let Λ_{CA} be a quantum channel on one qubit space such that

$$\Lambda_{CA}(\bar{\rho}) = \frac{(I + g_a(\bar{\rho})\sigma_3)}{2}$$

where I is the identity matrix and σ_3 is the z-component of the Pauli matrices. Let k be a positive integer; applying $(\Lambda_{CA})^k$ to $\bar{\rho}$, we have

$$(\Lambda_{CA})^k(\bar{\rho}) = \frac{(I + g_a^k(p)\sigma_3)}{2} = \rho_k$$

To find a proper value k we finally measure the value of σ_3 in the state ρ_k such that

$$x_k \equiv \mathrm{tr}\rho_k\sigma_3 .$$

The following theorems are proved in [6–8].

Theorem 2. *For the logistic map $x_{n+1} = ax_n(1 - x_n)$ with $a \in [0,4]$ and $x_0 \in [0,1]$, let x_0 be $\frac{1}{2^n}$ and a set J be $\{0,1,2,\ldots,n,\ldots,2n\}$. If a is 3.71, then there exists an integer k in J satisfying $x_k > \frac{1}{2}$.*

Theorem 3. *Let a and n be the same as in the theorem above. If there exists k in J such that $x_k > \frac{1}{2}$, then $k > \frac{n-1}{\log_2 3.71 - 1}$.*

Using these theorems, we can easily check whether the state $\rho = \left|0\right\rangle\left\langle 0\right|$ or not. Note that this amplification process can be written in the generalized Turing machine form [9], and it is related to the semigroup dynamics [14].

7 Application of the Quantum Search Algorithm

In this section, we show an application of the quantum searching algorithm described above. There are several problems in the class NP, for example, prime factorization, the SAT problem and the Hamilton path problem. Here we give an alternative quantum algorithm for prime factorization. The most famous quantum algorithm for prime factorization is Shor's one [4]. In this algorithm, he provided the quantum algorithm using a black box which calculates a modular exponentiation. In this section, we give the quantum algorithm without using it.

Let p and q be two prime numbers, and $N = p \times q$. We constructed the following algorithm [12].

Step1: Using Hadamard gates, we prepare an initial superposition vector of all combinations of p and q.

Step2: Create a unitary gate U to calculate $p \times q$, and apply it to the initial vector.

Step3: Create a unitary gate U_f checking $p \times q = N$, and run the quantum binary search with U_f.

After the quantum searching algorithm, we can find the solution of prime factorization. The computational complexity is polynomial of $\log N$ given in [12]. This quantum algorithm finds a solution directly, while Shor's one obtains the order with some probability. Moreover, Shor's algorithm does not run correctly with dust qubits. If there remain dust qubits, it cannot reduce the computational complexity. The precise discussion is given in [13].

References

1. Levin, L.A.: Universal sequential search problems. Probl. Inf. Transm. **9**(3), 265–266 (1973)
2. Levin, L.A.: Randomness conservation inequalities: information and independence in mathematical theories. Inf. Control **61**, 15–37 (1984)
3. Solomonoff, R.J.: Optimum sequential search. Memorandum, Oxbridge Research, Cambridge (1984)
4. Shor, P.W.: Algorithms for quantum computation: discrete logarithms and factoring. In: Proceedings of 35th Annual Symposium on Foundations of Computer Science. IEEE Press, Los Alamitos (1994)
5. Grover, L.K.: A fast quantum mechanical algorithm for database search. In: Proceedings of 28th Annual ACM Symposium on the Theory of Computing, p. 212 (1996)
6. Ohya, M., Volovich, I.V.: New quantum algorithm for studying NP-complete problems. Rep. Math. Phys. **52**(1), 25–33 (2003)
7. Ohya, M., Volovich, I.V.: Quantum computing and chaotic amplifier. J. Opt. B **5**(6), 639–642 (2003)
8. Ohya, M., Volovich, I.V.: Mathematical Foundation of Quantum Information and Quantum Computation. Springer, New York (2011)
9. Iriyama, S., Ohya, M.: Language classes defined by generalized quantum turing machine. Open Syst. Inf. Dyn. **15**(4), 383–396 (2008)

10. Iriyama, S., Ohya, M., Volovich, I.V.: Generalized quantum turing machine and its application to the SAT chaos algorithm. QP-PQ:Quantum Prob. White Noise Anal., Quantum Information and Computing, vol. 19, pp. 204–225. World Scientific Publishing, Singapore (2006)
11. Iriyama, S., Ohya, M., Volovich, I.V.: On quantum algorithm for binary search and its computational complexity. TUS preprint (2012)
12. Goto, K., Iriyama, S., Ohya, M., Volovich, I.V.: On quantum algorithm of prime factoring using quantum binary search. TUS preprint (2012)
13. Iriyama, S., Ohya, M., Volovich, I.V.: On computational complexity of shor's quantum factoring algorithm. TUS preprint (2012)
14. Iriyama, S., Ohya, M.: On semigroup dynaical representation of non-linear chaos dynamics (in preparation)

A Predicative Characterization of Quantum States and Matte Blanco's Bi-logic

G. Battilotti(✉)

Department of Mathematics, University of Padova, Padova, Italy
giulia@math.unipd.it

Abstract. We show a correspondence between a predicative characterization of quantum states, we have recently introduced, and bi-logic, proposed by the Chilean psychoanalyst I. Matte Blanco. In bi-logic, the logic of the unconscious is characterized by "infinite" objects and by the "symmetric mode", without negation and logical consequence. In the quantum model it is possible to define a class of first order domains, called virtual singletons, that are uncountable, and that allow a generalization of the notion of duality, called symmetry. Symmetry makes negation and logical consequence collapse, in favour of different links between judgements, that are due to quantum correlations.

Introduction

The relation between quantum physics and consciousness is a very controversial and challenging topic in quantum interaction. Even though its experimental basis is not stabilized yet, it is considered reasonable by several authors to investigate formal approaches connecting quantum physics and mind. Such approaches aim, as a first instance, to establish applications of quantum structures to psychological and cognitive fields, and, more specifically, to search isomorphisms which are compatible with the existence of a relation and could support it. There is an increasing number of proposals in such directions, witnessed in the proceedings [QI07, QI08, QI09, QI11, QI12] and the references therein; moreover we refer to the authors [Ae, ABGS, AGS, AS, AS2, BPFT, BB].

A unique opportunity for the development of a formal approach to the problem of consciousness is offered by *bi-logic*, namely, the logical system proposed in the 70's by the Chilean psychoanalyst Ignacio Matte Blanco [MB], to describe two logical sides of the human thinking, the rational thinking and the logic of the unconscious. His logical approach, inspired by set theory, is very effectively synthetized in the ideas of *infinite sets*, and of *symmetry*. This makes it possible to discuss a comparison with a model developed in a formal theory.

A topological approach has been proposed by A. Khrennikov (see e.g. [Kh1, Kh2], and see [Lg, Mu]). The present note would like to describe an isomorphism with a logical model, recently proposed by the author in the field of

Work partially supported by the ex-60 % funds of the University of Padova.

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 184–190, 2014.
DOI: 10.1007/978-3-642-54943-4_16,

quantum computational logics in [Ba], then developed in [Ba2, Ba3]. The basis for the model is proof-theoretical, in the approach of basic logic [SBF]. It considers equations, which define logical connectives and derive their rules in sequent calculus from metalinguistic links between logical judgements. The predicative interpretation of quantum states here considered results from the application of such a method to the case of some judgements in quantum physics, considering the equations defining the quantifiers introduced in [MS] and the equation defining the equality predicate given by Maietti (reported in [Ba2]).

We think that the approach we have chosen is particularly suited to the case of a logic arising from psychoanalysis, for it is based on a direct analysis of judgements, which can distinguish between the metalevel and the object level. Moreover, basic logic can discuss the notion of symmetry itself, in terms of logical consequence, represented, in sequent calculus, by the sequent sign $\vdash$. This allows a direct comparison with Matte Blanco's notion of symmetry, in terms of logical derivations and connectives.

In the following, we refer to results and definitions contained in [Ba2, Ba3] for the first section, and in [Ba3] for the second section.

1 Matte Blanco's Infinite Sets in Quantum Terms

Matte Blanco characterizes the objects of the unconscious as "infinite" sets. For, the unconscious treats the part as the whole thing. This means, in particular, that a bijection between a subset and the whole set is possible, namely the set one is considering is infinite.

We approach the infinite sets adopting a particular reading of the well known logical distinction between propositional formulae on closed terms and quantifiers. Let us consider any set D. We recall the following result, that links membership relation, quantifiers and equality relation:

Proposition 1. *The equivalence between the proposition $z \in D$ and the proposition $(\exists x \in D)x = z$ is provable from the equations defining the existential quantifier and of the equality relation. If $D = \{t_1, \ldots, t_n\}$, the sequent $z = t_1 \vee \ldots \vee z = t_n \vdash z \in D$ is provable from the definition of the additive disjunction $\vee$.*

The converse sequent $z \in D \vdash z = t_1 \vee \ldots \vee z = t_n$ is not provable. We adopt the intuitionistic interpretation of disjunction. With respect to it, we characterize a particular class of sets:

Definition 1. *A finite set $D = \{t_1, \ldots, t_n\}$ is* focused *by a certain logical system if and only if the sequent $z \in D \vdash z = t_1 \vee \ldots \vee z = t_n$ is true in that system, it is* unfocused *otherwise.*

A set $D = \{t_1, \ldots, t_n\}$, which is recognized as finite in the metalanguage, is not recognized as such inside a logical system, if it is unfocused in it, since its elements can be counted by the system only if, picking up a generic element $z \in D$, it can be recognized that $z = t_i$ for some index i.

One has the following characterization of focused sets, that is very significant for our model:

Proposition 2. *Given a finite set $D = \{t_1, \dots t_n\}$, the schema $(\forall x \in D)A(x)$ and the schema $A(t_1)\& \dots \& A(t_n)$ are equivalent if and only if the domain D is focused.*

Such a setting is suited to introduce a predicative representation of quantum states, as follows. Let us consider a particle $\mathcal{A}$ and fix an observable. Let us consider the random variable Z associated to the measurement of the particle with respect to that observable, and then the set

$$D_Z = \{(s(z), p\{Z = s(z)\})\}$$

of the outcomes of measurement with their probabilities. Moreover, let us consider the proposition

$$A(z): \text{ “The particle } \mathcal{A} \text{ is in state } s(z) \text{ with probability } p\{Z = s(z)\}\text{”}$$

describing the eventual state of the particle after measurement, in the first order variable z. If the measurement hypothesis, concerning the preparation, are denoted by Γ, the resulting metalinguistic assertion is:

$$\Gamma \ yield \ A(z) \ forall \ z \in D_Z,$$

translated into “*forall* $z \in D_Z$, $\Gamma \vdash A(z)$”, furtherly converted into the unique sequent $\Gamma, z \in D_Z \vdash A(z)$, where the sequent sign $\vdash$ translates the metalinguistic consequence *yield*, and where $z \in D$ is put as a premise of the sequent (see [MS][1]). Then one considers the definition of $\forall$:

$$\Gamma \vdash (\forall x \in D)A(x) \quad if \ and \ only \ if \quad \Gamma, z \in D \vdash A(z)$$

for all formulae $A(z)$ where z is a free variable in a domain D. It allows our mind to attribute a state to the particle by the proposition $(\forall x \in D_Z)A(x)$, that hence represents $\mathcal{A}$ in its state.

On the contrary, one can see that the propositional formula $A(t_1)\& \dots \& A(t_n)$, on the closed terms t_i describes the mixed state obtained after measurement, rather than the pure state. This follows from the definition of the propositional conjunction &. For, it is true for every i that the preparation Γ *yields* $A(t_i)$, that is written $\Gamma \vdash A(t_i)$. In logic, every sequent $\Gamma \vdash A(t_i)$ is derived from the sequent $\Gamma, z \in D_Z \vdash A(z)$ by the substitution t_i/z in the sequent, then cutting the true assumption $t_i \in D_Z$. Then, we say that substituting represents a measurement in logic.

Proposition 2 shows that the nature, pure or mixed, of states, can be told in terms of sets, since the set $D_Z = \{t_1, \dots t_n\}$ of the eventual outcomes of measurement has a finite description, by means of the propositional intuitionistic disjunction $\vee$, after quantum superposition has collapsed. So, quantum superposition corresponds to the infinite character of the set D_Z.

[1] Notice that the premises in Γ do not depend on the variable z, since the measurement hypothesis cannot depend on its eventual outcome.

2 Virtual Singletons: Converting Duality into Symmetry

Following Matte Blanco, the logical mode of the unconscious is symmetric. This means, primarily, that the unconscious treats every relation as if it were symmetric. The "symmetric mode" and the "infinite mode" are strongly related in Matte Blanco: one has an infinite cardinality since the unconscious considers the part as the whole thing, since the inclusion relation is treated as if it were symmetric. Notice that a set has only symmetric relations if and only if it is a singleton, and that the sets for which no nonempty subset is different from the whole are singletons as well. So, in order to treat the symmetric mode, we need to introduce infinite sets acting as singletons: then "normal" singletons will be the finite shadow of them.

By definition, singletons are sets V for which there is an $u \in V$ such that, if $z \in V$, then $z = u$. Then we write $V = \{u\}$. It is quite natural to assume the sequent $z \in V \vdash z = u$ (where u is a closed term denoting the same element), by extensionality. Then, in a normal logical setting, singletons are focused. However, singletons are quite close to unfocused sets, since they are not splitted by a disjunction. We assume that the particular odd logical behaviour of singletons is due to their borderline situation and we claim the existence of *virtual singletons*, namely unfocused sets characterized as singletons. Since unfocused sets are characterized as first order domains of quantifiers, we adopt a characterization of virtual singletons as domains of predicative formulae: we say that V is a virtual singleton when $(\forall x \in V)A(x) = (\exists x \in V)A(x)$ is true for every A.

Namely quantification is performed by a unique connective rather than by the couple ($\forall$, $\exists$) of dual connectives. We see that this produces other logical features discussed by Matte Blanco, such as paraconsistency, absence of negation and of logical consequence.

In our model, we have shown that non trivial virtual singletons are conceivable only if the logic we are considering does not admit the substitution rule for them. Since in the model substituting means measuring, virtual singletons can exist only prior to measurement. We consider, as an observable, the spin, that has no classical equivalent. Let us fix an axis, say the z axis. The measurement of the spin observable, on particles in the sharp states $\uparrow$ and $\downarrow$, determines, as domains, two singletons. The formulae quantified on such domains are equivalent to propositional formulae, since their domains are singletons. They are interpretable as pairs of opposites: one puts a duality $\perp$ that can switch $\uparrow$ and $\downarrow$. It is the logical translation of the application of the Pauli matrix σ_X, namely the *NOT* gate of computation. As is well known, a duality between couple of opposite literals can be inductively extended to all formulae, in order to obtain a negation (Girard's negation), which satisfies the equivalence $A \vdash B$ if and only if $B^\perp \vdash A^\perp$, for every formulae A, B.

On the other hand, in the same measurement context, the dual states (written $|+\rangle = 1/\sqrt{2}|\downarrow\rangle + 1/\sqrt{2}|\uparrow\rangle$ and $|-\rangle = 1/\sqrt{2}|\downarrow\rangle - 1/\sqrt{2}|\uparrow\rangle$, respectively, as vectors in the ket notation) are switched by the Pauli matrix σ_Z, and they are eigenvectors for σ_X. In turn, we can translate all this into logic. Then, in logic, particles in the dual states correspond to predicative formulae which are fixed points for the duality $\perp$. Then such formulae satisfy the equivalence $A \vdash B$ if and only if $B \vdash A$. This eliminates the direction of logical consequence, represented by $\vdash$. The set associated to the measurement of $|+\rangle$ and $|-\rangle$ contains the two pieces of information "up" and "down", one of which can be considered the negation of the other, as just seen. Changing the measurement context and measuring the spin w.r.t the x axis, $|+\rangle$ and $|-\rangle$ would produce an objective property, since the domains would be, provably, singletons. The corresponding formulae would be equivalent to propositional formulae that would be switched by the duality corresponding to σ_Z. However, different spin observables are incompatible, hence, once the measurement context has been fixed, the domain of particles in the dual states is a virtual singleton: for it is unfocused and one can see, by means of the duality originated by σ_Z, that the equality $(\forall x \in V)A(x) = (\exists x \in V)A(x)$ is true for the first order domains originated by $|+\rangle$ and $|-\rangle$.

So we find the logical framework expected by bi-logic: an asymmetric mode where negation is meaningful, a symmetric mode where negation is meaningless. The second is given by virtual singletons, that are infinite sets. In the virtual singletons of the spin model just seen, opposite pieces of information coexist. In the symmetric mode, negation is meaningless since the opposite coexist. This is one of the features of the unconscious thinking, termed condensation, that, as pointed out by Matte Blanco, is related to the absence of negation.

A further feature of the unconscious thinking is displacement. Following Matte Blanco, it also goes back to the symmetric mode, since two different subclasses of the same class are treated as identical by the unconscious, due to symmetry: both subclasses are identified with the larger class and then treated as identical. In such a case, symmetry is applied at the second order. In the predicative model, one can widen the action of virtual singletons to the second order, considering "virtual singletons of indexes of formulae" that allow to identify formulae whose index is in the same virtual singleton. This allows, in particular, to represent the quantum correlations of the Bell states, since the correlation takes place when the same variable is displaced elsewhere, considering another index. We think that the same kind of identification at the second order, namely at the level of formulae, could be exploited in order to justify displacement. Matte Blanco himself discusses the possibility of different logical links to be discovered for the logic of the unconscious. A further analysis of the consequences of virtual singletons of indexes of formulae in logic is in progress. However, we consider it very intriguing to find that what hides logical consequence supplies, at the same time, a different way to link judgements, by means of correlations of which we cannot be aware.

Acknowledgements. The present note has been written some years after the valuable suggestion by Stuart Hameroff, who proposed to compare quantum logics and Matte Blanco's bi-logic. His suggestion became more and more significant to me with the development of the predicative model here considered.

References

[Ae] Aerts, D.: Quantum structure in cognition. J. Math. Psychol. **53**, 314–348 (2009)

[ABGS] Aerts, D., Broekaert, J., Gabora, L., Sozzo, S.: Quantum structure and human thought. Behav. Brain Sci. **36**, 274–276 (2013)

[AGS] Aerts, D., Gabora, L., Sozzo, S.: Concepts and their dynamics: a quantum-theoretic modeling of human thought. Top. Cogn. Sci. ArXiv: 1206.1069 [cs.AI] (in print)

[AS] Aerts, D., Sozzo, S.: Quantum structure in cognition: why and how concepts are entangled. In: Song, D., Melucci, M., Frommholz, I., Zhang, P., Wang, L., Arafat, S. (eds.) QI 2011. LNCS, vol. 7052, pp. 116–127. Springer, Berlin (2011)

[AS2] Aerts, D., Sozzo, S: Quantum entanglement in concept combinations. Submitted in Int. J. Theor. Phys. ArXiv:1302.3831 [cs.AI] (2013)

[Ba] Battilotti, G.: Interpreting quantum parallelism by sequents. Int. J. Theor. Phys. **49**, 3022–3029 (2010)

[Ba2] Battilotti, G.: Characterization of quantum states in predicative logic. Int. J. Theor. Phys. **50**, 3669–3681 (2011)

[Ba3] Battilotti, G.: Quantum states as virtual singletons: converting duality into symmetry. Int. J. Theor. Phys. arXiv 1304.2788[math.LO] (to appear)

[QI07] Bruza, P.D., Lawless, W., van Rijsbergen, C.J., Sofge, D. (eds.): Proceedings of the AAAI Spring Symposium on Quantum Interaction, 27–29 March. Stanford University, Stanford (2007)

[QI08] Bruza, P.D., Lawless, W., van Rijsbergen, C.J., Sofge, D. (eds.): Quantum Interaction: Proceedings of the Second Quantum Interaction Symposium. College Publications, London (2008)

[QI09] Bruza, P.D., Sofge, D., Lawless, W., van Rijsbergen, K., Klusch, M. (eds.): QI 2009. LNCS (LNAI), vol. 5494. Springer, Heidelberg (2009)

[BPFT] Busemeyer, J.R., Pothos, E., Franco, R., Trueblood, J.S.: A quantum theoretical explanation for probability judgment 'errors'. Psychol. Rev. **118**, 193–218 (2011)

[BB] Busemeyer, J.R., Bruza, P.D.: Quantum Models of Cognition and Decision. Cambridge University Press, Cambridge (2012)

[QI12] Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.): QI 2012. LNCS, vol. 7620. Springer, Heidelberg (2012)

[Kh1] Khrennikov, A.: Human subconscious as the p-adic dynamical system. J. Theor. Biol. **193**, 179–196 (1998)

[Kh2] Khrennikov, A.: Modelling of psychological behavior on the basis of ultrametric mental space: encoding of categories by balls. P-Adic Numbers Ultrametric Anal. Appl. **2**(1), 1–20 (2010)

[Lg] Lauro-Grotto, R.: The unconscious as an ultrametric set. Am. Imago **64**(4), 535–543 (2008)

[MS] Maietti, M.E., Sambin, G.: Toward a minimalist foundation for constructive mathematics. In: Crosilla, L., Schuster, P. (eds.) From Sets and Types to Topology and Analysis: Towards Practicable Foundations for Constructive Mathematics. Oxford UP, Oxford (2005)

[MB] Matte Blanco, I.: The Unconscious as Infinite Sets. An Essay in Bi-Logic. Duckworth, London (1975)

[Mu] Murtagh, F.: Ultrametric model of mind, I: Review. P-Adic Numbers Ultrametric Anal. Appl. **4**(3), 193–206 (2012)
[SBF] Sambin, G., Battilotti, G., Faggian, C.: Basic logic: reflection, symmetry, visibility. J. Symbolic Logic **65**, 979–1013 (2000)
[QI11] Song, D., Melucci, M., Frommholz, I., Zhang, P., Wang, L., Arafat, S. (eds.): QI 2011. LNCS, vol. 7052. Springer, Heidelberg (2011)

Adaptive Dynamics and Optical Illusion on Schröder's Stair

Masanari Asano[1], Takahisa Hashimoto[2], Andrei Khrennikov[3], Masanori Ohya[2], and Yoshiharu Tanaka[2](✉)

[1] Liberal Arts Division, Tokuyama College of Technology Gakuendai, Shunan, Yamaguchi 745-8585, Japan
[2] Department of Information Sciences, Tokyo University of Science, Yamasaki 2641, Noda-shi, Chiba 278-8510, Japan
tanaka@is.noda.tus.ac.jp
[3] International Center for Mathematical Modeling in Physics and Cognitive Sciences, Linnaeus University, 35195 Växjö, Sweden

Abstract. Recently, various examples of non-Kolmogorovness in contextual dependent phenomena have been reported. In this study, we introduce non-Kolmogorovness in the measurement of depth inversion for the figure of Schröder's stair. Also we propose a model of the depth inversion, based on a non-Kolmogorovian probability theory which is called adaptive dynamics.

Keywords: Optical illusion · Non-Kolmogorovian probability · Adaptive dynamics

1 Introduction

The *Schröder's stair* is an ambiguous figure which induces optical illusion [1,10] (see the Fig. 1). Our brain can switch between the two alternative interpretations of this figure; (i) The surface of 'L' is front, and the surface of 'R' is back; (ii) The surface of 'R' is front, and the surface of 'L' is back. This switch-like process of human perception is called *depth inversion*, and many experimental proofs of this phenomenon have been reported. However, the details of its mechanism is not completely figured out even in recent studies.

It is a well-known fact that depth inversion depends on various contexts of figure; e.g. relative size of the surface 'L' for 'R', color or shadow in figure, angle to the horizon, etc. [10]. Therefore we must define the contextual dependent probability that a person answers either (i) or (ii) in the experiment if we discuss the adaptive system where such contextual dependences emerge. We have developed a contexual probabilistic model which is inspired by quantum mechanics. This is called *quantum-like models* (QLM) [13], and many authors have studied QLM for various contextual dependent phenomena [2–4,6–9,11,12,14]. The statistical probabilities in the experiments for such contextual dependent systems

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 191–200, 2014.
DOI: 10.1007/978-3-642-54943-4_17,

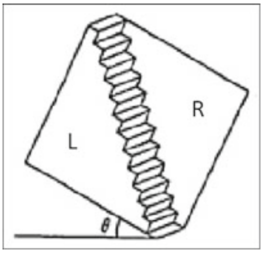


Fig. 1. Schröder's stair leaning at angle θ

violate the formulae of conventional probability theory, e.g. *the violation of total probability law* (TPL):

$$P(A = a) \neq \sum_b P(A = a|B = b)P(B = b).$$

The violation of TPL is one of important consequences of non-Kolmogorovness. Non-Kolmogorovian probabilistic approach for perception of ambiguous figures is discussed by Conte et al. [16], and Atmanspacher et al. [17].

Recently, we define the contextual joint probabilities which are based on a non-Kolmogorovian probability theory, called *Adaptive Dynamics*(AD) [2,4]. In this study, we proposed a model of the depth inversion in Schröder's stair with the non-Kolmogorovian joint probabilities. In Sect. 2, we explain the short introduction of AD and the new definition of the joint probability based on AD. In Sect. 3, we show the experimental results of depth inversion in different contexts. In Sect. 4, we discuss the violation of the total probability law in the experimental data. In Sect. 5, we propose the model of depth inversion where the concept of majority among N-agents plays an important role.

2 Adaptive Dynamics and Joint Probability

In this section, we briefly explain the definition of the non-Kolmogorovian joint probability which is based on the theory of Adaptive Dynamics (AD) [2,4].

Let $\mathcal{S}(\mathcal{H})$ be a set of all density operators on a Hilbert space $\mathcal{H}$, and let $\mathcal{O}(\mathcal{H})$ be a set of all observables on a Hilbert space of $\mathcal{H}$.

Here let us take another Hilbert space $\mathcal{K}$. A map from $\mathcal{S}(\mathcal{H})$ to the composite system $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ is called "lifting map" (or simply "lifting") [5]. In the paper of [2], we introduced a lifting from $\mathcal{S}(\mathcal{H})$ to $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$, say $\mathcal{E}^*_{\sigma Q}$. Here, σ is a state on $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$, and Q is an observable on $\mathcal{O}(\mathcal{H} \otimes \mathcal{K})$. The lifting $\mathcal{E}^*_{\sigma Q}$ is constructed by σ and Q. We consider the following dynamics.

$$\rho \Rightarrow \mathcal{E}^*_{\sigma Q}(\rho) \Rightarrow \mathrm{tr}_{\mathcal{H}} \mathcal{E}^*_{\sigma Q}(\rho) \equiv \rho_{\sigma Q} \in \mathcal{S}(\mathcal{K}).$$

The initial state ρ is defined in $\mathcal{S}(\mathcal{H})$ or $\mathcal{S}(\mathcal{K})$. We call this state change *the dynamics adaptive to the state σ and the observable Q or the dynamics adaptive to the context $C = \{\sigma, Q\}$.*

The compound state $\mathcal{E}^*_{\sigma Q}(\rho) = \mathcal{E}^*_C(\rho)$ will describe that an event system of interest is correlated with another event system. Now suppose two event systems $A = \{a_k \in \mathbb{R}, E_k \in \mathcal{O}(\mathcal{K})\}$ and $B = \{b_j \in \mathbb{R}, F_j \in \mathcal{O}(\mathcal{H})\}$, where E_k, F_j are not needed to be projections, but they should satisfy the conditions $\sum_k E_k = I$, $\sum_j F_j = I$ as POVM (positive operator valued measure).

Here, let us consider the *joint probability* as

$$P_C(a_k, b_j) = \mathrm{tr}\left\{(E_k \otimes F_j)\, \mathcal{E}^*_C(\rho)\right\}.$$

Further, the probability $P_C(a_k)$ is defined as

$$\begin{aligned}\sum_j P_C(a_k, b_j) &= \mathrm{tr} I \otimes E_k \mathcal{E}^*_C(\rho) \\ &= \mathrm{tr}_{\mathcal{K}} E_k \rho_C \\ &\equiv P_C(a_k).\end{aligned}$$

The violation of total probability law comes from a difference of two contexts, say $C = \{\sigma, Q\}$ and $\tilde{C} = \{\tilde{\sigma}, \tilde{Q}\}$. It is represented in the form as

$$P_C(a_k) = P_{\tilde{C}}(a_k) + \Delta = \sum_j P_{\tilde{C}}(a_k, b_j) + \Delta. \tag{1}$$

Generally, if $C \neq \tilde{C}$, then $\Delta \neq 0$. In order to discuss Δ mathematically, what we need is to define the probability $P_C(a_k)$, more precisely, to construct the liftings $\mathcal{E}^*_C$.

3 Depth Inversion on Optical Illusion of Schröder's Stairs

In this section, we explain the method of our experiment, and we show its results. We show the picture of Schröder's stair which is leaning at a certain angle θ (see Fig. 1) to the 151 subjects. We prepare the 11 pictures which are leaning at different angles: $\theta = 0, 10, 20, 30, 40, 45, 50, 60, 70, 80, 90$. A subject must answer either (i) "L is front" or (ii) "R is front" for every picture. We arrange the computer experiment to change the pictures and to record their answers.

Before the experiment, we divided the 151 subjects into three groups: (A) 55 persons, (B) 48 persons, (C) 48 persons. For the first group (A), the order of showing is randomly selected for each person. To assume statistically uniform randomness of this selection, we use the computer-implemented function (e.g. java.rand). For the second group (B), the angle θ is changed in order from small value: $0, 10, \cdots, 90$. Inversely, for the third group (C), the angle θ is changed in order from large value: $90, 80, \cdots, 0$.

The Fig. 2 shows the probability that a subject says (i) "L is front" with respect to angle θ. We denote it by $P(X_\theta = L)$. In all cases, the probability $P(X_\theta = L)$ is decreasing as the value of θ is increasing. However, the speed of decreasing is different among three cases. Moreover, we can see that the probability at $\theta = 0$ is nearly equal to one in all cases. That is, almost every subject says "L is front" at the angle $\theta = 0$. Conversely, the probability $\theta = 90$ is nearly equal to zero, that is, the almost every subject says "R is front" at the angle $\theta = 90$. Therefore it can be considered that subjects feel little ambiguity of the picture for $\theta = 0$ and 90. In fact, we observed that it takes only short time (less than one second) to answer his/her decision when angle θ is around 0 or 90°.

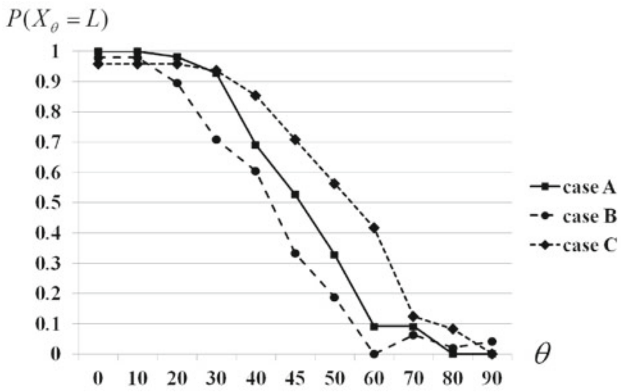


Fig. 2. Probability $P\left(X_{\theta} = L\right)$with respect to θ.

4 Violation of Formula of Total Probability

As explained in the previous section, we consider the three situations (A),(B) and (C) in the experiments. Similarly with the Eq. (1), we can write the violation of total probability law as follows:

$$\begin{aligned} P_A\left(X_\theta = L\right) &= P_B\left(X_\theta = L, X_{\theta'} = L\right) + P_B\left(X_\theta = L, X_{\theta'} = R\right) + \Delta_{AB}, \\ P_A\left(X_\theta = L\right) &= P_C\left(X_\theta = L, X_{\theta'} = L\right) + P_C\left(X_\theta = L, X_{\theta'} = R\right) + \Delta_{AC}, \\ P_C\left(X_\theta = L\right) &= P_B\left(X_\theta = L, X_{\theta'} = L\right) + P_B\left(X_\theta = L, X_{\theta'} = R\right) + \Delta_{CB}, \end{aligned}$$

where θ and θ' are different angles. Note that, with experimental results, we can estimate the degree of violation Δ_{AB}, Δ_{AC} and Δ_{CB}; for instance, Δ_{AB} at angle θ is given by

$$\begin{aligned} \Delta_{AB} &= P_A\left(X_\theta = L\right) - P_B\left(X_\theta = L, X_{\theta'} = L\right) + P_B\left(X_\theta = L, X_{\theta'} = L\right) \\ &= P_A\left(X_\theta = L\right) - P_B\left(X_\theta = L\right). \end{aligned}$$

You can see Δ_{AB} does not depends on θ' since only θ appears in the RHS of the above equation. This comes from Kolmogorovness within the context B s.t.

$$P_B\left(X_\theta = L\right) = P_B\left(X_\theta = L, X_{\theta'} = L\right) + P_B\left(X_\theta = L, X_{\theta'} = L\right).$$

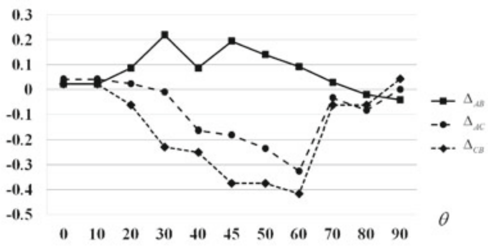


Fig. 3. Degree of violation of total probability law

From similar discussions, one can find that Δ_{AC} and Δ_{CB} depend on not θ' but only θ. As seen in Fig. 3, the strong violation occurs in the middle range of angle θ.

Here, let us compare the above Δ_{AB}, Δ_{AC} or Δ_{BC} with the conventional trigonometric interference in quantum mechanics. If Δ_{AB}, Δ_{AC} and Δ_{BC} have the form of trigonometric interference:

$$\begin{aligned}
\Delta_{AB} &= 2\cos(\phi_{AB})\sqrt{P_B(X_\theta = L, X_{\theta'} = L)P_B(X_\theta = L, X_{\theta'} = R)},\\
\Delta_{AC} &= 2\cos(\phi_{AC})\sqrt{P_C(X_\theta = L, X_{\theta'} = L)P_C(X_\theta = L, X_{\theta'} = R)},\\
\Delta_{BC} &= 2\cos(\phi_{BC})\sqrt{P_C(X_\theta = L, X_{\theta'} = L)P_C(X_\theta = L, X_{\theta'} = R)},
\end{aligned} \tag{2}$$

then the following quantities should take values between -1 and $+1$.

$$\begin{aligned}
\delta_{AB} &= \frac{\Delta_{AB}}{2\sqrt{P_B(X_\theta = L, X_{\theta'} = L)P_B(X_\theta = L, X_{\theta'} = R)}},\\
\delta_{AC} &= \frac{\Delta_{AC}}{2\sqrt{P_C(X_\theta = L, X_{\theta'} = L)P_C(X_\theta = L, X_{\theta'} = R)}},\\
\delta_{BC} &= \frac{\Delta_{BC}}{2\sqrt{P_C(X_\theta = L, X_{\theta'} = L)P_C(X_\theta = L, X_{\theta'} = R)}}
\end{aligned}$$

However, as shown in Fig. 4, the values of δ_{BC} exceeds over one at several point (θ, θ'). Therefore, the probability $P_B(X_\theta = L)$ can not be written as the conventional form of quantum mechanical interference:

$$P_B(X_\theta{=}L) \neq \left|\sqrt{P_C(X_\theta{=}L, X_{\theta'}{=}L)} + \exp(i\phi_{BC})\sqrt{P_C(X_\theta = L, X_{\theta'} = R)}\right|^2.$$

In order to explain δ_{BC} exceeding 1, we can use the generalized description of quantum mechanical interference, which has been discussed in the paper [13] or [15]. Instead of the trigonometric interference in Eq. (2), one can use hyper-trigonometric interference[13] for Δ_{BC} as follows:

$$\Delta_{BC} = 2\cosh(\phi_{BC})\sqrt{P_C(X_\theta = L, X_{\theta'} = L)P_C(X_\theta = L, X_{\theta'} = R)}. \tag{3}$$

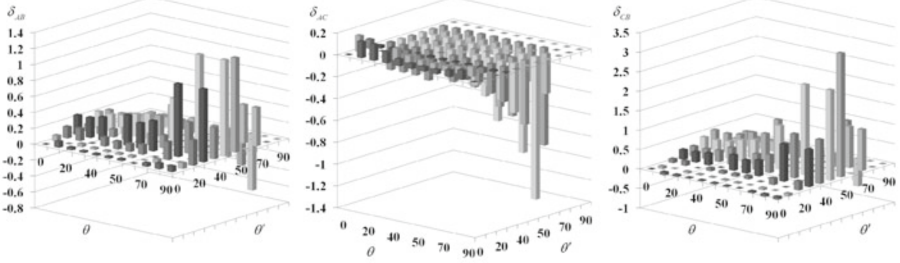


Fig. 4. Violation of total probability law

Then, the probability $P_B\left(X_\theta\right)$ is written as

$$P_B\left(X_\theta{=}L\right) = \left|\sqrt{P_C(X_\theta{=}L, X_{\theta'} = L)} + \exp\left(\varepsilon\phi_{BC}\right)\sqrt{P_C(X_\theta{=}L, X_{\theta'} = R)}\right|^2,$$

$$P_B\left(X_\theta = R\right) = \left|\sqrt{P_C(X_\theta = L, X_{\theta'} = L)} - \exp\left(\varepsilon\phi_{BC}\right)\sqrt{P_C(X_\theta = L, X_{\theta'} = R)}\right|^2$$

with a clifford number ε (i.e. $\varepsilon^2 = 1$ but not $\varepsilon \neq \pm 1$). The above equations can be considered as a generalization of Born's rule in quantum mechanics. These probabilities are written as

$$P_B\left(X_\theta = L\right) = \mathrm{tr} M_L \varrho,$$
$$P_B\left(X_\theta = R\right) = \mathrm{tr} M_R \varrho,$$

where ϱ is a pure state given by a density operator on two-dimensional clifford algebra:

$$\begin{aligned}\varrho &= \begin{pmatrix} \sqrt{P_C(X_{\theta'} = L)} \\ e^{\varepsilon\phi_{BC}}\sqrt{P_C(X_{\theta'} = R)} \end{pmatrix} \left(\sqrt{P_C(X_{\theta'} = L)}\; e^{-\varepsilon\phi_{BC}}\sqrt{P_C(X_{\theta'} = R)}\right) \\ &= \begin{pmatrix} P_C(X_{\theta'} = L) & e^{-\varepsilon\phi_{BC}}\sqrt{P_C(X_{\theta'} = L)P_C(X_{\theta'} = R)} \\ e^{\varepsilon\phi_{BC}}\sqrt{P_C(X_{\theta'} = L)P_C(X_{\theta'} = R)} & P_C(X_{\theta'} = R) \end{pmatrix},\end{aligned}$$

and M_L, M_R are Hermitian operators given by

$$M_L = \begin{pmatrix} P_C(X_\theta = L|X_{\theta'} = L) & \sqrt{P_C(X_\theta = R|X_{\theta'} = L)P_C(X_\theta = L|X_{\theta'} = L)} \\ \sqrt{P_C(X_\theta = R|X_{\theta'} = L)P_C(X_\theta = L|X_{\theta'} = L)} & P_C(X_\theta = R|X_{\theta'} = L) \end{pmatrix},$$

$$M_R = \begin{pmatrix} P_C(X_\theta = L|X_{\theta'} = R) & -\sqrt{P_C(X_\theta = L|X_{\theta'} = R)P_C(X_\theta = R|X_{\theta'} = R)} \\ -\sqrt{P_C(X_\theta = L|X_{\theta'} = R)P_C(X_\theta = R|X_{\theta'} = R)} & P_C(X_\theta = R|X_{\theta'} = R) \end{pmatrix}.$$

Note that $\{M_L, M_R\}$ is not positive operator valued measure since nondiagonal elements of $M_L + M_R$ are not equal to zero in general.

Another approach for the generalization of Born's rule is discussed in the paper [15]. The probability $P_B\left(X_\theta\right)$ is written in the following forms:

$$P_B\left(X_\theta = L\right) = \frac{\mathrm{tr}\Pi_L\rho}{\mathrm{tr}\Pi_L\rho + \mathrm{tr}\Pi_R\rho},$$

$$P_B\left(X_\theta = R\right) = \frac{\mathrm{tr}\Pi_R\rho}{\mathrm{tr}\Pi_L\rho + \mathrm{tr}\Pi_R\rho},$$

where Π_L, Π_R are positive operators but not satisfying $\Pi_L + \Pi_R = I$ (I is an identity operator on two-dimentional Hilbert space), and ρ is a pure state given by a density operator

$$\rho = \begin{pmatrix} \sqrt{P_C(X_{\theta'}=L)} \\ \exp\left(i\phi_{BC}\right)\sqrt{P_C(X_{\theta'}=R)} \end{pmatrix} \left(\sqrt{P_C(X_{\theta'}=L)}\ \exp\left(-i\phi_{BC}\right)\sqrt{P_C(X_{\theta'}=R)}\right)$$

$$= \begin{pmatrix} P_C(X_{\theta'} = L) & \exp\left(-i\phi\right)\sqrt{P_C(X_{\theta'}=L)P_C(X_{\theta'} = R)} \\ \exp\left(i\phi\right)\sqrt{P_C(X_{\theta'}=L)P_C(X_{\theta'}=R)} & P_C(X_{\theta'}=R) \end{pmatrix}.$$

These generalized Born's rules can apply to our experimental result. However, the interpretation of such a generalization is not figured out well. In the next section, we give more concrete model which is based on psychological process of self-dialog.

5 A Model of Depth Inversion: Majority Among N-Agent

The subjects have to answer either (i) "L is front" or (ii) "R is front", so that we take the Hilbert space $\mathcal{H} = \mathbb{C}^2$ for this model. Let $|L\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|R\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the orthogonal vectors describing the answers (i) and (ii), respectively. An initial state of a subject's mind is given by

$$\rho \equiv |x\rangle \langle x|,$$

where $|x\rangle$ is a state vector $\frac{1}{\sqrt{2}}\left(|L\rangle + |R\rangle\right)$, which represents the neutral mind for $|L\rangle$ and $|R\rangle$ before the decision making starts.

When a subject is shown a picture leaning at angle θ, the subject recognizes the lean of the picture. Such a recognition process is given by the operator

$$M\left(\theta\right) = \begin{pmatrix} \cos\theta & 0 \\ 0 & \sin\theta \end{pmatrix}.$$

After the recognition, the state of mind is changed from initial state ρ to an adaptive state

$$\rho_\theta = \Lambda_M^*(\rho) \equiv \frac{M^*\rho M}{\mathrm{tr}\left(|M|^2\rho\right)} = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}.$$

The fluctuation between (i) and (ii) is expressed as the above ρ_θ.

Here, we introduced the following assumption of a self-dialogue process by a subject. At the begining of the process, a subject imagines and creates an imaginary agent in the brain, and this agent has its own mind which is expressed as the adaptive state ρ_θ. A subject repeatedly create the imaginary agents during an experiment for X_θ. We can consider the adaptive state describing the whole agent set as the N-composite state of ρ_θ:

$$\sigma = \underbrace{\rho_\theta \otimes \cdots \otimes \rho_\theta}_{N},$$

where N is the number of agents which are created by a subject while making the answer (i) or (ii). If a subject takes a short time to answer, then N might be small. After the creation of agents (or on the way to creating them), the subject can talk with the imaginary agents in the brain, and he/she knows the answer of each agent. Through this dialogue, the subject can know the opinions of all the agents. For the N agents, there are 2^N possible opinions, and one of them is expressed. The subject's answer is determined by reference to the opinions of all the agents.

At the final step of this dialogue, we additionaly assumed that the σ is changed into $\left|L^{\otimes N}\right\rangle$ or $\left|R^{\otimes N}\right\rangle$ since every subject in this experiment can not answer anything except (i) or (ii). Along with this assumption, we introduce the observable-adaptive operator Q which describes the process of making a common decision. For example, in the case of $N = 2, 3$ and 4, the operator Q is

$$Q^{(2)} = |LL\rangle \langle LL| + |RR\rangle \langle RR|,$$

$$\begin{aligned} Q^{(3)} &= |LLL\rangle \left(\langle LLL| + \langle LLR| + \langle LRL| + \langle RLL|\right) \\ &\quad + |RRR\rangle \left(\langle RRR| + \langle RRL| + \langle RLR| + \langle LRR|\right), \end{aligned}$$

$$\begin{aligned} Q^{(4)} &= |LLLL\rangle \left(\langle LLLL| + \langle LLLR| + \langle LLRL| + \langle LRLL| + \langle RLLL|\right) \\ &\quad + |RRRR\rangle \left(\langle RRRR| + \langle RRRL| + \langle RRLR| + \langle RLRR| + \langle LRRR|\right). \end{aligned}$$

By applying the operator Q to the state of agents σ, the minority opinions of the agents are ignored, and changed to the majority ones. In this sense, our model is considered as the *majority system* among the N agents.

Here, the lifting $\mathcal{E}^*_{\sigma Q} : \mathcal{S}\left(\mathbb{C}^2\right) \to \mathcal{S}\left(\mathbb{C}^{2N}\right)$ is defined as

$$\mathcal{E}^*_{\sigma Q}(\rho) = \frac{Q\sigma Q^*}{\mathrm{tr}\left(|Q|^2 \sigma\right)} = \frac{Q\left\{\Lambda^*_M(\rho)\right\}^{\otimes N} Q^*}{\mathrm{tr}\left(|Q|^2 \left\{\Lambda^*_M(\rho)\right\}^{\otimes N}\right)} \in \mathcal{S}\left(\mathbb{C}^{2N}\right),$$

and the probabilities for X_θ are given with this lifting as

$$P(X_\theta = L) \equiv \mathrm{tr}\left(|L\rangle \langle L|^{\otimes N} \mathcal{E}^*_{\sigma Q}(\rho)\right),$$

$$P(X_\theta = R) \equiv \mathrm{tr}\left(|R\rangle \langle R|^{\otimes N} \mathcal{E}^*_{\sigma Q}(\rho)\right).$$

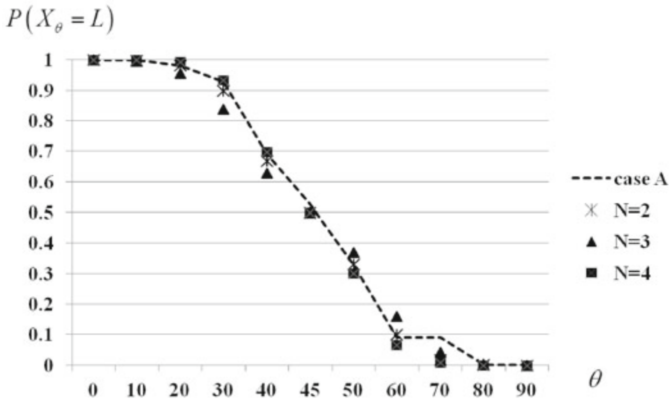


Fig. 5. Values of $P\left(X_\theta = L\right)$ in our model, its comparison with experimental data (A)

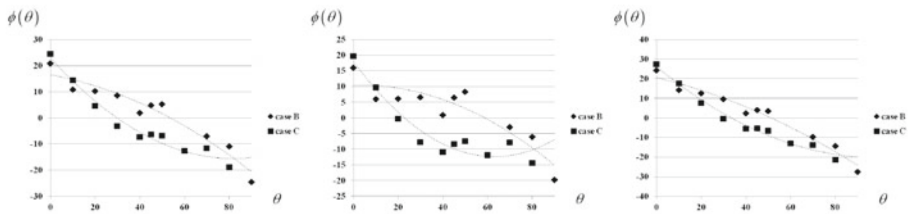


Fig. 6. Noise of biased phase; (Left) $N = 2$, (Center) $N = 3$, (right) $N = 4$

For $N = 2, 3$ and 4, the probability $P(X_\theta = L)$ has the following forms:

$$P^{(2)}\left(X_\theta = L\right) = \frac{\cos^4\theta}{\cos^4\theta + \sin^4\theta},$$

$$P^{(3)}\left(X_\theta = L\right) = \frac{\left(\cos^3\theta + 3\cos^2\sin\theta\right)^2}{\left(\cos^3\theta + 3\cos^2\sin\theta\right)^2 + \left(\sin^3\theta + 3\sin^2\cos\theta\right)^2},$$

$$P^{(4)}\left(X_\theta = L\right) = \frac{\left(\cos^4\theta + 4\cos^3\theta\sin\theta\right)^2}{\left(\cos^4\theta + 4\cos^3\theta\sin\theta\right)^2 + \left(\sin^4\theta + 4\sin^3\cos\theta\right)^2}.$$

We compared these probabilities with experimental data of case (A), see Fig. 5. One can find that the probabilities $P(X_\theta = L)$ in $N = 2, 3$ and 4 coincide with experimental data of A.

In the situation (B) or (C), we take another adaptive operator $M(\theta + \phi)$ in which the angle is shifted by an unknown psychological bias ϕ, instead of $M(\theta)$. This value of ϕ can be also calculated from experimental data. We show it in the Fig. 6. The values of ϕ in (B) are higher than those in (C), and especially the difference between $\phi^{(B)}$and $\phi^{(C)}$ is clearly seen in the middle range of angle θ . For increasing N, this difference become smaller.

References

1. Schröder, H.: Ueber eine optische inversion bei Betrachtung verkehrter, durch optische Vorrichtung entworfener physischer Bilder. Ann. Phys. **181**(10), 298–311 (1858)
2. Asano, M., Basieva, I., Khrennikov, A., Ohya, M., Yamato, I.: A general quantum information model for the contextual dependent systems breaking the classical probability law. Found. Phys. (submitted). arXiv:1105.4769
3. Basieva, I., Khrennikov, A., Ohya, M., Yamato, I.: Quantum-like interference effect in gene expression: glucose-lactose destructive interference. Syst. Synth. Biol. **5**, 59–68 (2011). (Springer)
4. Ohya, M., Volovich, I.: Mathematical Foundations of Quantum Information and Computation and Its Applications to Nano-and Bio-systems. Springer, New York (2011)
5. Accardi, L., Ohya, M.: Compound channels, transition expectations, and liftings. Appl. Math. Optim. **39**, 33–59 (1999)
6. Asano, M., Ohya, M., Khrennikov, A.: Quantum-like model for decision making process in two players game. Founds. of Phys. **41**, 538–548 (2010). (Springer)
7. Asano, M., Ohya, M., Tanaka, Y., Khrennikov, A., Basieva, I.: On application of gorini-kossakowski-sudarshan-lindblad equation in cognitive psychology. Open. Syst. Inf. Dyn. **17**, 1–15 (2010)
8. Asano, M., Basieva, I., Khrennikov, A., Ohya, M., Tanaka, Y.: Quantum-like generalization of the bayesian updating scheme for objective and subjective mental uncertainties. J. Math. Psycho. (2012). doi:10.1016/j.jmp.2012.02.003
9. Asano, M., Ohya, M., Tanaka, Y., Basieva, I., Khrennikov, A.: Quantum-like model of brain's functioning: decision making from decoherence. J. Theor. Biol. **281**(1), 56–64 (2011)
10. Ohya, M.: Optical illusions in view of phase transition. Suri-Kagaku (Japanese) **301**, 58–61 (1988)
11. Busemeyer, J.R., Bruza, P.: Quantum Models of Cognition and Decision. Cambridge Press, Cambridge (2012)
12. Dzhafarov, E.N., Kujala, J.V.: Quantum entanglement and the issue of selective influences in psychology: an overview. In: Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.) QI 2012. LNCS, vol. 7620, pp. 184–195. Springer, Heidelberg (2012)
13. Khrennikov, A.: Ubiquitous Quantum Structures: From Psychology to Finances. Springer, Berlin (2010)
14. Haven, E., Khrennikov, A.: Quantum Social Science. Cambridge Press, Cambridge (2013)
15. Basieva, I., Khrennikov, A.: Observables Generalizing Positive Operator Valued Measure. arXiv:1212.0758v1 (2012)
16. Conte, E., et al.: Mental states follow quantum mechanics during perception and cognition of ambiguous figure. Open Syst. Inf. Dyn. **16**, 85 (2009)
17. Atmanspacher, H., Filk, T., Rmer, H.: Complementarity in bistable perception. In: Atmanspacher, H., Primas, H. (eds.) Recasting Reality, pp. 135–150. Springer, Berlin (2009)

A Qualified Kolmogorovian Account of Probabilistic Contextuality

Ehtibar N. Dzhafarov[1(✉)] and Janne V. Kujala[2]

[1] Purdue University, West Lafayette, USA
ehtibar@purdue.edu
[2] University of Jyväskylä, Jyväskylä, Finland
jvk@iki.fi

Abstract. We describe a mathematical language for determining all possible patterns of contextuality in the dependence of stochastic outputs of a system on its deterministic inputs. The central principle (contextuality-by-default) is that the outputs indexed by mutually incompatible values of inputs are stochastically unrelated; but they can be coupled (imposed a joint distribution on) in a variety of ways. A system is characterized by a pattern of which outputs can be "directly influenced" by which inputs (a primitive relation, hypothetical or normative), and by certain constraints imposed on the outputs (such as Bell-type inequalities or their quantum analogues). The set of couplings compatible with these constraints determines the form of contextuality in the dependence of outputs on inputs.

Keywords: Bell-type inequalities · Cirelson inequalities · Context · Contextuality-by-default · Coupling · Direct influences · Determinism · EPR paradigm · Marginal selectivity · Sample spaces · Stochastically unrelated variables

1 Introduction

In this paper we describe a language for analyzing dependence of *stochastic outputs* of a system on *deterministic inputs*. This language applies to systems of all imaginable kinds: quantum physical, macroscopic physical, biological, psychological, and even purely mathematical, created on paper. The notion of "dependence," as well as related to it notions of "influence," "causality," and "context" may have different meanings in different areas. Even if not, we do not know how to define them. We circumvent the necessity of designing these definitions by simply accepting that some inputs are connected to some outputs by arrows called *direct influences*. We ignore the question of how these direct influences are determined, except for a certain necessary condition they must satisfy (*marginal selectivity*). A system is also characterized by certain *constraints* imposed on the joint distribution of its outputs across different inputs. A prominent example when both direct influences and constraints are justified by a well-developed

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 201–212, 2014.
DOI: 10.1007/978-3-642-54943-4_18,

theory is the EPR paradigm in quantum physics, where it is assumed that measurement settings for a given particle directly affect measurement outcomes in that particle only, and the joint distributions of the measurement outcomes on different particles satisfy certain inequalities or parametric equalities. If these constraints can be accounted for entirely in terms of the posited direct influences, the system can be viewed as "contextless." If this is not the case, we characterize *probabilistic contexts* by studying the deviations from the contextless behavior exhibited by the system.

Whether one deals with quantum contextuality or thinks of contextuality beyond even quantum bounds, our approach does squarely remains within the domain of the classical probability theory, which we refer to as *Kolmogorovian.* A caveat for using this attribution is that we do not mean the "naive" Kolmogorovian theory in which all random variables are thought of as defined on a single sample space (equivalently, as functions of a single random variable). Such a notion is no more tenable than the "set of all sets" of the naive set theory. The qualified Kolmogorovian approach we adopt is based on the principle of *contextuality-by-default:*

> *any two random variables recorded under mutually exclusive conditions are stochastically unrelated, defined on different sample spaces.*

This is a radical version of views previously expressed in the literature, e.g., in Khrennikov (2008a, b), where it is traced back to Andrei Kolmogorov himself and even to George Boole. Our emphasis, however, is on the fact that any set of stochastically unrelated variables (but never "all of them") can be *coupled*, or imposed a joint distribution upon, in many different ways (Thorisson 2000). In particular, the identity coupling is sometimes (but not always) possible, in which the two random variables defined under mutually exclusive conditions and "automatically" (by default) labeled as different and stochastically unrelated, merge into one and the same random variable.

The basics of this approach are presented in Sect. 2. In Sects. 3 and 4 we use it to investigate contextual influences with respect to a given pattern of direct influences. The theory and notation there closely follows Dzhafarov and Kujala (2013a). The departure point is that since different treatments (combinations of input values) are mutually exclusive, the joint distributions of the outputs corresponding to them, according to the principle of contextuality-by-default, are stochastically unrelated. We then consider all possible ways of coupling them across different treatments. From each such a coupling we extract stochastic relations that are "hidden," principally unobservable, because they correspond to outputs obtained under different treatments. We focus on the special kind of these hidden relations, those between random variables that share the same pattern of direct influences. We call these hidden relations *connections.* Given a certain constraint imposed on the system by a theory or empirical observations, we pose the question of what connections imply (or force) this constraint and what connections are implied by (or compatible with) it. Taken over all possible couplings, these relations between connections and constraints characterize the type of contextuality exhibited by the system. This view of contextuality is different from the existing approaches (Khrennikov 2009; Laudisa 1997).

2 Probability Theory: Multiple Sample Spaces

Given two probability spaces, (S, Σ, p) and (S_A, Σ_A, p_A), with standard meaning of the terms, a *random variable* is defined as a (Σ, Σ_A)-measurable function $A : S \to S_A$ subject to

$$p_A(X) = p\left(A^{-1}(X)\right), \tag{1}$$

for any $X \in \Sigma_A$. The probability space (S, Σ, p) is usually called a *sample space*, and we will refer to (S_A, Σ_A, p_A) as the *distribution* of A. The sample space itself is a distribution of the random variable R (let us call it a *basic variable*) which is the (Σ, Σ)-measurable identity function, $x \mapsto x$, $x \in \Sigma$. Any random variable A defined on this sample space can also be presented as a function $A = f(R)$, and (1) can be written as

$$p_A(X) = \Pr[A \in X] = \Pr\left[R \in f^{-1}(X)\right], \tag{2}$$

for any $X \in \Sigma_A$.

Let $\left(A^k = f_k(R) : k \in K\right)$ be a sequence[1] of random variables, all functions of one and the same basic variable R, with A^k distributed as $\left(S^k, \Sigma^k, p_k\right)$. Then $A = \left(A^k : k \in K\right) = f(R)$ too is a random variable that is a function of R, with the distribution

$$\left(S_A = \prod_{k\in K} S^k, \Sigma_A = \bigotimes_{k\in K} \Sigma^k, p_A\right). \tag{3}$$

Here, $\bigotimes_{k\in K} \Sigma^k$ is the minimal sigma-algebra containing sets of the form $X^k \times \prod_{i\in K-\{k\}} S^i$ for all $X^k \in \Sigma^k$, and p_A is defined by (2), with

$$f^{-1}(X) = \{x \in S : (f_k(x) : k \in K) \in X\}. \tag{4}$$

The distribution of A can also be given by (3) with no reference to its sample space, or basic variable. It can be viewed as a *joint distribution* of the components of a sequence $A = \left(A^k : k \in K\right)$, such that, for any nonempty $K' \subset K$, the subsequence $A' = \left(A^k : k \in K'\right)$ is a random variable distributed as

$$\left(S_{A'} = \prod_{k\in K'} S^k, \Sigma_{A'} = \bigotimes_{k\in K'} \Sigma^k, p_{A'}\right), \tag{5}$$

with

$$p_{A'}(X) = p_A\left(X \times \prod_{k\in K-K'} S^k\right), \tag{6}$$

[1] The term *sequence* in this paper is used in the generalized meaning, as any indexed family, a function from an index set into a set. Index sets need not be countable.

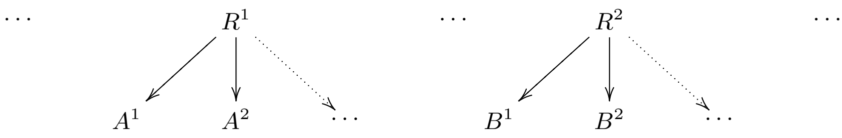
for any $X \in \Sigma_{A'}$. The distribution $\left(S^k, \Sigma^k, p_k\right)$ of a single A^k is determined by that of the one-element subsequence $\left(A^k\right)$ in the obvious way. All the random variables A^k obtained in this way from A can be viewed as functions on one and the same basic variable, e.g., $R = A$ itself.

We see that the relation "are jointly distributed" is synonymous to the relation "are functions of one and the same basic variable." But clearly there cannot be a single basic variable of which all imaginable random variables are functions. This is obvious from the cardinality considerations alone, as random variables may have arbitrarily large sets of possible values. But this is true even if one confines consideration to all imaginable random variables with any given distribution, provided it is not concentrated at a point. Let, e.g., $\mathcal{B}$ be a class (not necessarily a set) of all functions of R that are Bernoulli (0/1) variables with equiprobable values. That is, each $B \in \mathcal{B}$ is a function $f(R)$ with $f : S \to \{0, 1\}$, such that $\Pr\left(R \in f^{-1}(\{0\})\right) = 1/2$. Consider a Bernoulli variable B^* with equiprobable values such that for any $B \in \mathcal{B}$,

$$\Pr(B = 0, B^* = 0) = 1/4. \tag{7}$$

Then B^* cannot be a function of R because it is independent of (hence is not the same as) any of the elements of $\mathcal{B}$. If needed, however, one can redefine the basic variable, e.g., as $R^* = (R, B^*)$, with independent R and B^*, so that all elements of $\mathcal{B} \cup \{B^*\}$ become functions of R^*.

This simple demonstration shows that the Kolmogorovian approach to probability is not represented by a single sample space with measurable functions on it. Rather the true picture is an "open-ended" class (definitely not a set) of basic variables that are *stochastically unrelated* to each other, each with its own class of random variables defined as its functions: schematically,



(8)

If necessary, using some *coupling scheme* as discussed below, any sequence of stochastically unrelated basic variables $\left(R^k : k \in K\right)$ can be redefined into a random variable $H = \left(H^k : k \in K\right)$ such that H^k and R^k are identically distributed for all k. This amounts to considering all individual R^k, as well as their functions, as functions of H. But this procedure is not unique, and it cannot be performed for "all random variables."

The contextuality-by-default principle requires that any two random variables conditioned upon mutually exclusive values of some third variable are stochastically unrelated. Indeed, there is never a unique way for coupling their realizations. A simple example: I flip a coin and depending on the outcome weigh one of two lumps of clay, lump 1 (if "heads") or lump 2 (if "tails"). The random variables A = "weight reading for lump 1" and B = "weight reading for lump

2" do not a priori possess a joint distribution because there is no privileged way of deciding whether a given value of A *co-occurs* with a given value of B. If necessary, however, such a co-occurrence (or coupling) scheme can always be constructed. For instance, one can list the values of A and B chronologically and then couple the nth realization of A with the nth realization of B ($n = 1, 2, \ldots$). Or one could rank-order the values of A and B and couple the realizations of the same quantile rank (this would create positive correlation between the variables) or of the complementary ranks (negative correlation). One cannot say that one way of paring is better justified than another, each one represents "a point of view" and creates its own joint distribution of A and B.

3 All Possible Couplings Approach

Consider a sequence of random variables $A = (A_\phi : \phi \in \Phi)$. The elements of Φ are called *(allowable) treatments*. Two distinct treatments ϕ, ϕ' are mutually exclusive, so A_ϕ and $A_{\phi'}$ are stochastically unrelated. This means that A is not a random variable.

Let there be a sequence of nonempty sets $\alpha = \left(\alpha^k : k \in K\right)$ such that $\Phi \subset \prod_{k \in K} \alpha^k$. This means that every treatment is a sequence $\phi = (x^k : k \in K)$, with $x^k \in \alpha^k$. The sets α^k are called *inputs*, and their elements x^k *input values*. Note that generally $\Phi \neq \prod_{k \in K} \alpha^k$, that is, not all possible combinations of input values form treatments (hence the adjective "allowable").

For every treatment ϕ, let the random variable A_ϕ be a sequence of jointly distributed random variables $A_\phi = \left(A_\phi^\ell : \ell \in L\right)$. For each ℓ, the sequence $A^\ell = \left(A_\phi^\ell : \phi \in \Phi\right)$ is called an *output*. Its element A_ϕ^ℓ can then be referred to as *output* A^ℓ *at treatment* ϕ (or simply output A_ϕ^ℓ, when this does not create confusion). Note that A^ℓ is not a random variable, because its components are stochastically unrelated.

We postulate that, for every input α^k and every output A^ℓ, either α^k *directly influences* A^ℓ, and we write $A^\ell \leftarrow \alpha^k$, or this is not true, $A^\ell \not\leftarrow \alpha^k$. This relation is treated as primitive. Its intuitive meaning can be different in different applications. The only constraint imposed on this relation, *(complete) marginal selectivity*, is as follows (Dzhafarov 2003). Let index subsets $I \subset K$ and $J \subset L$ be such that if $A^\ell \leftarrow \alpha^k$ for some $\ell \in J$ then $k \in I$. That is, no input belonging to $\left(\alpha^k : k \in K - I\right)$ directly influences any output belonging to $\left(A^\ell : \ell \in J\right)$. Let $\phi = (x^k : k \in K)$ and $\phi' = (y^k : k \in K)$ be any allowable treatments such that

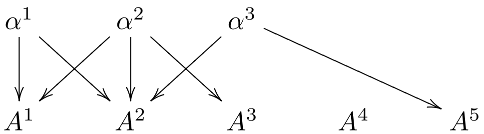
$$\phi|I = (x^k : k \in I) = (y^k : k \in I) = \phi'|I. \tag{9}$$

The slash here indicates restriction of a function (sequence) on a subset of arguments (indices). Marginal selectivity means that under these assumptions

$$\left(A_\phi^k : k \in J\right) \sim \left(A_{\phi'}^k : k \in J\right), \tag{10}$$

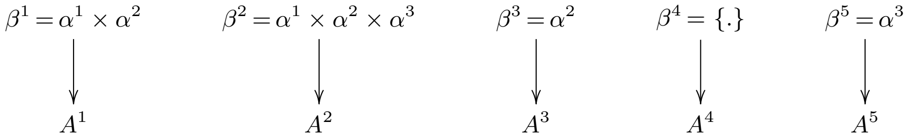
where $\sim$ means "has the same distribution as." In other words, the joint distribution of a subset of outputs does not depend on inputs that do not directly influence any of these outputs. This does not mean, however, that these inputs, $\left(\alpha^k : k \in K - I\right)$, can be ignored altogether when dealing with $\left(A^\ell : \ell \in J\right)$: generally, this will not allow one to account for its stochastic relation to other outputs, $\left(A^\ell : \ell \in L - J\right)$.

By appropriately redefining the inputs the relation of "being directly influenced by" can always be made bijective: each output is directly influenced by one and only one input. The procedure is easier to illustrate on an example. Let the diagram of direct influences be



(11)

Assume, for simplicity, that all combinations of input values are allowable, $\Phi = \alpha^1 \times \alpha^2 \times \alpha^3$. Then the redefined inputs are as shown:



(12)

The set $\{.\}$ represents a dummy (single-valued) input, it should be paired with any output that is not directly influenced by any inputs. The rest of the redefinition should be clear. The set of allowable treatments is redefined into a new set Ψ, which is not the Cartesian product of the new inputs but rather a proper subsequence thereof: e.g., if β^2 attains the value $\left(x^1, x^2, x^3\right)$, then the only treatment allowable is

$$\left(\left(x^1, x^2\right), \left(x^1, x^2, x^3\right), x^2, ., x^3\right). \tag{13}$$

We assume from now on that the direct influences are defined in a bijective form: $\alpha = \left(\alpha^k : k \in K\right)$, $\Phi \subset \prod_{k \in K} \alpha^k$, $A_\phi = \left(A^k_\phi : k \in K\right)$, $A^k \leftarrow \alpha^k$ for every $k \in K$, and there are no other direct influences.

Let us return to the sequence of random variables[2]

$$A = (A_\phi : \phi \in \Phi) = \left(A^k_\phi : k \in K, \phi \in \Phi\right), \tag{14}$$

with stochastically unrelated components. Consider a *complete coupling* for A,

$$H = \left(H^k_\phi : k \in K, \phi \in \Phi\right), \tag{15}$$

a random variable (that is, its components are jointly distributed) such that

$$H_\phi = \left(H^k_\phi : k \in K\right) \sim \left(A^k_\phi : k \in K\right) = A_\phi. \tag{16}$$

[2] In (14) and subsequently we are conveniently confusing differently grouped subsequences, such as (A, B, C), $((A, B), C)$, $(A, (B, C))$.

Such a random variable H always exists. It suffices, e.g., to consider every element of H_ϕ to be stochastically independent of every element in $H_{\phi'}$, for all $\phi \neq \phi'$. But generally, the complete couplings H for a given A can be chosen arbitrarily, except for the defining requirement (16).

Our approach consists in thinking of H, in addition to (16), in terms of "connections" it contains, by which we understand couplings for sequences of random variables that are indexed by different treatments sharing the same pattern of direct influences. Consider, e.g., the components A_ϕ^k for all ϕ whose kth element equals a given value $\phi(k) = x$. This subsequence can be written as

$$A_x^k = \left(A_\phi^k : \phi \in \Phi, \phi(k) = x\right). \tag{17}$$

Since $A^k \leftarrow \alpha^k$ only, all random variables A_ϕ^k are directly influenced by the same input value. Let

$$C_x^k = \left(C_\phi^k : \phi \in \Phi, \phi(k) = x\right) \tag{18}$$

be a coupling for $A_{x^k}^k$. This means that if $\phi(k) = x$,

$$C_\phi^k \sim A_\phi^k, \tag{19}$$

and it follows from the marginal selectivity property that the distribution of C_ϕ^k across all ϕ with $\phi(k) = x$ remains unchanged (and equal to the distribution of A_ϕ^k). There can be many joint distributions of (18) with this property. One possibility is that C_x^k is an *identity coupling*, meaning that for any two $C_\phi^k, C_{\phi'}^k$ in (18),

$$\Pr\left(C_\phi^k = C_{\phi'}^k\right) = 1. \tag{20}$$

If this is assumed for all $k \in K$ and $x \in \alpha^k$, then the complete coupling H in (15) can be written as the *reduced coupling*

$$R = \left(R_x^k : k \in K, x \in \alpha^k\right), \tag{21}$$

such that

$$R_\phi = \left(R_x^k : k \in K, \phi(k) = x\right) \sim A_\phi. \tag{22}$$

The existence of such a reduced coupling for a given A is the central theme of the theory of selective influences (Dzhafarov 2003; Dzhafarov and Kujala 2010, 2012a, b, 2013b, in press; Kujala and Dzhafarov 2008; Schweickert et al. 2012, Chap. 10), which includes the Bell-type theorems as special cases. Using the language of the present paper, if R exists, one can say that each A^k is influenced only by the input α^k that directly influences it. In other words, there are no influences that are not direct ("no context"). Other examples from behavioral sciences involve recent work on combination of concepts (Aerts et al., in press; Bruza et al. 2013; for a critical overview see Wang et al., in press; and Dzhafarov and Kujala, in press). In quantum physics the existence of the reduced coupling represents classical, pre-quantum determinism; it is the foundation of all Bell-type theorems (Basoalto and Percival 2003; Dzhafarov and Kujala 2012a).

We know, however, that Bell-type inequalities are violated in quantum physics. This leads us to explore alternatives to the assumption (20) and to the ensuing existence of a reduced coupling. This can be done by allowing C_x^k in (18) to be different from an identity coupling. The random variable C_x^k is called a *connection*. If its distribution is posited, we constrain the complete coupling (15) not just by (16), but also by its consistency with this connection:

$$H_x^k = \left(H_\phi^k : \phi \in \Phi, \phi(k) = x\right) \sim C_x^k. \tag{23}$$

With this additional constraint, the coupling H need not exist.

Generalizing, let I be a subset of K other than empty set and K itself. Then the (I, τ)-*connection* is defined as a random variable

$$C_\tau^I = \left(C_\phi^I : \phi \in \Phi, \phi|I = \tau\right) \tag{24}$$

such that for $\phi|I = \tau$,

$$C_\phi^I \sim A_\phi^I = \left(A_\phi^k : k \in I\right). \tag{25}$$

Recall that $\phi|I = \tau$ is the restriction of the treatment on a subset of its indices.[3] Note that the components of a given C_τ^I are jointly distributed, but if $(I, \tau) \neq (I', \tau')$, C_τ^I and $C_{\tau'}^{I'}$ are stochastically unrelated.

Given a sequence of outputs A in (14), denote the sequence of the connections C_τ^I for all I and τ by C_A (not a random variable). Assume that the distributions of all these connections are known. Then one can ask whether a complete coupling H for A is consistent with all connections in C_A, that is, whether in addition to (16) H also satisfies, for any $I \in 2^K - \{\emptyset, K\}$ and any $\tau \in \prod_{k \in I} \alpha^k$,

$$H_\tau^I = \left(H_\phi^I : \phi \in \Phi, \phi|I = \tau\right) \sim C_\tau^I, \tag{26}$$

where

$$H_\phi^I = \left(H_\phi^k : k \in I\right). \tag{27}$$

If this is true, then H is called an *Extended Joint Distribution Sequence* (EJDS) for (A, C_A). This notion is a generalization of the Joint Distribution Sequence (or "Joint Distribution Criterion set") that coincides with the reduced coupling (21) in the theory of selective influences (Dzhafarov and Kujala 2010, 2012a, 2013b). It is obtained from EJDS by requiring that all connections be identity ones, that is, for any ϕ, ϕ' in (24),

$$\Pr\left(C_\phi^I = C_{\phi'}^I\right) = 1. \tag{28}$$

[3] Strictly speaking, this notation makes the upper index I in C_τ^I redundant. But it is convenient as it allows one to abridge the presentation of τ. Thus, if $K = \{1, 2, 3\}$, $I = \{1, 3\}$, $\phi(1) = x$, $\phi(3) = y$, then a strict reading of C_τ^I is $C_{\{(1,x),(3,y)\}}^{\{1,3\}}$, but it is naturally abridged into $C_{x,y}^{1,3}$, which seems more convenient than $C_{\{(1,x),(3,y)\}}$. Note that our opening example of a connection, C_x^k, is an abridged form of $C_{\{(k,x)\}}^{\{k\}}$.

4 Characterizing Contextuality

The notion of an EJDS can be used to characterize contextuality in relation to constraints imposed on the outputs of a system. Suppose that it is known that the outputs A taken across all allowable treatments in (14) satisfy a certain property $\mathcal{P}(A)$. This property may be described by certain equations and inequalities relating to each other parameters of the outputs, such as Bell-type inequalities, or Cirelson-Landau's quantum inequalities (see below). One should investigate then the set of possible C_A in relation to this property $\mathcal{P}(A)$.

To understand this better, let us consider a simple example of A. Let K be $\{1,2\}$, the sequence of inputs $\left(\alpha^k : k \in K\right)$ be $\left(\alpha^1 = \{1,2\}, \alpha^2 = \{1,2\}\right)$, the sequence of allowable treatments be $\Phi = \alpha^1 \times \alpha^2$, and the sequence of outputs be $A = \left(\left(A^1_{ij}, A^2_{ij}\right) : i,j \in \{1,2\}\right)$ (where each subscript ij represents the treatment (i,j)). The diagram of direct influences is assumed to be

$$\begin{array}{ccc} \alpha^1 & & \alpha^2 \\ \big\downarrow & & \big\downarrow \\ A^1 & & A^2 \end{array} \tag{29}$$

The only choices of $I \subset K$ here other than $\emptyset$ and K are the singletons $\{1\}$ and $\{2\}$, so the only four connections are, for $i \in \{1,2\}$,

$$C^1_i = \left(C^1_{i1}, C^1_{i2}\right), C^2_i = \left(C^2_{1i}, C^2_{2i}\right), \tag{30}$$

where $C^k_{ij} \sim A^k_{ij}$ for all $i,j,k \in \{1,2\}$. Recall that the logic of forming $C^1_i = \left(C^1_{i1}, C^1_{i2}\right)$ is that A^1_{i1} and A^1_{i2}, while they are recorded at different treatments, $(i,1)$ and $(i,2)$, share the same pattern of direct influences, namely, both are directly influenced by the value i of α^1 (in our general notation, $\phi|\{1\} = (i)$). So if their joint distribution is described by anything other than $\Pr\left(C^1_{i1} = C^1_{i2}\right) = 1$, we can speak of indirect, contextual influences. The situation with C^2_i is analogous. The complete coupling for A here is the 8-vector

$$H = \left(H^1_{ij}, H^2_{ij} : i,j \in \{1,2\}\right). \tag{31}$$

Assume that each A^k_{ij} (hence also H^k_{ij} in the complete coupling, $i,j,k \in \{1,2\}$) is a binary random variable with equiprobable outcomes +1 and -1. Then A is represented by four probabilities $p = (p_{11}, p_{12}, p_{21}, p_{22})$, where

$$p_{ij} = \Pr\left[A^1_{ij} = +1, A^2_{ij} = +1\right] = \Pr\left[H^1_{ij} = +1, H^2_{ij} = +1\right]. \tag{32}$$

One prominent situation encompassed by this example is the Bohmian version of the EPR paradigm involving two spin-1/2 particles with two settings (spatial directions) per particle. As examples of a constraint $\mathcal{P}(A)$ consider the Bell/CH/Fine inequalities (Bell 1964; Clauser and Horne 1974; Fine 1982)

$$0 \le p_{ij} + p_{ij'} + p_{i'j'} - p_{i'j} \le 1 \tag{33}$$

and Cirel'son's (1980) inequalities

$$\frac{1-\sqrt{2}}{2} \le p_{ij} + p_{ij'} + p_{i'j'} - p_{i'j} \le \frac{1+\sqrt{2}}{2}, \tag{34}$$

where $i, j \in \{1,2\}$, $i' = 3-i$, $j' = 3-j$ (so each expression contains four double-inequalities). The Bell/CH/Fine inequalities are known to be necessary and sufficient for the existence of a classical explanation for the EPR paradigm in question (Fine 1982), whereas the Cirel'son inequalities are necessary for the existence of a quantum mechanical explanation (Landau 1987).

One question to pose about the connections is: what is the set of all C_A such that whenever $\mathcal{P}(A)$ is satisfied, an EJDS for (A, C_A) exists? We call any connection belonging to this C_A a *fitting connection* for $\mathcal{P}(A)$. A question can also be posed about the opposite implication: what is the set of all C_A such that whenever an EJDS for (A, C_A) exists, $\mathcal{P}(A)$ is satisfied? We call any connection in this C_A a *forcing connection* for $\mathcal{P}(A)$. In our example, C_A is the sequence of four connections C_i^k in (30), and they are uniquely characterized by the 4-vector $\varepsilon = \left(\varepsilon_1^1, \varepsilon_2^1, \varepsilon_1^2, \varepsilon_2^2\right)$, where

$$\varepsilon_i^1 = \Pr\left[C_{i1}^1 = +1, C_{i2}^1 = +1\right], \varepsilon_i^2 = \Pr\left[C_{1i}^2 = +1, C_{2i}^2 = +1\right]. \tag{35}$$

Hence the complete coupling H, in order to be an EJDS for (A, C_A), should satisfy not only (32), but also

$$\Pr\left[H_{i1}^1 = +1, H_{i2}^1 = +1\right] = \varepsilon_i^1, \Pr\left[C_{1i}^2 = +1, C_{2i}^2 = +1\right] = \varepsilon_i^2, \tag{36}$$

for $i \in \{1,2\}$.

To describe the fitting and forcing connections for our example, it is convenient to introduce the following abbreviations:

$$\begin{array}{l} s_0 = \max\left\{\pm\left(\varepsilon_1^1 - 1/4\right) \pm \left(\varepsilon_1^2 - 1/4\right) \pm \left(\varepsilon_2^1 - 1/4\right) \pm \left(\varepsilon_2^2 - 1/4\right) : \#\text{ of + signs is even}\right\}, \\ s_1 = \max\left\{\pm\left(\varepsilon_1^1 - 1/4\right) \pm \left(\varepsilon_1^2 - 1/4\right) \pm \left(\varepsilon_2^1 - 1/4\right) \pm \left(\varepsilon_2^2 - 1/4\right) : \#\text{ of + signs is odd}\right\}. \end{array} \tag{37}$$

It turns out (Dzhafarov and Kujala 2013a) that the sets of fitting connections for the Bell/CH/Fine and Cirel'son inequalities are described by, respectively,

$$s_1 \le 1/2, \tag{38}$$

and

$$s_0 \le \frac{3-\sqrt{2}}{2}, s_1 \le 1/2. \tag{39}$$

This means that if p satisfies (33), then any ε with $s_1 \le 1/2$ is compatible with it, that is, this p and this ε can be embedded in the same EJDS H. If p satisfies (34), the set of ε compatible with it is more narrow: they should additionally satisfy $s_0 \le \frac{3-\sqrt{2}}{2}$. Both sets include, of course, the vector $\varepsilon = (0,0,0,0)$, which represents no-contextuality and corresponds to the reduced coupling R in (21).

The sets of forcing connections for the Bell/CH/Fine and Cirel'son inequalities are described by, respectively,

$$s_0 = 1, \tag{40}$$

and

$$s_0 \geq \frac{3-\sqrt{2}}{2}. \tag{41}$$

The set of ε such that $s_0 = 1$ consists of $\varepsilon = (0,0,0,0)$, $\varepsilon = (1/2, 1/2, 1/2, 1/2)$, and vectors with two zeros and two $1/2$'s. All of them represent no-contextuality, with $+1$ and -1 interpreted differently in different connections. Only if ε is one of these vectors, p must satisfy the Bell/CH/Fine inequalities in order to be compatible with it. In other words, such an ε and no other "forces" p to satisfy these inequalities. The class of ε that force p to satisfy the Cirel'son inequalities should include these ε because every p satisfying (33) also satisfies (34). But there are other ε, all those with $s_0 \geq \frac{3-\sqrt{2}}{2}$, that too are compatible with p only if they satisfy the Cirel'son inequalities.

The above serves only as a demonstration of how one could characterize the constraints imposed on outputs (by a theory or empirical generalizations) through the connections compatible with them, in the sense of being embeddable in the same coupling. It should be noted, however, that connections generally do not characterize couplings uniquely. This opens ways for constructing qualified Kolmogorovian models more general than the one presented in this paper.

5 Conclusion

We have shown that the classical, if qualified, Kolmogorovian probability theory is not synonymous with the classical explanation of the input-output relations (especially, in the entanglement paradigm of quantum physics). The latter, since Bell's (1964) pioneering work, has been understood as the existence of a single sample space for all outputs when each output is indexed (identified) only by the inputs that directly influence it. In the qualified Kolmogorovian approach, however, this is only one of a potential infinity of possibilities. Different treatments (combinations of values of all inputs) correspond to stochastically unrelated random variables, and these can be coupled in many different ways. Only one of these ways, with identity connections, corresponds to John Bell's single sample space.

Acknowledgments. This research has been supported by the NSF grant SES-1155956. We are grateful to Jerome Busemeyer of Indiana University for critically reviewing this paper.

References

Aerts, D., Gabora, L., Sozzo, S.: Concepts and their dynamics: a quantum-theoretic modeling of human thought. Top. Cogn. Sci. **5**(4), 737–772 (2013). doi:10.111/tops.12042

Basoalto, R.M., Percival, I.C.: Belltest and chsh experiments with more than two settings. J. Phys. A: Math. Gen. **36**, 7411–7423 (2003)

Bell, J.: On the Einstein-Podolsky-Rosen paradox. Physics **1**, 195–200 (1964)
Bruza, P.D., Kitto, K., Ramm, B.J., Sitbon, L.: A probabilistic framework for analysing the compositionality of conceptual combinations. arXiv:1305.5753 (2013)
Cirel'son, B.S.: Quantum generalizations of bell's inequality. Lett. Math. Phys. **4**, 93–100 (1980)
Clauser, J.F., Horne, M.A.: Experimental consequences of objective local theories. Phys. Rev. D **10**, 526–535 (1974)
Dzhafarov, E.N.: Selective influence through conditional independence. Psychometrika **68**, 7–26 (2003)
Dzhafarov, E.N., Kujala, J.V.: The joint distribution criterion and the distance tests for selective probabilistic causality. Front. Quant. Psychol. Meas. **1**, 151 (2010). doi:10.3389/fpsyg.2010.0015
Dzhafarov, E.N., Kujala, J.V.: Selectivity in probabilistic causality: where psychology runs into quantum physics. J. Math. Psychol. **56**, 54–63 (2012a)
Dzhafarov, E.N., Kujala, J.V.: Quantum entanglement and the issue of selective influences in psychology: an overview. In: Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.) QI 2012. LNCS, vol. 7620, pp. 184–195. Springer, Heidelberg (2012)
Dzhafarov, E.N., Kujala, J.V.: All-possible-couplings approach to measuring probabilistic context. PLoS ONE **8**(5), e61712 (2013a). doi:10.1371/journal.pone.0061712
Dzhafarov, E.N., Kujala, J.V.: Order-distance and other metric-like functions on jointly distributed random variables. Proc. Am. Math. Soc. **141**, 3291–3301 (2013b)
Dzhafarov, E.N., Kujala, J.V.: On selective influences, marginal selectivity, and Bell/CHSH inequalities. Top. Cogn. Sci. **6**(1), 121–128 (2014). doi:10.1111/tops.12060
Fine, A.: Hidden variables, joint probability, and the bell inequalities. Phys. Rev. Lett. **48**, 291–295 (1982)
Khrennikov, A.Y.: Bell-boole inequality: nonlocality or probabilistic incompatibility of random variables? Entropy **10**, 19–32 (2008a)
Khrennikov, A.Y.: Epr-bohm experiment and bell's inequality: quantum physics meets probability theory. Theor. Math. Phys. **157**, 1448–1460 (2008b)
Khrennikov, A.Y.: Contextual Approach to Quantum Formalism. Springer, Heidelberg (2009)
Kujala, J.V., Dzhafarov, E.N.: Testing for selectivity in the dependence of random variables on external factors. J. Math. Psychol. **52**, 128–144 (2008)
Landau, L.J.: On the violation of bell's inequality in quantum theory. Phys. Lett. A **120**, 54–56 (1987)
Laudisa, F.: Contextualism and nonlocality in the algebra of epr observables. Philos. Sci. **64**, 478–496 (1997)
Schweickert, R., Fisher, D.L., Sung, K.: Discovering Cognitive Architecture by Selectively Influencing Mental Processes. World Scientific, New Jersey (2012)
Thorisson, H.: Coupling, Stationarity, and Regeneration. Springer, New York (2000)
Wang, Z., Busemeyer, J.R., Atmanspacher, H., Pothos, E.: The potential of using quantum theory to build models of cognition. Top. Cogn. Sci. **5**(4), 672–688 (2013). doi:10.1111/tops.12043

An Empirical Test of Type-Indeterminacy in the Prisoner's Dilemma

Peter D. Kvam[1(✉)], Jerome R. Busemeyer[2], and Ariane Lambert-Mogiliansky[3]

[1] Michigan State University, East Lansing, MI, USA
kvampete@msu.edu
[2] Indiana University, Bloomington, IN, USA
jbusemey@indiana.edu
[3] Paris School of Economics, Paris, France
alambert@pse.ens.fr

Abstract. In this paper, we test the type indeterminacy hypothesis by analyzing an experiment that examines the stability of preferences in a Prisoner Dilemma with respect to decisions made in a context that is both payoff and informationally unrelated to that Prisoner Dilemma. More precisely we carried out an experiment in which participants were permitted to make promises to cooperate to agents they saw, followed by playing a Prisoner's Dilemma game *with another, independent agent*. It was found that, after making a promise to the first agent, participants exhibited higher rates of cooperation with other agents. We show that a classical model does not account for this effect, while a type indeterminacy model which uses elements of the formalism of quantum mechanics is able to capture the observed effects reasonably well.

Keywords: Quantum probability · Type indeterminate · Prisoner's Dilemma · Cheap talk

1 Introduction

Many conventional accounts of behavior relate internal characteristics (e.g. beliefs, attitudes) to outward expressions in a deterministic way. That is, they assume that each person has stable traits that drive his actions. This is often accompanied by the assumption that the reverse is not true – that one's behaviors do not influence one's cognitive processes. We challenge this assumption, suggesting instead that a person constructs preferences according to the actions he takes. We propose that the motivational underpinning of behavior is intrinsically uncertain (i.e. indeterminate), and that a particular preference (or type) is actualized upon selecting an action. This view is formalized in the Type-Indeterminate (TI) model of decision-making [1], which offers testable predictions for decision-making behavior in sequential scenarios.

Our approach seeks to contribute to a growing body of literature in the social sciences, psychology, physics, computer science, and mathematics which is based on principles of non-classical indeterminacy found in quantum mechanics [2–8]. This line of research has been successfully applied to explain a variety of behavioral

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 213–224, 2014.
DOI: 10.1007/978-3-642-54943-4_19,

phenomena ranging from cognitive dissonance to preference reversals, the conjunction fallacy [6], disjunction [8], and other interference effects [5]. The critical connection between physical and psychological systems is that the object of investigation cannot (always) be separated from the process of investigation. Instead, our measurements result in a change to a person's internal state. As a person's preference is elicited, his preferences with respect to another, incompatible set of outcomes may be modified, resulting in inconsistent or order-dependent behavior.

We provide an experimental test of type indeterminacy, derived from the quantum probability framework, by comparing two sequential decision scenarios. In one, people are invited to play a Prisoner's Dilemma (PD) game. In the other, prior to playing the PD game, they are invited to make a promise to cooperate with another player. The promise is made to a third party – not the one the person interacts with in the focal PD game – and has no consequences for the outcome of that game. While this pre-play exchange has no strategic value, it may still play a role in terms of the person's relationship with himself. The nature of this relationship may be predicted by existing psychological literature.

1.1 Pre-play and Self-perception

Despite the view that there is an exclusively unidirectional influence of inner characteristics (preferences, attitudes, beliefs) on behavior, a long-standing and robust phenomenon in the psychological literature suggests that the opposite occurs. Self-perception [9] postulates two characteristics: (1) individuals come to "know" their own attitude and other internal states partially by inferring them from observations of their own behavior and/or the circumstances in which behavior occurs; and (2) "The individual is functionally in the same position as an outside observer, an observer who must necessarily rely upon those same external cues to infer the individual inner state." [10, p. 27].

Suppose, as above, we offer a person the chance to make a promise to cooperate with another player. If we expect self-perception to be effective after this exchange, then a person who (e.g.) makes a promise to cooperate should observe his own behavior as cooperative. Consequently, we should see him act in a cooperative way immediately following this promise.

As discussed in [11][1], self-perception does not necessitate a non-classical framework. However, its postulates are consistent with quantum indeterminacy, which overturns the classical postulates of pre-existing identity, attitudes, and preferences.

[1] As we show, with indeterminacy of the inner state, behavior (the action chosen in a decision situation, see below) shapes the state of preferences/attitude by force of a state transition process. Indeterminacy means intrinsic uncertainty about individual identity such that the individual may not know his own attitudes, preferences and beliefs. And as in self-perception theory, it is by observing his own action that he infers (learns) his state (of beliefs and preferences).

1.2 Prisoner's Dilemma

In order to examine the effect of cheap talk on subsequent decision-making, we have to be able to quantify and predict nontrivial decisions. A well-established framework for doing so is the non-zero-sum Prisoner's Dilemma game (see [12] for a review). The general form of the game can be seen in Fig. 1.

		Player 2	
		Cooperate	**Defect**
Player 1	**Cooperate**	(R,R)	(S,T)
	Defect	(U,V)	(P,P)

Fig. 1. Payoff matrix in the form (Player 1 payoff, Player 2 payoff). Note that for the game to be a Prisoner's Dilemma, the values are set such that $U = T > R > P > V = S$

The well-known result is that the Nash equilibrium entails defection from both players. However, substantial deviation from this outcome has been cited as evidence that players have subjective valuation of the outcomes that is distinct from the monetary payoffs [13]. In order to capture the distinction between material payoffs and subjective payoffs, we introduce "types" as follows. Uncertainty about the subjective payoffs translates into uncertainty about the type of the player which in the Prisoner's Dilemma game may be one of the 3 following types:

- PD type 1 (t_1), "cooperator" type who always prefers to cooperate in a PD game;
- PD type 2 (t_2), "moderate" type who prefers to cooperate when he expects the opponent to cooperate with probability $p \geq q^*$; and
- PD type 3 (t_3), "materialistic" type who prefers to always defect.

Similarly, a player's preferences in a promise exchange game are characterized by the 3 analogous types:

- Promise type 1 (τ_1), "promise-maker" type who always promises to cooperate
- Promise type 2 (τ_2), "moderate" type who makes a promise to cooperate when he expects the opponent to cooperate with probability $p \geq q^*$; and
- Promise type 3 (τ_3), "honest" type who never makes a promise to cooperate.

We frequently treat each of these types as bases of a vector space – type 1 as the x-axis [1 0 0], type 2 as the y-axis [0 1 0], and type 3 as the z-axis [0 0 1].

1.3 Models of Behavior

Classical Model

In [9], it was shown that the classical model predicts the same outcome of a PD game whether it was preceded by an independent promise exchange or not. This model implies that a player is of one specific type when entering into a PD game, and that the behavior of all types is known once q^* is determined. When the promise exchange takes place, it provides no new information for the player to use in subsequent games, as it has no payoff implications. Thus, if this model is correct, we should see the same behavior in PD games that are and are not preceded by an independent promise exchange.

Type-Indeterminate Model

The basic Type-Indeterminacy model is formulated in [1]. In this model, an individual is represented by a state vector which captures all information about his preferences. A decision situation (DS) is viewed as a measurement of this property. The outcome of the measurement is the choice made, which informs the decision-maker about his preferences. This decision is modeled as an (projection) operator that acts upon the individual (i.e. changes his state). It is a well-known result that if all DS (operators) are commuting, the TI-model is equivalent to a standard incomplete information model. The novelty comes when allowing for non-commuting DS. This implies that since the DS are affecting the state, the order in which non-commuting DS are presented to the individual matters to the preferences that are being revealed. This feature captures the intrinsic indeterminacy of preferences. The formalism of TI-model provides a well-defined state transition process borrowed from the mathematical formalism of quantum mechanics.[2]

For the present purpose we only need to know that the TI-model works as follows. Before making a decision in a given PD, Agent A cannot be represented as a single preference order over the two choices (Cooperate or Defect). Instead we say that he is in a state of superposition, a state of simultaneous selves. With a certain probability he may express anyone of the 3 possible types. This initially uncertain state can be disturbed by engaging in the promise exchange. The decision whether or not to make the promise is such a decision that is we assume that it does not commute with the PD. This simply means that the individual who makes his decision in the PD is not in the same state at the beginning and end of this pre-play. Hence the TI-model predicts that his choice behavior will not be the same if he engages in pre-play. Below we provide a more detailed account for the calculation of predicted behavior. For now we just note that the prediction of the TI-model contrasts with the classical model which predicts the same behavior in the PD in the two scenarios. This distinction (in its details) is the object of the experimental test.

1.4 Synthesis and Summary

Having developed a suitable framework for assessing the effect of cheap talk promise-making to one player (A) on games played with another player (B), we suspect that the effect will be a different rate of cooperation with player B after having made a promise to cooperate with player A. We have both qualitative [10] and formal quantitative [1, 2] accounts that can explain such an effect.

[2] In Danilov and Lambert-Mogiliansky 2008, we investigate the axioms behind the Hilbert space model of QM and the implied state transition process to find that they have a natural interpretation in social sciences [13].

2 Methods

2.1 Participants

A total of 31 participants were recruited (17 female, 14 male), all students aged 18–30. The study was conducted at Indiana University, Bloomington. The task portion of the study took place in an interdisciplinary laboratory, with participants seated spaced apart at computers. Each participant completed one session of the experiment. They were paid $5 for participating and were awarded up to an additional $10 based on how many points they accumulated during the experiment. For each set of payoffs in the experiment, a participant could earn 0–45 points based on their and their opponent's decision. The fraction of the $10 they received was based on the number of points they had accumulated divided by the total number of points possible across all payoff matrices in the experiment (maximum number for each trial). Average earnings were approximately $11.

2.2 Agents

In the main task, participants played a series of games with computer agents. They were aware that these agents were not other people, but each agent had a unique face, name, and a tendency to cooperate or defect[3]. Those agents who tended to cooperate were described as cooperative (75 % C) and those that tended to defect were described as opportunistic (75 % D). Participants were not informed of these exact rates – only that an agent was one type or the other, and hence more likely to cooperate or defect, respectively.

2.3 Payoffs

The main set of payoffs that participants encountered during the task were standard Prisoner's Dilemma payoffs ($T = U > R > P > S = V$, see Fig. 1), where the payoffs are presented as [Participant's payoff, computer agent's payoff].

In addition, there were several sets of payoffs that were used as "filler" to keep participants engaged and to prevent them from developing a set strategy. These payoffs were either Chicken ($T = U > R > S = V > P$, again see Fig. 1), Battle of the Sexes ($T = U > S = V > R = P = 0$), Free Rider ($U > R > V > P = S = T = 0$), or Coordination ($R = P > S = T = U = V = 0$) payoffs.

All payoffs varied between point values of 0 and 45, and different sets of the same type of payoffs were approximately proportional to one another.

2.4 Task Progression

The task consisted of 9 iterations of the same 4 stage sequence, using agents and payoff matrices described above. These 4 stages were:

---

[3] An example of a screen participants would see can be found in the online supplementary material at www.msu.edu/~kvampete/resources.html

1. Pre-play cheap talk promise exchange: Participant sees a new agent (Agent A) and set of PD payoffs and is told they will play this game with this agent in the future. They choose to either promise to cooperate or not make any promise
2. Independent intermediate PD game: Participant sees new agent (Agent B) and set of PD payoffs. They play the game, either cooperating or defecting.
3. Cheap talk follow-up: Participant plays the game with the agent they saw in the first stage (Agent A, same payoffs), choosing to cooperate or defect.
4. Filler: Participant plays 8 games, all with new agents (Agents C-J), some of which are PD games and some of which are the non-PD payoffs described above.

During the filler stage of each iteration, participants received feedback on how many points they had accumulated up to that point, but never received any feedback on the outcomes of individual trials.

In our analysis, we focus on predicting the rates of cooperation in the second stage as a function of the type of agent they encounter there as well as their behavior and the type of agent they encountered in the first stage.

3 Results

No systematic gender or other demographic differences were found, and so participants' data were combined in the analyses presented. We review the results first from the pre-play cheap talk exchange, then the effect that this stage had on behavior in the subsequent independent PD game. We compare this to behavior in PD games that took place during the filler stage (i.e. without a preceding pre-play promise exchange), and finally derive and compare model predictions for the rates of cooperation during stage 2. For brevity, we omit the results from stage 3, as neither model makes clear predictions for behavior during this stage.

Additionally, we introduce some notation to simplify communication of results:

- $P_A =$ promise made to Agent A in Stage 1
- $NP_A =$ no promise was made to Agent A
- $Coop_A =$ Agent A is /was a cooperative agent
- $Opp_B =$ Agent B is /was opportunistic
- $\Pr(C_B) =$ probability of cooperating with agent B
- $\Pr(D) = 1 - \Pr(C)$ probability of defecting
- $\#C_n =$ number of times players cooperated in condition n
- $\#D_n =$ number of times players defected in condition n.

3.1 Cheap Talk Promise Exchange (Stage 1)

When participants were presented with a cooperative computer agent and given the chance to make a promise to cooperate, they opted to do so on 85 out of 134 pre-play trials $(Pr(P_A|Coop_A) = 0.634)$. When the computer agent was described as opportunistic, they made promises on 51 out of 145 $(Pr(P_A|Opp_A) = 0.352)$. This difference occurred across all stages of the task – participants tended to behave cooperatively (made promises and cooperated) more often when the agent was cooperative as opposed to opportunistic.

3.2 Prisoner's Dilemma Games with and Without Pre-play (Stages 2 and 4)

The rate of cooperation in the PD filler trials – those without any pre-play cheap talk preceding them (phase 4) – with cooperative agents was 95 out of 235 ($Pr(C|Coop) = 0.404$) and with opportunistic agents was 42 out of 267 ($Pr(C|Opp) = 0.157$).

The rates of cooperation following pre-play (phase 2) were 58 out of 123 for cooperative agents ($Pr(C_B|Coop_B = 0.472$) and 25 out of 156 for opportunistic agents ($Pr(C_B|Coop_B = 0.160$). For those participants that had made a promise to cooperate in the first stage and encountered a cooperative agent, the rate of cooperation was 39 out of 58 ($Pr(C_B|P_A, Coop_B) = 0.672$); when they had made a promise and encountered an opportunistic agent, the rate of cooperation was 21 out of 76 ($Pr(C_B|P_A, Opp_B) = 0.276$). The comparisons matched by agent type were significant ($p < 0.05$ for each pairwise t-test), indicating an increased rate of cooperation when participants had made a promise in the independent pre-play cheap talk exchange.

Rates of cooperation for no-promise conditions can be seen in Table 1 and the supplementary material. Modeling also accounts for the type of agent encountered in the cheap talk stage, as this contributes to determining the type that a participant expresses. Including the type of agent from the first stage, the decision from the first stage (promise/no promise), and the type of agent from the second stage, there are 8 different rates of cooperation in the second stage. These were the rates of cooperation that each model attempted to predict.

The argument can be made that those participants who were more likely to cooperate had a higher propensity to make promises, and that this would inflate the rate of cooperation given a pre-play promise because more cooperative participants would be in the 'promise made' group. To account for this, we re-weighted the rate of cooperation from the fourth stage (originally 26.25 %, collapsed across agent type)

Table 1. Summary table of rates of cooperation during phases 1, 2 and 4 of the task

Agent A is:	Cooperative (Coop)				Opportunistic (Opp)			
Phase 1: Promise exchange (Agent A)	Promise		No promise		Promise		No promise	
	63.4%		36.6%		35.2%		64.8%	
Agent B is:	Coop	Opp	Coop	Opp	Coop	Opp	Coop	Opp
Phase 2: Independent PD (Agent B)	69.1%	62.5%	10.5%	37.0%	30.2%	24.2%	9.4%	1.8%
Phase 4: Other PD games (Agent C)	40.4% (p = .011, H_0: equal to post-promise rate of cooperation)				15.7% (p = .037, H_0: equal to post-promise rate of cooperation)			

according to the number of promises each participant made. This ensured that each participant had equal weight in calculating both the non-pre-play rates of cooperation and pre-play rates. The re-weighted rate of cooperation for the fourth stage is 33.6 %, which is still significantly lower ($p < 0.05$) than the rate of cooperation given a promise had been made in the pre-play cheap talk stage, which was 44.8 % (again collapsing across agent types). This suggests that while there was some effect of self-selection from making promises, this was not enough to account for the increased rate of cooperation when a promise had been made in the cheap talk stage.

3.3 Modeling Results

One key assumption is made in order to fit both models. Recall that a player of PD or promise type 2 (t_2 or τ_2) will cooperate or make a promise, respectively, if he believes that the agent he is interacting with will cooperate with probability $p > q^*$, where q* is a threshold. The assumption we make is that telling a participant that an agent is cooperative will insure that $p > q^*$, and telling a participant that an agent is opportunistic will insure $p < q^*$, so a type 2 player will be differentiated from the others based on the agent type he encounters. Such a manipulation has been successful in previous experiments [14] and is necessary for fitting either model. Without making such a claim, we could never distinguish between type 2 and type 1 players if a player cooperated/made a promise, and we could never differentiate between type 2 and type 3 players if a player defected/made no promise.

Classical/Standard Model

The standard model, as we described previously, assumes that a person's type is determined prior to starting a particular game, and that the payoff matrix and opponent type for that game resolves any residual uncertainty about their behavior. Given this information, we can extract what type a player is based on his actions in a particular game. For the PD games contained in the fourth stage, the relative frequency of these types occurring is:

$$\begin{aligned} \Pr(t_1) &= \Pr(C|Opp, filler) = 0.1573 \\ \Pr(t_3) &= \Pr(D|Coop, filler) = 0.5957 \\ \Pr(t_2) &= 1 - (\Pr(t_1) + \Pr(t_2)) = 0.2740 \end{aligned} \tag{1}$$

Since the types are mutually exclusive and their frequencies in the classical model add up to one, we can extract the type distribution for the fourth stage. Since the second stage (independent PD with pre-play) is treated the same as the fourth stage (independent PD without pre-play), we should observe the same type frequencies appearing in the intermediate PD games. For example:

$$\Pr(C_B|Coop_B, Coop_A, P_A) = \Pr(C_B|Coop_B) = \Pr(t_1 \cup t_2) = 0.4043 \tag{2}$$

$$\Pr(C_B|Opp_B, Coop_A, NP_A) = \Pr(C_B|Opp_B) = \Pr(t_1) = 0.1573$$

This implies that across all 8 conditions in stage 2 (see Table 1), there should only be 2 different rates of cooperation, based only on the immediate agent type. These rates should be 40.43 % for cooperative agents, and 15.73 % for opportunistic agents, shown above.

We can use these two parameter inputs to estimate the log likelihood of the model using the log of a multinomial function:

$$\sum_{n=1}^{8} \#C_n * \ln(\Pr(C|n)) + \#D_n * \ln(\Pr(D|n)) \tag{3}$$

where n corresponds to each specific combination of agent type in the promise exchange, player decision in the promise exchange, and agent type in the intermediate Prisoner's Dilemma game. The log likelihood (LL) for this model is -155.8373. Using this value, we compute an approximate Bayesian information criterion (BIC):

$$BIC = -2 * LL + \#\ parameters * \ln(\#\ data\ points) = 322.9371 \tag{4}$$

The BIC for the classical model uses the 279 total data points across all conditions in the intermediate PD stage, and 2 parameters based on estimates from the filler stage. Note that the BIC includes a term that depends on the number of parameters of the model – it penalizes models with a larger number of parameters relative to the number of data points, which allows us to fairly compare the classical model against the more complex TI model. Later, this is used to approximate a Bayes factor between the two models.

Type-Indeterminate Model

The type-indeterminate model uses a state vector to represent a particular player's types at any given time. Each type is noted in bra(|)-ket(⟩) notation, as it corresponds to one of a set of basis vectors that spans $\mathbb{R}^3$. In our case, we set $t_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $t_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $t_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ in the PD game space and $\tau_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $\tau_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\tau_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ in the promise game space. Since a player can be any of the 3 types, we can represent his state φ as a linear combination of the three possible types, called a superposition state. For example, in the promise game, his type would begin as

$$|\varphi\rangle = a_1|\tau_1\rangle + a_2|\tau_2\rangle + a_3|\tau_3\rangle \tag{5}$$

The coefficients a_1, a_2, a_3 are probability amplitudes whose squared length gives the probability of expressing the corresponding type $\left(\text{e.g. } \Pr(\tau_1) = |a_1|^2\right)$. These squared amplitudes must add up to 1, referred to as the unit length property of the superposition state. We can infer estimates of what the values of these coefficients are based on players' behavior in the promise exchange game, using the assumption about the threshold value ($p > q^*$ when facing a cooperative agent, $p < q^*$ when facing an opportunistic agent) and the unit length property of the coefficients:

$$
\begin{aligned}
|a_1| &= \sqrt{\Pr(P_A|Opp_A)} = 0.5854 \\
|a_3| &= \sqrt{\Pr(NP_A|Coop_A)} = 0.6123 \\
|a_2| &= \sqrt{1-\left(a_1^2+a_3^2\right)} = 0.5314
\end{aligned} \tag{6}
$$

Substituting these values into (5) gives us the initial distribution of types during the promise exchange game, and so we have a superposition state reflecting all player types. Once a player makes a decision, however, he must project down onto the corresponding basis state(s). For example, if he promises to cooperate with an opportunistic agent, he must be promise type 1. However, if he does not make a promise, he must be either promise type 2 or promise type 3. We model this by projecting the state vector onto the bases that correspond to the types a player could be. However, this results in a vector whose length is less than 1, so we must rescale the new state vector to preserve unity. For example, the new state vector after making a promise to cooperate with a cooperative agent is:

$$
\left|\varphi_{P,Coop}{}'\right\rangle = \frac{a_1}{\sqrt{a_1^2+a_2^2}}|\tau_1\rangle + \frac{a_2}{\sqrt{a_1^2+a_2^2}}|\tau_2\rangle = 0.7404|\tau_1 + 0.6721|\tau_2\rangle \tag{7}
$$

The new state vector after making no promise to an opportunistic agent is:

$$
\left|\varphi_{NP,Opp}{}'\right\rangle = \frac{a_2}{\sqrt{a_2^2+a_3^2}}|\tau_2 + \frac{a_3}{\sqrt{a_2^2+a_3^2}}|\tau_3\rangle = 0.6554|\tau_2\rangle + 0.7553|\tau_3\rangle \tag{8}
$$

The calculation is much simpler when making a promise to an opportunistic agent or making no promise to a cooperative agent.

$$
\left|\varphi_{NP,Coop}{}'\right\rangle = |\tau_3\rangle \quad \left|\varphi_{P,Opp}{}'\right\rangle = |\tau_1\rangle \tag{9}
$$

Depending on the combination of agents a player faces and the decision the player made, we can extract which of these four states they are in after the promise exchange. In order to map the state vector from its position in the promise game to its position in the PD game a transformation must be applied. We represent this transformation as a change-of-bases unitary matrix. For a complete account of how this is done, see the online supplementary material. The resulting unitary matrix is parameterized using rotations about the x, y, and z axis: θ_1, θ_2, and θ_3, respectively.

$$
U = \begin{bmatrix}
\cos\theta_2\cos\theta_3 & \cos\theta_1\sin\theta_3+\sin\theta_1\sin\theta_2\cos\theta_3 & \sin\theta_1\sin\theta_3-\cos\theta_1\sin\theta_2\cos\theta_3 \\
-\cos\theta_2\sin\theta_3 & \cos\theta_1\cos\theta_3-\sin\theta_1\sin\theta_2\sin\theta_3 & \sin\theta_1\cos\theta_3+\cos\theta_1\sin\theta_2\sin\theta_3 \\
\sin\theta_2 & -\sin\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_2
\end{bmatrix} \tag{10}
$$

Using maximum likelihood estimation to obtain the log likelihood of the model using (3), we find best-fit parameters of $\theta_1 = 6.2257$, $\theta_2 = 3.8794$, and $\theta_3 = 4.2796$. Substituting these values gives

$$U = \begin{bmatrix} 0.3103 & -0.9225 & -0.2295 \\ -0.6717 & -0.3836 & 0.6337 \\ -0.6727 & -0.0425 & -0.7387 \end{bmatrix} \quad (11)$$

To obtain the final type superposition for the intermediate PD game, we multiply through to obtain a 3-dimensional vector, now in the PD game space.

$$U * \begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \text{ where } \Pr(t_1) = |b_1|^2, \ \Pr(t_2) = |b_2|^2, \text{ and } \Pr(t_3) = |b_3|^2 \quad (12)$$

Given the values for each b obtained from (7), (8), (9), (11) and (12), we can determine a player's behavior in the PD game based on the type of agent he faces. The log likelihood for this mode, computed using (3), is -149.1063. This gives a BIC [from (4)] of

$$BIC(TI) = -2 * (-149.1063) + 3 * \ln(179) = 315.1008 \quad (13)$$

A comparison of model predictions for each of the 8 conditions can be seen in the supplementary material. It may not be immediately clear that the TI model has overall better predictions; in order to truly compare the two models, we approximate a Bayes Factor using the BICs from the two models:

$$BF(Classical, TI) = \exp\left(\frac{BIC(classical) - BIC(TI)}{2}\right) = 50.3073 \quad (14)$$

This Bayes Factor can be interpreted as 'strong' evidence for the type indeterminate model over the classical model in accounting for the data [15].

4 Discussion

The qualitative results – showing a different rate of cooperation after making a promise, even after readjusting the base rate – suggest that the type-indeterminate model better predicts behavior in the Prisoner's Dilemma game following the promise exchange. The quantitative model fits provide decisive support of this conclusion as well – the Bayes factor computed between the models, even when penalizing the TI model for being more complex, clearly favors it over the classical model. It certainly seems that the promise exchange does have a distinct effect on behavior in the Prisoner's Dilemma, and that the TI model provides the better account of this effect.

The success of the TI model in this task was more than reasonable, and it presents a mathematical formalization of the self-perception effect [10]. Previously, this effect had mainly been a qualitative one, but descriptive game theoretic quantum models (such as in [5, 8]) can offer fully quantitative reinterpretations of two- or multi-player interactions. More generally, the quantum class of models can offer formalizations of many existing effects as well as predictions of new ones [4]. We hope to have offered not simply a prediction of interference from a promise exchange, but also to have presented a framework that has wide, varied, and effective applications.

References

1. Lambert Mogiliansky, A., Zamir, S., Zwirn, H.: Type indeterminacy: a model of the KT (Kahneman–Tversky)-man. J. Math. Psychol. **53**(5), 349–361 (2009)
2. Busemeyer, J., Lambert-Mogiliansky, A.: TI-games I: an exploration of type indeterminacy in strategic decision-making. PSE Working Papers 2009-07, PSE (2009)
3. Aerts, D., Brokert, J., Gabora, L.: A case for applying an abstracted quantum formalism to cognition. New Ideas Psychol. **29**(2), 136–146 (2011)
4. Busemeyer, J.R., Bruza, P.D.: Quantum Models of Cognition and Decision. Cambridge University Press, Cambridge (2012)
5. Haven, E., Khrennikov, A.: Quantum Social Science. Cambridge University Press, Cambridge (2013)
6. Franco, R.: The conjunction fallacy and interference effects. J. Math. Psychol. **53**(5), 415–422 (2009)
7. Trueblood, J.S., Busemeyer, J.R.: A quantum probability account of order effects in inference. Cogn. Sci. **35**(8), 1518–1552 (2011)
8. Busemeyer, J.R., Matthew, M., Wang, Z.: An information processing explanation of the disjunction effect. In: Sun, R., Miyake, N. (eds.) The 28th Annual Conference of the Cognitive Science Society and the 5th International Conference of Cognitive Science, pp. 131–135 (2006)
9. Bem, D.J.: Self-perception theory. In: Berkowitz, L. (ed.) Advances in Experimental Social Psychology, vol. 6, pp. 1–62. Academic Press, (1972)
10. Bem, D.J.: Self-perception: an alternative interpretation of cognitive dissonance phenomena. Psychol. Rev. **74**(3), 183–200 (1967)
11. Lambert-Mogiliansky, A., Busemeyer, J.R.: Quantum type indeterminacy in dynamic decision-making: self-control through identity management. Games **3**(2), 97–118 (2012)
12. Poundstone, W.: Prisoner's Dilemma. Random House Digital Inc, New York (1993)
13. Danilov, V.I., Lambert-Mogiliansky, A.: Measurable systems and behavioral sciences. Math. Soc. Sci. **55**(3), 315–340 (2008)
14. Matthew, M.R., Busemeyer, J.R.: The disjunction effect on a sliding scale (Unpublished manuscript, Indiana University) (2010)
15. Raftery, A.E.: Bayesian model selection in social research. Sociol. Methodol. **25**, 111–163 (1995)

Interference in Choice and Confidence: Using the Quantum Random Walk to Model Distributions of Confidence

Peter D. Kvam[1(✉)], Timothy J. Pleskac[1], Jerome R. Busemeyer[2], and Shuli Yu[1]

[1] Michigan State University, East Lansing, MI, USA
{kvampete, pleskact, yushuli}@msu.edu
[2] Indiana University, Bloomington, IN, USA
jbusemey@indiana.edu

Abstract. In this paper, we examine the effect of making a choice on subsequent confidence. Using a simple binary forced-choice perceptual task, we show that committing to a decision results in perturbed probability (confidence) judgments compared to a control condition where no decision is elicited, suggesting that committing to a decision interferes with subsequent information processing. Current classical probability models of decision-making and confidence do not predict this interference effect, but it arises naturally out of a quantum random walk model. We show that this model provides a better fit to the data and provides novel predictions of interference as well as improved confidence accuracy following a decision.

Keywords: Quantum probability · Decision-making · Confidence · Interference

1 Introduction

The degree of belief people express in the likelihood of different outcomes informs our actions, transmitting information about uncertainty and risk as well as the weight that information should carry. From confidence in military intelligence to assessments of eyewitness testimony, judgments about the likelihood of events can carry tremendous impact, and in many cases it is critical for them to be as accurate and unbiased as possible.

One of the more common assumptions about these judgments is that they are read out of internal states. This assumption, which we refer to as the read-out assumption, implies that asking for responses should not affect ongoing cognitive processes. Instead, it suggests that responses simply externalize information [1, 2]. If this is true, then taking measurements such as decisions should not affect subsequent data.

There seems to be evidence for violations of the read-out assumption, perhaps most notably in the form of confirmation bias (see [3] for a review). According to this bias, decision-makers change their beliefs to be consistent with their previous actions [4]. Another study [5] found the opposite effect – were less confident about an

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 225–230, 2014.
DOI: 10.1007/978-3-642-54943-4_20,

alternative after having previously chosen it, compared to when they rated confidence without prior choice. In both cases, the act of choosing interacts with the production of subsequent judgments – a clear violation of the read-out assumption.

Despite evidence for violations of the read-out assumption, cognitive models of judgment and decision-making continue to make it. This is largely because they are constructed on assumptions from classical probability theory. However, models based on the quantum probability framework can capture these effects by invoking a projection operator to model choice. This operator changes the state of the system, moving it into a state consistent with the choice that a person has made. With this in mind, the purpose of this paper is two-fold: first, we empirically test the read-out assumption in a simple perceptual task where participants express their confidence in the characteristics of a stimulus, comparing conditions where this is and is not preceded by a decision about those characteristics; and second, we evaluate how well a quantum random walk model of choice and confidence [6] captures these effects. We compare its performance against its classical counterpart, the Markov random walk model [7–9].

2 Methods

2.1 Participants

A total of 9 participants were recruited from a pool of Michigan State University students. Each completed 5 1-h sessions of the experiment.

Task
Participants were instructed to watch a field of moving dots, most of which were moving randomly. A subset of the dots (coherence levels: 2/4/8/16 %) was consistently moving either left or right. On half the trials (choice condition), participants were first asked to choose whether the dots were moving left or right. On the remaining trials (click condition), participants were instructed to either press a left or right mouse button. After the choice or click, participants were asked to rate their confidence that the dots were moving rightward.

The time course of an individual trial is shown in Fig. 1. At the beginning of each trial participants would click on a shape to start the trial. The particular shape indicated whether the trial was from the choice (square) or click (circle) condition.

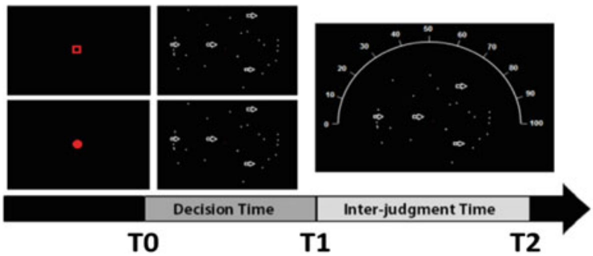


Fig. 1. Time course of random dot motion task: Click/Choice at T1, Confidence at T2

After clicking the fixation the dot stimulus appeared. After 500 ms (T1), participants were prompted with a beep to indicate using a button press on the mouse which direction they believed the dots were moving (choice condition) or to just click the right or left button (click condition). We included the click in the non-choice condition in order to mimic the motor action required to make a choice and eliminate it as a possible confounding factor.

To prompt responses, a low frequency beep was used in the choice condition (400 Hz) and a high frequency beep in the click condition (800 Hz). Following the choice or click, participants were prompted with a second, 400 Hz beep at T2, which sounded 50/750/1500 ms later. They would then enter their confidence that the dots were moving right by clicking a semi-circular scale that ranged from 0 (certain left) to 100 (certain right) scale. Feedback was given after each trial indicating the number of points that a participant received for that trial as well as how fast their click/choice and confidence responses were.

Individual trials of this task were organized into blocks. Each block included 2 trials of each combination of 4 coherence levels and 3 inter-judgment times, for a total of 24 trials. The choice and click manipulation occurred between blocks in an alternating pattern. In the click condition, whether the participant was supposed to click left or right was randomly selected. Each participant completed a total of 112 blocks (56 choice and 56 click blocks) of the task across the 5 experimental sessions.

3 Results

We rescaled all confidence ratings relative to the correct answer, i.e. 0 % confidence means a person was sure the dots are moving in the incorrect direction.

At the distribution level, participants gave confidence ratings that were closer to 0 and 100 in the click condition than in the choice condition, indicating more extreme beliefs when they were not asked for a choice (Fig. 2 shows this in a cumulative distribution of confidence judgments for an example condition). To better characterize these differences, we submitted the data to a 4-factor (choice/click × inter-judgment time × coherence × correct/incorrect for choice) hierarchical Bayesian linear model (see [9]). This analysis revealed a main effect of choice condition (choice/click) on

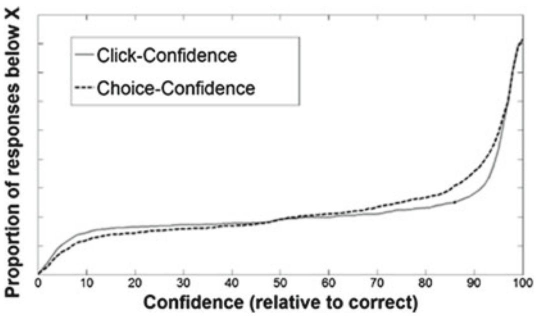


Fig. 2. Cumulative distribution of confidence judgments for a typical participant/condition

confidence, implying that making a choice impacted subsequent confidence judgments. The modal value of the posterior distribution for the standardized predictor was −0.443, HDI = [−0.975 −0.071], which indiates that making a choice actually lowered participants average confidence ratings. This effect also scaled (interacted) with coherence, increasing in magnitude with improved coherence.

We also found that the interference due to choice resulted in better resolution in confidence judgments, i.e. participants could better tell whether their answer was correct or incorrect. The standardized difference between mean confidence for correct versus incorrect confidence judgments (DI‘, a sensitivity measure; see [10]) was larger in the choice condition. The mean DI‘ for the choice condition was 0.444, while it was 0.342 in the click condition. A Bayesian means comparison suggests this is a reliable difference with the modal posterior estimated difference between the means being 0.101, HDI = [0.006 0.193].

3.1 Models

Each of the models we use describes an internal cognitive state as a vector that moves across different bases over time. These bases represent different levels of confidence, with bases 0-50 corresponding to states that favor an incorrect response, and bases 50-100 corresponding to states that favor a correct response. The internal state begins at a position around 50, indicating uncertainty about the dot motion direction. As participants gather samples of the stimulus over time, this state shifts out over the other possible positions and, on average, toward the correct answer. This new state gives the stochastic structure to the possible responses, leading to both choice probabilities (based on the proportion above /below 50) and predictions for confidence distributions. While both models share this same 101-base state representation and basic dynamics, there are a few differences that lead to diverging predictions, particularly regarding interference.

State Representation

The classical Markov random walk model represents the internal state as a mixed state $P(t)$, a probability distribution across the 101 bases. The value along each basis gives the probability of being in that state at time t.

The quantum random walk also represents the internal state as a vector, but its state vector $\psi(t)$ is a distribution of probability amplitudes, where the probability of being in a particular base is the squared value of amplitudes along that base.

Dynamics

The Markov random walk moves the state from its initial distribution $P(0)$ to its later positions by using a transition matrix $T(t) = \exp(Q * t)$, where t is the amount of time that passes. Q is defined by a drift rate δ, which represents the average rate of evidence accumulation toward the correct answer, and a variance parameter σ^2, which corresponds to the error in evidence accumulation. They form the matrix Qsuch that each entry $q_{row,column}$ is

$$q_{j,j} = -\sigma^2, q_{j-1,j} = (\sigma^2 - \delta)/2 \; and \; q_{j+1,j} = (\sigma^2 - \delta)/2 \tag{1}$$

This gives the distribution at T1 as$P(T1) = e^{Q*T1} * P(T0)$.

The quantum random walk model uses 2 similar parameters to define its state transition, but is further restricted by the requirement that the transition is done by a Unitary matrix. The state at T1 is given by $\psi(T1) = U(T1) * \psi(T0)$, where $\psi(T0)$ is the initial state and U is defined by 2 analogous parameters: potential (μ), which is analogous to drift, pushing the distribution toward the correct answer; and diffusion (σ^2), which moves the probability amplitude out across the basis states, much like variance. They define the Hamiltonian H such that each entry $h_{row,column}$ is

$$h_{j,j} = -(\mu * \frac{j}{11}) \text{ and } h_{j-1,j} = -\sigma^2 = h_{j+1,j} \quad (2)$$

The correct choice proportion at T1 is given by the summed proportion of the state vector above 50 in the Markov model, and the sum of squared amplitudes of the vector above 50 in the quantum model. The critical difference between the two models arises in what happens at choice: in the quantum model, the state is projected onto the corresponding states: 0-50 if the incorrect answer is chosen, or 50-100 if the correct answer is chosen. In this task, this results in 3 possible states at T1: one conditional on a correct answer, another on the incorrect answer, and a third conditional on having not chosen any answer (the state that would result at T1 in the click condition). As the dynamics are applied to move the states from time T1 to T2, these 3 different states persist, yielding different distributions of confidence for the choice and click conditions at T2 where the Markov model, owing to its classical probability structure, produces only one marginal distribution.

Fits

We can see how the quantum and Markov predictions diverge –the quantum model predicts different marginal distributions of confidence for the click and choice conditions, while the Markov model does not. However, these are just qualitative predictions. We fit both models to the data using maximum likelihood estimation, factoring in its predictions for distributions of confidence as well as its predictions for choice proportions in the choice condition.

This is done for each of the 24 conditions described above. The models use the same number of (analogous) parameters: 4 drift rates/potentials (1 for each coherence level), 1 diffusion parameter, 1 parameter for non-decision components of response time, 1 parameter for non-judgment components of inter-judgment time, and 1 final parameter that sets the width of the distribution of the initial state (e.g. 5 would indicate a uniform distribution over states 48-52). This allows for direct comparison of the models using the number of parameters as an index of complexity. However, we still transform the log likelihood into an approximate Bayesian Information Criterion (BIC), which is a measure of model performance as a function of the fit to the data and number of parameter.

When we fit both models to the aggregate data, the BIC for the quantum model is 357,902, while the BIC for the Markov model is 367,802, indicating very strong support for the quantum over the Markov model in predicting our data.

4 Conclusion

Above we have shown that a qualitative prediction of the quantum random walk model – interference due to choice on subsequent confidence – holds true across levels of difficulty and processing time. This finding is inconsistent with the read-out assumption made by classical probability models of choice and confidence. Instead, we find that the quantum random walk model predicts this effect as well as providing better fits to the data than its classical counterpart.

In addition, the quantum random walk posits new predictions: notably, higher confidence resolution and better confidence calibration in the choice condition. The model suggests that such an effect occurs because the action of making a choice usually aligns a person's cognitive state with the true state of the world. Our findings suggest that these and other predictions of the model, such as double stochasticity and oscillating confidence, are worth investigating and extending to other domains.

References

1. Ericsson, K.A., Simon, H.A.: Verbal reports as data. Psychol. Rev. **87**(3), 215–251 (1980)
2. Payne, J.W., Bettman, J.R., Johnson, E.J.: The Adaptive Decision Maker. Cambridge University Press, Cambridge (1993)
3. Nickerson, R.S.: Confirmation bias: a ubiquitous phenomenon in many guises. Rev. Gen. Psychol. **2**(2), 175–220 (1998)
4. Rosenfeld, P., Kennedy, J.G., Giacalone, R.A.: Decision making: a demonstration of the postdecision dissonance effect. J. Soc. Psychol. **126**(5), 663–665 (1986)
5. Sniezek, J.A., Paese, P.W., Switzer, F.S.: The effect of choosing on confidence in choice. Organ. Behav. Hum. Decis. Process. **46**(2), 264–282 (1990)
6. Busemeyer, J.R., Wang, Z., Townsend, J.T.: Quantum dynamics of human decision-making. J. Math. Psychol. **50**(3), 220–241 (2006)
7. Ratcliff, R.: A theory of memory retrieval. Psychol. Rev. **85**(2), 59–108 (1978)
8. Pleskac, T.J., Busemeyer, J.R.: Two-stage dynamic signal detection: a theory of choice, decision time, and confidence. Psychol. Rev. **117**(3), 864–901 (2010)
9. Kruschke, J.: Doing Bayesian Data Analysis: A Tutorial with R. Academic Press, Burlington (2010)
10. Wallsten, T.S., Budescu, D.V., Erev, I., Diederich, A.: Evaluating and combining subjective probability estimates. J. Behav. Decis. Mak. **10**(3), 243–268 (1997)

The Relevance of Bell-Type Inequalities for Mental Systems

Harald Atmanspacher[1,2(✉)] and Thomas Filk[1,3,4]

[1] Institute for Frontier Areas of Psychology, Freiburg, Germany
[2] Collegium Helveticum ETH Zürich, Zürich, Switzerland
atmanspacher@collegium.ethz.ch
[3] Institute of Physics, University of Freiburg, Baden-Württemberg, Germany
[4] Parmenides Center for the Study of Thinking, Munich, Germany

Abstract. We compare some basic inequalities due to Bell and others, originally proposed to test hidden variable models in quantum theory, and explore their implications for mental systems. We find that violations of such inequalities outside quantum systems may exceed the quantum bound. We propose (conscious or unconscious) priming as a most intuitive and plausible interpretation for such violations and sketch a few examples supporting this conjecture.

Keywords: Bell-type inequalities · Entanglement · Invasiveness · Nonlocality · Priming

1 Introduction

Basic notions of quantum theory do not only represent essential features of physical nature, but they also represent essential features of how our modes of knowledge are organized. These two sides of one coin have been inherent in discussions about the interpretation of quantum theory from its early days on. So it is not surprising that quite a few among its founding fathers, most emphatically Niels Bohr, were convinced that its central notions will prove meaningful not only in physics, but also in areas such as psychology and even philosophy.

Intuitively, it is quite easy to understand why quantum concepts should be relevant, maybe even inevitable, for mental systems. Simply speaking, the non-commutativity of operations means nothing else than that the sequence, in which operations are applied, matters for the final result. This is in close connection with the elementary observation that interactions with a mental state always change this state, generally in an uncontrollable fashion. Since the 1990s, several research groups have started exploring whether and how this can be tied down to concrete psychological scenarios, both empirically and theoretically.[1] One among

[1] For recent overviews see Atmanspacher (2011, Chap. 4.7), Busemeyer and Bruza (2012) and Wang *et al.* (2013). Note that the topics of these overviews must be delineated from various proposals concerning a literal quantum physics of the brain. See the review by Atmanspacher (2011) for a critical assessment of such approaches.

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 231–243, 2014.
DOI: 10.1007/978-3-642-54943-4_21,

several areas of such applications refers to mental instantiations of entanglement, a notion originally coined by Schrödinger (1935). For a historical account of entanglement in quantum physics see Gilder (2008).

In quantum physics, entangled systems can arise due to all kinds of interactions, so entanglement is a generic feature of the quantum world. Measuring a specific local observable of an entangled system in a pure state leads to its decomposition into subsystems. For instance, spin measurements on a pair of subsystems lead to two separate states of subsystems 1 and 2. If the spin observable in subsystem 1 takes on a definite (but unpredictable) value, e.g. spin up, this implies its anti-correlated value spin down for subsystem 2. This correlation between the results of measurements is nonlocal in the sense that it is not generated by local interactions between the subsystems.

Ironically, Einstein *et al.* (1935), who pointed out such correlations for the first time, concluded that this be so absurd that quantum theory must miss essential elements of reality. This conclusion is unavoidable indeed if reality is assumed to be local, as in classical physics. An ingenious theorem by Bell (1966) proposed how this issue can be resolved by measuring correlations between entangled quantum systems. Bell's inequalities set a bound for classical correlations which is violated for entangled quantum systems. Since Bell's seminal work, a number of related inequalities, called Bell-type inequalities, have been proposed.

These inequalities have been formulated in various ways, and different quantities have been used for expressing them. Furthermore, depending on the intended application, there are different types of proofs for different inequalities. In this article we concentrate on four inequalities: the Wigner inequality (Wigner 1970, d'Espagnat 1979), the original Bell inequality (Bell 1966), the CHSH inequality due to Clauser *et al.* (1969), and the Fine inequality (Fine 1982). They all differ by the quantities in which they are expressed. Essentially all proofs of Bell-type inequalities can be classified in two groups:

1. One assumes that all observables which, in principle, can be measured on a system and which are relevant for the inequality, are local "elements of reality". In other words, for each individual system the possible outcome for each measurement, if it were performed, is pre-determined and has a specific value. Of course, this value is not known prior to its measurement, but it is assumed that, in some ontological sense, it "exists". For the proof of a particular inequality one simply lists all possible combinations of values and checks for each case whether the inequality is satisfied. If this is the case, it must also be satisfied for ensembles of corresponding systems.

   The advantage of this type of proof is that it is extremely elementary and allows a clear identification of the assumptions made. The disadvantage is that one talks about joint probabilities of values of observables which cannot be measured simultaneously. The epistemically accessible probabilities (or correlations) are derived from joint probabilities.
2. One starts with probabilities of values of observables that can be measured and are expressed as an integral over a hidden variable with a general distribution function. One then derives the particular inequality under certain

assumptions (typically a factorization of the distribution function with respect to the subsystems).

The advantage of this type of proof is that, more or less right from the beginning, one is dealing with observable probabilities. The disadvantage is that the empirical assumptions made are often not so obvious.

In this article, we mainly focus on proofs of the first type for the considered inequalities, although most of the original proofs were based on a strategy of the second type.

Quantum entanglement is not a necessary condition for the formulation of the inequalities, but it seems to be necessary (or at least extremely conducive) for experimental tests with quantum systems. The reason is that in almost all cases the inequalities are expressed in terms of correlation functions (or joint probabilities) of non-commuting observables which, in the framework of quantum theory, cannot be measured simultaneously. Due to quantum correlations in entangled systems one can measure the two observables simultaneously on two different subsystems. Violations of Bell-type inequalities were for the first time conclusively demonstrated by Aspect *et al.* (1982) and have been found in numerous experiments since then.

In the following Sect. 2 we introduce inequalities due to Wigner (1970) and due to Bell (1966) together with applications in quantum and non-quantum systems. Section 3 discusses the CHSH inequality and the Fine inequality, and Sect. 4 shows how they can be violated in a simple classical OR gate. Section 5 concludes the article with tentative applications, including the conjecture that priming is a key mechanism for violations of Bell-type inequalities in mental systems.

2 Inequalities due to Wigner and due to Bell

2.1 The Wigner Inequality

Before we will discuss the historically original Bell inequality introduced by Bell himself in the early 1960s, let us start with a most general version employed by Wigner (1970), and later by d'Espagnat (1979) to illustrate quantum entanglement. This so-called Wigner inequality is essentially a set theoretical result (see d'Espagnat 1979) about probabilities p^- of measuring different values of two observables a and b which can only assume two possible outcomes, say + or −.

Denoting these probabilities as $p^-(a, b)$, $p^-(a, c)$, and $p^-(b, c)$, the inequality reads:

$$p^-(a, c) \leq p^-(a, b) + p^-(b, c)\,, \tag{1}$$

The observables a, b, c correspond to three different measurement prescriptions which, e.g., in photon polarization experiments refer to planar polarizations along three different directions. Inequality (1) holds for any pairwise permutation of the observables a, b, c.

We now prove this inequality by checking all possible combinations for the outcomes of a, b, and c (see Table 1). One sees immediately that inequality (1)

Table 1. Eight combinatorial possibilities for three observables a, b, c with two possible outcomes $+, -$ each. The probabilities in the right columns correspond to an ensemble consisting only of systems with these particular combinations.

a b c	$p^-(a,b)$	$p^-(b,c)$	$p^-(a,c)$
$+ + +$	0	0	0
$+ + -$	0	1	1
$+ - +$	1	1	0
$+ - -$	1	0	1
$- + +$	1	0	1
$- + -$	1	1	0
$- - +$	0	1	1
$- - -$	0	0	0

holds for each case separately and, therefore, also for an arbitrary mixture. The sum of any $p^-(a,b)$ and $p^-(b,c)$ is always greater or equal to $p^-(a,c)$. In other words, there are never two zero entries in a row without the third entry also being zero. This is a simple consequence of the transitivity of "being equal": If two quantities are equal to a third one they are equal to each other.

The assumptions on which inequality (1) is based depend crucially on how we try to verify it. In all cases we assume that we are given an ensemble of systems to be used for statistical analysis. If all three observables can be measured simultaneously, the outcome has to satisfy the inequality, because the outcome necessarily falls into one of the eight classes enumerated in Table 1. This situation does not require any additional assumptions.

However, inequality (1) requires us to know only two of the values of the three observables (it requires even less, namely only the correlation between two of the observables, but this shall not concern us here). In this case the inequality can be violated. The reason for such a violation is that the first measurement has an influence on the result of the second measurement. For such "invasive" measurements it becomes irrelevant whether or not all three outcomes are predetermined before the actual measurements are performed.

An example of a violation in this case is given by the following rules:

- Whenever observable a or c is measured first with the result x ($+$ or $-$), the other observable, c or a, is set to the opposite value (such that $p^-(a,c) = 1$) and b is set to x (such that $p^-(a,b) = p^-(b,c) = 0$).
- Whenever observable b is measured first with result x, the other two observables a and c are set to the same value x (such that, again, $p^-(a,b) = p^-(b,c) = 0$).

Obviously, the first (invasive) measurement has an influence on the result of the second measurement. Therefore, in psychological experiments, a violation of this inequality can be interpreted as a form of "priming".[2]

[2] Briefly, the concept of priming means that exposure to a stimulus influences a response to another, usually later, stimulus. It is widely used in social and personality psychology; see Bargh and Chartrand (2000) for a review.

Another way to violate inequality (1) in classical systems is by "information transfer" between the preparation of the system and the measurements. If, e.g., it is already known during the preparation phase which measurements will be performed at each single system, one can easily arrange configurations which violate the inequality. On the other hand, if the experimenter "knows" the exact values of all three observables in advance, he can select the two measured observables such that the inequality is also violated.

These two loop-holes are often associated with either (a) a limitation of the experimenter's "free will" in choosing experimental conditions, or with (b) a "premonition" of the experimenter concerning the outcome of measurements not yet performed. Conway and Kochen (2006) touch scenario (a) in what they call (somewhat hyperbolically) a "free will theorem". With respect to scenario (b) we will refer to recent claims of this kind and suggest an alternative deflationary interpretation in Sect. 5.

In recent work (Atmanspacher and Filk 2010), we employed inequality (1) for temporal correlations, in order to provide evidence that mental states may be nonlocal in time. Here, the probabilities refer to the values of one observable at three different time instances. With proper models for particular mental operations it is possible to violate inequality (1). However, the problem of invasivity becomes almost insuperable empirically, because every measurement will be capable of priming other measurements of the same observable.[3] (As long as a final result is not read out, priming may even be efficacious if it follows rather than precedes the other measurement.)

2.2 The Original Bell Inequality

Quantum theory predicts a violation of inequality (1). However, this violation cannot be tested "directly", because in systems that are candidates for a violation only one of the observables can be measured at a time. It is possible, though, to exploit an intrinsic feature of quantum theory itself: entangled states. Most commonly known among these states is the so-called EPR-state (named after the seminal paper by Einstein *et al.* 1935), for which the results for measurements of the same observable in both subsystems are strictly anti-correlated.

So, in principle, six different values of observables are now involved, but only pairs of them (one for each subsystem) can be measured simultaneously. Because of the strict anti-correlation with respect to the same observable, exactly the same eight possible combinations of measured results as in Table 1 can occur.

Table 2 lists the corresponding cases with expectation values for the correlations ($x_i = a_i, b_i, c_i$) rather than probabilities:

$$C(x_1, x_2) = \begin{cases} +1 & \text{if the two measured values are identical,} \\ -1 & \text{if the two measured values are different.} \end{cases} \quad (2)$$

[3] In psychology, invasivity is related to the lack of "selective influence" (Dzhafarov 2003). "Weak" measurements (Aharonov *et al.* 1988) might be an option to avoid invasivity. Is has also been suggested (Wilde and Mizel 2012) that "adroit" measurements may be used to test invasivity.

Table 2. Eight possible combinations of values for anti-correlated systems. Indices 1 and 2 refer to the two subsystems, and C values are correlations rather than probabilities.

a_1 b_1 c_1	a_2 b_2 c_2	$C(a_1, b_2)$	$C(b_1, c_2)$	$C(a_1, c_2)$
$+ + +$	$- - -$	-1	-1	-1
$+ + -$	$- - +$	-1	$+1$	$+1$
$+ - +$	$- + -$	$+1$	$+1$	-1
$+ - -$	$- + +$	$+1$	-1	$+1$
$- + +$	$+ - -$	$+1$	-1	$+1$
$- + -$	$+ - +$	$+1$	$+1$	-1
$- - +$	$+ + -$	-1	$+1$	$+1$
$- - -$	$- - -$	-1	-1	-1

Note that now the first measurement refers to subsystem 1 and the second measurement to subsystem 2, so that – for instance (see line 2 in Table 2) – $C(a_1, c_2) = +1$ if and only if a_1 and c_1 or a_2 and c_2 within the same subsystem are different. From Table 2 one can easily check that:

$$|C(a_1, b_2) - C(a_1, c_2)| \leq 1 + C(b_1, c_2)\,, \tag{3}$$

which is the standard form of the Bell inequality according to Bell (1966).

As we are now dealing with another system than in Sect. 2.1, there is an additional assumption: The strict anti-correlation between the same observables for each subsystem has to hold also in those cases where the measurement is performed for two different observables. This means that the eight possibilities listed above in fact cover all possible cases. On the other hand, a violation of inequality (3) is possible under the following two conditions:

1. The outcome for all three possible measurements was not predetermined before the measurement is actually performed but is "generated" (with an intrinsic, ontic dispersion) during the process of measurement.
2. In order to explain the complete anti-correlation with respect to the same observables, condition 1 requires that the first measurement (and its result) performed at one of the subsystems has an "influence" on the second measurement (and its result) performed on the other subsystem.

The second condition expresses the requirement of non-locality indispensable for any hidden variables theory of quantum mechanics. Any exchange of information (i.e., interaction) between the two subsystems after the first and before the second measurement naturally may violate the inequality.

3 Inequalities due to Clauser-Horne-Shimony-Holt, and due to Fine

3.1 CHSH Inequality

From an experimental point of view, inequality (3) has the disadvantage that three possible measurements can be performed on a single subsystem, and it is very difficult to switch between three possibilities on nano-second time-scales (which is necessary in order to rule out the possibility of a "relativistic signaling" between the two subsystems). Moreover, it is conceptually prejudicial that one has to assume the anti-correlations between the two subsystems also if they are not explicitly measured. Quantum theory has taught us to be careful in assuming certain features as "real" if they cannot be explicitly tested.

For these (and other) reasons, Clauser *et al.* (1969) derived a different type of inequality which avoids both disadvantages just mentioned. They propose four different observables (a, a', b, b'), grouped into two classes (a, a') and (b, b'), and each with two possible outcomes only, say $+$ and $-$. The class (a, a') refers to two measurements performed on subsystem 1, and the measurements of class (b, b') are performed on subsystem 2. However, the assumption of two separate subsystems is not necessary for the actual derivation of the inequality.

We first define the CHSH-quantity S as a sum of pairwise correlations $C(a, b)$, etc., between the measured values of the observables involved:

$$S = C(a,b) - C(a,b') + C(a',b) + C(a',b')\,, \tag{4}$$

As can be seen from Table 3, in each single case the quantity S is either $+2$ or -2. Therefore, the expectation value of this quantity in an arbitrary ensemble of systems satisfies the so-called CHSH inequality:

$$-2 \leq S \leq +2\,. \tag{5}$$

A typical experimental design to extract S uses entangled photons (Aspect *et al.* 1982) and measure a or a' at one of the photons and b or b' at the other.

Table 3. The sixteen combinatorial possibilities for the measured results of four binary observables and their correlations C. The quantity S in the rightmost column is the CHSH-quantity for which the inequality is formulated.

n	a	a'	b	b'	$C(a,b)$	$C(a,b')$	$C(a',b)$	$C(a',b')$	S
1	+	+	+	+	+1	+1	+1	+1	+2
2	+	+	+	−	+1	−1	+1	−1	+2
3	+	+	−	+	−1	+1	−1	+1	−2
4	+	+	−	−	−1	−1	−1	−1	−2
5	+	−	+	+	+1	+1	−1	−1	−2
6	+	−	+	−	+1	−1	−1	+1	+2
7	+	−	−	+	−1	+1	+1	−1	−2
8	+	−	−	−	−1	−1	+1	+1	+2
9	−	+	+	+	−1	−1	+1	+1	+2
10	−	+	+	−	−1	+1	+1	−1	−2
11	−	+	−	+	+1	−1	−1	+1	+2
12	−	+	−	−	+1	+1	−1	−1	−2
13	−	−	+	+	−1	−1	−1	−1	−2
14	−	−	+	−	−1	+1	−1	+1	−2
15	−	−	−	+	+1	−1	+1	−1	+2
16	−	−	−	−	+1	+1	+1	+1	+2

If these four observables are particular angles of photon polarizations ($a \sim 0°, b \sim 25.5°, a' \sim 45°, b' \sim 67.5°$) the inequality is violated: the expectation value of S approaches $S = 2\sqrt{2}$, the so-called quantum bound which cannot be exceeded in quantum systems. In actual experiments the expectation value of S is smaller than this bound due to unavoidable noise and detection errors.

It is clear from Table 3 that, without the restrictions imposed by quantum physics, S can become as large as 4, e.g., if $C(a, b') = -1$ and all other C terms are $+1$. In Sect. 4 we will introduce and discuss an example of a non-quantum system violating inequality (5) maximally in this sense.

3.2 The Fine Inequality

The CHSH inequality was reexamined from a slightly different point of view by Fine (1982); see also Clauser and Horne (1974). He defined the probability $P(a)$ as the probability that a measurement of the observable a yields the result + (this probability is the sum of the probabilities of the upper eight cases in Table 3), and similarly $P(a')$, $P(b)$, and $P(b')$. Furthermore, he defined the probability $P(ab)$ as the joint probability that both a and b assume the value +, and similarly for all other combinations ab', $a'b$ and $a'b'$. Introducing a quantity

$$F = P(ab) - P(a, b') + P(a'b') + P(a'b) - P(a') - P(b) \; , \qquad (6)$$

he ends up with the inequality:

$$-1 \leq F \leq 0 \, . \qquad (7)$$

Table 4 lists all probabilities and the Fine quantity F for each of the sixteen possible cases. F assumes either the value 0 or the value -1 and, therefore, its expectation value satisfies inequality (7).

Fine's own proof of this inequality does not use Table 4 though. His original proof (which we do not repeat here) does not belong to type 1, but to type 2 in Sect. 1. This way, Fine could prove a very strong result: If inequality (7) is satisfied then there exists a hidden variable model which describes the observed probabilities. In order to show this, Fine assumes that the probability distribution $P(AB, \lambda)$ can be factorized such that

$$P(AB, \lambda) = P(A, \lambda) P(B, \lambda) \qquad (8)$$

for any $A = a$ or a' and $B = b$ or b', where λ refers to some hidden variable (or variables) and all expectation values are obtained by summing (or integrating) over all possible λ.

Expressed in our type 1 formulation, Eq. (8) is fulfilled for each of the sixteen combinations listed in Table 4. This is the case because by definition $P(AB)$ is 1 if and only if A is + (i.e. $P(A) = 1$) and B is + (i.e. $P(B) = 1$). This condition is weaker than assuming factorizability with respect to arbitrary functions of A and B. Indeed, Fine emphasizes in his article that "... the idea of deterministic

Table 4. The sixteen combinatorial possibilities for the measured results of four binary observables and the probabilities used in Fine's inequality. F is the Fine-quantity for which the inequality is formulated.

n	$a\ a'\ b\ b'$	$P(ab)$	$P(ab')$	$P(a'b')$	$P(a'b)$	$P(a')$	$P(b)$	F
1	$++++$	1	1	1	1	1	1	0
2	$+++-$	1	0	0	1	1	1	0
3	$++-+$	0	1	1	0	1	0	-1
4	$++--$	0	0	0	0	1	0	-1
5	$+-++$	1	1	0	0	0	1	-1
6	$+-+-$	1	0	0	0	0	1	0
7	$+--+$	0	1	0	0	0	0	-1
8	$+---$	0	0	0	0	0	0	0
9	$-+++$	0	0	1	1	1	1	0
10	$-++-$	0	0	0	1	1	1	-1
11	$-+-+$	0	0	1	0	1	0	0
12	$-+--$	0	0	0	0	1	0	-1
13	$--++$	0	0	0	0	0	1	-1
14	$--+-$	0	0	0	0	0	1	-1
15	$---+$	0	0	0	0	0	0	0
16	$----$	0	0	0	0	0	0	0

hidden variables is just the idea of a suitable joint probability function" (Fine 1982).

4 Violating CHSH and Fine Inequalities Outside Quantum Physics

One may wonder how it is possible to violate the CHSH or Fine's inequality even beyond the violations known for quantum systems. Essentially, the conditions have been specified in Sect. 2.2 for the original Bell inequality. The following example illustrates a mechanism which we consider as relevant in particular for psychological experiments with mental systems. Technically speaking, the example consists of two subsystems coupled by a logical OR gate.

The device consists of an upper part, where measurements of a and a' are performed (system A), and a lower part where measurements of b and b' are performed (system B). The two parts are separated in Fig. 1 by a dashed line. Both systems yield outputs which are either $+$ or $-$. The systems also contain "interpreters" which transform the measurement of a into a 1, a' into a 0, b' into a 0, and b into a 1. Additional "interpreters" near the output transform a 0 into a "$-$" and a 1 into a "$+$".

Obviously, system B always yields $+$ as an output, independent of the measurements performed. This is just for simplicity and can easily be generalized. System A yields $-$ only if the combination (a, b') is performed (in which case the input at the OR-gate in system A is $(0, 0)$ and its output 0). In all other cases the output from system A is $+$. The corresponding expectation values for the correlations in the CHSH inequality are

$$C(a,b) = C(a',b) = C(a',b') = +1, \quad C(a,b') = -1\,, \tag{9}$$

which yields $S = 4$ and violates inequality (5) maximally.

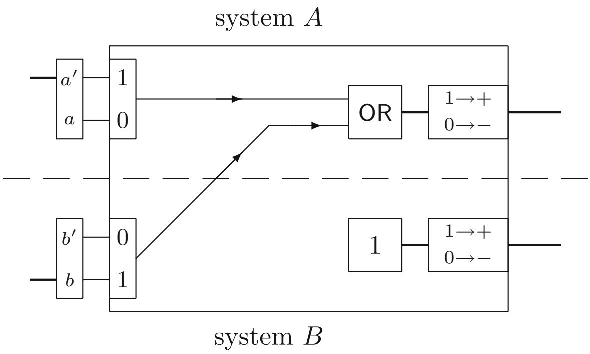


Fig. 1. A simple device which violates both the CHSH and Fine inequalities. The experimental result of a measurement of a or a' and b or b' is translated into a number (1 or 0). "OR" refers to a gate which performs a logical OR-operation. Finally the logical outputs 0 and 1 are transformed into $-$ und $+$, respectively.

With respect to Fine's inequality we notice that in all cases with a' and b involved, the probabilities are 1:

$$P(ab) = P(a'b) = P(a'b') = 1\ , \quad P(ab') = 0 \quad \text{and} \quad P(a') = P(b) = 1\,, \tag{10}$$

yielding $F = +1$, hence a violation of the Fine inequality as well.[4]

How can we understand this? First of all, the output from B is a "dummy variable" which in our example is always $+$. Only the output from A is relevant, and it is $+$ if a' is measured. However, if a is measured, the output depends on which observable is probed at system B. This illustrates the non-locality inevitably entailed by the violation of both CHSH and Fine inequalities. In this particular example, the specific output from A (which may differ) is not correlated with the specific output from B (which is always 1), but it is correlated with which measurement, b or b', is performed at B. The non-locality of the device as a whole is also evident from the internal wiring connecting B with A.

[4] Note that we do not intend to apply the Fine inequality to identify a joint probability distribution over the four variables. We simply show that a violation of Fine's inequality is possible in a purely classical system.

5 Summary and Concluding Remarks

We discussed a number of similarities and differences of four Bell-type inequalities: the Wigner inequality, the original Bell inequality, the CHSH inequality (due to Clauser, Horne, Shimony, Holt) and the Fine inequality. All these inequalities were designed to test versions of hidden variable models in quantum theory, and their violation for entangled quantum systems is usually interpreted as the ultimate nail in the coffin of local realism.

Recent developments in psychology, cognitive science, and consciousness studies have witnessed the fruitful application of quantum ideas in selected examples for mental systems. Some areas concerned are decision theory, bistable perception, order effects in surveys, and others, where violations of Bell-type inequalities have been predicted from models or even been claimed empirically.

In this paper we argue that quantum-like entanglement and nonlocality are not the only possible implications of such violations. A viable, more conservative alternative may be priming effects or, more generally, other kinds of contextuality. Such effects are especially influential whenever invasive priming effects naturally violate Bell-type inequalities, possibly even beyond the quantum bound.

This might be exemplified by an example where violations of Bell-type inequalities are suggested to provide evidence against the compositionality of concept combinations, an old dogma in cognitive linguistics.[5] In typical word association experiments one can think of B as a priming expression. Let us consider measurements consisting of the following questions:

a: What do you think of when you hear the word "apple"?
b': What do you think of when you hear the word "processor"?
b: What do you think of when you hear the word "banana"?

Then the answer to question a will most likely be "computer" if primed by b', and it will be "fruit" if primed by b. This corresponds to the example of Sect. 5, which is entirely classical, without any quantum entanglement involved.

Now, any violation of the CHSH or Fine inequality would be surprising if A and B referred to two different individuals out of communication, for instance if someone in Brisbane always associates with "apple" a computer when someone in Bloomington is primed for "processor", while someone in Brisbane associates a fruit when someone in Bloomington is primed by "banana". This would resemble the astonishing results of what happens in experiments with entangled quantum systems. But in the cognitive case, when A and B refer to the same individual, the violation of Bell-type inequalities could rather be a direct consequence of plain priming, the strength of which is measured by the quantities S or F.

In this sense, violations of Bell-type inequalities in psychological scenarios could be instrumental to identify (conscious or unconscious) priming effects. Furthermore, violations of the CHSH inequality in word association experiments are related to multiple possible meanings of the same concept (like "apple"). Such

[5] Intriguing work on the non-compositionality of concept combinations is due to Bruza *et al.* (2013), whose examples we use to illustrate our own hypotheses.

multiple possible meanings could be interpreted as a "non-locality" of concepts represented in a conceptual space: Several meanings of the same word are not part of the same convex set in conceptual space (see, e.g., Gärdenfors 2000). In such cases, priming actualizes one particular meaning among those that are possible.

Both this example and the classical OR gate in Sect. 4 show that holistic features may arise in classical systems. Yet one should bear in mind that this kind of holism differs from cases of entanglement in genuine quantum systems. Unfortunately, we do not know of any conclusive test which could firmly distinguish between genuine quantum holism and holistic features in the classical systems discussed above.

Although the term "priming" suggests that the priming stimulus causally precedes the target stimulus to be primed, the structure of Bell-type inequalities tells us that this need not be the case. Correlations due to a priming stimulus following a target stimulus may imply violations as well. In fact, the experiments by Bruza *et al.* (2013) show that "post-priming" does a pretty good job for violating CHSH and Fine inequalities.

In a recent publication, Bem (2011) reported post-priming correlations in nine experiments with more than 1000 participants and interpreted them as "anomalous retroactive influences". In our view, it would be interesting to explore whether such apparently spectacular observations can be deflated in terms of post-selective priming effects, quantified by violations of Bell-type inequalities.

References

Aharonov, Y., Albert, D.Z., Vaidman, L.: How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100. Phys. Rev. Lett. **60**, 1351–1354 (1988)

Aspect, A., Dalibard, J., Roger, G.: Experimental test of bell's inequalities using time-varying analyzers. Phys. Rev. Lett. **49**, 1804–1807 (1982)

Atmanspacher, H.: Quantum approaches to consciousness. In: Stanford Encyclopedia of Philosophy. http://plato.stanford.edu/entries/qt-consciousness/ (2011)

Atmanspacher, H., Filk, T.: A proposed test of temporal nonlocality in bistable perception. J. Math. Psychol. **54**, 314–321 (2010)

Bargh, J.A., Chartrand, T.L.: Studying the mind in the middle: a practical guide to priming and automaticity research. In: Reis, H., Judd, C. (eds.) Handbook of Research Methods in Social and Personality Psychology, pp. 253–285. Cambridge University Press, Cambridge (2000)

Bell, J.S.: On the problem of hidden variables in quantum theory. Rev. Mod. Phys. **38**, 447–452 (1966)

Bem, D.J.: Feeling the future: experimental evidence for anomalous retroactive influences on cognition and affect. J. Pers. Soc. Psychol. **100**, 407–425 (2011)

Busemeyer, J.R., Bruza, P.D.: Quantum Models of Cognition and Decision. Cambridge University Press, Cambridge (2012)

Bruza, P.D., Kitto, K., Ramm, B., Sitbon, L.: A probabilistic framework for analyzing the compositionality of conceptual combinations (2013) (Preprint)

Clauser, J.F., Horne, M.A., Shimony, A., Holt, R.A.: Proposed experiment to test local hidden-variable theories. Phys. Rev. Lett. **23**, 880–884 (1969)

Clauser, J.F., Horne, M.A.: Experimental consequences of objective local theories. Phys. Rev. D **10**, 526–535 (1974)

Conway, J.H., Kochen, S.: The free will theorem. Found. Phys. **36**, 1441–1473 (2006)

d'Espagnat, B.: The quantum theory and reality. Sci. Am. **241**, 158–181 (1979)

Dzhafarov, E.N.: Selective influence through conditional independence. Psychometrika **68**, 7–25 (2003)

Einstein, A., Podolsky, B., Rosen, N.: Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **47**, 777–780 (1935)

Fine, A.: Hidden variables, joint probability, and the bell inequalities. Phys. Rev. Lett. **48**, 291–295 (1982)

Gärdenfors, P.: Conceptual Spaces. MIT Press, Cambridge (2000)

Gilder, L.: The Age of Entanglement. Alfred Knopf, New York (2008)

Schrödinger, E.: Discussion of probability relations between separated systems. Proc. Camb. Philos. Soc. **31**, 555–563 (1935)

Wang, Z.J., Busemeyer, J.R., Atmanspacher, H., Pothos, E.M.: The potential for using quantum theory to build models of cognition. Top. Cogn. Sci. (2013) (in press)

Wigner, E.P.: On hidden variables and quantum mechanical probabilities. Am. J. Phys. **38**, 1005–1009 (1970)

Wilde, M.M., Mizel, A.: Addressing the clumsiness loophole in a leggett-garg test of macrorealism. Found. Phys. **42**, 256–265 (2012)

Purposeful Choice and Point-of-View
A Generalized Quantum Approach

Dominic Widdows(✉)

Microsoft Bing, Bellevue, WA, USA
dwiddows@gmail.com

Abstract. This paper presents a generalized quantum model for describing purposes or goals of individual agents, and the way choices can be made that enable these goals to be achieved. The underlying model is a semantic vector space model, which is turned into a purposeful choice model by labelling some axes as objectives, and describing choices as transformations on the vector spaces that enable agents in the model to set these objective axes in sight.

We introduce this framework using a simplified example model of a dog trying to get food. Many parts of what has become the standard generalized quantum toolkit become apparent in this model, including learning, superposition, the importance of the metric used for normalization, classification, and a generalized uncertainty principle. The incorporation of purpose or goal into semantic vectors models also enables the analysis of traditional areas that are relatively new to artificial intelligence, including rhetoric, political science, and some of the philosophical questions touched by quantum theorists.

1 Introduction

This paper presents a vector model for the everyday notions of 'purpose' and 'choice'. While the initial example model requires as background only the basics of Cartesian coordinate geometry, as it progresses, the work builds on progress in quantum interaction and generalized quantum structures, which have in recent years been used with success to address many classic problems. The basic approach in this research program involves representing information systems using vector space models, and as such has been applied to information discovery and retrieval [1,2], cognition and decision-making [3], economics [4, Chap. 10] and organizational dynamics [5]. The 'quantum' qualities of these systems evaluated in the literature to date include non-locality of logical connectives in information retrieval [6, Chap. 7], non-commutativity of observables in psychological tests [3], the violation of the Bell inequalities in concept combinations [7,8], and entanglement in concept combinations [1,9].

In spite of the success of vector models in large-scale practical tasks such as the creation of information retrieval systems, the notions of purpose and choice in such models is comparatively unexplored, sometimes being ignored, and sometimes being entirely denied as a valid ingredient of the model in the first place.

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 244–256, 2014.
DOI: 10.1007/978-3-642-54943-4_22,

Scientific works where purpose is largely ignored include studies of the parallels between information retrieval and quantum theory (see e.g., [10], where the epistemological and ontological status of items in the system is considered, but the motivations for creating or using the system are not), and distributional models of concepts and their semantics derived from natural language corpora (see e.g., [11,12]). Of course, the relationship of items in a retrieval system to one another and to objects in the world, and the distribution of terms and topics in a corpus, are important and valuable areas of study in understanding language and meaning: however, they do not attempt to explain anything about what authors are trying to accomplish by writing documents, or what users or a retrieval system are trying to accomplish by issuing search queries. Scientific examples where purpose is denied as a valid ingredient of the model are much more general, and are a hallmark of many classical mechanical approached. In broad strokes, Francis Bacon's philosophy and the success of Newtonian mechanics in the 17th century led to a broad consensus in the 18th century that the only notion of cause that can be discussed scientifically is 'efficient cause' or cause in the mechanical sense: one of the few things Hume (a great empiricist) and Kant (a great rationalist) agreed upon was the notion that causes must precede their effects in time. This overrode Aristotle's much older analysis in which included 'final cause' or purpose among the natural causes of things (see in particular *Physics* Book II Chap. 3; for modern consequences see [13, Chap. 1]).

Whether or not the apparent notion of future purpose can be explained in terms of temporally prior mechanical causes (for example, by a generalization of field theories in which the notion of potential is explained in terms of force-carrying particles), it is noticeable that classic models motivated by cause-precedes-effect determinism have been found wanting in many fields in which the systems under consideration are too complex or subtle to be described as closed mechanical systems evolving predictably [14], and the reader will observe that many of the successes of generalized quantum approaches cited above are precisely in fields that are not (yet) amenable to mechanistic prediction. In simpler terms, as soon as we consider systems involving living things, especially people, we see that purpose and choice are fundamental factors in any thorough explanation. These cannot (yet) be explained in terms of more mechanical primitives, but cannot be neglected if effective scientific models of such systems are to be discovered. This is by no means a purely abstract exercise: appropriate models for understanding the goals and choices of authors and readers could (for example) enable engineers to build better search engines.

It should be noted that models for purpose and choice are not absent from the scientific literature: one of the most famous approaches is the use of 'belief, desire, intention networks', which [15] have been particularly influential in modelling agency in artificial intelligence [16]. Generalized quantum methods are potentially a complementary innovation to such discrete network models, because the continuous vector representation automatically enables robust or inexact inference, in ways that are naturally amenable to learning from experience [17,18]. Decision-making is also by now deep-rooted in the quantum interaction community (see particularly [3] and associated works). Here we note that most of the decision-making situations discussed in this literature are about deciding between possible information states or beliefs: so arguably, the innovative part of the purposeful choice model presented in this paper is that it applies vector representations to desires and intentions as well as beliefs. However, it is also our hope that this research area is by now mature enough that the contribution of this paper is not that it supersedes prior work, but that it simplifies, generalises and extends ideas that are already available.

With these goals in mind, this paper proceeds as follows. In Sect. 2, we introduce a first, extremely simple example that explains the behaviour of an agent (in this case, a family pet) in a model with one objective axis and two axes for expressing behavioural choices. The semantic space introduced in this model is similar to the distributional vector models used widely in information retrieval, computational linguistics, and cognitive science, but unlike many semantic models in these fields, the purposeful choice model presented here distinguishes 'ends' and 'means' directions.

In spite of its simplicity, this model is enough to motivate definitions for several important cognitive processes, including learning and classification: some of these developments are discussed in Sect. 3. Many further topics and developments are suggested by this discussion, but due to space constraints, they cannot be included in this paper. Section 4 outlines some of these topics. They include the modelling of rhetoric, applications to political theory, and the relationship between purposeful choice models and some standard areas of discussion in the philosophy of quantum mechanics.

2 First Example: A Dog's Life

"Look cute, get fed!" may be the motto of fortunate, well-kept pet dogs throughout many parts of the world. Many readers who own dogs are probably well aware of this trait: for those who are not, it is sufficient to note that:

1. Most dogs (especially those rescued from situations of hunger) are tirelessly devoted to the purpose of getting food.
2. Pet dogs devote themselves to this purpose by seeking out humans who might give them food, and doing their best to look cute, cuddly, hungry, pitiful, and attractive to humans as best they can.

In the wild, these behavioural traits are not especially useful compared with the basic hunting skills of (say) being able to run fast to catch prey. However, pet dogs have successfully transformed their strategy for getting food from running fast to looking cute. Experiments with the domestication of the silver fox, a closely related species to the dog, have demonstrated that profound behavioural changes can take place within a matter of a few decades, or 30 to 35 generations, leaving the animals "eager to please and unmistakably domesticated" [19]. By the same token, many tame dogs can make use of their wilder traits at a moment's notice: ill-trained or uncontrolled sheepdogs may chase herds, and most terriers

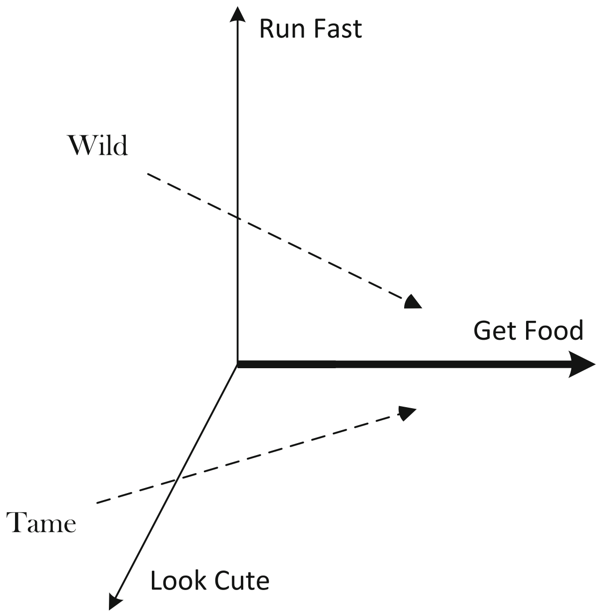


Fig. 1. A purposeful choice model for a dog in three dimensions

and hounds will kill any small furry creature given the opportunity. We may sum up by noting that all dogs wish to get food, different strategies are available (even to an individual dog), and the choice between these strategies is sometimes made quite quickly and fluidly.

A *purposeful choice model* for this (obviously simplified) description of a pet dog's objective and behaviours is represented in Fig. 1. The model uses a 3-dimensional vector space, whose axes are labelled *Get Food*, *Run Fast*, and *Look Cute*. The *Get Food* axis is represented by a thicker line because it represents a purpose, otherwise described as a goal, end, or objective. Such an axes will be called an *objective axis*. The other two axes, *Run Fast* and *Look Cute*, represent different possible behaviours that a dog may choose between to achieve its goal.

The choice between behaviours is now modelled as a change of *point-of-view*. The use of this term in the model is just a formalization of its normal conversational use: a point-of-view is a place from which the concepts around are observed. So an agent adopts a point-of-view in the model. The wild dog adopts a point-of-view which keeps the *Get Fed* objective axis in sight and approximately aligns this axis with the *Run Fast* choice. The tame dog instead adopts a point-of-view which approximately aligns the *Get Fed* objective axis with the *Look Cute* axis.

The views from the point-of-view of the wild and the tame dog are shown in Fig. 2. For the wild dog, the objective axis *Get Fed* is aligned with the *Run Fast* axis, whereas for the tame dog, the *Get Fed* axis is aligned with the *Look Cute* axis. The key point to see is that by adopting a different point-of-view,

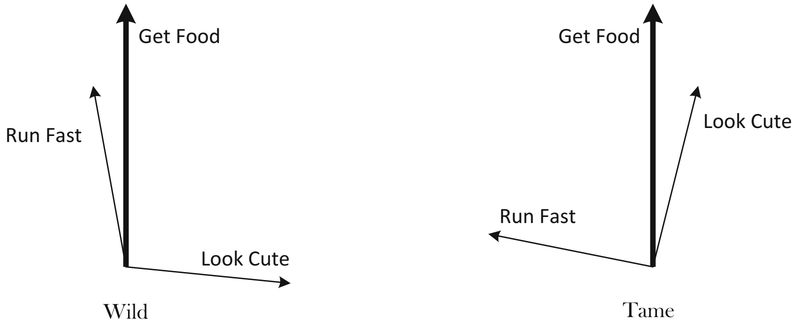


Fig. 2. The dog model again, from the points-of-view of a wild dog (left) and a tame dog (right)

the relationships between different axes change, and that this alignment can be made very deliberately to align behaviours with objectives.

The rest of this paper, one way or another, will be devoted to fleshing out this basic model, and explaining how more sophisticated (and to the research community, more familiar) structures arise in this framework.

The reader should note before progressing further that we have made no claims or assumptions of orthogonality, or assumed any particular metric function in the purposeful choice model. Many well-kept pet dogs will agree that looking cute and running fast are not necessarily orthogonal, and indeed, the different strategies for accomplishing a particular objective are rarely entirely unrelated to one another. To model a purposeful choice as a change in point-of-view gives considerable freedom here: even if axes are orthogonal to each other in the underlying model given a particular metric, they will not usually appear orthogonal to each other once a point-of-view is selected.

The investigation of similarity methods with respect to a point-of-view has already been introduced in [20], and again, the main single innovation introduced in this paper is the use of these ideas to model behaviour directed towards particular purposes or objective. The transformation in similarity measurements resulting from a change in point-of-view could also be modelled using a change in the metric function on the vector space, or a rescaling of some of the axes. This is also suggested in the cognitive science literature: changing the weights assigned to different axes can change the way items are classified in experiments [21].

From a historical point-of-view, we note that orthogonal coordinate systems are not assumed in Descartes' pioneering work on analytic geometry [22] in the 1600's, though it is explicitly discussed in Grassmann's extension of the theory to higher dimensions in the 1800's [23].

Another point to note is that, whatever the relationship between the *Run Fast* and *Look Cute* axes, most dogs use a superposition of these states anyway. Wolves may be single-minded to running fast, shih tzus may be single-minded to looking cute, but most real dogs have a foot in each camp. As will be discussed in the next section, vector models are particularly well-suited to representing hybrid strategies of this sort.

Of course, this description is purely a mathematical model, and as such is a simplification and abstraction. We are not attempting to model the physiological or neuronal patterns and changes involved in transitioning from one strategy to another (as discussed in the case of canids in [19] and mentioned briefly from a cognitive point-of-view in [17]). Just as vector models for information retrieval do not describe the physical formats of documents (typefaces, character encodings, etc.), our vector model for purposeful choice is so far independent of its physical manifestation.

This concludes our initial presentation of the purposeful choice model, using the simplest possible nontrivial number of dimensions. The key parts to emphasize are that:

- Purposeful choice models are semantic vector models where some of the axes are marked as goals or objective axes.
- Other axes can be brought into line with these objective axes using a suitable transformation of point-of-view.

3 Common Structures in the Purposeful Choice Model

This section develops the ideas of the purposeful choice model introduced above. This serves two principal purposes. Firstly, it demonstrates how several well-established techniques can be incorporated and described in terms of these models. Secondly, where it presents itself, we take the opportunity to compare classic and generalized quantum models: in some cases they are similar, but it some cases they lead to strikingly different paths.

3.1 Learning in Purposeful Choice Models

Vector models are particularly well-suited to learning, and this is one of the key reasons they have been a key model in information retrieval [2,24] and have become so successful in statistical machine learning [25]. This is easy to explain, at least anecdotally, referring back to the dog model of Fig. 1. Suppose that a piece of food is available through hunting: then a dog who gets fed in this way learns that hunting characteristics are useful. Such an example is modelled as a point somewhere close to the plane spanned by the *Run Fast* and *Get Food* axes. If the dog gets food in this way, it ends up satisfying an objective, and in so doing, the dog's point-of-view is updated to align these axes more closely. Alternatively, if food is available through begging, this may be modelled as a point somewhere near to the *Look Cute* and *Get Food* axes, and a dog that successfully fills its belly through begging will have its point-of-view updated to align these axes. (Which of several possible update functions is used is not discussed here, suffice it to say that several are available [25,26].)

Such flexibility to combine learning with action would in itself not be remarkable, were it not so lacking in many classic models. Consider, for example, Quine's now famous example of the problem of deducing whether the word *gavagai* coupled with the stimulus of a rabbit-sighting, corresponds to the set of rabbits, or to (say) edible animals [27]. Given the practical advances of empiricism in artificial intelligence in the intervening decades, many researchers today would disagree with the conclusion that language-learning cannot be explained logically, but would instead argue that to model language-learning, one should use a more appropriate logic. (See [18,28,29] for more details on this point: it is also appropriate to note that George Boole, the inventor of so-called classical logic, intended 'The Laws of Thought' to be used for deduction, and apparently never intended to apply them to learning [30].)

3.2 Objective Axes and Objective Functions

The most typical way to compare directions in a vector model would be to use cosine similarity (that is, the similarity between two vectors is measured using the cosine of the angle between these vectors [6, Chap. 5]. Cosine similarity with an objective axis behaves as a simple *objective function* in the classic sense of mathematical optimization, as used in economics, logistics, management science, etc. At this level of generality, with a single objective axis and an entirely specified point-of-view, there appears to be no difference between a classic and a generalized quantum model. What is perhaps more surprising is that in mathematical optimization, classic models tend to be continuous, whereas in logical semantics, classic models are discrete. This mixture is partly informed by the observation that in classical mechanics, the set of states is continuous but the logic for inference is discrete (Boolean), whereas in quantum mechanics, the set of states is discrete but the logic for inference (the set of projectors onto vector subspaces [31]) is continuous. For more details on this point, see [32]. This serves as a reminder that definitions of a 'classic' or 'classical' model vary even more than definitions of 'quantum' or 'generalized quantum' models.

3.3 Superposition, or Hybrid Strategies

It has already been noted that most real dogs are not devoted exclusively to running fast or looking cute as a means to get food, but easily combine both strategies. Suppose that x is the amount of attention devoted to scenarios where looking cute is helpful, and y is the amount of attention devoted to scenarios where running fast is helpful. In a classic probabilistic model, the coordinates must add to unity, so $x + y = 1$. For (say) a wild dog accustomed purely to running fast (the strategy where $y = 1$ and $x = 0$, giving any attention to looking cute therefore immediately detracts from the attention devoted to running fast. By contrast, one key different in a generalized quantum model is that the *squares* of the coordinates must add to unity, so that $x^2 + y^2 = 1$. An immediate consequence of this is that beginning to pay attention to another axis introduces *no*

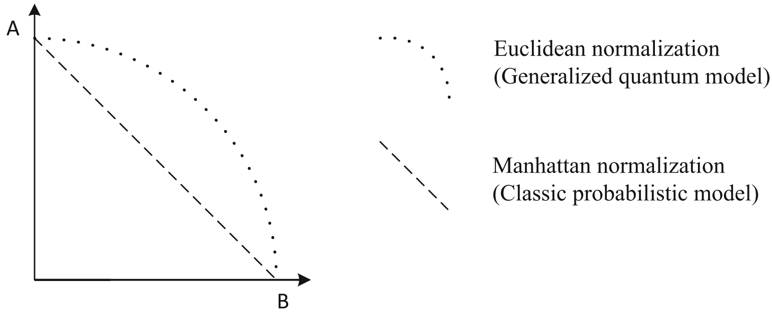


Fig. 3. The set of normalized states between two axes in classic and generalized quantum models

loss[1] to the axes that are already preferred. These two frameworks are depicted in Fig. 3: in mathematical terms, the vectors in the classic model are normalized using a Manhattan metric, whereas in a generalized quantum model, the vectors are normalized using a Euclidean metric (see [6, Chap. 4]).

The upshot of this is that in generalized quantum models, there is very little cost involved in departing from a pure state to a somewhat mixed state. The result is a dog whose ideal strategy is modelled as a superposition of the two pure strategies. This is a difficult idea to incorporate into classic models: in classic models, we may model a dog who sometimes tries one thing, and sometimes tries another, but not a dog who is simultaneously trying both. However, such a description has become standard in the quantum modelling literature, and has been used to accurately predict experimental observations about how people behave in situations where many potential outcomes are simultaneously possible: see in particular the work of Busemeyer et al ([33],[3, Chap. 9]) in which shows that difficult situations in psychology that have been traditionally regarded as paradoxes can be resolved using these methods.

We must of course choose words carefully in this case: clearly the dog cannot normally be pursuing prey and begging from a human at the same time, and we make no suggestion that the benefits of Euclidean normalization of a point-of-view vector can somehow enable an agent to outwit conservation of energy. The claim here is that agents can easily *consider* several strategies at once without diluting the attention devoted to any one of these strategies as much as may be expected in a classic model. A more familiar version of this principle arises when

[1] The claim that there is no loss at all to existing priorities when a new axis is considered is strictly true only in the continuous limit. For example, using the standard polar-coordinates parametrization where $x = \cos(\theta)$ and $y = \sin(\theta)$, when $\theta = 0$, $x = 1$, $y = 0$, $\frac{dy}{d\theta} = 1$ and $\frac{dx}{d\theta} = 0$. In practice, we assume that all models will be quantized, and so to make an actual change, there will be some *very small* cost. The issues involved in quantizing vector models of information and cognition is not a focus of this paper: we note briefly that this example implies that it is advantageous for the smallest 'representational quantum' to be small.

it is translated to the information retrieval literature: here we may remark that in vector models for information retrieval, if a document is about two topics, its relevance to each of the individual topics as measured by cosine similarity will be $\frac{\sqrt{2}}{2}$ instead of just 0.5. Furthermore, this property becomes even more pronounced in higher dimensions: in very high dimensions, many many vectors can be surperposed without losing the identity of any of the original summands [17].[2]

The immediate consequence in the simple purposeful choice model of Fig. 1 is that it costs very little in terms of cognitive attention for a wild dog devoted to hunting to consider begging as a once-in-a-while alternative.[3] This potential for an individual to break free from the rest of the population and find a new strategy has already been modelled successfully in quantum-inspired models (see [34]), so we hope that the purposeful choice model presented here contributes to this strand of research.

3.4 Classification — Fight or Flight?

The purposeful choice model of Fig. 1 is oversimplified in many ways, one of which is that the dog only has one goal, *Get Food*. Obviously a real dog has several other objectives, including *Avoid Injury*. For example, in the case of a dog trying out a *Look Cute* begging strategy for the first time, there is a tradeoff between the possibilities that a human offering a piece of food may give the piece of food, or use it as bait to capture or injure the dog, and there are good (if anecdotal) reasons to believe that the first dogs to become domesticated were the first to overcome this fear.

Consider also the case of a dog who has successfully obtained a piece of food, but is challenged by an antagonist before the food is eaten. There are two choices: to stay and safeguard the food at the risk of injury, or to run away, at the risk of hunger. This is commonly known as the *flight or fight* decision.

It is extremely easy to begin to add such tradeoffs to a purposeful choice model. We would simply add a *fourth* axis to Fig. 1, marked *Suffer Injury*, and with some label to denote that fact that it is a *negative* objective axis: that is, an axis that the agent will try to *avoid* aligning with behaviours in the purposeful choice model. (Due to lack of space and our inability to draw in four dimensions, we have not included a diagram of such a system.) Then, for each situation, the agent judges the extent to which *Get Food* and *Suffer Injury* are

[2] The difference between normalized coordinates of evenly-balanced vectors using Manhattan and Euclidean metrics is greatest in dimension 4. The proof is elementary, and consists of finding $x \in [0,1]$ such that $f(x) = \sqrt{x} - x$ is maximized, so $f'(x) = \frac{x^{-\frac{1}{2}}}{2} - 1 = 0$, implying $x^{\frac{1}{2}} = \frac{1}{2}$ and so $x = \frac{1}{4}$. We are not sure if this number has any special significance.

[3] Simulations demonstrate that a system with Euclidean normalization is more open to learning and, depending on the distribution of "food opportunities", gets to eat more food in the long run. These results are however quite preliminary: please contact the author for more details.

likely outcomes, and, based on some decision boundary, will choose either fight or flight (alternatively, beg from human or retreat) accordingly. Again, several algorithmic strategies for learning such decision boundaries are available in the machine learning literature [25,26].

3.5 A Generalized Uncertainty Principle

An important consequence of choosing *flight* in a *flight or fight* decision is that the agent is unable to observe the outcome of the other decision. This is very obvious in everyday situations, and leads to natural sayings such as “There’s only one way to find out!”, “If you don’t try it you’ll never know!”, etc.

More generally, in navigating a purposeful choice model, an agent will be aware that each time a choice is made, this affects which observations will be distorted or become completely unavailable at subsequent stages. This idea follows the work of Busemeyer and Bruza on the effects of ordering on attitude (see [3, Chap. 3] and related work). In particular, the effect of making a *flight* choice may be modelled as a projection orthogonal to the *Suffer Injury* axis, which, while it guarantees that the agent will avoid injury, also results in information being lost.

4 Further Work and Related Areas: A Grab-Bag of Ideas

Many traditional ideas can be defined and described in purposeful choice models. We use the remaining space in this paper to outline some of these in a preliminary fashion.

4.1 Persuasion or Rhetoric

It is well-known that information is often presented in a way designed to persuade or influence the point-of-view of others. Traditionally studied as rhetoric, the scientific discussion of this hugely important process has been largely neglected in computational linguistics and information retrieval.

In a purposeful choice model, persuasion or rhetoric can be defined as the presentation of information in a way designed to influence the agent’s point-of-view. Given the approach to training and classification outlined above, it is clearly possible to arrange data so that some points-of-view become reinforced, and others become less likely or unobtainable.

4.2 Application to Political and Organizational Theory

Generalized models have already been applied to political theory (see [35]) and organizational theory (see [36] and related work). As with work on quantum approaches to cognition and decision, the focus of this work is largely on describing how agents make decisions: a further step would be to model the ways other agents act in order to influence these decisions. Such influencing actions can be decribed in purposeful choice models as:

- A careful choice of issues by some author to design an appropriate classification boundary (e.g., in politics, a bill before the legislature is designed to accomplish as many of the author's desired goals, while maximizing the bill's chances of being voted into law).
- A careful choice of rhetoric designed to bring others to a point-of-view from which they are likely to agree with the author.

For example, President Lincoln's 1861 State of the Union Address makes an admirable case study of the use of rhetoric to align many points-of-view towards a common goal. Further work would be to model parts of this speech and its goals explicitly using a purposeful choice model.

4.3 The Purposeful Choice Model and Quantum Mechanics: Some Philosophical Conclusions

The notion of choice and decision is of course intimately connected to the idea of will in the sense of freedom of the will, a topic that is often associated with quantum theory, due largely to the probabilistic nature of quantum mechanical results. The notion of point-of-view is also relevant in quantum mechanics because in quantum mechanics, the observer is usually considered as part of the system, though the meaning and implications of this broad statement remain much-discussed [13].

It may be thought that a purposeful choice model implies an assumption of free-will at the expense of determinism. This is not necessarily the case. What *is* necessarily the case is that purposeful choice models agree with the basic Aristotelean principle that *final cause* is a valid and valuable kind of causation or explanation when studying natural processes ('natural' including human behaviour for these purposes). That is, a dedicated determinist may postulate that human behaviour including the notion of purpose could in principle be reduced to efficient or mechanical cause (just as the apparent 'goal' of an electron and a proton to be near each other can be explained mechanically by the exchange of photons). The problem with this approach is that it remains a very incomplete postulate. Social and information sciences have needed models that explain more of the phenomena observed in these disciplines, and this is what originally motivated many researchers to turn to generalized quantum models.

We cannot currently explain human (or even canine!) behaviour without the notion of purpose and choice: and even if it could be demonstrated in the end that such notions can be reduced to mechanical or efficient cause, models that successfully incorporate purpose into informatics would be a necessary stepping-stone. Thus, whether we are completing classical science or starting a new generalized science, the notion of purpose will be a key part of the explanation, and we suggest that the purposeful choice models introduced in this paper can play a valuable and practical role in this project.

References

1. Cohen, T., Widdows, D., de Vine, L., Schvaneveldt, R., Rindflesch, T.C.: Many paths lead to discovery: analogical retrieval of cancer therapies. In: [37]
2. van Rijsbergen, K.: The Geometry of Information Retrieval. Cambridge University Press, Cambridge (2004)
3. Busemeyer, J., Bruza, P.D.: Quantum models of cognition and decision. Cambridge University Press, Cambridge (2012)
4. Khrennikov, A.: Ubiquitous Quantum Structure: From Psychology to Finance. Springer, Heidelberg (2010)
5. Lawless, W.F., Bergman, M., Lou, J., Kriegel, N.N., Feltovich, N.: A quantum metric of organizational performance: terrorism and counterterrorism. Comput. Math. Organ. Theory **13**, 241–281 (2007)
6. Widdows, D.: Geometry and Meaning. CSLI Publications, Stanford (2004)
7. Aerts, D., Aerts, S., Broekaert, J., Gabora, L.: The violation of bell inequalities in the macroworld. Found. Phys. **30**, 1387–1414 (2000)
8. Aerts, D., Aerts, S., Gabora, L.: Experimental evidence for quantum structure in cognition. In: Bruza, P., Sofge, D., Lawless, W., van Rijsbergen, K., Klusch, M. (eds.) QI 2009. LNCS, vol. 5494, pp. 59–70. Springer, Heidelberg (2009)
9. Galea, D., Bruza, P., Kitto, K., Nelson, D., McEvoy, C.: Modelling the acitivation of words in human memory: the spreading activation, spooky-activation-at-a-distance and the entanglement models compared. In: Song, D., Melucci, M., Frommholz, I., Zhang, P., Wang, L., Arafat, S. (eds.) QI 2011. LNCS, vol. 7052, pp. 149–160. Springer, Heidelberg (2011)
10. Arafat, S.: Senses in which quantum theory is an analogy for information retrieval. In: [38]
11. Cohen, T., Widdows, D.: Empirical distributional semantics: methods and biomedical applications. J. Biomed. Inform. **42**(2), 390–405 (2009)
12. Blei, D.M.: Probabilistic topic models. Commun. ACM **55**, 77–84 (2012)
13. Bohm, D.: Wholeness and the Implicate Order. Routledge Classics, republished 2002, Routledge, 1980
14. Widdows, D., Bruza, P.: Quantum information dynamics and open world science. In: Proceedings of the First International Symposium on Quantum Interaction, Stanford, California (2007)
15. Bratman, M.E.: Intention, Plans, and Practical Reason. Harvard University Press, Cambridge (1999)
16. Georgeff, M., Pell, B., Pollack, M., Tambe, M., Wooldridge, M.: The belief-desire-intention model of agency. In: Papadimitriou, C., Singh, M.P., Müller, J. (eds.) ATAL 1998. LNCS (LNAI), vol. 1555, pp. 1–10. Springer, Heidelberg (1999)
17. Kanerva, P.: Hyperdimensional computing: an introduction to computing in distributed representation with high-dimensional random vectors. Cogn. Comput. **1**(2), 139–159 (2009)
18. Bruza, P.D., Widdows, D., Woods, J.: A quantum logic of down below. In: Engesser, K., Gabbay, D., Lehmann, D. (eds.) Handbook of Quantum Logic and Quantum Structures: Quantum Logic, pp. 625–660. Elsevier, Amsterdam (2009)
19. Trut, L.: Early canid domestication: the farm-fox experiment foxes bred for tamability in a 40-year experiment exhibit remarkable transformations that suggest an interplay between behavioral genetics and development. Am. Sci. **87**(2), 160–169 (1999)
20. Aerts, S., Kitto, K., Sitbon, L.: Similarity metrics within a point of view. In: [38]

21. Gärdenfors, P.: Conceptual Spaces: The Geometry of Thought. Bradford Books, MIT Press, Cambridge/London (2000)
22. Descartes, R.: The Geometry of René Descartes. Dover, New York (1637). Dover edn., with facsimile of the original
23. Grassmann, H.: Extension Theory. History of Mathematics Sources, American Mathematical Society, London Mathematical Society (1862). Translated by L.C. Kannenberg (2000)
24. Baeza-Yates, R., Ribiero-Neto, B.: Modern Information Retrieval. Addison Wesley/ACM Press, New York (1999)
25. Hastie, T., Tibshirani, R., Friedman, J.H.: The Elements of Statistical Learning. Springer Series in Statistics. Springer New York Inc., New York (2001)
26. Mitchell, T.: Machine Learning. McGraw-Hill, Boston (1997)
27. Quine, W.V.O.: Word and object. MIT Press (MA), 2013 (orig. 1960)
28. Gabbay, D., Woods, J.: The new logic. Logic J. IGPL **9**(2), 141–174 (2001)
29. Widdows, D., Higgins, M.: Geometric ordering of concepts, logical disjunction, learning by induction, and spatial indexing. In: Compositional Connectionism in Cognitive Science, Washington, DC. AAAI Fall Symposium Series, October 2004
30. Boole, G.: An Investigation of the Laws of Thought. Macmillan, London (1854). Dover edn., 1958
31. Birkhoff, G., von Neumann, J.: The logic of quantum mechanics. Ann. Math. **37**, 823–843 (1936)
32. Varadarajan, V.S.: Geometry of Quantum Theory. Springer, Berlin (1985)
33. Busemeyer, J.R., Pothos, E.M., Franco, R., Trueblood, J.S.: A quantum theoretical explanation for probability judgment errors. Psychol. Rev. **118**(2), 193 (2011)
34. Kitto, K., Boschetti, F.: The quantum inspired modelling of changing attitudes and self-organising societies. In: [37]
35. Zorn, C., Smith, C.E.: Pseudo-classical non-separability and mass politics in two-party systems. In: [38]
36. Lawless, W.F., Sofge, D.A.: Social-psychological harmonic oscillators in the self-regulation of organizations and systems. In: [37]
37. Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.): QI 2012. LNCS, vol. 7620. Springer, Heidelberg (2012)
38. Song, D., Melucci, M., Frommholz, I., Zhang, P., Wang, L., Arafat, S. (eds.): QI 2011. LNCS, vol. 7052. Springer, Heidelberg (2011)

Decision Making for Inconsistent Expert Judgments Using Negative Probabilities

Jose Acacio de Barros(✉)

Liberal Studies Program, San Francisco State University,
1600 Holloway Ave, San Francisco, CA 94132, USA
barros@sfsu.edu

Abstract. In this paper we provide a simple random-variable example of inconsistent information, and analyze it using three different approaches: Bayesian, quantum-like, and negative probabilities. We then show that, at least for this particular example, both the Bayesian and the quantum-like approaches have less normative power than the negative probabilities one.

1 Introduction

In recent years the quantum-mechanical formalism (mainly from non-relativistic quantum mechanics) has been used to model economic and decision-making processes (see [1,2] and references therein). The success of such models may originate from several related issues. First, the quantum formalism leads to a propositional structure that does not conform to classical logic [3]. Second, the probabilities of quantum observables do not satisfy Kolmogorov's axioms [4]. Third, quantum mechanics describes experimental outcomes that are highly contextual [5–9]. Such issues are connected because the logic of quantum mechanics, represented by a quantum lattice structure [3], leads to upper probability distributions and thus to non-Kolmogorovian measures [10–12], while contextuality leads the nonexistence of a joint probability distribution [13,14].

Both from a foundational and from a practical point of view, it is important to ask which aspects of quantum mechanics are actually needed for social-science models. For instance, the Hilbert space formalism leads to non-standard logic and probabilities, but the converse is not true: one cannot derive the Hilbert space formalism solely from weaker axioms of probabilities or from quantum lattices. Furthermore, the quantum formalism yields non-trivial results such as the impossibility of superluminal signaling with entangled states [15]. These types of results are not necessary for a theory of social phenomena [16], and we should ask what are the minimalistic mathematical structures suggested by quantum mechanics that reproduce the relevant features of quantum-like behavior.

In a previous article, we used reasonable neurophysiological assumptions to created a neural-oscillator model of behavioral Stimulus-Response theory [17]. We then showed how to use such model to reproduce quantum-like behavior [18]. Finally, in a subsequent article, we remarked that the same neural-oscillator

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 257–269, 2014.
DOI: 10.1007/978-3-642-54943-4_23,

model could be used to represent a set of observables that could not correspond to quantum mechanical observables [19], in a sense that we later on formalize in Sect. 3. These results suggest that one of the main quantum features relevant to social modeling is contextuality, represented by a non-Kolmogorovian probability measure, and that imposing a quantum formalism may be too restrictive. This non-Kolmogorovian characteristic would come when two contexts providing incompatible information about observable quantities were present.

Here we focus on the incompatibility of contexts as the source of a violation of standard probability theory. We then ask the following question: what formalisms are normative with respect to such incompatibility? This question comes from the fact that, in its origin, probability was devised as a normative theory, and not descriptive. For instance, Richard Jeffrey [20] explains that "the term 'probable' (Latin *probable*) meant *approvable*, and was applied in that sense, univocally, to opinion and to action. A probable action or opinion was one such as sensible people would undertake or hold, in the circumstances." Thus, it should come as no surprise that humans actually violate the rules of probability, as shown in many psychology experiments. However, if a person is to be considered "rational," according to Boole, he/she should follow the rules of probability theory.

Since inconsistent information, as above mentioned, violates the theory of probability, how do we provide a normative theory of rational decision-making? There are many approaches, such as Bayesian models or the Dempster-Shaffer theory, but here we focus on two non-standard ones: quantum-like and negative probability models. We start first by presenting a simple case where expert judgments lead to inconsistencies. Then, we approach this problem first with a standard Bayesian probabilistic method, followed by a quantum model. Finally, we use negative probability distributions as a third alternative. We then compare the different outcomes of each approach, and show that the use of negative probabilities seems to provide the most normative power among the three. We end this paper with some comments.

2 Inconsistent Information

As mentioned, the use of the quantum formalism in the social sciences originates from the observation that Kolmogorov's axioms are violated in many situations [1,2]. Such violations in decision-making seem to indicate a departure from a rational view, and in particular to though-processes that may involve irrational or contradictory reasoning, as is the case in non-monotonic reasoning. Thus, when dealing with quantum-like social phenomena, we are frequently dealing with some type of inconsistent information, usually arrived at as the end result of some non-classical (or incorrect, to some) reasoning. In this section we examine the case where inconsistency is present from the beginning.

Though in everyday life inconsistent information abounds, standard classical logic has difficulties dealing with it. For instance, it is a well know fact that if we have a contradiction, i.e. $A\&(\neg A)$, then the logic becomes trivial, in the sense that any formula in such logic is a theorem. To deal with such difficulty, logicians have proposed modified logical systems (e.g. paraconsistent logics [21]). Here, we will discuss how to deal with inconsistencies not from a logical point of view, but instead from a probabilistic one.

Inconsistencies of expert judgments are often represented in the probability literature by measures corresponding to the experts' subjective beliefs [22]. It is frequently argued that this subjective nature is necessary, as each expert makes statements about outcomes that are, in principle, available to all experts, and disagreements come not from sampling a certain probability space, but from personal beliefs. For example, let us assume that two experts, Alice and Bob, are examining whether to recommend the purchase of stocks in company X, and each gives different recommendations. Such differences do not emerge from an objective data (i.e. the actual future prices of X), but from each expert's interpretations of current market conditions and of company X. In some cases the inconsistencies are evident, as when, say, Alice recommends buy, and Bob recommends sell; in this case the decision maker would have to reconcile the discrepancies.

The above example provides a simple case. A more subtle one is when the experts have inconsistent beliefs that seem to be consistent. For example, each expert, with a limited access to information, may form, based on different contexts, locally consistent beliefs without directly contradicting other experts. But when we take the totality of the information provided by all of them and try to arrive at possible inferences, we reach contradictions. Here we want to create a simple random-variable model that incorporates expert judgments that are locally consistent but globally inconsistent. This model, inspired by quantum entanglement, will be used to show the main features of negative probabilities as applied to decision making.

Let us start with three ± 1-valued random variables, $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$, with zero expectation. If such random variables have correlations that are too strong then there is no joint probability distribution [13]. To see this, imagine the extreme case where the correlations between the random variables are $E(\mathbf{XY}) = E(\mathbf{YZ}) = E(\mathbf{XZ}) = -1$. Imagine that in a given trial we draw $\mathbf{X} = 1$. From $E(\mathbf{XY}) = -1$ it follows that $\mathbf{Y} = -1$, and from $E(\mathbf{YZ}) = -1$ that $\mathbf{Z} = 1$. But this is in contradiction with $E(\mathbf{XZ}) = -1$, which requires $\mathbf{Z} = -1$. Of course, the problem is not that there is a mathematical inconsistency, but that it is not possible to find a probabilistic sample space for which the variables $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ have such strong correlations. Another way to think about this is that the $\mathbf{X}$ measured together with $\mathbf{Y}$ is not the same one as the $\mathbf{X}$ measured with $\mathbf{Z}$: values of $\mathbf{X}$ depend on its context.

The above example posits a deterministic relationship between all random variables, but the inconsistencies persist even when weaker correlations exist. In fact, Suppes and Zanotti [13] proved that a joint probability distribution for $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ exists if and only if

$$\begin{aligned} -1 &\leq E(\mathbf{XY}) + E(\mathbf{YZ}) + E(\mathbf{XZ}) \\ &\leq 1 + 2\min\left\{E(\mathbf{XY}), E(\mathbf{YZ}), E(\mathbf{XZ})\right\}. \end{aligned} \tag{1}$$

The above case violates inequality (1).

Let's us now consider the example we want to analyze in detail. Imagine $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ as corresponding to future outcomes in a company's stocks. For instance, $\mathbf{X} = 1$ corresponds to an increase of the stock value of company X in the following day, while $\mathbf{X} = -1$ a decrease, and so on. Three experts, Alice (A), Bob (B), and Carlos (C), have the following beliefs about those stocks. Alice is an expert on companies X and Y, but knows little or nothing about Z, so she only tells us what we don't know: her expected correlation $E_A(\mathbf{XY})$. Bob (Carlos), on the other hand, is only an expert in companies X and Z (Y and Z), and he too only tells us about their correlations. Let us take the case where

$$E_A(\mathbf{XY}) = -1, \tag{2}$$

$$E_B(\mathbf{XZ}) = -\frac{1}{2}, \tag{3}$$

$$E_C(\mathbf{YZ}) = 0, \tag{4}$$

where the subscripts refer to each experts. For such case, the sum of the correlations is $-1\frac{1}{2}$, and according to (1) no joint probability distribution exists. Since there is no joint, how can a rational decision-maker decide what to do when faced with the question of how to bet in the market? In particular, how can she get information about the joint probability, and in particular the unknown triple moment $E(\mathbf{XYZ})$? In the next sections we will show how we can try to answer these questions using three possible approaches: quantum, Bayesian, and signed probabilities.

3 Quantum Approach

We start with a comment about the quantum-like nature of correlations (2)–(4). The random variables $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ with correlations (2)–(4) cannot be represented by a quantum state in a Hilbert space for the observables corresponding to $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$. This claim can be expressed in the form of a simple proposition.

Proposition 1. *Let $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ be three observables in a Hilbert space $\mathcal{H}$ with eigenvalues ± 1, let them pairwise commute, and let the ± 1-valued random variable $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ represent the outcomes of possible experiments performed on a quantum system $|\psi\rangle \in \mathcal{H}$. Then, there exists a joint probability distribution consistent with all the possible outcomes of $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$.*

Proof. Because $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ are observables and they pairwise commute, it follows that their combinations, $\hat{X}\hat{Y}$, $\hat{Y}\hat{Z}$, $\hat{X}\hat{Z}$, and $\hat{X}\hat{Y}\hat{Z}$ are also observables, and they commute with each other. For instance,

$$\left(\hat{X}\hat{Y}\hat{Z}\right)^{\dagger} = \hat{Z}^{\dagger}\hat{Y}^{\dagger}\hat{X}^{\dagger} = \hat{X}\hat{Y}\hat{Z}.$$

Furthermore,

$$[\hat{X}\hat{Y}\hat{Z}, \hat{X}] = [\hat{X}\hat{Y}\hat{Z}, \hat{Y}] = \cdots = [\hat{X}\hat{Y}\hat{Z}, \hat{X}\hat{Z}] = 0.$$

Therefore, quantum mechanics implies that all three observables $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ can be simultaneously measured. Since this is true, for the same state $|\psi\rangle$ we can create a full data table with all three values of $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ (i.e., no missing values), which implies the existence of a joint.

So, how would a quantum-like model of correlations (2)–(4) be like? The above result depends on the use of the same quantum state $|\psi\rangle$ throughout the many runs of the experiment, and to circumvent it we would need to use different states for the system. In other words, if we want to use a quantum formalism to describe the correlations (2)–(4), a $|\psi\rangle$ would have to be selected for each run such that a different state would be used when we measure $\hat{X}\hat{Y}$, e.g. $|\psi\rangle_{xy}$, than when we measure $\hat{X}\hat{Z}$, e.g. $|\psi\rangle_{xz}$. Then, the quantum description could be accomplished by the state

$$|\psi\rangle = c_A|A\rangle \otimes |\psi\rangle_{xy} + c_B|B\rangle \otimes |\psi\rangle_{xz} + c_C|C\rangle \otimes |\psi\rangle_{yz}.$$

This state would model the correlations the following way. When Alice makes her choice, she uses a projector into her "state of knowledge" $\hat{P}_A = |A\rangle\langle A|$, and gets the correlation $E_A(\mathbf{XY})$, and similarly for Bob and Carlos.

In the above example, all correlations and expectations are given, and the only unknown is the triple moment $E(\mathbf{XYZ})$. Furthermore, since we do not have a joint probability distribution, we cannot compute the range of values for such moment based on the expert's beliefs. But the question still remains as to what would be our best bet given what we know, i.e., what is our best guess for $E(\mathbf{XYZ})$. The quantum mechanical approach does not address this question, as it is not clear how to get it from the formalism given that any superposition of the states preferred by Alice, Bob, and Carlos are acceptable (i.e., we can choose any values of c_A, c_B, and c_C).

4 Bayesian Approach

Here we focus again on the unknown triple moment. As we mentioned before, there are many different ways to approach this problem, such as paraconsistent logics, consensus reaching, or information revision to restore consistency. Common to all those approaches is the complexity of how to resolve the inconsistencies, often with the aid of *ad hoc* assumptions [22]. Here we show how a Bayesian approach would deal with the issue [23,24].

In the Bayesian approach, a decision maker, Deanna (D), needs to access what is the joint probability distribution from a set of inconsistent expectations. To set the notation, let us first look at the case when there is only one expert. Let $P_A(x) = P_A(\mathbf{X} = x|\delta_A)$ be the probability assigned to event x by Alice conditioned on Alice's knowledge δ_A, and let $P_D(x) = P_D(\mathbf{X} = x|\delta_D)$ be

Deanna's prior distribution, also conditioned on her knowledge δ_D. Furthermore, let $\mathbf{P}_A = P_A(x)$ be a continuous random variable, $\mathbf{P}_A \in [0,1]$, such that its outcome is $P_A(x)$. The idea behind $\mathbf{P}_A$ is that consulting an expert is similar to conducting an experiment where we sample the experts opinion by observing a distribution function, and therefore we can talk about the probability that an expert will give an answer for a specific sample point. Then, for this case, Bayes's theorem can be written as

$$P'_D(x|\mathbf{P}_A = P_A(x)) = \frac{P_D(\mathbf{P}_A = P_A(x))\, P_D(x)}{P_D(\mathbf{P}_A = P_A(x))},$$

where $P'_D(x|\mathbf{P}_A = P_A(x))$ is Deanna's posterior distribution revised to take into account the expert's opinion. As is the case with Bayes's theorem, the difficulty lies on determining the likelihood function $P_D(\mathbf{P}_A)$, as well as the prior. This likelihood function is, in a certain sense, Deanna's model of Alice, as it is what Deanna believes are the likelihoods of each of Alice's beliefs. In other words, she should have a model of the experts. Such model of experts is akin to giving each expert a certain measure of credibility, since an expert whose model doesn't fit Deanna's would be assigned lower probability than an expert whose model fits.

The extension for our case of three experts and three random variables is cumbersome but straightforward. For Alice, Bob, and Carlos, Deanna needs to have a model for each one of them, based on her prior knowledge about $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$, as well as Alice, Bob, and Carlos. Following Morris [23], we construct a set E consisting of our three experts joint priors:

$$E = \{P_A(x,y), P_B(y,z), P_C(x,z)\}.$$

Deanna's is now faced with the problem of determining the posterior $P'_D(x|E)$, using Bayes's theorem, given her new knowledge of the expert's priors.

In a Bayesian approach, the decision maker should start with a prior belief on the stocks of X, Y, and Z, based on her knowledge. There is no recipe for choosing a prior, but let us start with the simple case where Deanna's lack of knowledge about X, Y, and Z means she starts with the initial belief that all combinations of values for $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are equiprobable. Let us use the following notation for the probabilities of each atom: $p_{xyz} = P(\mathbf{X} = +1, \mathbf{Y} = +1, \mathbf{Z} = +1)$, $p_{xy\overline{z}} = P(\mathbf{X} = +1, \mathbf{Y} = +1, \mathbf{Z} = -1)$, $p_{\overline{x}y\overline{z}} = P(\mathbf{X} = -1, \mathbf{Y} = +1, \mathbf{Z} = -1)$, and so on. Then Deanna's prior probabilities for the atoms are

$$p^D_{xyz} = p^D_{\overline{x}yz} = \cdots = p^D_{\overline{xyz}} = \frac{1}{16},$$

where the superscript D refers to Deanna.

When reasoning about the likelihood function, Deanna asks what would be the probable distribution of responses of Alice if somehow she (Deanna) could see the future (say, by consulting an Oracle) and find out that $E(XY) = -1$. For such case, it would be reasonable for Alice to think it more probable to have, say, $\overline{x}y$ than xy, since she was consulted as an expert. So, in terms of the correlation ϵ_A, Deanna could assign the following likelihood function:

$$P_D\left(\epsilon_A|\overline{x}y\right) = P_D\left(\epsilon_A|x\overline{y}\right) = \frac{1}{4}\left(1-\epsilon_A\right)^2, \tag{5}$$

$$P_D\left(\epsilon_A|xy\right) = P_D\left(\epsilon_A|\overline{xy}\right) = 1 - \frac{1}{4}\left(1-\epsilon_A\right)^2, \tag{6}$$

where the minus sign represents the negative, i.e. $p^A_{xy\cdot} = p_{\overline{xy}\cdot} = \frac{1}{4}\left(1+\epsilon_A\right)$ and $p_{\overline{x}y\cdot} = p_{x\overline{y}\cdot} = \frac{1}{4}\left(1-\epsilon_A\right)$. Notice that the choice of likelihood function is arbitrary.

Deanna's posterior, once she knows that Alice thought the correlation to be zero (cf. (2)), constitutes, as we mentioned above, an experiment. To illustrate the computation, we find its value below, from Alice's expectation $E_A\left(\mathbf{XY}\right) = -1$. From Bayes's theorem

$$\begin{aligned} p^{D|A}_{xyz} &= k\left[1-\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8} \\ &= \frac{1}{4}\left[1-\frac{1}{4}\left(1-\epsilon_A\right)^2\right] = \frac{3}{16}, \end{aligned}$$

where the normalization constant k is given by

$$\begin{aligned} k^{-1} = &\left[1-\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8} + \left[\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8} + \left[\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8} \\ &+ \left[1-\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8} + \left[\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8} + \left[\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8} \\ &+ \left[1-\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8} + \left[1-\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8}, \end{aligned}$$

and we use the notation $p^{D|A}$ to explicitly indicate that this is Deanna's posterior probability informed by Alice's expectation. Similarly, we have

$$p^{D|A}_{\overline{x}yz} = p^{D|A}_{\overline{x}y\overline{z}} = p^{D|A}_{x\overline{y}z} = p^{D|A}_{x\overline{y}\overline{z}} = \frac{1}{16},$$

and

$$p^{D|A}_{xyz} = p^{D|A}_{xy\overline{z}} = p^{D|A}_{\overline{xy}z} = p^{D|A}_{\overline{xyz}} = \frac{3}{16}.$$

If we apply Bayes's theorem twice more, to take into account Bob's and Carlos's opinions given by correlations (3) and (4), using likelihood functions similar to the one above, we compute the following posterior joint probability distribution,

$$p^{D|ABC}_{xyz} = p^{D|ABC}_{x\overline{y}z} = p^{D|ABC}_{\overline{x}y\overline{z}} = p^{D|ABC}_{\overline{xyz}} = 0,$$

$$p^{D|ABC}_{\overline{x}yz} = p^{D|ABC}_{x\overline{yz}} = \frac{7}{68},$$

and

$$p_{xy\overline{z}}^{D|ABC} = p_{\overline{xy}z}^{D|ABC} = \frac{27}{68}.$$

Finally, from the joint, we can compute all the moments, including the triple moment, and obtain $E(\mathbf{XYZ}) = 0$.

It is interesting to notice that the triple moment from the posterior is the same as the one from the prior. This is no coincidence. Because the revisions from Bayes's theorem only modify the values of the correlations, nothing is changed with respect to the triple moment. In fact, if we compute Deanna's posterior distribution for any values of the correlations ϵ_A, ϵ_B, and ϵ_C, we obtain the same triple moment, as it comes solely from Deanna's prior distribution. Thus, the Bayesian approach, though providing a proper distribution for the atoms, does not in any way provide further insights on the triple moment.

5 Negative Probabilities

We now want to see how we can use negative probabilities to approach the inconsistencies from Alice, Bob, and Carlos. The first person to seriously consider using negative probabilities was Dirac in his Bakerian Lectures on the physical interpretation of relativistic quantum mechanics [25]. Ever since, many physicists, most notably Feynman [26], tried to use them, with limited success, to describe physical processes (see [27] or [28] and references therein). The main problem with negative probabilities is its lack of a clear interpretation, which limits its use as a purely computational tool. It is the goal of this section to show that, at least in the context of a simple example, negative probabilities can provide useful normative information.

Before we discuss the example, let us introduce negative probabilities in a more formal way[1]. Let us propose the following modifications to Kolmogorov's axioms.

Definition 1. *Let Ω be a finite set, $\mathcal{F}$ an algebra over Ω, p and p' real-valued functions, $p : \mathcal{F} \to \mathbb{R}$, $p' : \mathcal{F} \to \mathbb{R}$, and $M^- = \sum_{\omega_i \in \Omega} |p(\{\omega_i\})|$. Then $(\Omega, \mathcal{F}, p)$ is a negative probability space if and only if:*

$$A.\ \forall p' \left(M^- \leq \sum_{\omega_i \in \Omega} |p'(\{\omega_i\})| \right)$$

$$B.\ \sum_{\omega_i \in \Omega} p(\{\omega_i\}) = 1$$

$$C.\ p(\{\omega_i, \omega_j\}) = p(\{\omega_i\}) + p(\{\omega_j\}), \quad i \neq j.$$

Remark 1. If it is possible to define a proper joint probability distribution, then $M^- = 1$, and A-C are equivalent to Kolmogorov's axioms.

[1] We limit our discussion to finite spaces.

Going back to our example, we have the following equations for the atoms.

$$p_{xyz} + p_{\overline{x}yz} + p_{x\overline{y}z} + p_{xy\overline{z}} + p_{x\overline{yz}} + p_{\overline{x}y\overline{z}} + p_{\overline{xy}z} + p_{\overline{xyz}} = 1, \tag{7}$$

$$p_{xyz} + p_{\overline{x}yz} + p_{x\overline{y}z} + p_{xy\overline{z}} - p_{x\overline{yz}} - p_{\overline{x}y\overline{z}} - p_{\overline{xy}z} - p_{\overline{xyz}} = 0, \tag{8}$$

$$p_{xyz} + p_{\overline{x}yz} - p_{x\overline{y}z} + p_{xy\overline{z}} - p_{x\overline{yz}} + p_{\overline{x}y\overline{z}} - p_{\overline{xy}z} - p_{\overline{xyz}} = 0, \tag{9}$$

$$p_{xyz} + p_{\overline{x}yz} + p_{x\overline{y}z} - p_{xy\overline{z}} - p_{x\overline{yz}} - p_{\overline{x}y\overline{z}} + p_{\overline{xy}z} - p_{\overline{xyz}} = 0, \tag{10}$$

$$p_{xyz} - p_{\overline{x}yz} - p_{x\overline{y}z} + p_{xy\overline{z}} - p_{x\overline{yz}} - p_{\overline{x}y\overline{z}} + p_{\overline{xy}z} + p_{\overline{xyz}} = 0, \tag{11}$$

$$p_{xyz} - p_{\overline{x}yz} + p_{x\overline{y}z} - p_{xy\overline{z}} - p_{x\overline{yz}} + p_{\overline{x}y\overline{z}} - p_{\overline{xy}z} + p_{\overline{xyz}} = -\frac{1}{2}, \tag{12}$$

$$p_{xyz} + p_{\overline{x}yz} - p_{x\overline{y}z} - p_{xy\overline{z}} + p_{x\overline{yz}} - p_{\overline{x}y\overline{z}} - p_{\overline{xy}z} + p_{\overline{xyz}} = -1, \tag{13}$$

where (7) comes from the fact that all probabilities must sum to one, (8)–(10) from the zero expectations for **X**, **Y**, and **Z**, and (11)–(13) from the pairwise correlations. Of course, this problem is underdetermined, as we have seven equations and eight unknowns (we don't know the unobserved triple moment). A general solution to (7)–(10) is

$$p_{xyz} = -p_{\overline{x}yz} = -\frac{1}{8} - \delta, \tag{14}$$

$$p_{x\overline{y}z} = p_{\overline{x}y\overline{z}} = \frac{3}{16}, \tag{15}$$

$$p_{xy\overline{z}} = p_{\overline{xy}z} = \frac{5}{16}, \tag{16}$$

$$p_{x\overline{yz}} = -p_{\overline{xyz}} = -\delta, \tag{17}$$

where δ is a real number. From (14)–(17) it follows that, for any δ, some probabilities are negative. First, we notice that we can use the joint probability distribution to compute the expectation of the triple moment, which is $E(\mathbf{XYZ}) = -\frac{1}{4} - 4\delta$. Since $-1 \leq E(\mathbf{XYZ}) \leq -1$, it follows that $-1\frac{1}{4} \leq \delta \leq \frac{3}{4}$. Of course, δ is not determined by the lower moments, as we should expect, but axiom A requires M^- to be minimized. So, to minimize M^-, we focus only on the terms that contribute to it: the negative ones. To do so, let us split the problem into several different sections. Let us start with $\delta \geq 0$, which gives $M^-_{\delta\geq 0} = -\frac{1}{8} - 2\delta$, having a minimum of $-\frac{1}{8}$ when $\delta = 0$. For $-1/8 \leq \delta < 0$, $M^-_{-\frac{1}{8}\leq\delta<0} = \delta - \frac{1}{8} + \delta = -\frac{1}{8}$, which is a constant value. Finally, for $\delta < -1/8$, the mass for the negative terms is given by $M^-_{\delta<-\frac{1}{8}} = \frac{1}{8} - 2\delta$. Therefore, negative mass is minimized when δ is in the following range

$$-\frac{1}{8} \leq \delta \leq 0.$$

Now, going back to the triple correlation, we see that by imposing a minimization of the negative mass we restrict its values to the following range:

$$-\frac{1}{4} \leq E\left(\mathbf{XYZ}\right) \leq \frac{1}{2}.$$

But Eqs. (7)–(13) and the fact that the random variables are ±1-valued allow any correlation between −1 and 1, and we see that the minimization of the negative mass offers further constraints to a decision maker.

Before we proceed, we need to address the meaning of negative probabilities, as well as the minimization of M^{-}. We saw from Remark 1 that when M^{-} is zero we obtain a standard probability measure. Thus, the value of M^{-} is a measure of how far p is from a proper joint probability distribution, and minimizing it is equivalent to asking p to be as close as possible to a proper joint, while at the same time keeping the marginals. This point in itself should be sufficient to suggest some normative use to negative probabilities: a negative probability (with M^{-} minimized) gives us the most rational bet we can make given inconsistent information. But the question remains as to the meaning of negative probabilities.

To give them meaning, let us redefine the probabilities from p to p^{*} such that $p^{*}\left(\{\omega_i\}\right) = 0$ when $p\left(\{\omega_i\}\right) \leq 0$. It follows from this redefinition that $\sum_{\omega_i \in \Omega} p^{*}\left(\{\omega_i\}\right) \geq 1$. This newly defined probability would not violate Kolmogorov's nonnegativity axiom, but instead would violate B above. The p^{*}'s corresponds to de Finetti's upper probability measures, and axiom A above guarantees that such upper is as close to a proper distribution as possible. Thus, according to a subjective interpretation, the negative probability atoms correspond to impossible events, and the positive ones to an upper probability measure consistent with the marginals. Once again, the triple moment corresponds to our best bet.

6 Conclusions

The quantum mechanical formalism has been successful in the social sciences. However, one of the questions we raised elsewhere was whether some minimalist versions of the quantum formalism which do not include a full version of Hilbert spaces and observables could be relevant [19]. In this paper we adapted the example modeled with neural oscillators in [19] to a different case where each random variable could be interpreted as outcomes of a market, and where the inconsistencies between the correlations could be interpreted as inconsistencies between experts' beliefs. Such inconsistencies result in the impossibility to define a standard probability measure that allows a decision-maker to select an expectation for the triple moment. The computation of the triple moment from the inconsistent information was done in this paper using three different approaches: Bayesian, quantum-like, and negative probabilities.

With the Bayesian approach, we showed that not only does it rely on a model of the experts (the likelihood function), but also that no new information is gained from it, as the triple moment from the prior is not changed by the application of Bayes's rules. Therefore, the Bayesian approach had nothing to say about the triple moment.

Similar to the Bayesian, the quantum approach also had nothing to say about the triple moment, as the arbitrariness of choices for quantum superpositions (without any additional constraints) results in all values of triple moments being possible. In fact, the quantum approach above could be similarly implemented using a contextual theory. For instance, Dzhafarov [29] proposes the use of an extended probability space where different random variables (say, $\mathbf{X}_z$ and $\mathbf{X}_y$) are used, and where we then ask how similar they are to each other (for instance, what is the value of $P(\mathbf{X}_z \neq \mathbf{X}_y)$). However, as with the quantum case, the meaning given to $P(\mathbf{X} = 1)$ in our example does not fit with this model, as it corresponds to the expectation of an increase in the stock value of company X in the future, and the X that Alice is talking about is exactly the same one for Bob and Carlos, as it corresponds to the increase in the objective value (in the future) of a stock in the same company. Furthermore, as expected due to its similar features, this approach has the same problem as the quantum one in terms of dealing with the triple moment, but it has the advantage of making it clearer what the problem is: the triple moment does not exist because we have nine random variables instead of three, as we have three different contexts.

The negative probability approach, on the other hand, led to a nontrivial constraint to the possible values of the triple moment. When used as a computational tool, a joint probability distribution, and with it the triple moment, could be obtained. Together with the minimization of the negative mass M^-, this joint leads to a nontrivial range of possible values for the triple moment. Given the interpretation of negative probabilities with respect to uppers, it follows that this range is our best guess as to where the values of the triple moment should lie, given our inconsistent information. Thus, negative probabilities provide the decision maker with some normative information that is unavailable in either the Bayesian or the quantum-like approaches.

Acknowledgments. Many of the details about negative probabilities were developed in collaboration with Patrick Suppes, Gary Oas, and Claudio Carvalhaes on the context of a seminar held at Stanford University in Spring 2011. I am indebted to them as well as the seminar participants for fruitful discussions. I also like to thank Tania Magdinier, Niklas Damiris, Newton da Costa, and the anonymous referees for comments and suggestions.

References

1. Khrennikov, A.: Ubiquitous Quantum Structure. Springer, Heidelberg (2010)
2. Haven, E., Khrennikov, A.: Quantum Social Science. Cambridge University Press, Cambridge (2013)
3. Birkhoff, G., von Neumann, J.: The logic of quantum mechanics. Ann. Math. **37**(4), 823–843 (1936)

4. von Weizsäcker, C.F.: Probability and quantum mechanics. Br. J. Philos. Sci. **24**(4), 321–337 (1973)
5. Bell, J.S.: On the einstein-podolsky-rosen paradox. Physics **1**(3), 195–200 (1964)
6. Bell, J.S.: On the problem of hidden variables in quantum mechanics. Rev. Mod. Phys. **38**(3), 447–452 (1966)
7. Kochen, S., Specker, E.P.: The problem of hidden variables in quantum mechanics. In: Hooker, C.A. (ed.) The Logico-Algebraic Approach to Quantum Mechanics, pp. 293–328. D. Reidel Publishing Co., Dordrecht (1975)
8. Greenberger, D.M., Horne, M.A., Zeilinger, A.: Going beyond bell's theorem. In: Kafatos, M. (ed.) Bell's Theorem, Quantum Theory, and Conceptions of the Universe. Fundamental Theories of Physics, vol. 37, pp. 69–72. Kluwer Academic, Dordrecht (1989)
9. de Barros, J.A., Suppes, P.: Inequalities for dealing with detector inefficiencies in greenberger-horne-zeilinger type experiments. Phys. Rev. Lett. **84**, 793–797 (2000)
10. Holik, F., Plastino, A., Sánz, M.: A discussion on the origin of quantum probabilities. arXiv:1211.4952, November 2012
11. de Barros, J.A., Suppes, P.: Probabilistic results for six detectors in a three-particle GHZ experiment. In: Bricmont, J., Dürr, D., Galavotti, M.C., Ghirardi, G., Petruccione, F., Zanghi, N. (eds.) Chance in Physics. Lectures Notes in Physics, vol. 574, pp. 213–223. Springer, Berlin (2001)
12. de Barros, J.A., Suppes, P.: Probabilistic inequalities and upper probabilities in quantum mechanical entanglement. Manuscrito **33**, 55–71 (2010)
13. Suppes, P., Zanotti, M.: When are probabilistic explanations possible? Synthese **48**(2), 191–199 (1981)
14. Suppes, P., de Barros, J.A., Oas, G.: A collection of probabilistic hidden-variable theorems and counterexamples. In: Pratesi, R., Ronchi, L. (eds.) Waves, Information, and Foundations of Physics: A Tribute to Giuliano Toraldo di Francia on his 80th Birthday. Italian Physical Society, Italy (1996)
15. Dieks, D.: Communication by EPR devices. Phys. Lett. A **92**(6), 271–272 (1982)
16. de Barros, J.A., Suppes, P.: Quantum mechanics, interference, and the brain. J. Math. Psychol. **53**(5), 306–313 (2009)
17. Suppes, P., de Barros, J.A., Oas, G.: Phase-oscillator computations as neural models of stimulus–response conditioning and response selection. J. Math. Psychol. **56**(2), 95–117 (2012)
18. de Barros, J.A.: Quantum-like model of behavioral response computation using neural oscillators. Biosystems **110**(3), 171–182 (2012)
19. de Barros, J.A.: Joint probabilities and quantum cognition. In: Khrennikov, A., Migdall, A.L., Polyakov, S., Atmanspacher, H. (eds.) AIP Conference Proceedings, vol. 1508, pp. 98–107. American Institute of Physics, Sweden (2012)
20. Jeffrey, R.: Probability and the Art of Judgment. Cambridge University Press, Cambridge (1992)
21. da Costa, N.C.A.: On the theory of inconsistent formal systems. Notre Dame J. Formal Logic **15**(4), 497–510 (1974)
22. Genest, C., Zidek, J.V.: Combining probability distributions: a critique and an annotated bibliography. Stat. Sci. **1**(1), 114–135 (1986). (Mathematical Reviews number (MathSciNet): MR833278)
23. Morris, P.A.: Decision analysis expert use. Manage. Sci. **20**(9), 1233–1241 (1974)
24. Morris, P.A.: Combining expert judgments: a Bayesian approach. Manage. Sci. **23**(7), 679–693 (1977)
25. Dirac, P.A.M.: Bakerian lecture. The physical interpretation of quantum mechanics. Proc. R. Soc. Lond. B **A180**, 1–40 (1942)

26. Feynman, R.P.: Negative probability. In: Hiley, B.J., Peat, F.D. (eds.) Quantum Implications: Essays in Honour of David Bohm, pp. 235–248. Routledge, London (1987)
27. Mückenheim, G.: A review of extended probabilities. Phys. Rep. **133**(6), 337–401 (1986)
28. Khrennikov, A.: Interpretations of Probability. Walter de Gruyter, New York (2009)
29. Dzhafarov, E.N.: Selective influence through conditional independence. Psychometrika **68**(1), 7–25 (2003)

A Quantum Framework for 'Sour Grapes' in Cognitive Dissonance

Polina Khrennikova(✉)

School of Management, University of Leicester,
University Road, Leicester LE1 7RH, England, UK
pkl98@leicester.ac.uk

Abstract. This paper elaborates on a well-known and widespread bias -'cognitive dissonance'. The bias occurs when a person has conflicting cognitions framed by his/her values, beliefs etc. In such complex situations the individual choices and actions become emotionally tinted and thus inconsistent with the postulates of rational homo economicus. We evidence that classical probabilistic updating of information does not work correctly: hidden factors and motivations come into play to balance the conflicting cognitions and restore mental harmony. To support this inference we present a scheme of a 'gedanken experiment' in combination with known statistical data from a real experiment, the forbidden toy paradigm. Our findings show that the phenomenon of cognitive dissonance is a source of probabilistic non-classicality directly violating Bayes formula for conditional probability and so the law of total probability. Furthermore, we aim to show with the help of the quantum framework that the quantum probability formula and Hilbert space state representation of observables can well account for the 'incorrect behavior' among participants.

Keywords: Cognitive dissonance · Context · Bayes theorem · Law of total probability · Quantum like modeling · Quantum probability · Born's rule

1 Introduction

The 'cognitive dissonance' phenomenon was firstly discovered and brought to the public by a famous American psychologist Leon Festinger in his book 'A theory of cognitive dissonance' [9]. This theory had a wide resonance in society and was believed to be one of the most influential and fruitful discoveries in social and experimental psychology [3, 24]. Festinger found out through his numerous experiments that in many situations an individual has conflicting or 'dissonant' cognitions that do not fit together, e.g.:

> one of these is the knowledge that he believes "X," the other the knowledge that he has publicly stated that he believes "not X." If no factors other than his private opinion are considered it would follow, at least in our culture, that if he believes "X" he would publicly

---

Sour grapes: Is a metaphorical illustration of the cognitive dissonance bias, from the fable "The fox and the grapes" by Aesop [17]. The fox cannot reach the grapes on the tree and convinces himself that he doesn't want them after all because they are sour.

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 270–280, 2014.
DOI: 10.1007/978-3-642-54943-4_24,

state “X.” Hence, his cognition of his private belief is dissonant with his cognition concerning his actual public statement. [10], p. 203.

This cognitive discrepancy results in a tendency to changing behavior, through inner motivation, self-justification, and by selectively searching for new information or attitudes.

At the same time, modern economic theory of the 20th Century postulates that humans are by their nature rational. They carefully process all obtained information and strive to minimize losses and maximize their own gain in any given situation. Coming back to the subject of cognitive dissonance we can notice that people mainly act on an emotional basis, trying to minimize the mental discomfort or pain, reinstate the self-esteem and even search to shape their identity[1] through their behavior. In our article we will focus on cognitive dissonance occurring mainly through incorrect Bayesian updating of new information and the violation of the classical probabilistic framework. Individuals in this situation are processing information incompletely, making excuses and lowering the significance of the dissonant element. Firstly, this phenomenon will be explored in more depth, by illustrating it in a framework of a so called ‘gedanken experiment’ to draw a parallel with quantum physics.

As background data analysis, we choose an illustrative forbidden toy experiment [2][2]. For similar experiments see the classical counter–attitudinal experiment on students asked to perform and then promote various boring tasks [10]; for other cognitive dissonance experiments see [1, 6]. After presenting the experimental data we will propose a solution of the cognitive dissonance problem with the help of the quantum framework. The motivation for introducing quantum like models (quantum calculus applied to domains outside quantum physics e.g. decision making) is that they have a notable prediction accuracy that was successfully shown in similar types of ‘irrational’ behavior experiments, that is: type indeterminacy in self-perception theory [18, 19]; inverse fallacy [4], the disjunction effect and conjunction effect [7, 11, 14, 21]. The above mentioned works are exploring the mathematical formalism of quantum mechanics in cognitive psychology and economics. They emphasize the role of the violation of classical probability theory, in particular its core formalism the Law of Total Probability.

2 Bayes Rule and the Law of Total Probability

The notion of Bayesian probabilistic reasoning was applied as the essential ingredient of decision theory during the 20th century in economics as well as in the behavioral sciences [8]. Bayesian probability according to [20] is elucidating a particular concept of rational choice. Truly, Pierre Laplace assumed that all probabilities of our judgment depend on our knowledge and ignorance, and

[1] Self-perception theory is an alternative explanation for the biased behavior, based on the idea that we mainly act to observe our actions, hereby forming our identity. The term was first introduced by Bem, see [5].

[2] We do not elaborate on the ethical side of the experiment where the word ‘punishment’ could possibly cause anxiety and stress of small children. The data is chosen purely for illustrative purpose.

the completion of knowledge in the limit simply eliminates them' transmitting our estimation to a state of an absolute confidence. [23], p. 165.

In Kolmogorov's probabilistic framework [16], the Bayes formula is used as the definition of conditional probability. Say, two mutually exclusive events A and A− are in the same sample space N1, where for instance P(A) ≠ 0. In a decision making situation those probabilities depict prior beliefs of a decision maker [4]. As additional information is obtained, a decision maker updates his/her beliefs with a new probability of A respective A−, given that events B or B− occur (corresponding to another sample space N2). Such an updating of beliefs is defined by Bayes formula (2) and (3). It can be expressed with the help of the formula of total probability (FTP) (1):

$$\boldsymbol{P}(\boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{B}|\boldsymbol{A})\boldsymbol{P}(\boldsymbol{A}) + \boldsymbol{P}(\boldsymbol{B}|\boldsymbol{A}-)\boldsymbol{P}(\boldsymbol{A}-) \tag{1}$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{\boldsymbol{P}(\boldsymbol{B}|\boldsymbol{A})\boldsymbol{P}(\boldsymbol{A}) + \boldsymbol{P}(\boldsymbol{B}|\boldsymbol{A}-)\boldsymbol{P}(\boldsymbol{A}-)} \tag{2}$$

Respectively, the conditional probability of A− will be updated as follows:

$$P(A-|B) = \frac{P(B|A-)P(A-)}{P(B)} = \frac{P(B|A-)P(A-)}{\boldsymbol{P}(\boldsymbol{B}|\boldsymbol{A})\boldsymbol{P}(\boldsymbol{A}) + \boldsymbol{P}(\boldsymbol{B}|\boldsymbol{A}-)\boldsymbol{P}(\boldsymbol{A}-)} \tag{3}$$

3 Gedanken Experiment

For the purpose of this article we first present a simulation model. We depict it as following: 2 groups of children are left one by one in a room with a variety of toys, including a so called favorite toy.

Part A: The experimenter tells one group of children that there will be a severe punishment (S) if they play with this particular toy and the experimenter tells the other group that there will be a milder punishment if they play with it (M). Thus, the children are placed in two decision making contexts under the condition S and M. After the experimenter leaves the room for say 10–15 min and covertly observes the children in the room. We note that: P(M) = P(S) = ½, i.e. these are the probabilities of "realizations of contexts" in our model experiment.

Within the line of rational reasoning, those children that experienced a danger of severe punishment (a greater risk to be punished) would be less inclined to play (G+) with the toy, hence, the following "inequality of rationality" would hold[3]:

$$P(G+|M) > P(G+|S) \tag{4}$$

[3] We make an important note that in the real forbidden toy experiment P(G+|S) = 0 and P(G+|M) = 0. This is a disadvantage for our theoretical study, since "nobody playing'' result implies zero/one probabilities, which are not very convenient for quantum-like analysis. For that reason the probabilities in this part are 'theoretical' to illustrate the principle of Bayesian updating scheme. In further works we consider to perform analogous studies for statistical data with nonzero probabilities.

Part B: the experimenter returns to the room and tells the children that they can freely play with any toy, including the forbidden one. Next, the children who decided to play with the favorite toy in the part A are asked whether they were in M or S contexts. The respective probabilities for (M|G+) and (S|G+) contexts are collected.[4] With the help of Bayes formula, we express the children's conditional preference probabilities structure:

$$P(M|G+) = \frac{P(G + |M)P(M)}{P(G+)} \tag{5}$$

$$P(S|G+) = \frac{P(G + |S)P(S)}{P(G+)} \tag{6}$$

We can easily recognize that the denominator P(G+) is the same for both equations. As it is a favorite toy, every child wants to play with it in the initial context, thus P(G+) = 1 and we also know that P(M) = P(S). Accordingly, we can 'switch' the conditional probabilities, and in line with the Bayesian analysis the "inequality of rationality" (4) would predict that following "contextual behavior inequality" would hold as well:

$$P(M|G+) > P(S|G+) \tag{7}$$

However, in case of cognitive dissonance, the group in the Severe prohibitory context is more likely to play with the desired toy than the group with the Mild context, where (4) and (7) are violated. Cognitive dissonance causes a mismatching of the theoretical Bayesian updating with the actual probability updating procedure. This gives an indication of an impossibility to use the apparatus of classical probability theory to predict unknown probabilities on the basis of experimental results. Our analysis motivates us to use non-classical theories of probabilities, in particular quantum probability. We will support our inference with data from the actual forbidden toy experiment.

4 Experimental Data

This experiment [2] illustrates the cognitive dissonance exhibition, in a slightly different and multipart context compared to the model experiment. Cognitive dissonance is measured indirectly via the change of the toy's attractiveness among the participants. The authors aim to support a hypothesis that cognitive dissonance exists and can be enhanced or reduced in this case by the level of prohibition. They advocate:

[4] The same data can be collected by taking the record of children's distribution between experimental contexts. By employing the direct question scheme we bring a closer analogy with the measurement in quantum physics. In our "gedanken experiment'' in part B the experimenter searches to find out the contextual structure of the part A. To make our experiment more real, we could proceed with two different experimenters: one person that is setting up part A of the experiment and the second experimenter that comes into the room in part B (after the children were allowed to play) and gets to know about the contextual structure of part A from the children's behavior.

'The greater the threat of punishment the less the dissonance – since a severe threat is consonant with ceasing to perform an action' [2], p. 584.

At the same time when the danger of punishment is mild, then a person would seek for some inner justifications to abstain from performing it. We depict this inconsistency with Bayes' conditional frequencies obtained from the experimental data, see (Table 1).

Table 1 Experimental data

P(L+| M)	0.636
P(L−| M)	0.364
P(L+|S)	1
P(L−|S)	0

4.1 Method

The experiment was run on 22 preschool children (11 boys and 11 girls ranging in age from 3.8 to 4.6 years) that took part in both experimental conditions (Mild and Severe) with a time interval of 45 days that according to the authors enabled them to restore the initial attitude towards the toys.

- Preparation of the participants:

The experimenter led each subject into the experimental room closed the door and showed the subject the toys. He demonstrated how each toy worked and allowed the subject to play with it briefly before moving on the next one. After the subject was familiar with all the toys the experimenter suggested a "question game, following which the subject would have chance to play the toys (again)" [2], p. 585.

The "question game" involved a ranking of the toys two by two, until a choice between 10 pairs of toys was made. Then a ranking list for each child was accomplished. After the ranking procedure was completed the experimenter took a N2[5] (second ranked toy) and placed it on a low board in the center of the room and applied a condition of either Mild or Severe punishment.[6]

- Observation of the participants during the experiment: After the experimenter left the room the children were observed for 10 min, through a special one-way mirror. No one of the subjects played with the toys neither in Mild nor in Severe threat context.

[5] We remark that the observers choose for their purposes not the absolute favorite but the second ranked toy (N2). The authors explain their choice by giving the children space to increase the desirability of the toy. The obtained frequencies could be very different if the children had to abstain from playing with the most favorite toy e.g. they would still play with it despite the threat condition.

[6] Detailed depiction of the experimental conditions and experimenter's phrases can be found in the original text [2].

After the observation the experimenter came back into the room and allowed each child to play briefly with all the toys including the N2 toy. Next the question game was played again to establish the new ranking of the toys. Finally the experimenter played with every participant for a short while. As result new rankings of children's desirability for the N2 toy were elicited.

The results reveal an increase of attractiveness of the toy after the threat in the Severe condition and a decrease in the Mild condition. The decrease of the toy's desirability in Mild condition is consistent with the authors' conjecture that children in such context experience cognitive dissonance: the cognition that the toy is desirable is dissonant with the cognition that the child shouldn't play with it. For that reason the children are lowering the toy's attractiveness to restore their mental harmony. At the same time the Severe condition gives by itself a good reason for not playing with the toy without affecting the desirability of the toy per see.

At the same time it was not clear from the results, why the effect of toy's increased desirability took place among some children placed in the Severe context. The authors asked themselves whether this increase was a function of precisely the strong level of prohibition or other factors. For that reason an additional test was performed namely, 11 participants (selected randomly) ran through an analogous experiment with a sole control condition 'No Threat' to establish a baseline. The experimenter simply took the toy N2 with him when leaving the room, without announcing anything to the children. The results showed that about the same number of participants increased their liking for the toy as in the Severe condition, implying that the increased desirability was not a consequence of a particular condition.[7]

4.2 Analysis

We will analyze the data from the experiment to see if the occurrence of cognitive dissonance caused a violation of classical probability. We depict: P(M) = P(S) = ½, since each group consisted of the same number of participants.[8] As mentioned, in the threat condition none of the participants played with the toy (8), but we witness that the reasons behind not- playing differ.

$$P(G+|S) = P(G+|M) = 0 \tag{8}$$

[7] Other possible factors behind the increased desirability highlighted in the paper: the children paid more attention to the N2 toy, because the experimenter enhanced its value by the impossibility to play with it. Also the participants were likely to become bored playing with other toys thus favoring the key toy more.

[8] The ½ frequency is a common type of group division within experiments e.g., Hawaii experiment, Gambling experiment by Tversky and Shafir [22]. The participants are either equally divided in two groups or the same number of participants is firstly placed in one group and after a time period in the second group. It enables to compare face to face the change of preferences. In the toy experiment we directly see that liking for the toy decreases with ca. 36 % for those kids who are in the M group. Another group division would give a similar result.

To depict updating of the toy's attractiveness, we apply following symbols: L+ for not decreasing attractiveness of the toy[9] and L− for decreasing attractiveness of the toy.

We insert our data in (1) and check whether the FTP holds for this experiment. We know that P(L+) = 1, as all children like the toy in the beginning of the experiment.

$$\mathrm{P}(L+) = \mathrm{P}(S)\mathrm{P}(L+|S) + P(M)P(L+|M) = 0.818 \tag{9}$$

We observe that the FTP is violated (9). This indicates that we cannot predict the unconditional probability $P(L+)$ with the aid of conditional probabilities $P(L+|S)$ and $P(L+|M)$ as one can do in the classical probabilistic framework. The situation is very similar to experiments on violation of rationality in games of the Prisoner Dilemma type and violations of the Savage sure thing principle where a so called disjunction effect takes place; see [21] and [14].

We recap that FTP is derived from two fundamental principles of classical probability theory, the Bayes formula for conditional probability and the additivity principle of probabilities. By finding a violation of the FTP we see that the principle of additivity is definitely violated, since the experimentally obtained probabilities don't sum up to 1 in (9). However, we cannot prove whether the Bayes formula is also violated as we could demonstrate in the gedanken experiment. In the future the illustrated gedanken experiment could be performed in a real setting to confirm its findings.

5 Quantum Paradigm

The motivation for applying the formalism of quantum physics to explain the above discussed biased behavior has a number of justifications. From a probabilistic viewpoint quantum physics is about the generalization of the Bayesian formula of conditional probability. With that in mind [13] suggests that 'Quantum Bayesian analysis' is based on an analogue of quantum interference effect which cannot be described by FTP of Kolmogorov's probability model. As is known, in quantum physics incompatible observational contexts lead to complementarity of quantum measurements. In our case we can regard the cognitive dissonance phenomenon as an interference of children's cognitions that are incompatible. As the authors of the toy experiment state:

> When an individual experiences dissonance he attempts to reduce it by changing one or both of his cognitions. [2], p. 584.

That for example cognition and emotion should also be treated as complementary is discussed in [7]. In mathematical language it means that a child who simultaneously likes a toy but is not allowed to play with it has different probability amplitudes for these events and they interfere with each other.

[9] Contains those participants, who ranked the toy higher or the same after the threat context. Since the increased desirability of the toy is not a consequence of a particular condition, we treat it similarly as the 'not-decreasing desirability'.

Some authors propose that we can go one step further and build a special model for cognitive dissonance behavior, where two groups of subjects can be distinguished: those who are comfortable with cognitive dissonance or those who want to change it.

> The agents that are comfortable with dissonance will likely be able to maintain attitudes that do not conform to their actions while those who prefer a consistent cognitive state will experience a significant swing in attitude as a result of actions that they choose to take. [15], p. 8.

The data in our example shows that many children decided to take an action and therefore experienced a swing in attitude towards the N2 toy. The reduction of incompatibilities of children's cognitions to 'like the toy' and 'not to be allowed to play with it' was reduced through resolution from superposition of (L + |M) and (L − |M) To depict it we apply the quantum probability formula (10) as an extension of the traditional probability equation with the help of the so called 'interference term'.

$$P(B) = P(B|A)P(A) + P(B|A-)P(A-) + 2cos\theta\sqrt{P(A)P(B|A)P(A-)P(B|A-)} \tag{10}$$

Instead of classical updating of P(A|B) we use the state transition from A to B indicating it as a complex number ‹A|B› in Hilbert space. Its squared absolute value gives us the final choice probabilities. The amplitudes of choice probabilities can exhibit positive or negative interference. When the interference is zero we can apply the classical Bayes formula. The right hand-side probability is calculated as a quantum probability with respect to the state: the probability amplitude, represented through vector Ψ in the complex Hilbert space. This is the state of the group of children after the preparatory stage of the experiment (initial state).

For our experimental data, the interference extension of FTP has the form:

$$P(L+) = P(M)P(L+|M) + P(S)P(L+|S) + 2cos\theta\sqrt{P(M)P(L+|M)P(S)P(L+|S)} \tag{11}$$

We obtain cos $\theta = 0.228$ and angle $\theta = 1.34$ radian. Thus, our decision probabilities exhibit positive interference. The positive interference of probability amplitudes could possibly explain an increased desirability of the toy through the interference of probability amplitudes for the severe punishment condition and the liking of the toy.

We represent P(L+) as the probability amplitude of Ψ to check if the Born's rule (determination of quantum probabilities from probability amplitudes) can be applied:

$$P(L+) = |\Psi|^2 \tag{12}$$

$$\Psi = \sqrt{P(M)P(L+|M)} + e^{i\theta}\sqrt{P(S)P(L+|S),} \tag{13}$$

$$e^{i\theta} = cos\theta + isin\theta = 0.228 + 0.974i \tag{14}$$

$$\Psi = \sqrt{0.5 * 0.636} + \left(0.228 + 0.974\text{i}\sqrt{0.5 * 1}\right) = 0.721 + 0.689\text{i} \tag{15}$$

$$|\Psi|^2 = |0.721 + 0.689i|^2 = 0.52 + 0.475 = 0.995 \approx 1 \tag{16}$$

By Born's rule one can reconstruct individual initial mental state Ψ using the matrix of transition probabilities (17) from the experimental data and knowing the context (represented by basis vectors).

$$\begin{pmatrix} 0.636 & 0.364 \\ 1 & 0 \end{pmatrix} \tag{17}$$

At the same time we should note that (17) in this example is not double stochastic thus not showing entirely quantum features. Possible explanation proposed in [12] is that:

- Statistics of the experiment is neither quantum nor classical. The 'Quantumness' is merely present in the phenomenological application of mathematical calculus.
- Observables (in our case the choices L+/L− and events S/M) are not completely captured by the two dimensional Hilbert space and a state space of higher dimension would be needed.

6 Concluding Remarks

We have presented a simulated model, the gedanken experiment thereafter supporting our conjectures with data obtained from a real forbidden toy paradigm. It has been shown that cognitive dissonance is inconsistent with the traditional probabilistic framework. The novelty of our findings from the gedanken experiment is an illustration of a direct violation of the Bayes formula. This gives us an even deeper root of cognitive dissonance's non-classicality than just the demonstrated violation of FTP.

We propose an alternative representation, based on the quantum probability formula to account for the observed non-classicality. In addition, we were able to find complex probability amplitude which can be represented by Born's rule. The possibility to construct such a quantum representation for the probabilities in the cognitive dissonance context stimulates us to think about other applications of quantum probabilistic calculus. For instance, as a second step the dynamical equations from quantum physics could be applied to capture the evolution of a particular decision maker's mental state in the context of cognitive dissonance.

Lastly, we recap that quantum-like approach does not claim that the classical probabilistic description doesn't work at all. For each case of inconsistent behavior (such as cognitive dissonance) there might be found classical models. One could elaborate that there is a similarity with hidden variables descriptions in quantum physics. At the same time it is generally stated in quantum physics that for some situations the classical stochastic variables are in principle impossible to introduce. In general such models can be very complicated and not easy to find. The quantum like description is attractive by providing a more general mathematical framework that could be applied for any case and for any data.

Acknowledgments. I would like to thank the participants of Quantum Interaction 2013 Conference and Prof. Ehtibar Dzhafarov for fruitful discussion and useful comments that contributed to the accomplishment of this paper.

References:

1. Aronson, E., Mills, J.: The effect of severity of initiation on liking for a group. J. Abnorm. Soc. Psychol. **59**(2), 177–181 (1959)
2. Aronson, E., Carlsmith, J.: Effect of the severity of threat on the devaluation of forbidden behavior. J. Abnorm. Soc. Psychol. **66**(6), 584–588 (1963)
3. Aronson, E.: Leon Festinger and the art of audacity. Psychol. Sci. **2**(4), 213–217 (1991)
4. Asano, M., Basieva, I., Khrennikov, A., Ohya, M., Tanaka, Y.: Quantum-like representation of irrational inference. In: Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.) QI 2012. LNCS, vol. 7620, pp. 138–147. Springer, Heidelberg (2012)
5. Bem, D.J.: Self-perception: an alternative interpretation of cognitive dissonance phenomena. Psychol. Rev. **74**, 183–200 (1967)
6. Brehm, J.W: Post decision changes in the desirability of alternatives. J. Abnorm. Soc. Psychol. **52**(3), 384–389 (1956)
7. Busemeyer, J.R, Bruza, P.D.: Quantum Models of Cognition and Decision. Cambridge University Press, Cambridge (2012)
8. De Finetti, B.: Theory of Probability, 2 vols. Wiley, New York (1974)
9. Festinger, L.: A Theory of Cognitive Dissonance. Stanford University Press, Stanford (1957)
10. Festinger, L., Carlsmith, J.: Cognitive consequences of forced compliance. J. Abnorm. Soc. Psychol. **58**(2), 203–210 (1959)
11. Haven, E., Khrennikov, A.: Quantum Social Science. Cambridge University Press, Cambridge (2013)
12. Khrennikov, A.: Ubiquitous Quantum Structure. Springer, Berlin (2010)
13. Khrennikov, A.: Contextual Approach to Quantum Formalism. Springer, New York (2009)
14. Khrennikov, A., Haven, E.: Quantum mechanics and violations of the sure-thing principle: the use of probability interference and other concepts. J. Math. Psychol. **53**, 378–388 (2009)
15. Kitto, K., Boschetti, F., Bruza, P.: The quantum inspired modelling of changing attitudes and self-organising societies. In: Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.) QI 2012. LNCS, vol. 7620, pp. 1–12. Springer, Heidelberg (2012)
16. Kolmogorov, A.N.: Foundations of Probability. Chelsea Publishing Company, New York (1950)
17. Kurke, L: Aesop and the contestation of delphic authority. Dougherty, C., Kurke, L. (eds.) The Cultures Within Ancient Greek Culture: Contact, Conflict, Collaboration. Cambridge University Press, Cambridge (2003)
18. Lambert-Mogilansky, A., Zamir, S., Zwirn, H.: Type indeterminacy- A model of the KT (KahnemanTversky) – man. J. Math. Psychol. **53**, 349–362 (2009)
19. Lambert-Mogiliansky, A., Busemeyer, J.R.: Emergence and instability of individual identity. In: Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.) QI 2012. LNCS, vol. 7620, pp. 102–113. Springer, Heidelberg (2012)
20. Maher, P.: Bayesian probability. Synthese **172**, 119–127 (2010)
21. Pothos, E.M., Busemeyer, J.: A quantum probability explanation for violations of 'rational' decision theory. Proc. R. Soc. B **276**(1165), 2171–2178 (2009)

22. Tversky, A., Shafir, E.: The disjunction effect in choice under uncertainty. Psychol. Sci. **3**(5), 305–309 (1992)
23. von Plato, J.: Creating Modern Probability. Cambridge University Press, Cambridge (1994)
24. Zuckier, H.: Introduction. In: Schachter, S., Gazzaniga, M.S. (eds.) Extending Psychological Frontiers: Selected Works of Leon Festinger, pp. xi–xxiv. Russell Sage Foundation, New York (1989)

A Quantum State Metanalysis of the Dynamic Inconsistency Effect

George Economides(✉)

St Peter's College, University of Oxford, New Inn Hall Street, Oxford OX1 2DL, UK
george.economides@spc.ox.ac.uk

Abstract. The approach of Quantum State analysis for experimental data is presented and used for a metanalysis of the dynamic inconsistency effect experiment of ref. [1]. The results of this experiment have already been demonstrated to be in the field of quantum cognition by ref. [2]. Here the quantitative collective state of the participants is retrieved at each stage and a value-level diagram is presented. This has led to additional insights about the dynamic inconsistency effect, as well as suggestions and a way to compare with future research. Through the Quantum State analysis, we have been able to construct a quantitative wavefunction from psychological experiment data.

Keywords: Disjunction effect · Quantum state · Dynamic inconsistency · Quantum cognition

1 Introduction

Previously, Barkan and Busemeyer [1] studied a specific version of the disjunction effect, the dynamic inconsistency effect. Later Busemeyer, Wang and Trueblood [2] argued that the results are better modeled by a Quantum decision model rather than a Markov one. However, the better fit obtained can be partly attributed to the additional degree of freedom, γ, which is representing the ability "*for changing beliefs to align with actions*" [2]. This was taken as a measure of "*quantum interference*", and as such an indication of validity of a quantum analysis.

Here a different approach is taken, introducing a Quantum State analysis of the problem. The argument of representing the experiment's values as the expectation value of some state is taken from previous literature [4] and is assumed valid here.

$$Observable = \frac{< \psi|\widehat{O}|\psi >}{< \psi|\psi >} \tag{1}$$

The added benefit of the Quantum State approach taken here is that it provides an intuitive way that could potentially allow for predictions and is also expandable.

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 281–285, 2014.
DOI: 10.1007/978-3-642-54943-4_25,

2 The Quantum State Analysis

The main assumption is that the observed (experimental) data depend on the mental state of the people participating on the experiment and that this state may be expressed as a quantum state in Dirac's Bra-(c-)ket notation [3]. Representing the individual mental states as states that have a preference ordering (some more preferable than others) as well as representing interactions as operators acting on these states, is the basis of the Quantum State analysis.

The mental states of the participants could be represented and added in a theoretical Hilbert-like space, thus creating a wavepacket. This can then be treated as a single state. It should be clearly stated that this does not assume any actual interaction of the individual mental states between them, just a close-enough similarity to be described collectively.

In other words, if the mental states of the people are very close, then the experiment may be considered as multiple measurements of a collective state. If the mental states of the individuals are too different to define a collective state, then the experiment is flawed (as the individual cannot be grouped together), and this could be used to explain outliners in quantum cognition studies. Again, both a collective state and an individual state are implicitly assumed in the original papers [2].

In this specific example, two different type of operations are recognised as acting on that state. One is the question ($\widehat{Q}$) and the other is the experience ($\widehat{E}$).

3 Metanalysis of the Disjunction Effect

In [1], 100 participants were asked whether they would take a gamble a second time, depending on either winning or losing the first time but before knowing the outcome of the first gamble. Once they played the first gamble, their actual behaviour (if they played again) was recorded and there was a clear discrepancy depending on whether they lost or won the first gamble from their planned response. People that won the first gamble were notably less keen to gamble again, while people that lost were more keen.

Respecting the statistical nature of measurement in quantum mechanics, only the average probabilities of the predicted and actual behaviour are considered. The possible answer space for the participants is spanned only by "*yes*" and "*no*", and as no "*maybe*" or other feedback was accepted, this can be thought of as a projection of the participants' mental state to a "*yes/no*" answer-space. As the "*yes/no*" states form a orthogonal and complete set, any state in that space can be represented as a linear combination of a function of these answer-states. Following the Born interpretation of the wavefunction, the modulus squared of the coefficients in this linear combination corresponds to the probability density of the state of that outcome.

$$|\psi >= c_{yes}|\psi_{yes} > +c_{no}|\psi_{no} > \tag{2}$$

Now, the $|c_i|^2$ can be identified with the measured probability, assuming the experiment is providing complete enough representation. To use the data of ref. [1] further, they are re-arranged in the following table by simply taking the average, to the decimal figures provided originally.

Average probability after winning	Plan	0.62
Average probability after losing	Plan	0.62
Average probability after winning	Action	0.53
Average probability after losing	Action	0.68

One insight comes instantly by presenting the results in such a way: according to their plans all participants had a 0.62 probability to play again, irrespective of if they won or lost the first time. This is surprising as it seems that the first experience is not expected by the participants to effect their behaviour in any way.

Now, the collective mental state of the participants might be analysed. Before any question is asked, the answer-states "*yes*" and "no" are degenerate. This is an assumption, but implicitly taken in the original research as well-the participants are answering the question, not because they have a tendency to say "*yes*". The question, with only two possible answers, collapses the mental state on the "*yes/no*" space, and causes a splitting between the answer-states. Now, the collective mental state can be written as:

$$|\psi_q >= \sqrt{0.62}|\psi_{yes} > +\sqrt{0.38}|\psi_{no} > \tag{3}$$

The experience (of losing or winning the first gamble) then acts on that state. Now, there are two new states depending on whether the participant lost or won the first gamble:

$$|\psi_w >= \sqrt{0.53}|\psi_{yes} > +\sqrt{0.47}|\psi_{no} > \tag{4}$$

$$|\psi_l >= \sqrt{0.68}|\psi_{yes} > +\sqrt{0.32}|\psi_{no} > \tag{5}$$

What is important to note is that this quantitative description of a mental state wavefunction is directly related to all the data and we have not made any assumptions about the mathematical form of the question, thinking or experience operators, which frankly one cannot yet know about.

4 The Value-Level Diagram

Here we can go further and find the ordering and separation of "*yes/no*" states by writing out and diagonalising the operator matrix for $(\widehat{Q})$ and $(\widehat{E}\widehat{Q})$ operations. Since this is not a physical system, the *energy* is replaced by *value*, representing how much value the participants place on a choice [3], i.e. what is the preference ordering. Obviously no units are assigned, but the splittings are quantitative.

This means that the splitting between the levels (their ratio) is precise, even though that may be multiplied by any constant. For a given operator ($\widehat{O}$):

$$\begin{vmatrix} O_{yy} - V & O_{yn} \\ O_{ny} & O_{nn} - V \end{vmatrix} = \begin{vmatrix} |c_{yes}|^2 - V & 0 \\ 0 & |c_{no}|^2 - V \end{vmatrix}$$

This leads to the following value-level diagram (analogous to an energy-level diagram, but the highest eigenstate is taken as the preferred one):

	After question	After a win	After a loss
$\|\psi_{no}>$	0.26	0.44	0.14
$\|\psi_{yes}>$	0.74	0.56	0.86

This is depicted in Fig. 1. The initial point of larger probability of taking a second gamble after a loss and the lower probability of taking a second gamble after a win is again illustrated. However now each effect is separated and also the final description of the state is available. If more data were available one could test that as a predictor of future behaviour. This is a highly expandable formulation where effects can be added.

An additional point can be made: the win seems to partially negate the initial ordering of states, thus making both choices (play or don't play a second time) almost equally probable. Apart from the risk aversion and risk seeking

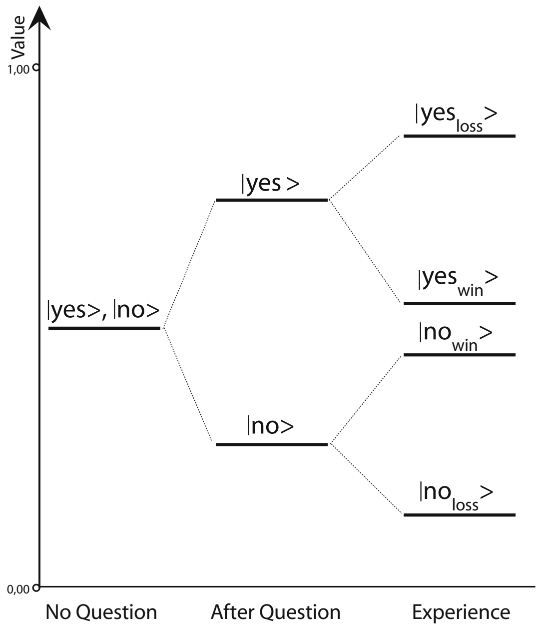


Fig. 1. The value-level diagram at each stage of the experiment.

tendencies, now one could view the results as amplifying the initial preference or partly canceling it.

In terms of the dynamic inconsistency, now instead of a generic γ factor to represent any change of beliefs, this is explicitly modeled as a preference towards one or another state. Therefore, in this formulation another reasoning can be argued: after a loss, there is further commitment to the original Value System [3] ordering, while after a win the choice is more equal. Therefore, the effect of experience in the dynamic inconsistency can be argued to be to amplify the initial tendency in the case of a loss, while to make the next choice more random in case of a win.

In further experiments one can use the Quantum State analysis to break down the effects once more. To illustrate the point further, let us assume that the coeffiecients in the collective state after the initial question were reversed:

$$|\psi_q >= \sqrt{0.38}|\psi_{yes} > +\sqrt{0.62}|\psi_{no} > \tag{6}$$

Now, by repeating the experiment and constructing a new value-level diagram, one can compare with the one presented here to see the effect of experience, i.e. if a win reduces risk-seeking (in which case the coefficient of $|\psi_{no} >$ should be increased) or makes the following choice more random (in which case the coefficient of $|\psi_{no} >$ should be decreased).

A similar analysis is possible for a variety of problems and it is hoped that the Quantum State analysis will prove a useful tool in the field of quantum cognition.

5 Conclusions

The approach of Quantum State analysis was presented and a metanalysis of the data originally presented at [1] was carried out. This led to an new way of representing the dynamic inconsistency and offered additional insights to the effect, as well as a way to compare with future research. Moreover, the quantitative wavefunction at each stage was retrieved and the quantitative diagram of internal states was produced.

References

1. Barkan, R., Busemeyer, J.R.: Modeling dynamic inconsistency with a changing reference point. J. Behav. Decis. Mak. **16**, 235–255 (2003)
2. Busemeyer, J.R., Wang, Z., Trueblood, J.S.: Hierarchical Bayesian estimation of quantum decision model parameters. In: Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.) QI 2012. LNCS, vol. 7620, pp. 80–89. Springer, Heidelberg (2012)
3. Economides, G.: On the nature of the human mind: the cognit space theory. In: Song, D., Melucci, M., Frommholz, I., Zhang, P., Wang, L., Arafat, S. (eds.) QI 2011. LNCS, vol. 7052, pp. 199–204. Springer, Heidelberg (2011)
4. Khrennikov, A.: Ubiquitous Quantum Structure. Springer, Heidelberg (2010)

On Quantum Models for Opinion and Voting Intention Polls

François Dubois[1,2](✉)

[1] Conservatoire National des Arts et Métiers, Paris, France
[2] Association Française de Science des Systèmes, Paris, France
francois.dubois@cnam.fr

Abstract. In this contribution, we construct a connection between two quantum voting models presented previously. We propose to try to determine the result of a vote from associated given opinion polls. We introduce a density operator relative to the family of all candidates to a particular election. From an hypothesis of proportionality between a family of coefficients which characterize the density matrix and the probabilities of vote for all the candidates, we propose a numerical method for the entire determination of the density operator. This approach is a direct consequence of the Perron-Frobenius theorem for irreductible positive matrices. We apply our algorithm to synthetic data and to operational results issued from the French presidential election of April 2012.

Keywords: Density matrix · Perron-Frobenius theorem · French presidential election

1 Introduction

Electoral periods are favorable to opinion polls. We keep in mind that opinion polls are intrinsically complex (see e.g. Gallup [14]) and give an approximates picture of a possible social reality. They are traditionally of two types: popularity polls for various outstanding political personalities and voting intention polls when a list of candidates is known. We have two different informations and to construct a link between them is not an easy task. In particular, the determination of the voting intentions is a quasi intractable problem! Predictions of votes classically use of so-called "voting functions". Voting functions have been developed for the prediction of presidential elections in the United States. They are based on correlations between economical parameters, popularity polls and other technical parameters. We refer to Abramowitz [1], Lewis-Beck [22], Campbell [10] and Lafay [20].

We do not detail here the mathematical difficulties associated with the question of voting when the number of candidates is greater than three [2,6,9]. They conduct to present-day researches like range voting, independently

AMS classification: 65F15 · 81Q99 · 91C99

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 286–295, 2014.
DOI: 10.1007/978-3-642-54943-4_26,

proposed by Balinski and Laraki [3,4] and by Rivest and Smith [24,25]. It is composed by two steps: grading and ranking. In the grading step, all the candidates are evaluated by all the electors. This first step is quite analogous to a popularity investigations and we will merge the two notions in this contribution. The second step of range voting is a majority ranking; it consists of a successive extraction of medians.

In this contribution, we adopt quantum modelling (see e.g. Bitbol *et al.* [5] for an introduction), in the spirit of authors like Khrennikov and Haven [16,17], La Mura and Swiatczak [21] and Zorn and Smith [28] concerning voting processes. Moreover, Wang *et al.* [27] present a quantum model for question order effects found with Gallup polls. The fact of considering quantum modelling induces a specific vision of probabilities. We refer e.g. to the classical treatise on quantum mechanics of Cohen-Tannoudji *et al.* [8], to the so-called contextual objectivity proposed by Grangier [15], or to the elementary introduction proposed by Busemeyer and Trueblood [7] in the context of statistical inference.

This contribution is organized as follows: we recall in Sects. 2 and 3 two quantum models for the vote developed previously [11,12] and a first tentative [13] to connect these two models (Sect. 4). In Sect. 5, we develop the main idea of this paper. We construct a link between opinion polls and voting. This idea is tested numerically in Sects. 6 and 7 for synthetic data and a "real life" election.

2 A Fundamental Elementary Model

In a first tentative [11], we have proposed to introduce an Hilbert space V_Γ formally generated by the candidates $\gamma_j \in \Gamma$. In this space, a candidate γ_j is represented by a unitary vector $\mid \gamma_j >$ and this family of n vectors is supposed to be orthogonal. Then an elector ℓ can be decomposed in the space V_Γ of candidates according to

$$\mid \ell > = \sum_{j=1}^{n} \theta_j \mid \gamma_j > \ . \tag{1}$$

The vector $\mid \ell > \in V_\Gamma$ is supposed to be a unitary vector to fix the ideas. According to Born's rule, the probability for a given elector ℓ to give his voice to the particular candidate γ_j is equal to $\mid \theta_j \mid^2$. The violence of the quantum measure is clearly visible with this example: the opinions of an elector ℓ never coincidate with the program of any candidate. But with a voting system where an elector has to choice only one candidate among n, his social opinion is *reduced* to the one of a particular candidate.

3 A Quantum Model for Range Voting

Our second model [12] is adapted to the grading step of range voting [3,24]. We consider a grid G of m types of opinions as one of the two following ones.

We have $m = 5$ for the first grid (2) and $m = 3$ for the second one (3):

$$++ \;\succ\; + \;\succ\; 0 \;\succ\; - \;\succ\; -- \tag{2}$$

$$+ \;\succ\; 0 \;\succ\; - \,. \tag{3}$$

These ordered grids are typically used for popularity polls. We assume also that a ranking grid like (2) or (3) is a basic tool to represent a social state of the opinion. We introduce a specific grading space $W_{\rm G}$ of political appreciations associated with a grading family G. The space $W_{\rm G}$ is formally generated by the m orthogonal vectors $\,|\, \zeta_i >$ relative to the opinions. Then we suppose that the candidates γ_j are now decomposed by each elector on the basis $\,|\, \zeta_i >$ for $1 \leq i \leq m$:

$$|\, \gamma_j > = \sum_{i=1}^{m} \alpha_j^i \,|\, \zeta_i > , \quad \gamma_j \in \varGamma \,, \quad 1 \leq j \leq n \,. \tag{4}$$

Moreover the vector $\,|\, \gamma_j >$ in (4) is supposed to be by a unitary:

$$\sum_{i=1}^{m} |\, \alpha_j^i \,|^2 \;=\; 1 \,, \quad \gamma_j \in \varGamma \,, \quad 1 \leq j \leq n \,. \tag{5}$$

With this notation, the probability for a given elector to appreciate a candidate γ_j with an opinion ζ_i is simply a consequence of the Born rule. The mean statistical expectation of a given opinion ζ_i for a candidate γ_j is equal to $|\, \alpha_j^i \,|^2$ on one hand and is given by the popularity polls $S_{j\,i}$ on the other hand. Consequently,

$$|\, \alpha_j^i \,|^2 \;=\; S_{j\,i} \,, \quad \gamma_j \in \varGamma \,,\; \zeta_i \in {\rm G} \,, \quad 1 \leq j \leq n \,, \quad 1 \leq i \leq m.$$

4 A First Link Between the Two Previous Models

In [13], we have proposed a first link between the two previous models. We simplify the approach (1) and suppose that there exists some equivalent candidate $|\, \xi > \in V_\varGamma$ such that the voting intention for each particular candidate $\gamma_j \in \varGamma$ is equal to $\,|\, < \xi \,, \gamma_j \, > |^2$. We interpret the relation (4) in the following way: for each candidate $\gamma_j \in \varGamma$, there exists a political decomposition $A \,|\, \gamma_j > \in W_{\rm G}$ in terms of the grid G. By linearity, we construct in this way a linear operator $A : \, V_\varGamma \longrightarrow W_{\rm G}$ between two different Hilbert spaces. Preliminary results have been presented, in the context of the 2012 French presidential election.

5 From Opinion Polls to the Prediction of the Vote

In the space $W_{\rm G}$ of political appreciations described in Sect. 3 of this contribution, the opinion polls allow through the relation (4) to determine some knowledge about each candidate $\gamma_j \in \varGamma$ in the space $W_{\rm G}$. We suppose that each candidate is represented by a unitary vector and the relation (5) still holds. The question is now to evaluate the probability for an arbitrary elector to vote for the various candidates.

We denote by $\Pi_j \equiv |\,\gamma_j><\gamma_j\,|$ the orthogonal projector onto the direction of the state $|\,\gamma_j>$. Then we introduce a density matrix ρ associated to a statistical representation of the voting population:

$$\rho = \sum_{j=1}^{n} \alpha_j \, \Pi_j \equiv \sum_{j=1}^{n} \alpha_j \,|\,\gamma_j><\gamma_j\,| \,. \tag{6}$$

It is classical that $\mathrm{tr}\rho = \sum_{j=1}^{n} \alpha_j$ and if $\alpha_j \geq 0$ for each index j, the auto-adjoint operator ρ is non-negative:

$$<\rho\zeta \,,\, \zeta> \geq 0 \,, \qquad \forall \zeta \in W_{\mathrm{G}} \,.$$

It is then natural to search the coefficients α_j such that

$$\begin{cases} \alpha_j \geq 0 \,, & 1 \leq j \leq n \\ \displaystyle\sum_{j=1}^{n} \alpha_j = 1 \,. \end{cases} \tag{7}$$

In these conditions, the Esperance of election $<\gamma_j>$ of the candidate γ_j is given through the relation

$$<\gamma_j> = \mathrm{tr}\big(\rho \, \Pi_j\big) \,, \quad 1 \leq j \leq n \,. \tag{8}$$

We have the following calculus:

$$\begin{aligned} \mathrm{tr}\big(\rho\,\Pi_j\big) &= \sum_{k=1}^{n} <\zeta_k \,,\, \rho\,\Pi_j\,\zeta_k> = \sum_{k=1}^{n}\sum_{\ell=1}^{n} <\zeta_k \,,\, \alpha_\ell\,\gamma_\ell><\gamma_\ell \,,\, \gamma_j><\gamma_j \,,\, \zeta_k> \\ &= \sum_{\ell=1}^{n} \alpha_\ell <\gamma_\ell \,,\, \gamma_j> \sum_{k=1}^{n} <\zeta_k \,,\, \gamma_\ell><\gamma_j \,,\, \zeta_k> \\ &= \sum_{\ell=1}^{n} \alpha_\ell <\gamma_\ell \,,\, \gamma_j> \sum_{k=1}^{n} <\gamma_j \,,\, \zeta_k><\zeta_k \,,\, \gamma_\ell> \\ &= \sum_{\ell=1}^{n} \alpha_\ell <\gamma_\ell \,,\, \gamma_j><\gamma_j \,,\, \gamma_\ell> = \sum_{\ell=1}^{n} \alpha_\ell \, |<\gamma_\ell \,,\, \gamma_j>|^2 \,. \end{aligned}$$

We introduce the matrix A composed by the squares of the scalar products of the vectors of candidates:

$$A_{j\,\ell} = |<\gamma_j \,,\, \gamma_\ell>|^2 \,, \quad 1 \leq j, \ell \leq n \,. \tag{9}$$

Then the previous calculus establishes that

$$<\gamma_j> = \sum_{\ell=1}^{n} A_{j\,\ell} \, \alpha_\ell \,. \tag{10}$$

It is interesting to imagine a link between the Esperance of election $< \gamma_j >$ of the candidate γ_j and the coefficient α_j of the density matrix introduced in (6). In general they differ. In the following, we focus our attention to the particular case where these two quantities are proportional, *id est*

$$\exists\, \lambda \in \mathbb{C} \,, \quad \forall\, j = 1,\, 2,\, \ldots\,,\, n\,, \quad < \gamma_j >= \lambda\, \alpha_j \,. \tag{11}$$

Because both $< \gamma_j >$ and α_j are positive, the coefficient λ must be a positive number. Moreover, due to (10), the relation (11) express that the non-null vector $\alpha \in \mathbb{R}^n$ composed by the coefficients α_j is an eigenvector of the matrix A. Then, due to the hypothesis (7), we have $\alpha_j \geq 0$ and this eigenvector has non-negative components. If we suppose that the matrix A is irreductible (see e.g. in the book of Meyer [23] or Serre [26]), the Perron-Frobenius theorem states that there exists a **unique** eigenvalue (equal to the spectral radius of the matrix A) such that the corresponding eigenvector has all non-negative components. Moreover, all the components of this eigenvector are strictly positive. In other words, if the matrix A defined in (9) is irreductible and if the hypothesis of proportionality (11) is satisfied, the coefficients α_j of the density matrix are, due to the second relation of (7), completely defined. In the following, we propose to determine the coefficients α_j of the density matrix (6) and satisfying the conditions (7) as proportional to the positive eigenvector of the matrix A defined by (9).

The above model is not completely satisfactory for the following reason. The underlying order associated to the grading family G has not been taken into account. To fix the ideas, we suppose that each grade ν_i is associated to a number σ_i such that

$$\sigma_1 \,>\, \sigma_2 \,>\, \cdots \,>\, \sigma_m \,. \tag{12}$$

We introduce a "popularity operator" P_j associated to the jth candidate γ_j:

$$P_j \,\equiv\, \sum_{i=1}^{m} \sigma_i \,\,|<\gamma_j\,,\, \zeta_i>|^2\, |\,\zeta_i><\zeta_i\,| \,. \tag{13}$$

We can determine without difficulty the mean value of the operator P_j for the density configuration ρ defined in (6):

$$\begin{aligned}
\mathrm{tr}\big(\rho\, P_j\big) &= \sum_{k=1}^{m} \,<\zeta_k\,,\, \rho\, P_j\, \zeta_k> \\
&= \sum_{k=1}^{m} \sum_{\ell=1}^{n} \,<\zeta_k\,,\, \alpha_\ell\, \gamma_\ell><\gamma_\ell\,,\, \sum_{i=1}^{m} \sigma_i \,\,|<\gamma_j\,,\, \zeta_i>|^2\, |\,\zeta_i><\zeta_i\,,\, \zeta_k> \\
&= \sum_{i=1}^{m} \sum_{\ell=1}^{n} \sigma_i\, \alpha_\ell \,\,<\zeta_i\,,\, \gamma_\ell><\gamma_\ell\,,\, \zeta_i> \,\,|<\gamma_j\,,\, \zeta_i>|^2 \\
&= \sum_{i=1}^{m} \sum_{\ell=1}^{n} \sigma_i\, \alpha_\ell\, |<\gamma_\ell\,,\, \zeta_i>|^2\, |<\gamma_j\,,\, \zeta_i>|^2 \,.
\end{aligned}$$

In other words, if we set

$$B_{j\,\ell} \equiv \sum_{i=1}^{m} \sigma_i \, | < \gamma_j \, , \, \zeta_i > |^2 \; | < \gamma_\ell \, , \, \zeta_i > |^2 \; , \tag{14}$$

we have:

$$< P_j > \equiv \operatorname{tr}\big(\rho \, P_j\big) \; = \; \sum_{\ell=1}^{n} B_{j\,\ell} \; \alpha_\ell \, . \tag{15}$$

We use a positive parameter t and search the coefficients α_j in such a way that the mean value of the candidate γ_j with some "upwinding" associated to its popularity is proportional to the above coefficients. In other words, due to (10) and (15), the mean value $< \gamma_j > + t < P_j >$ takes the algebraic form

$$< \gamma_j > \; + t < P_j > = \sum_{\ell=1}^{n} \big(A_{j\,\ell} + t \, B_{j\,\ell} \big) \, \alpha_\ell \, . \tag{16}$$

Under the condition that all the coefficients $A_{j\,\ell} + t \, B_{j\,\ell}$ are positive, *id est* that the parameter t is small enough, we compute the coefficients α_ℓ with the help of the Perron-Frobenius theorem as presented previously.

6 A First Numerical Test Case

Our first model uses synthetic data. We suppose that we have three candidates ($n = 3$) and two ($m = 2$) levels of "political" appreciation. We suppose that

$$\begin{cases} |\gamma_1 > = \; \cos\left(\dfrac{\pi}{6}\right) |\zeta_1 > \; + \sin\left(\dfrac{\pi}{6}\right) |\zeta_2 > \\[2ex] |\gamma_2 > = \; \cos\left(\dfrac{\pi}{4}\right) |\zeta_1 > \; + \sin\left(\dfrac{\pi}{4}\right) |\zeta_2 > \\[2ex] |\gamma_3 > = \; \cos\left(\dfrac{\pi}{3}\right) |\zeta_1 > \; + \sin\left(\dfrac{\pi}{3}\right) |\zeta_2 > \; . \end{cases} \tag{17}$$

With the choice $\sigma_1 = 1$ and $\sigma_2 = 0$ in a way suggested at the relation (12), we can simulate numerically the process presented in the Sect. 5. The results are presented in Fig. 1. When the variable t is increasing, the first candidate has a better score, due to his best results in the grading evaluation (17).

7 Test of the Method with Real Data

We have also used data coming from the "first tour" of the French presidential election of April 2012. Popularity data [18] and result of voting intentions [19] are displayed in Table 1. The names of the principal candidates to the French presidential election are proposed in alphabetic order with the following abbreviations: "Ba" for François Bayrou, "Ho" for François Hollande, "Jo" for Eva Joly,

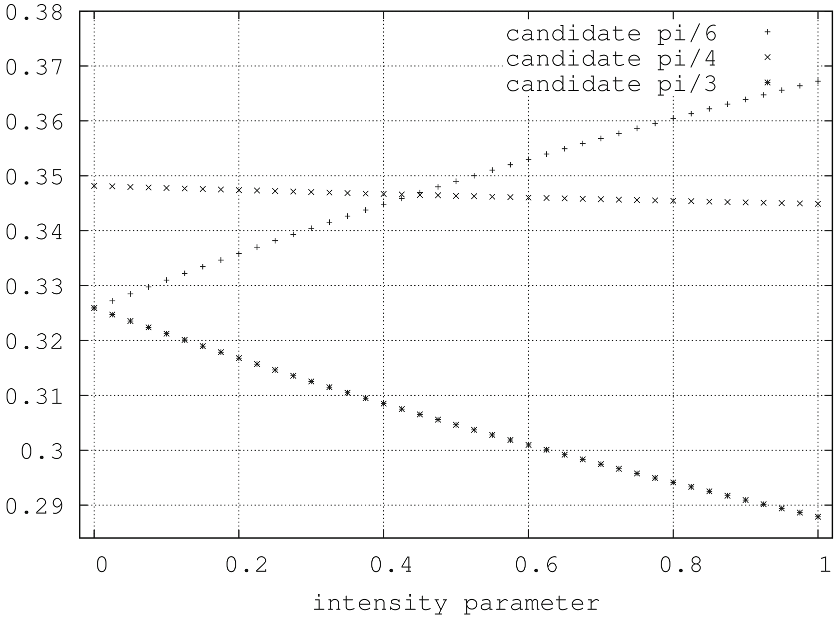


Fig. 1. Result of the vote obtained by a quantum model from the opinion poll, with synthetic data proposed in (17).

"LP" for Marine Le Pen, "Mé" for Jean-Luc Mélanchon and "Sa" for Nicolas Sarkozy. In Table 1, we have also reported the result of the election of 22 April 2012.

This test case corresponds to $n = 6$ and $m = 3$. The numerical data relative to the relation (12) are chosen such that $\sigma_1 = 1$, $\sigma_2 = 0$, and $\sigma_3 = -1$. Then the above Perron-Frobenius methodology is available up to $t = 2.2$. The numerical result are presented in Fig. 2. It reflects some big tendances of the real election. But the correlation between the popularity and the result is not always satisfied, as shown clearly by comparison between our simulation in Fig. 2 and the result of the election shown in the last column of Table 1.

Table 1. Popularity, sounding polls and result, April 2012 [18,19].

	+	0	−	Voting	Result
Ba	0.56	0.07	0.37	0.095	0.091
Ho	0.57	0.03	0.40	0.285	0.286
Jo	0.35	0.10	0.55	0.015	0.023
LP	0.26	0.05	0.69	0.15	0.179
Mé	0.47	0.10	0.43	0.145	0.111
Sa	0.49	0.05	0.46	0.29	0.272

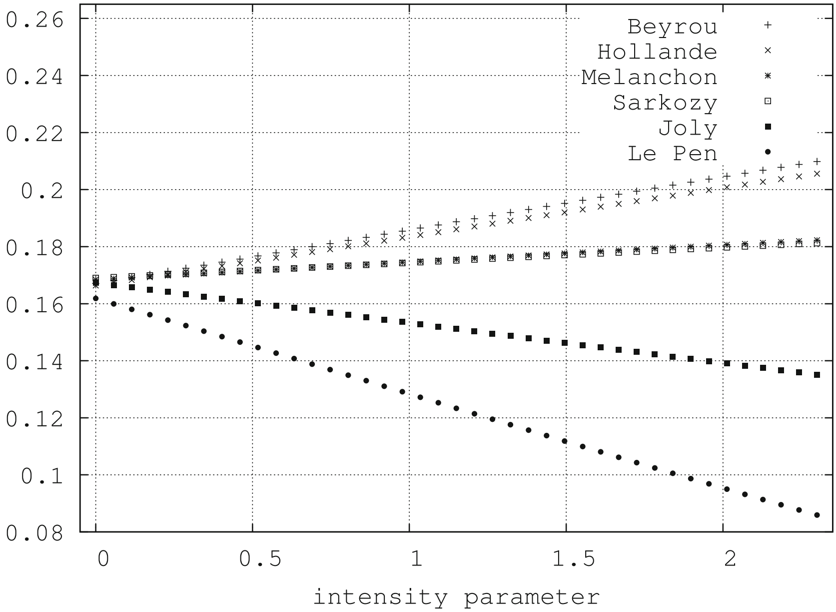


Fig. 2. Result of the vote obtained by a quantum model from the opinion poll. Data issued from the April 2012 French presidential election.

8 Conclusion

In this contribution, we have used a given quantum model for range voting in the context of opinion polls. From these data, we have proposed a quantum methodology for predicting the vote. We introduce a density operator associated to the candidates. The mathematical key point is the determination of a positive eigenvector for a real matrix with non-negative coefficients. Our results are encouraging, even if the confrontation to real life data shows explicitly that other parameters have to be taken into account.

Acknowledgments. The author thanks the referees for helpful comments on the first edition (April 2013) of this contribution. Some of them have been incorporated into the present edition of the article.

References

1. Abramowitz, A.: An improved model for predicting the outcomes of presidential elections. PS: Pol. Sci. Pol. **21**, 843–847 (1988)
2. Arrow, K.J.: Social Choice and Individual Values. Wiley, New York (1951)
3. Balinski, M., Laraki, R.: A theory of measuring, electing and ranking. Proc. Natl. Acad. Sci. U S A **104**(21), 8720–8725 (2007). doi:10.1073/pnas.0702634104

4. Balinski, M., Laraki, R.: Le Jugement Majoritaire: l'Expérience d'Orsay. Commentaire **30**(118), 413–420 (2007)
5. Bitbol, M. (ed.): Théorie Quantique et Sciences Humaines. CNRS Editions, Paris (2009)
6. de Borda, J.C.: Mémoire sur les élections au scrutin. Histoirede l'Académie Royale des Sciences, Paris (1781)
7. Busemeyer, J.R., Trueblood, J.: Comparison of quantum and Bayesian inference models. In: Bruza, P., Sofge, D., Lawless, W., van Rijsbergen, K., Klusch, M. (eds.) QI 2009. LNCS, vol. 5494, pp. 29–43. Springer, Heidelberg (2009)
8. Cohen-Tannoudji, C., Diu, B., Laloë, F.: Mécanique Quantique. Hermann, Paris (1977)
9. de Condorcet, N.: Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralié des voix. Imprimerie Royale, Paris (1785)
10. Campbell, J.E.: Forecasting the presidential vote in the states. Am. J. Polit. Sci. **36**, 386–407 (1992)
11. Dubois, F.: On the measure process between different scales. In: Res-Systemica, 7th European Congress of System Science, Lisboa, vol. 7, December 2008
12. Dubois, F.: On voting process and quantum mechanics. In: Bruza, P., Sofge, D., Lawless, W., van Rijsbergen, K., Klusch, M. (eds.) QI 2009. LNCS, vol. 5494, pp. 200–210. Springer, Heidelberg (2009)
13. Dubois, F.: A quantum approach for determining a state of the opinion. Workshop on Quantum Decision Theory. Symposium on Foundations and Applications of Utility, Risk and Decision Theory, Atlanta, Georgia, USA, 30 June–03 July 2012, unpublished
14. Gallup, G.H.: Guide to Public Opinion Polls. Princeton University Press, Princeton (1944)
15. Grangier, P.: Contextual objectivity: a realistic interpretation of quantum mechanics. Eur. J. Phys. **23**, 331–337 (2002). doi:10.1088/0143-0807/23/3/312
16. Haven, E., Khrennikov, A.: Quantum Social Science. Cambridge University Press, Cambridge (2013)
17. Khrennikov, A.Y., Haven, E.: The importance of probability interference in social science: rationale and experiment. arXiv:0709.2802, September 2007
18. IPSOS: Le baromètre de l'action politique, for Le Point, 09 April 2012
19. IPSOS-Logica: Baromètres d'intentions de vote pour l'élection présidentielle, vague 16, 10 April 2012
20. Lafay, J.D.: L'analyse économique de la politique: raisons d'être, vrais problèmes et fausses critiques. Rev. Fran. Sociol. **38**, 229–244 (1997)
21. La Mura, P., Swiatczak, L.: Markovian Entanglement Networks. Leizig Graduate School of Management, Leipzig (2007)
22. Lewis-Beck, M.S.: French national elections: political economic forecasts. Eur. J. Polit. Econ. **7**, 487–496 (1991)
23. Meyer, C.: Matrix Analysis and Applied Linear Algebra. SIAM, Philadelphia (2000). ISBN 0-89871-454-0
24. Smith, W.D.: Range Voting, December 2000
25. Rivest, R.L., Smith, W.D.: Three voting protocols: Threeballot, VAV, and Twin. In: Proceedings of the Electronic Voting Technology'07, Boston, MA, 6 August 2007

26. Serre, D.: Matrices: Theory and Applications. Springer, New York (2002)
27. Wang, Z., Busemeyer, J.R., Atmanspacher, H., Pothos, E.M.: The potential to use quantum theory to build models of cognition. Top. Cogn. Sci. **5**(4), 672–688 (2013). doi:10.1111/tops.12043
28. Zorn, C., Smith, C.E.: Pseudo-classical nonseparability and mass politics in two-party systems. In: Song, D., Melucci, M., Frommholz, I., Zhang, P., Wang, L., Arafat, S. (eds.) QI 2011. LNCS, vol. 7052, pp. 83–94. Springer, Heidelberg (2011)

Quantum Probabilistic Description of Dealing with Risk and Ambiguity in Foraging Decisions

Peter Wittek[1(✉)], Ik Soo Lim[2], and Xavier Rubio-Campillo[3]

[1] University of Borås, Borås, Sweden
peterwittek@acm.org
[2] Bangor University, Bangor, UK
[3] Barcelona Supercomputing Centre, Barcelona, Spain

Abstract. A forager in a patchy environment faces two types of uncertainty: ambiguity regarding the quality of the current patch and risk associated with the background opportunities. We argue that the order in which the forager deals with these uncertainties has an impact on the decision whether to stay at the current patch. The order effect is formalised with a context-dependent quantum probabilistic framework. Using Heisenberg's uncertainty principle, we demonstrate the two types of uncertainty cannot be simultaneously minimised, hence putting a formal limit on rationality in decision making. We show the applicability of the contextual decision function with agent-based modelling. The simulations reveal order-dependence. Given that foraging is a universal pattern that goes beyond animal behaviour, the findings help understand similar phenomena in other fields.

1 Introduction

Studying foraging strategies has been a successful approach in understanding animal decision making [1]. Foraging patterns are not restricted to animal behaviour, similar patterns occur in other scenarios, such as searching in semantic memory [2], evaluating options by humans [3], and consumer behaviour [4].

Optimal foraging theory (OFT) studies foraging behaviour claiming that organisms aim to maximise their net energy intake per unit time [5,6]. Food sources are available in patches, which vary in quality. Furthermore, switching between patches comes with a cost. The forager faces uncertainty while making decisions about staying at a patch or moving on to the next one. Following Knight, 1921 and Ellsberg, 1961, we distinguish between two fundamental types of uncertainty: ambiguity and risk [7,8]. We associate ambiguity with the estimation of the quality of a patch. Risk, on the other hand, means the potential of the quality of other patches, the loss or gain by not foraging elsewhere as opposed to foraging in the current patch.

Decisions in this model are bound to be sequential: the forager must make decisions patch by patch; this assumption is not uncommon [9–11]. We argue that the order in which the forager deals with risk and ambiguity has an impact

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 296–307, 2014.
DOI: 10.1007/978-3-642-54943-4_27,

on the decision, which in turn influences net energy intake. We introduce a contextual probabilistic framework familiar from quantum mechanics to model the decision making process.

A growing number of projects in Social Sciences use computer simulation as their main research tool. A simulation using an agent-based model (ABM) defines the behaviour of any entity of a system that involves decision-making processes known as agents. The generation of emergent properties that arise from the definition of individual agents include both quantitative and qualitative concepts, combining behaviour aspects and data. Thus, the explanation provided by an ABM is closer to how knowledge is acquired in Social Sciences. We rely on an ABM simulation to find evidence of order dependence and context sensitivity in decision making.

2 Foraging Decisions, Uncertainty, and Context Dependence

Risk and ambiguity are factors in various extensions of OFT, and they have been experimentally verified (Sect. 2.1). Context dependence, preferring one factor over the other is also a common behaviour pattern in various animal species (Sect. 2.2). These observations provide the foundations for our model.

2.1 Risk and Ambiguity

Stochastic variants of OFT are successful in describing strategies that deal with ambiguity, with numerous experimental validations [1,12]. Actual foraging strategies include simple heuristics such as the fixed-time strategy, in which the forager devotes the same amount of time to each patch irrespective of the patch quality. More intricate models of patch utilization include the Bayesian decision process. In this model, animals have an a priori assessments of food distributions, and their foraging decisions are influenced by experience [1,13].

The Bayesian foraging strategy relies on the following assumptions [14]:

1. Perfect knowledge of patch-type distribution (a priori).
2. No instant identification of the quality of a particular patch, resulting in a sample.

The second assumption corresponds to *ambiguity* as a form of uncertainty.

Foraging decision is formulated by an a posteriori distribution made using the sample. An estimator keeps track of the mean value of the current patch, which is either under- or oversampled compared to the actual patch quality. This decision making process leads to density-dependent resource harvest. In the Bayesian model, the forager is allowed to make sequential decisions that vary according its current state, which is affected by the outcome of previous decisions.

The other aspect of uncertainty, *risk*, is also present in OFT. If foraging decisions are influenced by past history, the variations of any foraging parameter affect the expected rate of food gain, and hence the optimal foraging strategy [1]. Variance in a parameter is associated with risk. Foraging decisions are risk-sensitive, as empirical and theoretical proofs show [15,16]. Risk sensitivity should have a sequential component, but it is often overlooked [9]. Whether a simultaneous or a sequential decision making model follows reality closer depends on the degree to which a forager commits itself when making a choice. For instance, in a sequence of choices, immediate rewards are more valuable than delayed ones: the time saved is used to pursue further rewards [10].

2.2 Contextuality

Context-dependent decision rules consider both aspects of uncertainty. We understand the human decision making is context-dependent [17,18], but the phenomenon is less understood in animals. As Freidin and Kacelnik, 2011 put it, "context-dependent utility results from the fact that perceived utility depends on background opportunities" [11]. Spatially or temporally distinct patches are contexts that differ in quality. The sequence of optimal decisions depends on the attributes of the present opportunity and its background options.

Ample experimental evidence shows context-dependent decisions. Honey bees have an intransitive behaviour pattern, the perform a comparative evaluation of flowers depending on the context [19]. Gray jays have a similar behaviour, hinting at a complex decision process involving context-dependent assessment of each fitness-related value of the options [20]. Rufous hummingbirds change their risk preferences depending on whether binary or trinary choices are available to them [21]. The foraging choices of European starlings are better explained by context-dependent utility [22], the birds being more risk averse at lower temperatures [23]. Another study confirmed these findings, pointing out that sequential food encounters are more likely in an animal's natural environment [11]. Simultaneous decision making is important in many species, for instance, humans are able to alternate between the two models of choice [24].

3 Quantum Probabilistic Description of Foraging Decisions

We turn to quantum probability theory to derive a formal decision model of a context-dependent optimal foraging strategy that considers both risk and ambiguity in a sequential setting. We use the quantum probabilistic description purely as a mathematical device, we do not conjecture that a forager's context-dependent decision making process is the result of a macroscopic escalation of quantum effects starting at a sub-atomic level. This investigation is part of a recent trend which claims quantum-like behaviour of systems is in fact not uncommon [25,26].

Our efforts are not the first to bridge contextual probability and animal behaviour. Competing lizard communities and population dynamics also show quantum-like behaviour [27,28].

3.1 Quantum Probability and Decisions

The fundamental difference between classical and quantum probability is that the event algebra in the latter is non-commutative. That is, given two events, A and B, $p(A \cap B) \neq p(B \cap A)$. To understand how non-commutativity leads to context-dependence, let us consider two hypotheses that describe the decision space of a forager:

- h_1: Stay at the current patch.
- h_2: Leave the patch.

These are complementary and mutually exclusive (this example resembles the one described in Trueblood and Busemeyer, 2011 about a medical decision on whether a patient has urinary tract infection [29]).

Consider the following events:

- A: Current patch quality with two possible outcomes: a_1 – the patch quality is good; a_2 – the patch quality is bad.
- B: Quality of other patches. A collective observation across all other patches with two possible outcomes.

A corresponds to ambiguity, whereas B corresponds to risk, the opportunity cost. We will argue that A and B are incompatible observations on a system, leading to non-commutativity and context-dependence. To achieve that, first we demonstrate how quantum probability theory derives probabilistic outcomes. In a quantum framework, the forager's state of belief is described by a state vector in superposition. Under observation A, this superposition is written as

$$|\psi\rangle = \sum_{i,j} \alpha_{ij} |A_{ij}\rangle \tag{1}$$

In this case, for instance, A_{h_1,a_1} means the event that the forager will stay at the patch and the current patch quality is good, and A_{h_2,a_2} corresponds to the event in which the forager will leave the patch and the patch has low quality. These elementary events form an orthonormal basis. The α_{ij} values are the corresponding probability amplitudes. Since the norm of the state vector must be one, we have

$$\sum_{i,j} |\alpha_{ij}|^2 = 1. \tag{2}$$

To measure the probability of one combination, we use a projection operator. The projector onto the event $h_1 \wedge a_1$ is given by

$$P_{11} = P(h_1, a_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

If we apply the projector on the state vector, the square norm of the projected vector will be the quantum probability of $h_1 \wedge a_1$: $||P_{11}|\psi\rangle|| = |\alpha_{11}|^2$. Similarly, if we are interested in the probability of whether the forager would stay at the current spot, we need to project onto the first two basis vectors, and the result will be $|\alpha_{11}|^2 + |\alpha_{12}|^2$.

Switching to observation B, the state of belief is a superposition of four basis vectors: $|\psi\rangle = \sum_{i,j} \beta_{ij}|B_{ij}\rangle$. These basis vectors are not identical to previous ones: the same hypotheses, the decision whether to stay or leave, are now expressed in a different basis. This is equivalent to saying that the same problem is studied from a different perspective: under observation A, the forager bases its decision on local information, whereas under B, it looks at a global perspective. If the two perspectives can be expressed in the same basis, they are called compatible, otherwise they are incompatible; in the present case we take A and B incompatible. Since the state vector can be expressed in both basis, we have

$$|\psi\rangle = \sum_{i,j} \alpha_{ij}|A_{ij}\rangle = \sum_{i,j} \beta_{ij}|B_{ij}\rangle. \quad (4)$$

To switch from one perspective to the other, to change the basis, we need to apply a unitary rotation. To change from perspective A to B, we need to apply the unitary transformation U_{AB}, whereas to change from B to A, we need to apply U_{BA}.

Context sensitivity arises from the quantum equivalent of Bayes' rule, also known as Lüders' rule [30]. Suppose that event A is true, the patch is of good quality. This changes the original state $|\psi\rangle$ to $|\psi_A\rangle = P_{a1}|\psi\rangle/||P_{a1}|\psi\rangle||$. The denominator re-normalises the state vector to meet the condition described in Eq. (2). The probability of event B given that A is true will be $||P_{b1}|\psi_{a1}\rangle||^2 = ||P_{b1}P_{a1}|\psi\rangle||^2/||P_{a1}|\psi\rangle||^2$. Generally speaking, two projectors are not commutative, that is, $P_{a1}P_{b1} \neq P_{b1}P_{a1}$. Therefore the probability of event A given that B is true will be different. Measuring the perceived quality of other patches, the forager may deem those more desirable, and the quality of the current patch looses its importance.

Since the result of an earlier decision changes the context of a new decision, a simple Bayesian inference model has difficulty accounting for order effects [29]. A key concept of quantum probability is *incompatibility*. Not all elementary events can be measured simultaneously, incompatible events can only be measured in a sequence. The first assumption in the Bayesian decision process, which assumes the forager has an a priori knowledge of the probability distribution of the entire event space (Sect. 2.1), is too strong. For instance, the forager has an assessment of the quality of the current patch, which updates its a priori estimate of the distribution. To sample the frequency of such patches, it must move on to the next patch, abandoning the current one. The quality of the current patch and the quality of the other patches are incompatible observations. The latter is related to the risk the forager faces. Incompatibility is a source of order effects on judgements, and this is the critical point at which quantum probabilities differ from classic probabilities [30]. We show that this approach formally introduces a limit on rational decisions by applying the uncertainty principle to foraging decisions involving risk and ambiguity.

3.2 Inherent Uncertainty in Sequential Decisions

If two observables do not commute, a state cannot be a simultaneous eigenvector of the two observables in general [31, p. 233]. This leads to a form of uncertainty relation, similar to the one the inequality found by Heisenberg in his analysis of sequential measurements of position and momentum. This original relation states that there is a fundamental limit to the precision with which the position and momentum of a particle can be known.

Since observations A and B do not commute, there must be a similar limit in foraging decisions corresponding to Heisenberg's uncertainty principle. The forager needs to leave the current patch to assess the quality of other patches: there is an inherent uncertainty in the decision irrespective of the quantity of information gained about either A or B. Since the observables in foraging do not have a strict physical meaning, the physical constants of the canonical commutation relation are not present. Yet, as long as the operators do not commute, an uncertainty principle with a similar lower bound will hold. With regards to risk and ambiguity, we state that

$$\sigma_A \sigma_B \geq c, \tag{5}$$

where $c > 0$ is a constant. The constant itself will depend on actual foraging scenarios, we do not believe there is a universal way of determining its value.

4 Simulation by Agent-Based Modelling

We use an ABM simulation to model contextuality of a decision function, which is not an entirely novel idea. Kitto, Boschetti, and Bruza (2012) proposed an ABM to simulate changing attitudes in social decision making [32]. The context in that case evolves with time, global attitudes and the individuals' own local attitudes change over time, making a case for a contextual decision making function. We do not require temporal evolution of contexts, we assume an order dependence emphasising the two different aspects of dealing with uncertainty.

We use Pandora [33,34], an open-source ABM framework designed to accomplish a realistic simulation environment for social scientists. The source code of the simulation is available online[1].

4.1 Model

In the simulation model, we generate a small map with heterogeneous resources available to the forager. An agent is placed at a random position with certain *requirements*: the quantity of resources needed at every time step. The *starvation rate* is the agent's accumulated percentage of past time steps when requirements were not achieved. The inverse of this value is the *net food intake*. The agent

[1] Code is available at https://github.com/xrubio/pandora. This example is in the pandora/examples/quantumForaging folder.

has a limited time *horizon*, the number of time steps into the future the agent is exploring possible scenarios. The agent has a *knowledge* of the patch distribution, it keeps track of the values and knowledge quality of patches already visited.

An agent is parametrized by the following characteristics:

– Ambiguity aversion: a value between 0 to 1 specifying the preference of the agent to avoid ambiguity. Ambiguity aversion is related to observation A.
– Risk aversion: a value between 0 to 1 specifying the preference of the agent to avoid risk. Risk aversion is related to observation B.

The sum of risk and ambiguity preferences is equal to one. An agent at every time step chooses between two actions: either explore and forage in the current patch, resulting in a decrease in ambiguity; or move and explore an adjacent patch, resulting in a decrease in risk.

Possible scenarios are explored using a Markov decision process that allows to explore the decisions of an agent within a fully observable stochastic state model. In our simulation, the model was solved using the UCT algorithm [35], capable of defining optimal policies for problems with a large set of possible states and finite horizon. The parameters of the algorithm were the horizon of the agent and the width (i.e. the number of states explored during the process), the latter was defined at 300 for all the experiments.

The risk and ambiguity of the resources are varied, and also the agent's preference to minimizing risk (R) or minimizing ambiguity (A). The decision function is a normalised linear combination of the two options for minimizing uncertainty, combined with the cost associated with starving. Thus, the final cost C of an action a given current state s is as follows:

$$C_{s,a} = \frac{m}{M} + (1 - a_A)A + (1 - a_R)R \tag{6}$$

where m is the resources consumed in this time step, M corresponds to the resource requirements, a_A is the decrease of ambiguity of the action, and a_R is the decrease of risk of the action. The value of decrease is 1 if the knowledge of the patch is not complete, and 0 all the other cases.

Once an action is chosen, the knowledge of the patch where it occurs is increased by 1 if the maximum is not already reached. At the same time, the quality of the patch in the knowledge map of the agent is updated to the real value if a random value between 0 and 10 is lesser than current knowledge. Finally, the starvation rate of the agent is updated comparing the requirements with foraged resources; this will be always 0 when choosing a Movement action.

4.2 Experiments

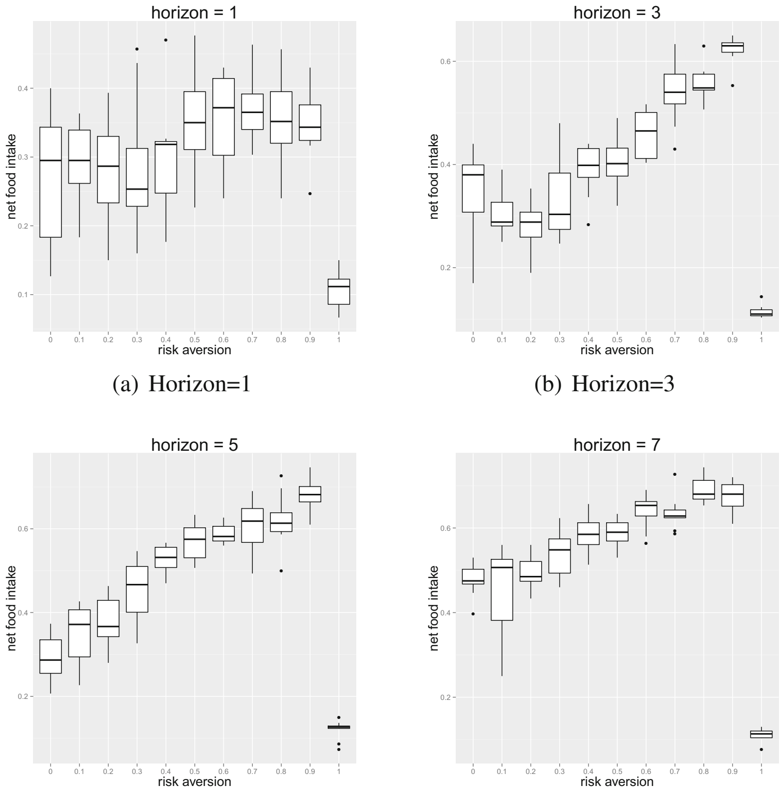
We analysed the trade-off between ambiguity and risk with different time horizon thresholds for a resource map of 20 × 20 cells, and a time span of 1,000 steps. Every scenario contains a single agent, and the exploration of the parameter space is defined terms of time horizon values (1, 3, 5, and 7 steps), and risk aversion ranging from 0 to 1 in 0.1 increments. Each scenario was run 10 times for a total of 440 experiments. This configuration was used in two different experiments:

1. Gradual resource map. The value of each patch is its x relative coordinate multiplied by 10.
2. Random resource map. The value of each patch is defined using a random uniform distribution.

4.3 Discussion of Results

Figure 1 shows net food intakes for the gradual resource maps, and Fig. 2 shows net food intakes for the random distribution of resources.

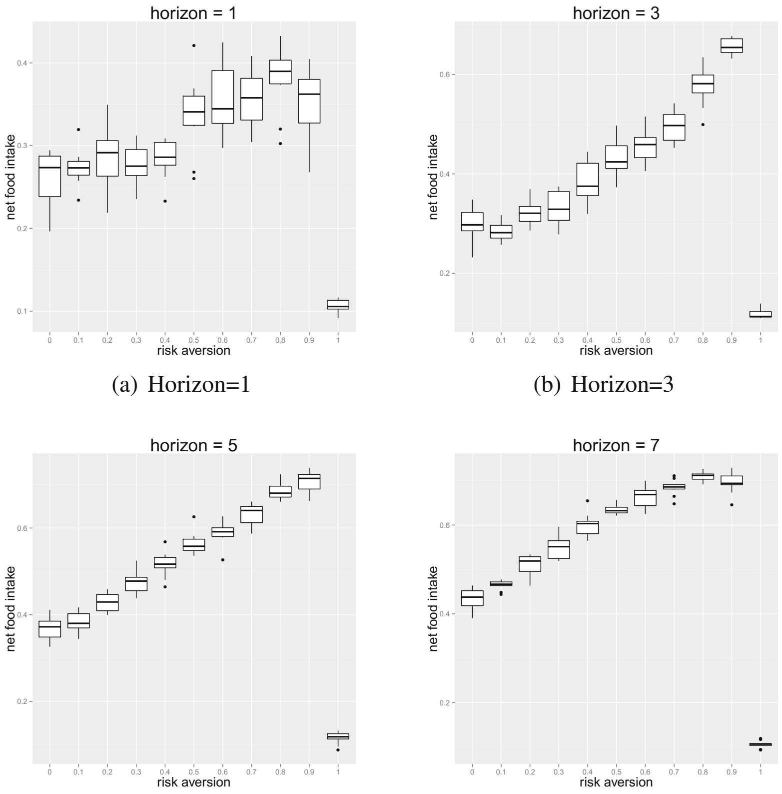
A common pattern to all the experiments is a distinct phase transition at complete risk aversion. At risk aversion value of one, the net food intake sharply



(a) Horizon=1 (b) Horizon=3

(c) Horizon=5 (d) Horizon=7

Fig. 1. Net food intakes for different scenarios using the gradual resource map



(a) Horizon=1

(b) Horizon=3

(c) Horizon=5

(d) Horizon=7

Fig. 2. Net food intakes for different scenarios using the random resource map

drops compared to all other values of this parameter. The phenomenon is not surprising: this behaviour means that the forager hardly ever consumes resources, and keeps visiting patch after patch to learn the overall distribution. While hardly a realistic scenario, it is worth taking note of.

For a time horizon of one time step, there is a different pattern for risk aversion below 0.5. The variance of net food intake is lowest at 0.1 in both types of resource distribution. Whereas the mean value is higher for larger values of risk aversion, a real-life forager would probably prefer reducing the variance. This behaviour indicates fairly long visits to a patch before exploring other options. To some extent, the observation holds for a time horizon of 3 steps. For risk aversion values higher than 0.5, the pattern is less obvious. While generally it pays off to reduce risk first, the exact extent of risk aversion is ambiguous.

If the time horizon is at least 5 steps, reducing risk first becomes far more important and leads to much higher rewards. A patch is easily depleted in five steps, so thinking ahead means a willingness to reduce risk, and the pay-off is clearly visible.

The values of risk aversion of zero and one are the extreme cases of order dependence, whereas the values in between correspond to a case of a more balanced decision making process. For a given parameter setting in the simulated environment, the highest point of the food intake is an empirical limit on the completeness of knowledge, and corresponds to the inequality in the uncertainty principle, as shown in Eq. (5).

5 Conclusions

Relying on a classical Markovian decision model, we simulated the behaviour of a forager in an environment where food resources are available in patches of varying quality. Well-defined patterns emerged that show order-dependence of decisions, and the decisions have a significant impact on net food intake. Our observations coincide with a quantum probabilistic model that considers two aspects of uncertainty, risk and ambiguity, and states that decisions relating to these two aspects are order-dependent.

Acknowledgement. This work was partially supported by the European Commission Seventh Framework Programme under Grant Agreement Number FP7-601138 PERICLES. Xavier Rubio-Campillo is supported by the SimulPast Project (CSD2010-00034), funded by the CONSOLIDER-INGENIO2010 program of the Ministry of Science and Innovation – Spain. We also thank the reviewers for their insights, they helped us clarify conceptual issues.

References

1. Pyke, G.: Optimal foraging theory: a critical review. Annu. Rev. Ecol. Syst. **15**, 523–575 (1984)
2. Hills, T., Jones, M., Todd, P.: Optimal foraging in semantic memory. Psychol. Rev. **119**, 431–440 (2012)
3. Rushworth, M., Kolling, N., Sallet, J., Mars, R.: Valuation and decision-making in frontal cortex: one or many serial or parallel systems? Curr. Opin. Neurobiol. **22**, 946–955 (2012)
4. Rajala, A., Hantula, D.: Towards a behavioral ecology of consumption: delay-reduction effects on foraging in a simulated Internet mall. Manag. Decis. Econ. **21**(3–4), 145–158 (2000)
5. MacArthur, R., Pianka, E.: On optimal use of a patchy environment. Am. Naturalist **100**(916), 603–609 (1966)
6. Charnov, E.: Optimal foraging, the marginal value theorem. Theor. Popul. Biol. **9**(2), 129–136 (1976)
7. Knight, F.: Risk, Uncertainty and Profit. Houghton Mifflin, Boston (1921)
8. Ellsberg, D.: Risk, ambiguity, and the Savage axioms. Q. J. Econ. **75**(4), 643–669 (1961)
9. Houston, A., McNamara, J.: A sequential approach to risk-taking. Anim. Behav. **30**(4), 1260–1261 (1982)
10. Kacelnik, A., Bateson, M.: Risky theories - the effects of variance on foraging decisions. Am. Zool. **36**(4), 402–434 (1996)

11. Freidin, E., Kacelnik, A.: Rational choice, context dependence, and the value of information in European starlings (Sturnus vulgaris). Science **334**(6058), 1000–1002 (2011)
12. Oaten, A.: Optimal foraging in patches: a case for stochasticity. Theor. Popul. Biol. **12**(3), 263–285 (1977)
13. Green, R.: Bayesian birds: a simple example of Oaten's stochastic model of optimal foraging. Theor. Popul. Biol. **18**(2), 244–256 (1980)
14. Valone, T., Brown, J.: Measuring patch assessment abilities of desert granivores. Ecology **70**(6), 1800–1810 (1989)
15. Caraco, T., Martindale, S., Whittam, T.: An empirical demonstration of risk-sensitive foraging preferences. Anim. Behav. **28**(3), 820–830 (1980)
16. Stephens, D.: The logic of risk-sensitive foraging preferences. Anim. Behav. **29**(2), 628–629 (1981)
17. Anand, P.: Foundations of Rational Choice Under Risk. Oxford University Press, Oxford (1995)
18. Waksberg, A., Smith, A., Burd, M.: Can irrational behaviour maximise fitness? Behav. Ecol. Sociobiol. **63**(3), 461–471 (2009)
19. Shafir, S.: Intransitivity of preferences in honey bees: support for 'comparative' evaluation of foraging options. Anim. Behav. **48**(1), 55–67 (1994)
20. Waite, T.: Intransitive preferences in hoarding gray jays (Perisoreus canadensis). Behav. Ecol. Sociobiol. **50**(2), 116–121 (2001)
21. Hurly, T., Oseen, M.: Context-dependent, risk-sensitive foraging preferences in wild rufous hummingbirds. Anim. Behav. **58**(1), 59–66 (1999)
22. Pompilio, L., Kacelnik, A.: Context-dependent utility overrides absolute memory as a determinant of choice. Proc. Natl. Acad. Sci. U S A **107**(1), 508–512 (2010)
23. Bateson, M.: Context-dependent foraging choices in risk-sensitive starlings. Anim. Behav. **64**(2), 251–260 (2002)
24. Kolling, N., Behrens, T., Mars, R., Rushworth, M.: Neural mechanisms of foraging. Science **336**(6077), 95–98 (2012)
25. Pitowsky, I.: Quantum Probability - Quantum Logic. Springer, Heidelberg (1989)
26. Khrennikov, A.: Ubiquitous Quantum Structure: From Psychology to Finance. Springer, Berlin (2010)
27. Aerts, D., Czachor, M., Kuna, M., Sinervo, B., Sozzo, S.: Quantum probabilistic structures in competing lizard communities. arXiv preprint arXiv:1212.0109 (2012)
28. Aerts, D., Czachor, M., Kuna, M., Sozzo, S.: Systems, environments, and soliton rate equations: a non-Kolmogorovian framework for population dynamics. arXiv preprint arXiv:1303.0281 (2013)
29. Trueblood, J., Busemeyer, J.: A quantum probability account of order effects in inference. Cogn. Sci. **35**, 1518–1552 (2011)
30. Busemeyer, J., Pothos, E., Franco, R., Trueblood, J.: A quantum theoretical explanation for probability judgment errors. Psychol. Rev. **118**(2), 193–218 (2011)
31. Cohen-Tannoudji, C., Diu, B., Laloë, F.: Quantum Mechanics. Wiley, New York (1996)
32. Kitto, K., Boschetti, F., Bruza, P.: The quantum inspired modelling of changing attitudes and self-organising societies. In: Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.) QI 2012. LNCS, vol. 7620, pp. 1–12. Springer, Heidelberg (2012)
33. Rubio-Campillo, X.: Pandora: an HPC agent-based modelling framework. https://github.com/xrubio/pandora/ (2013). Accessed 01 April 2013

34. Wittek, P., Rubio-Campillo, X.: Scalable agent-based modelling with cloud HPC resources for social simulations. In: Proceedings of CloudCom-12, 4th IEEE International Conference on Cloud Computing Technology and Science, Taipei, Taiwan, pp. 355–362, December 2012
35. Bonet, B., Geffner, H.: Action selection for MDPs: anytime AO* vs. UCT. In: Proceedings of AAAI-12, 26th Conference on Artificial Intelligence, Toronto, Canada, July 2012

A Methodological Call for a Quantum Econophysics

Christophe Schinckus(✉)

School of Management, University of Leicester,
Leicester, England, UK
Cs354@le.ac.uk

Abstract. Econophysics is an emerging field dealing with complex systems. A deeper analysis of themes studied by econophysicists shows that research conducted in this field can be decomposed into two different computational techniques: statistical econophysics and agent-based econophysics. Both perspectives import the classical idea of randomness coming from statistical physics. This methodological paper calls for the development of a more quantum-oriented econophysics which could complete the approach provided by classical econophysics by enlarging our way of thinking randomness and, therefore, economic uncertainty.

Keywords: Statistical econophysics · Agent-based econophysics · Quantum econophysics

1 Introduction

In the 1990s physicists turned their attention to economics, and particularly financial economics, giving rise to econophysics[1]. A general agreement in the specialized literature seems to exist concerning the fact that econophysics deals with complex economic systems [1–4]. However, an analysis of the themes studied by econophysicists shows that research conducted in this field can be decomposed into two different computational approaches: a macro-perspective and a more micro-oriented one.

The next section will present these two components of classical econophysics: a macro-perspective (statistical econophysics) and a micro-perspective (agent-based econophysics). Although these approaches share the same objective (analysis of socioeconomic complex systems), they use a very different computational methodology. These two econophysical approaches also share the same perception of randomness which is implicitly considered as a lack of knowledge that could be improved. Quantum econophysics has been developed in a different epistemological framework since randomness is considered as deeply embedded in the studied processes. Therefore, randomness does not reflect a specific ignorance about these phenomena, it is rather a fully-fledged component of them. The last section of this paper will emphasize the complementary nature between classical econophysics and quantum econophysics.

[1] For further information about the emergence and the institutional dimension of econophysics, see Gingras and Schinckus [1].

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 308–316, 2014.
DOI: 10.1007/978-3-642-54943-4_28,

2 Classical Econophysics

Econophysics is a recent field whose works can be divided into two major methodological approaches: a macro econophysics (usually associated with "statistical econophysics") and a micro econophysics (associated with "agent-based econophysics"). These two subfields have common foundations since they describe socio-economic systems as complex systems suggesting the unavoidable result of bringing together numerous components in a non-simple manner. Moreover, these two approaches avoid arbitrary assumptions[2] and they base their methodology on empirical verifications. However, some differences exist since these approaches do not use the same computational methodology. While agent-based econophysics deals with microscopic models applied to heterogeneous and learning agents, statistical econophysics rather uses "zero-intelligence agents" (with no learning abilities because particles do not think) whose interactions are random. Whereas agent-based econophysics tries to reproduce the statistical regularities observed in economic or financial systems, statistical econophysics tries to describe these regularities directly from the evolution of these systems. This distinction between these two subfields of econophysics has been suggested by Bouchaud [7], evoked by Chakraborti et al. [8, 9] and explained in detail by Schinckus [10] who suggested that there is only one econophysics (whose major objective is to study statistical patterns observed in the financial\economic data). This section is not a review of literature on statistical and agent-based econophysics (see [8–10] for this kind of literature review) but rather an introduction on the methodological differences between these two computational approaches.

Statistical econophysics is often associated with "stylized facts" in the economics, which mainly refer to "empirical facts that arose in statistical studies of financial (or economic) time series and that seem to be persistent across various time periods, places, markets, assets etc" [8, p. 994]. Statistical econophysics requires a lot of past data on prices, volumes, or transactions from which models describe the fat-tailed empirical distributions of returns, the absence of autocorrelation of returns or the volatility clustering. For statistical econophysics, economic systems are composed of multiple components (no learning agents) interacting in such a way as to generate the macro-properties for systems [11, p. 4]. In this perspective, econophysics aims to associate these macro-properties with statistical regularities[3]. In opposition to economics (or agent-based econophysics), statistical econophysics considers that only the macro-level of the system can be observed and analyzed.

While economists and agent-based econophysicists are based on a microscopic methodology, statistical econophysics is rather founded on a macro-approach in which atoms do not think implying the fact that all "market components" (including traders, speculators, and hedgers) obey statistical properties. In this perspective, statistical

[2] This arbitrary dimension refers to agents's perfect rationality. In opposition to mainstream economics, see Farmer and Foley [5] or Samanidou et al. [6].

[3] See McCauley [3] for further information about the importance of power law regularities in econophysics.

econophysicists avoid the difficult task of theorizing about the individual psychology (or rationality) of investors [12]. In terms of the agent-based modeling approach, statistical econophysicists also avoid the difficult process of calibration of agent-based models (which appears to be the main drawback of this approach [13]) because it refers to a macroscopic approach describing macro-statistical regularities. In a sense, the main objective of statistical econophysics is *to describe* the past financial and economic data through models.

At the opposite, agent-based econophysics is founded on a micro-approach. Agent-based modelling can be looked on as an interdisciplinary approach [14] and it refers to so many fields[4] that it is not possible to number them in this paper whose objective is to focus on agent-based models used in econophysics. Agent-based econophysics comes from computational physics [21] and, this field has mainly developed models of order-driven markets (related to microstructure), game theory models (by redefining the minority problems and related problems) or models using kinetic theory [9].

All agent-based models (in economics or in econophysics) focus on the mathematical modeling of atomistic agents but this atomistic approach is very different from what is usually developed in mainstream economics mainly based on a methodological individualism focusing on personal characteristics (utility function, risk aversion, and so on) and rationality. Rationality is not a necessary condition in agent-based econophysics. Indeed, agents are considered as interacting particles whose adaptative behaviours create different structures (such as molecules, crystals, etc). The notion of "agent" must be thought as the starting point for the reasoning and therefore, it is an "elementary particle" whose interacting behaviour is an epistemic basis for the understanding of the macro-phenomenon. In this view, the concept of agent cannot be associated with the representative agent method [22]. Since learning agents are seen as heterogeneous and "they may differ in myriad ways – genetically, culturally, by social networks, by preferences etc" [14, p. 6]. This agent-based econophysics generates a very interesting literature for the understanding of complex economic system. Pickhardt and Seibold [23], for example, explained that income tax evasion dynamics can be modelled through an "agent-based econophysics model" based on the Ising model of ferromagnetism whereas Donangelo and Sneppen [24] or Shinohara and Gunji [25], approached the emergence of money through studying the dynamics of exchange in a system composed of many interacting and learning agents.

The main difference between statistical econophysics and agent-based econophysics refers to the kind of data these two subfields use. While the first use historical data with (eventually) zero-intelligent agent, the latter rather produce data through a complex modeling of self-evolving systems (with learning agents). In this perspective, the main objective of agent-based econophysics is the "reproduction of the phenomenon" (and not the description like statistical econophysics) [9].

---

[4] The literature about the agent-based models is huge and published in several disciplines. Agent-based approach appeared in the 1990s as a new tool for empirical research in a lot of fields such as economics [15], voting behaviors [16], military tactics [17], organizational behaviors [18], epidemics [19], traffic congestion patterns [20], etc.

3 What About Quantum Econophysics?

Models used in classical econophysics are supposed to capture the main regularities observed in complex phenomena through a methodology coming from statistical physics. Although classical econophysics provides detailed description of the evolution of these statistical regularities through the two computational approaches described previously, the big issue is to explain how these statistical regularities emerge[5]. Classical econophysics does not answer this question. While statistical econophysics only focuses on the macro regularities, agent-based econophysics provides a bottom-up "trial-error" process to fill the "in-between step" between micro and macro level. In other words, these two computational approaches failed to clarify the gap between the micro and the macro level. Basically, econophysicists use the word "emergence" (or concepts evoking the idea of emergence) as, what Craver [27, p. 360] called a "filler terms such as *activate*, *cause*, *produce*, *represent*"[6].

This puzzle about the concept of emergence is implicitly related to the way of thinking of randomness. Statistical and agent-based models share the same understanding of randomness whose identification of statistical characteristics can reduce our ignorance of a more detailed complex situation[7]. In this perspective, randomness reflects our inability to deeply describe a complex situation and the concept of "emergence" veiled our inability to understand the operating mechanism producing the observed statistical macro properties (whose description is the only epistemological access to these emerging phenomena).

In opposition, randomness observed in quantum processes (such as spontaneous emission of light, radioactive decay or state reduction, for example), is considered as a fundamental feature of nature which is independent of our ignorance. This way of dealing with randomness paves the way to another characterization of emerging phenomena [28] which could be very interesting in finance or economics. Despite the existence of a "natural indeterminism", individual quantum events are constrained by statistical laws which make them interesting for analogy with financial phenomena.

This necessity to explain the emergence of statistical regularities in financial/economic systems is a good methodological justification for the development of a quantum econophysics. There is a specific literature[8] dedicated to quantum econophysics whose works often provide a technical justification of their development by emphasizing the lacks of classical econophysics [30]). In this section, I would like to complete these technical justifications with a more methodological one. More precisely, I will explain that a quantum econophysics can improve our bottom-up

[5] For further information about this specific point, see Schinckus [4].

[6] Craver [26] explained that these filler terms can play two roles in knowledge: "they can stand as place-holder for future work" or, in contrast, "they can barriers to progress when they veil failures of understanding".

[7] By adopting this perspective, Schinckus [27] explained that ecnophysics can be looked on as a Knightian reduction of economic uncertainty. See Schinckus [27] for further information about economic uncertainty and econophysics.

[8] The literature dedicated to quantum econophysics is not so large. See Saptsin and Soloviev [29] for a detailed literature review.

understanding of complex systems related to financial and economic spheres and, moreover, how this contribution can be seen as a complementary approach to the existing econophysics.

Although the application of quantum ideas to the economic phenomena appeared in the 1990s [31], the emergence of a quantum econophysics is very recent[9]. Basically, quantum econophysics can be defined through three major characteristics [29]:

– Use of mathematical apparatus of quantum mechanics in order to model economic/ financial processes.
– Application of quantum mechanical models and analogies
– Application of quantum mechanical ideology.

While the two first characteristics are well debated in the existing literature dedicated to quantum physics [29, 30, 34], the importance of quantum-mechanical ideology is sometimes undervalued in the literature especially regarding its potential contribution to classical econophysics. Indeed, the quantum mechanical ideology contributes to econophysics by paving the way to a new theoretical framework for unifying key econophysical concepts such as randomness, emergence and interaction.

In a quantum perspective, randomness and emergence are fundamental features of nature which are independent of our ignorance or knowledge[10]. That is very interesting for economics since randomness is a fully-fledged component of economic activity. Economists are aware about this necessity to develop more complex conceptual tools in order to better understand the complexity of phenomena: Shubik [36], for example, wrote that "modern finance…has not yet provided us with either the appropriate concepts or measures for the bounds on the minimal overall uncertainty that have to be present in an economy". Econophysics can contribute to improve our understanding of economic uncertainty [27]. While classical econophysics studies our description of randomness by assuming we can reduce it by improving our conceptual tools, quantum econophysics rather tries to integrate this randomness in the developed conceptual tools. In a sense, the first is still in line with classical perception of randomness whereas the latter develops another epistemic way of dealing with randomness. These two movements are not the same but they can be complementary in our understanding of complex systems.

Many phenomena in quantum systems require no-intuitive concepts to be described properly. In this perspective, quantum econophysics calls for new rules of thinking inspired from quantum mechanics and its key concepts (indeterminacy principle, system wave function, superposition principle etc). In that perspective, Maslov [34], for example, proposed an interesting analogy between classical econophysics and quantum econophysics. More precisely, he associated the notions of entropy, temperature, free energy, and Hamiltonian used in classical econophysics with a system of identical objects so that modern methods of quantum statistics can be applied to describe the dynamics of financial markets.

[9] See Haven [32] or Haven and Khrennikov [33] for a good introduction to applications of quantum physics to social sciences.

[10] See Khrennikov [35] for information about classical and quantum randomness.

The epistemic way of dealing with randomness used in quantum econophysics also requires another kind of characterization for agents' interactions. As mentioned above, agent-based econophysics provides inductive models describing agents' interactions through a great number of iterations from which macro-regularities are supposed to emerge. In this perspective, theoreticians must only define some rules determining the elementary interactions between the actors. However, agent-based econophysics do not really offer a modeling of agents' consciousness. Actors are defined through basic abilities to interact and interactions are perceived as the result of their mental states. These actions can therefore be algorithmically traced thanks to the initially defined interactions rules and the current state of actors [13]. In a sense, this inductive methodology is more related to the "action making process" than the "decision making process". In this perspective, quantum econophysics could be combined with the work of Busemeyer and collaborators [37] on decision theory which could therefore provide quantum micro-level foundations for a more general quantum econophysics.

The application of quantum concepts in finance and economics deeply changes not only the way of thinking economic phenomena but also the epistemological access to these complex phenomena. In other words, quantum concepts challenge *what* we know and *how* we can get this knowledge. The indeterminacy principle, for example, implicitly means that there is no conception of the particle paths implying therefore that it is impossible to trace algorithmically the actors' interactions. Guevara [30], for example, provided a quantum bridge between the micro and macro level since the dynamics of the whole systems can be explained through a microscopically cooperation between quantum objects trying to improve their states with the purpose of reaching or maintaining the equilibrium of the system. Bagarello [38] also developed this kind of analysis by describing the evolution of a stock market in terms of Heisenberg dynamics while all micro components were defined through quantum Hamiltonians.

In line with this kind of work, Filk and Mueller [28] suggested, some applications of quantum to consciousness could be combined with quantum econophysics in order to develop a real theory explaining the "decision making process". In this perspective, existing literature on quantum decision theory [38–40] can be seen as perfect complementary approach to classical econophysics for a better understanding of the economic phenomena in whole. Although this literature still mainly focuses on the description of the individuals' decision making process, we can find the first generalization for all society in whole (meaning that this kind of analysis could be applied for studying financial markets).

There is no generalized theory of quantum systems [29]. All applications of quantum mechanics in economics and finance challenge the classical perception of uncertainty but also the classical modelling of the decision making process. Although quantum econophysics is still in its infancy, we can mention two major methodological contributions of its field: on one hand, it improves our way of valuing economic uncertainty and on the other hand, it enlarges our epistemic access to complex phenomena by providing new conceptual tools for studying the dynamics of financial markets. More concretely, quantum econophysicists currently develop diversified bottom-up perspectives[11] as a perfect complementary approach to classical econophysics because this approach could offer conceptual tools (or complete the existing ones) in order to bridge the gap between micro and macro level from which emerging properties appear.

In this perspective, by considering that an integrative movement could exist among econophysicists, the econophysical landscape would be composed of three different methodologies: a macro-approach improving our statistical descriptions of financial/economic phenomena (statistical econophysics), a micro-inductive approach improving our ability to reproduce (and then anticipate) macro-laws observed in these systems (agent-based econophysics) and a micro-quantum approach improving our understanding (and explanation) of how these statistical laws emerged in economic/ financial systems (quantum econophysics).

References:

1. Gingras, Y., Schinckus, C.: The institutionalization of econophysics in the shadow of physics. J. Hist. Econ. Thought. **34**, 109–130 (2012)
2. Mantegna, R., Stanley, E.: An Introduction to Econophysics. Cambridge University Press, New York (1999)
3. McCauley, J.: Dynamics of Markets: Econophysics and Finance. Cambridge University Press, Cambridge (2004)
4. Bouchaud, J.-P., Potters, M.: Theory of Financial Risk and Derivative Pricing. Cambridge University Press, Cambridge (2009) (Initial first edition 2003)
5. Farmer, D., Foley, D.: The economy needs agent-based modelling. Nature **460**, 685–686 (2009)
6. Samanidou, E., Zschischang, E., Lux, Th: Agent-based models of financial markets. Rep. Prog. Phys. **70**, 409–450 (2007)
7. Bouchaud, J.-P.: Introduction to statistical physics. Physica A **313**, 338–351 (2002)
8. Chakraborti, A., Muni Toke, I., Patriarca, M., Abergel, F.: Econophysics review: I. Empirical facts. Quant. Financ. **11**(7), 991–1012 (2011a)
9. Chakraborti, A., Muni Toke, I., Patriarca, M., Abergel, F.: Econophysics review: I. Empirical facts. Quant. Financ. **11**(7), 1013–1041 (2011b)
10. Schinckus, C.: Introduction to econophysics towards a new step in the evolution of physical sciences. Contemp. Phys. **54**(1), 17–32 (2013)
11. Rickles, D.: Econophysics for philosophers. Stud. Hist. Philos. Mod. Phys. **38**(4), 948–978 (2007)
12. Brandouy, O.: Complexité et phénomènes critiques en finance. In: Bourghelle, D., Brandouy, O., Gillet, R., Orléan, A. (eds.) Croyances, Représentations Collectives et Conventions en Finance. Economica, Paris (2005)
13. LeBaron, B.: Agent-based computational finance. In: Tesfatsion, L., Judd, K.L. (eds.) Handbook of Computational Economics: Agent-Based Computational Economics, vol. 2, pp. 1187–1233. North-Holland, Amsterdam (2006)

[11] Bitbol [42] used the word “transcendental deduction” to characterize reasoning used in the different quantum perspectives.

14. Epstein, J.M.: Generative Social Science: Studies in Agent-Based Computational Modeling. Princeton University Press, Princeton (2006)
15. Axtell, R.: The emergence of firms in a population of agents: local increasing returns, unstable nash equilibria, and power law size distributions. Santa Fe Institute Working Paper, 19 March 1999
16. Lindgren, K., Nordahl, M.G.: Cooperation and community structure in artificial ecosystems. Artif. Life **1**, 15–37 (1994)
17. Ilachinski, A.: Irreducible Semi-Autonomous Adaptive Combat (ISAAC): An Artificial-Life Approach to Land Warfare. Center for Naval Analyses Research Memorandum, CRM 97-61.10 (1997)
18. Prietula, M., Carley, K., Gasser, L. (eds.): Simulating Organizations: Computational Models of Institutions and Groups. AAAI Press, Menlo Park (1998)
19. Epstein, J.M., Axtell, R.: Growing Artificial Societies: Social Science form Bottom Up. MIT Press, Cambridge (1996)
20. Rasmussen, S., Nagel, K.: Traffic at the edge of chaos. In: Brooks, R. (eds.) Artificial Life IV. MIT Press, Cambridge (1994)
21. Seybold, P.: Agent-Based Models for Physics and Chemistry. In: Proceeding of the American Physical Society (2008)
22. Kirman, A.: Whom or what does the representative individual represent. J. Econ. Perspect. **6**, 117–136 (1992)
23. Pickhardt, M., Seibold, G.: Income tax evasion dynamics: evidence from an agent-based econophysics model. Working paper of the University of Cottbus (2011)
24. Donangelo, R., Sneppen, K.: Self-organization of value and demand. Physica A **276**, 572–580 (2000)
25. Shinohara, S., Gunji, Y.: Emergence and collapse of money through reciprocity. Appl. Math. Comput. **117**, 131–150 (2001)
26. Craver, C.: When models explain. Synthese **153**, 355–376 (2006)
27. Schinckus, C.: Economic uncertainty and econophysics. Physica A **388**(20), 4414–4423 (2009)
28. Filk, T., Mueller, A.: Quantum physics and consciousness: the quest for a common conceptual foundation. Mind Matter **7**(1), 59–79 (2009)
29. Saptsin, V., Soloviev, V.: Relativistic quantum econophysics. new paradigms of complex systems modeling. Monograph http://kafek.at.ua/solsapmonogr.rar. Brama-Ukraine, Cherkassy (2009)
30. Guevara, E.: Quantum econophysics. arXiv:physics/0609245, September 2007
31. Vaga, T.: The coherent market hypothesis. Financ. Anal. J. **4**, 36–49 (1990)
32. Haven, E.: Elementary quantum mechanical principles and social science: is there a connection? Rom. J. Econ. Forecast. **9**(1), 41–58 (2008)
33. Haven, E., Khrennikov, A.: Quantum Social Science. Cambridge University Press, Cambridge (2013)
34. Maslov, M.: Econophysics and quantum statistics. Math. Notes **72**(6), 811–818 (2002)
35. Khrennikov, A.: Classical and quantum randomness and the financial market. arXiv: 0704.2865v1 [math.PR] (2007)
36. Shubik, M.: Quantum economics, uncertainty and the optimal grid size. Econ.Lett. **64**(3), 277–278 (1999)
37. Busemeyer, J.R., Wang, Z., Townsend, J.T.: Quantum dynamics of human decision making. J. Math. Psychol. **50**, 220–241 (2006)
38. Bagarello, F.: Stock markets and quantum dynamics: a second quantized description. Physica A **386**, 283–302 (2007)

39. Yukalov, V.I., Sornette, D.: Quantum decision theory as quantum theory of measurement. Phys. Lett. A **372**, 6867–6871 (2008)
40. Yukalov, V.I., Sornette, D.: Physics of risk and uncertainty in quantum decision making. Eur. Phys. J. B **71**, 533–548 (2009)
41. Yukalov, V.I., Sornette, D.: Mathematical structure of quantum decision theory. Adv. Complex Syst. **13**, 659–698 (2010)
42. Bitbol, M.: Some steps towards a transcendental deduction of quantum mechanics. Philosophia Naturalis **35**, 253–280 (1998)

Hydrodynamic Equations and Finance

Emmanuel Haven(✉)

University of Leicester, Leicester LE1 7RH, UK
eh76@leicester.ac.uk

Abstract. This paper attempts to argue that hydrodynamic equations, a mainstay of quantum mechanics, can in fact find use in elementary financial theory.

1 Introduction

The link between physics and social science is at the present time still a highly unconfirmed relation. If we position ourselves from the social science 'camp' point of view, then it is not so clear how physics concepts have been contributory in the understanding of some of the conundrum problems social science has been served with. At the most, it seems that it is particularly within financial option pricing theory, that physics may have entered. This can be easily seen in the use such theory makes of Brownian motion. But economists will be quick to point out that option pricing theory is in fact not even that closely related to economics anyhow, since in fact it does completely disregard the use of preferences, a mainstay tool of applied and theoretical economics. If we position ourselves from the physics 'camp' point of view then it may well be that the picture is somewhat different. The econophysics community has worked hard over the last ten years to augment the understanding of problems posed by economics and finance. However, it is probably not an exaggeration to claim that the econophysics community is barely recognized by the economics and finance community. This paper has surely not as its objective to delve any deeper in attempting to understand this precarious relationship between two 'sides' of intellectual endeavour. We have two good reasons for not doing so: (i) this paper does not even deal with statistical physics applications to economics or finance; it actually deals with trying to show that hydrodynamical equations (which have a link to quantum mechanics) may have a good use in elementary financial theory; (ii) if econophysics can not be recognized in the economics/finance community, why do we then even bother being disturbed by such a fact if we are considering to investigate a link, which prima facie, is even more remote, i.e. the one between quantum mechanics and social science.

This paper continues a line of work which elaborates on the use of quantum-like concepts in social science. It is important to stress, that we are not at all claiming that quantum phenomena are happening at macroscopic scales like they are encountered in the social sciences. Instead, we argue for the use of technical tools, borrowed from quantum mechanics, in social science, more specifically in

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 317–323, 2014.
DOI: 10.1007/978-3-642-54943-4_29,

economics and finance. Early work on this link is contained in the paper by Khrennikov [17]. We could supply the novice reader in this area, with a plethora of other references. The paper by Segal and Segal [23] is a key paper which launched the connection between quantum mechanics and option pricing. Non-stochastic approaches can be found in the work by Baaquie [1,2]. Important inroads in psychology and quantum mechanics have also been made. See for instance the book by Busemeyer and Bruza [5]. For more work in the area of quantum mechanical tools in finance and economics, see also Haven and Khrennikov [11].

In the next section, we highlight with some examples of how we can measure information in finance. In Sect. 3, we outline the basics of the hydrodynamic approach, following the work by Nelson [21]. Section 4, shows how financial information can be linked to hydrodynamic equations. We show the approach of Hawkins and Frieden [13]. In Sect. 5, we show how the Bellman function, which is a well known equation used in macroeconomic dynamics, relates to the hydrodynamic equation. We use the approach by Gondron and Lepaul [9].

We thank participants at the Seminar 'Quantum Thinking' held at the Institute for Mathematical Behavioral Sciences (February 22 and 23 - 2013) at the University of California - Irvine, for their excellent remarks on some of the ideas contained in this paper.

2 Information in Finance: Some Examples

2.1 (Non) Dependence on Preferences for Risk in Standard Option Pricing Theory

A financial option is a contract which allows the buyer of such contract to either buy or sell the asset (for instance a stock) mentioned in the contract for a certain price during a specified time period (or *at* a date in the future). The Black-Scholes [4] option pricing formula is a celebrated approach which allows for the calculation of the price of an option under the assumption that the volatility of the price of the asset is constant. It can be shown that standard Black-Scholes theory is completely independent of preferences for risk. Those preferences are captured by the expected return parameter which sits in the drift of the geometric Brownian motion. The construction of the Black-Scholes portfolio allows for the elimination of that parameter. This is an important achievement, as preferences for risk are notoriously difficult to model. It is possible to show that the basic 'amplitude function - wave function' framework can be used to give financial sense to the key parameter in that setup: the wave number. This framework can then aid us in understanding how the dependence on preferences for risk can actually be introduced depending on the narrowness (width) of the amplitude function. We do not pursue this further in the current paper. Please see Haven and Khrennikov [12].

2.2 Fisher Information in Finance

In Hawkins and Frieden [13] the fluctuations x around the price of an asset x_0, observed at time t_0, are modelled by a probability amplitude $\psi(x)$ (which the authors explicitly say is independent of any quantum mechanical interpretation). Fisher information (see Frieden [7,8]) is then defined by the authors as:

$$I = E\left[\frac{\partial \ln(p(x_{obs}|x_0,t))}{\partial x_0}\right]^2 = \int \left|\frac{d\psi(x)}{dx}\right|^2 dx; \tag{1}$$

where $x_{obs} = x_0 + x$; i.e. the measured or observed price. Hawkins and Frieden [13] show that with this measure, the level of information is high when the density function is tight, i.e. when the fluctuations round x_0 are small. This is a very reasonable argument.

3 Basics of the Hydrodynamic Framework

We need to be careful in how we want to present the so called 'basics' of the hydrodynamic framework. We will not refer back to the original Madelung [20] paper. There are two avenues we can follow in trying to present the equational expressions of this framework. One way is to consider the Bohm - Hiley- de Broglie framework (Bohm and Hiley [3]). See also Dubois [6] (especially Sect. 3). We can of course also consider the approach Nelson [21] proposes. Holland [14] has made important remarks about the fact that one needs to carefully distinguish between the hydrodynamic interpretation and the Bohm- Hiley de Broglie framework. We do not expand on it further in this paper. See Haven and Khrennikov [11].

In the Bohm - Hiley - de Broglie framework, if we insert a quantum mechanical wave function of the form $\Psi(x,t) = e^{R(x,t)}e^{\frac{i}{\hbar}S(x,t)}$, with R and S respectively the amplitude and phase functions of Ψ into the Schrödinger equation, then it is easy to show (see for instance Holland [15]) when separating real and imaginary parts, one equation (for the real part) becomes:

$$\frac{\partial S}{\partial t} + V + \frac{1}{2m}(\nabla S)^2 - \frac{\hbar^2}{2m}\left[(\nabla R)^2 + \Delta R\right] = 0; \tag{2}$$

where V is the real potential; m is mass; $\hbar$ is the Planck constant and ∇ and Δ are respectively the first and second derivatives of the amplitude function towards position x. We note that the continuity equation is also obtained from this separation.

In a non-quantum context, Nelson [21] shows that one can obtain:

$$\frac{\partial S}{\partial t} + V + \frac{1}{2m}(\nabla S)^2 - \frac{m\sigma^4}{2}\left[(\nabla R)^2 + \Delta R\right] = 0; \tag{3}$$

where a probability density function $f(x,t) = e^{2R(x,t)}$, with $R(x,t)$ now being a scalar field and $\sigma^2 = \hbar/m$. The above equation is obtained with what Nelson [21] defines are mean forward and mean backward derivatives. In this paper, we do not expand any further on this important development. The hydrodynamic framework consists of the above equation joined with the continuity equation.

4 Financial Information and Hydrodynamic Equations

Considering Sect. 2.2 again, we note that Hawkins and Frieden [13] take the information perspective further, by defining a level of so called 'intrinsic information', J, which is akin to 'perfectly knowable' information. What is very important in their set up is the definition of a measure of information asymmetry:

$$I - J; \tag{4}$$

where I follows Eq. (1).

The above Eq. (4) is then interpreted as a Lagrangian, $\mathcal{L}$. A result, which for the purposes of finance, we believe to be of *very* promising value, is the physics result, which Hawkins and Frieden [13] cite as obtained by Reginatto [22]. This result says that the variation of $\mathcal{L}$, notably with respect to the density function, leads to the hydrodynamic equations which we discussed in the above section. Hence, this connection shows how closely one can argue between a financial and physical interpretation of the hydrodynamic equations.

5 The Bellman Function and Hydrodynamic Equations

The Bellman function in macroeconomics is a very heavily used mathematical tool. The key reference on dynamic programming, which uses the Bellman equation is by Stokey and Lucas [24]. Gondran and Lepaul [9] provide for some very beautiful ideas on how the Bellman function can be actively related to population issues in a mean field framework. We provide in this paper for an example which is based on their approach.

Let us consider the simple situation where we compare the 'value' of an asset A with the price of the asset A. We imagine a scenario where a 'consensus' value of an asset A is set relative to a variety of prices, P_A that the asset A can take. We assume there is a density function on the price of asset A: $f(P_A)$. Assume we allow a geometric Brownian motion on the price of asset A to exist:

$$dP_A = \mu P_A dt + \sigma P_A dz; \tag{5}$$

where μ is the expected return of the asset A; σ is the volatility of the price of the asset and dz is the Wiener process. In fully similar fashion as in the Gondran and Lepaul [9] paper, we assume the economic decision maker derives a level of satisfaction ('utility' in economics vocabulary), on the basis of the position of the set 'consensus' valuation of asset A relative to the price of asset P_A. The utility u can then be defined as: $u(f(P_A, t))$, where f is the density function; P_A follows a geometric Brownian motion, $dP_A = \mu P_A dt + \sigma P_A dz$.

Let us now assume that costs are incurred (we call those costs 'dis-utility', u_d) in having to estimate the 'consensus' valuation in terms of the expected return,

μ, of asset A. For instance: a higher expected return may require a higher cost in estimating the 'consensus' valuation. In similar fashion as in Gondran and Lepaul [9], we can write the Bellman function $U(P,t)$, as a control problem:

$$\max_{\mu, P_A = P} E\left[\int_t^T (u(f(P_{As}, s)) - u_d)\exp(-\lambda(s-t))ds\right]; \tag{6}$$

where t, T and s are respectively beginning time, maturity time and the time integration constant; λ is the discount rate and E is the expectation operator. The above equation is also proposed, in a population framework by Gondran and Lepaul [9] and Guéant [10].

What is now important is to note the remark that Gondran and Lepaul [9] make by saying that Lasry and Lions [18,19] find that the associated PDE's to the above control problem, contains terms:

$$(i) : \frac{\partial U}{\partial t}; (ii) : \frac{1}{2}(\frac{\partial U}{\partial P})^2; (iii) : -\lambda U + u(f) \tag{7}$$

and

$$(iv) : \frac{\partial f}{\partial t}; (v) : \frac{\partial}{\partial P}\left(f\frac{\partial U}{\partial P}\right) \tag{8}$$

Terms (i); (ii); (iii) are part of a Hamilton Jacobi equation. Such an equation, when rendered in the form of Bohm - Hiley - de Broglie, will contain a real and quantum potential. Terms (iv) and (v) will figure in the continuity equation (describing the dynamics of the pdf f), but the associated partial differential equations are not exactly like in the Hamilton-Jacobi and continuity equations though. Gondran and Lepaul [9] remark, $U(P,t)$ plays the role of the action in the Madelung (hydrodynamic) equations [20]. Therefore, $\frac{\partial U}{\partial P}$ must indicate in our simple asset valuation toy model, some sort of velocity of a utility flow.

6 Conclusion

As Hawkins and Frieden [13] have indicated in their paper, stochastic dynamics provided for an excellent vehicle to connect financial economics and statistical mechanics. Clearly, the same can surely not be said about the potential link between quantum mechanical techniques and financial economics. It needs stressing again, to avoid any mis-understanding, that while we endeavour to bringing quantum mechanics to use in macroscopic environment, we surely are NOT claiming there are quantum mechanical phenomena happening in a macroeconomic context.

We want to re-iterate the concluding arguments we mentioned at a talk within the 'Quantum Thinking' seminar held at the Institute for Mathematical Behavioral Sciences (February 22 and 23 - 2013) at the University of California - Irvine. Why do we bother to use the quantum mechanical toolkit in finance or economics? Here are three possible reasons:

- Reason 1: to get possible 'mileage' out of additional variables: what can we say about the real (financial) but also quantum potential in Eqs. (2) or (3)?
- Reason 2: we believe there are (possibly) interesting avenues to look for quantum mechanical characteristics in finance. Haven and Khrennikov [11] give a flavour.
- Reason 3: there seems to be more and more evidence (see Busemeyer [5] and Khrennikov [16]) that decision making processes have quantum mechanical features and hence (heuristically speaking then) it may not be such a huge jump to argue that such features may then be found back in economic and financial modelling.

References

1. Baaquie, B., Liang, C.: Feynman perturbation expansion for the price of coupon bond options and swaptions in quantum finance II. empirical. Phys. Rev. E **75**, 016704 (2007)
2. Baaquie, B.: Quantum Finance. Cambridge University Press, Cambridge (2004)
3. Bohm, D., Hiley, B.: The Undivided Universe. Routledge, New York (2005)
4. Black, F., Scholes, M.: The pricing of options and corporate liabilities. J. Polit. Econ. **81**, 637–659 (1973)
5. Busemeyer, J., Bruza, P.: Quantum Models of Cognition and Decision. Cambridge University Press, New York (2012)
6. Dubois, F., Greff, I., Hélie, T.: On least action principles for discrete quantum scales. In: Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.) QI 2012. LNCS, vol. 7620, pp. 13–23. Springer, Heidelberg (2012)
7. Frieden, B.R.: Physics from Fisher Information: A Unification. Cambridge University Press, Cambridge (1998)
8. Frieden, B.R.: Science from Fisher Information: A Unification. Cambridge University Press, Cambridge (2004)
9. Gondran, M., Lepaul, S.: Indiscernability and mean field, a base of quantum interaction. In: Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.) QI 2012. LNCS, vol. 7620, pp. 218–226. Springer, Heidelberg (2012)
10. Guéant, O.: A reference case for mean field game models. Cahier de la Chaire Finance et Développement Durable 10
11. Haven, E., Khrennikov, A.: Quantum Social Science. Cambridge University Press, Cambridge (2013)
12. Haven, E., Khrennikov, A.: Quantum-like tunnelling and levels of arbitrage. Int. J. Theor. Phys. **52**, 4083–4099 (2013)
13. Hawkins, R.J., Frieden, B.R.: Asymmetric information and quantization in financial economics. Int. J. Math. Math. Sci. (2012). doi:10.1155/2012/470293
14. Holland, P.: Computing the wavefunction from trajectories: particle and wave pictures in quantum mechanics and their relation. Ann. Phys. (New York) **315**, 505–531 (2005)
15. Holland, P.: The quantum Theory of Motion: An Account of the de Broglie - Bohm Causal Interpretation of Quantum Mechanics. Cambridge University Press, Cambridge (2000)
16. Khrennikov, A.: Ubiquitous Quantum Structure: From Psychology to Finance. Springer, Heidelberg (2010)

17. Khrennikov, AYu.: Classical and quantum mechanics on information spaces with applications to cognitive, psychological, social and anomalous phenomena. Found. Phys. **29**, 1065–1098 (1999)
18. Lasry, J.M., Lions, P.L.: Jeux à champs moyen: I. le cas stationnaire. Comptes Rendus de l'Académie des Sciences de Paris **343**(9), 619–625 (2006)
19. Lasry, J.M., Lions, P.L.: Jeux à champs moyen: II. horizon fini et contrôle optimal. Comptes Rendus de l'Académie des Sciences de Paris **343**(10), 679–684 (2006)
20. Madelung, E.: Quantentheorie in hydrodynamischer form. Zeitschrift für Physik **40**(3–4), 322–326 (1926)
21. Nelson, E.: Derivation of the Schrödinger equation from newtonian mechanics. Phys. Rev. **150**, 1079–1085 (1966)
22. Reginatto, M.: Derivation of the equations of nonrelativistic quantum mechanics using the principle of minimum fisher information. Phys. Rev. A **58**(3), 1775–1778 (1998)
23. Segal, W., Segal, I.E.: The black-scholes pricing formula in the quantum context. Proc. Natl. Acad. Sci. USA **95**, 4072–4075 (1998)
24. Stokey, N., Lucas, R.: Recursive Methods in Economic Dynamics. Harvard University Press, Cambridge (1989)

Lamarckian Evolution of Epigenome from Open Quantum Systems and Entanglement

Masanari Asano[1], Irina Basieva[2], Andrei Khrennikov[2](✉), Masanori Ohya[1], Yoshiharu Tanaka[1], and Ichiro Yamato[3]

[1] Department of Information Sciences, Tokyo University of Science, Yamasaki 2641, Noda-shi, Chiba 278-8510, Japan

[2] International Center for Mathematical Modeling in Physics and Cognitive Sciences, Linnaeus University, 35195 Växjö, Sweden

Andrei.Khrennikov@lnu.se

[3] Department of Biological Science and Technology, Tokyo University of Science, Yamasaki 2641, Noda-shi, Chiba 278-8510, Japan

Abstract. We develop a quantum-like (QL) model of cellular evolution based on the theory of open quantum systems and entanglement between epigenetic markers in a cell. This approach is applied to modeling of epigenetic evolution of cellular populations. We point out that recently experimental genetics discovered numerous phenomena of cellular evolution adaptive to the pressure of the environment. In such phenomena epigenetic changes are fixed in one generation and, hence, the Darwinian natural selection model cannot be applied. A number of prominent genetists stress the Lamarckian character of epigenetic evolution. In quantum physics the dynamics of the state of a system (e.g. electron) contacting with an environment (bath) is described by the theory of open quantum systems. Therefore it is natural to apply this theory to model adaptive changes in the epigenome. Since evolution of the Lamarckian type is very rapid – changes in the epigenome have to be inherited in one generation – we have to find a proper mathematical description of such a speed up. In our model this is the entanglement of different epigenetic markers.

Keywords: Entanglement · Open quantum systems · Epigenetics · Cellular evolution · Neo-Lamarckism · Quantum master equation · Markov approximation

1 Introduction

Recently the theory of open quantum systems, i.e. systems contacting[1] with various environments (baths), was applied to cognitive science, psychology, decision

[1] One may in principle speak about interaction with an environment. However, "interaction" is not the right term, since it assumes the presence of an interaction force. This is natural in classical physics, but not in quantum. In some sense a quantum system just "feels" the presence of the environment. There is no real force acting to

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 324–334, 2014.
DOI: 10.1007/978-3-642-54943-4_30,

making and cellular biology [3–7]. We start with an illustration from decision making in game theory (e.g. in games of the Prisoners Dilemma type). Before starting decision making one party, say Alice, has the mental state which is characterized by a very high (often maximal) degree of uncertainty about the intentions of another party, say Bob, and Alice's own possible actions. In the process of decision making Alice has to resolve this state of uncertainty and select a strategy. Since in general a strategy has to be understood as a mixed strategy, by resolving uncertainty Alice has to generate a classical probability distribution (mixed strategy). In [3–5] we described such a process by encoding the state of uncertainty as a superposition of all possible states corresponding to the strategies of Alice and her conjectures about the strategies of Bob, cf. [9]. To resolve such quantum-like (QL) uncertainty the state has to evolve to a mixed quantum state: the off-diagonal elements have to be killed. This process is described properly by the theory of open quantum systems, by the quantum master equation – the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation, see, e.g. the book of Ohya and Volovich [10]. In the example about decision making in game theory an environment is, in fact, a mental environment of Alice, including her own memory.

We now apply the same scheme to the process of resolution of uncertainty of possible epigenetic[2] changes in a cell contacting with an environment. From the very beginning we emphasize that such an environment always has a nontrivial information component; in particular, in the form of cells signaling. The epigenetic evolution is described by quantum(-like) master equation and the final steady state (of the attractor type) represents the distribution of epimutations fixed in the cellular population which was exposed to the pressure of an environment. In physics such a process is called decoherence. Hence, we can speak about the epigenetic inheritance as decoherence of superposition of possibilities for various epimutations in a cell.

Since evolution of the (neo-)Lamarckian type[3] is very rapid – changes in the epigenome have to be inherited in one generation – we have to find a proper mathematical description of such a speed up. In our model this is the entanglement of different epigenetic markers; cf. [8].

---

a system from the side of the environment. In fact, in for cognitive phenomena we have the similar situation. By using "mental forces" one would make a model too mechanistic (of Newtonian type), see, however, [1,2] for attempts to apply Bohmian mechanics to cognitive phenomena and, especially, finance.

[2] Everywhere below we shall use the term *epimutation*: a heritable change in gene expression that does not affect the actual base pair sequence of DNA, see, e.g., Jablonka and Raz [11] for an extended review of heritable epimutations, theory and experiment.

[3] Lamarck was one of the firsts who presented evolution in biology as a scientific theory. One of the basic ideas of J. B. Lamarck is that an organism can pass on characteristics that it acquired during its lifetime to its offspring. According to Lamarck evolution is fundamentally *adaptive* in its nature. Extension of Lamarckism to cellular evolution is known as neo-Lamarckism.

In principle, we can discuss a closer analogy with the above example of QL modeling of decision making. By making the assumption that any cell has some form of cognition we can just map the previously developed model of cognitive dynamics in the process of decision making onto cellular epigenetic evolution: a cell "decides" which epimutations are useful to match with the environment and which are not. However, the aforementioned assumption is too strong and too speculative and, although it may have revolutionary consequence for cellular biology and genetics, we prefer to proceed without it.

We point to other applications of the mathematical apparatus of quantum mechanics to cognitive science and decision making: see the monographs [1,2,9] and extended bibliography in these books.

We call our model *quantum-like* (QL) to distinguish it from really quantum models in cell biology. Such models are based on "*quantum reductionism*", i.e. reduction of a cell's behavior to the behavior of quantum particles inside a cell, e.g., Ogryzko [13,14]; McFadden and Al-Khalili [15], McFadden [16][4]. In particular, superpositions of a cell's states are induced by real quantum superpositions of quantum particles inside the cell. These (really quantum) models were strongly criticized, e.g. Donald [17], since quantum physics does not support speculations on the possibility to create macroscopic superpositions and entanglements in cells (as physical systems) and moreover to preserve it for sufficiently long time. Even the idea that a cell interacting with an environment can process information in the same vein as a quantum computer has no physical support. Unitary evolution characterizing quantum computations is a feature of isolated physical systems; see Donald [17] for the complete presentation of critical arguments.

2 Adaptive Dynamics of the Epigenetic State

Denote the space of the states of a cell's epigenome by the symbol H. These states represent statistical information about possible observations on the phenotype's changes. The structure of this space will be discussed in Sect. 3. For a moment, this is simply a complex Hilbert space. (Thus we use the quantum-like formalism.) The space of states of the environment is denoted by K. Since, finally, we shall be interested only in the dynamics of cell's epigenetic state, the degrees of freedom of the environment will be excluded from the direct consideration by tracing with respect to the space K. The state space of the compound system is the tensor product $H \otimes K$.

Our proposal is to use the machinery of the theory of *open quantum systems* and to describe the dynamics of the epigenetic QL-state by using the quantum master equation (the GKSL-equation). This equation describes transitions from states of uncertainty given by superpositions to classical probability

[4] These studies belong to the domain of quantum biology. Our QL model is not a part of quantum biology. We can play with terminology and call the domain of our research *bio-quantumiology* – to distinguish from research in quantum biology.

distributions. Hence, such an equation cannot be an equation with respect to the quantum state represented as a *vector* belonging to complex Hilbert space (normalized to one) – a *pure state.* (We recall that Schrödinger's equation describes the dynamics of pure states.) Such vectors represent superpositions of possibilities which have to disappear at the end of the epigenetic evolution. We have to use more general quantum states represented by *density operators.* Both purely quantum superposition describing uncertainty and the final classical probability distribution can be represented by density operators.

In the quantum Markovian approximation the dynamics of the state of a system interacting with an environment is described by the GKSL-equation:

$$\gamma \frac{d\rho_{\rm epi}}{dt} = -i[\mathcal{H}, \rho_{\rm epi}(t)] + \mathcal{W}\rho_{\rm epi}(t), \rho_{\rm epi}(0) = \rho^0_{\rm epi}, \tag{1}$$

where $\mathcal{H}$ is a Hermitian operator determining the internal dynamics of epimutational changes in cells which are isolated from environmental pressure ("cell's Hamiltonian") and the linear operator $\mathcal{W}$ describes the environmental pressure. Opposite to $\mathcal{H}$, in general the operator $\mathcal{W}$ has a complex mathematical structure. It has such a form that starting with a density operator ρ^0_C we shall get density operators at all instances of time. For a moment, the concrete structure of $\mathcal{W}$ is not important for us; see, e.g., Ohya and Volovich [10] for mathematical details. Biologically this operator is determined by the properties of the environment, including the initial state of the environment. Here γ is the time scale constant, it determines the temporal dimension of the epigenetic evolution. By using such a scaling factor of the dimension of time, we are able to proceed with dimensionless Hamiltonian $\mathcal{H}$ and environmental operator $\mathcal{W}$. In quantum physics, there is the Planck constant h in place of γ. The Planck constant has the physical dimension of action and, hence, the operators $\mathcal{H}$ and $\mathcal{W}$ have the dimension of energy. In quantum biology one has to proceed with the Planck constant and to use operators of real physical energy. However, in our "bio-quantumiology" all operators have only operational meaning and we do not consider the real physical processes of the cellular energy transfer.

For a very general class of GKSL-equations, the environmental operator $\mathcal{W}$ drives (in the limit $t \to \infty$) the epigenetic state of an ensemble of cells, $\rho_{\rm epi}(t)$, to the *steady solution*: $\rho_{\rm epi}(t) \to \rho_{\rm epi;st}$. Typically the uncertainty (in the form of superposition) is eliminated from the asymptotic state $\rho_{\rm epi;st}$. In our model such a steady state is considered as the result of the epigenetic evolution in the environment (mathematically represented by the operator $\mathcal{W}$). The limiting probability distribution $\rho_{\rm epi;st}$ describes the probability distribution of epimutations which took place in a cell population as a consequence of interaction with the environment. Internal uncertainty, to (epi)mutate or not mutate, was resolved and a stable *phenotype* was created.

This model can be considered as a sort of (neo-)Lamarckism; *epigenetic Lamarckism.*

3 Dynamics of Superposition: Mutate or Not Mutate

Consider the concrete gene g in cell's genome. Suppose that this cell interacts with an environment such that some type of epigenetic mutation, say μ, in g can happen. This epimutation changes the level of expression of g.

By ignoring the presence of other genes and corresponding gene expressions we can model the μ-mutation by considering simply the two dimensional state space $H_{\rm epi}$ (qubit space). States of no mutation and mutation are represented by two orthogonal vectors $|0\rangle$ and $|1\rangle$. Hence, a (pure) QL-state can be represented as superposition

$$|\psi_{\rm epi}\rangle = c_0|0\rangle + c_1|1\rangle, \tag{2}$$

where $c_0, c_1 \in \mathbf{C}, |c_0|^2 + |c_1|^2 = 1$. As was remarked, the quantum master equation does not respect pure states, so sooner or later superposition (2) will be transferred into the statistical mixture given by a density matrix. Thus nontrivial superposition is characterized by the presence of the nonzero off-diagonal terms. We remark that the absolute value of the off-diagonal terms is maximal and equals 1/2 for the uniform superposition $|\psi_{\rm epi}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, representing the maximal uncertainty. The dynamics (1) suppresses the off-diagonal terms and, finally, a diagonal density matrix (steady state) arises, $\rho_{\rm st}$. Its elements $\rho_{00;\rm st}$ and $\rho_{11;\rm st}$ give probabilities of the events: no μ-epimutation and μ-epimutation. Thus in a large population of cells, say M cells, $M \gg 1$, the number of, e.g. cells with mutation is given (approximately) by $N_m \approx \rho_{11;\rm st} M$. The limiting QL-state (represented by the diagonal matrix) obtained stability with respect to the influence of this (concrete) environment. We remark that mathematically a population needs infinite time to stabilize completely to the steady state. Therefore in reality one can expect fluctuations (of decreasing amplitude) on a finite interval of time.

We remark that under a special interrelation between operators $\mathcal{H}$ and $\mathcal{W}$ the stabilization is achieved with the state $\rho_{\rm st}$ such that $\rho_{11;\rm st} \gg \rho_{00;\rm st}$. In such a case the epimutation μ spreads to practically the whole population and, moreover, it will be inherited. Thus the quantum master equation is sufficiently general to represent (on the epigenetic level) the regime which is similar to one represented by *Fisher's equation* that was used to describe the spreading of biological populations. The main distinguishing feature of the epigenetic situation is that the epimutation spreads in a single generation of cells and then it is inherited by the next generation.

4 Entanglement of Genes' Expressions

We construct quantum-like representation of the information state of the epigenome expressing epigenetic inheritance of the chromatin-marking type. Consider a cell with genome consisting of m genes $g_1, ..., g_m$. Let us assign to each gene g all its possible epimutations (of the chromatin-marking type); we simply enumerate them by numbers[5]: $j_g = 1, ..., k_g$.

The state of all potential epimutations in the gene g is represented as a superposition

$$|\psi_g\rangle = \sum_j c_{g;j}|j_g\rangle, \tag{3}$$

where $\sum_j |c_{g;j}|^2 = 1$.

What is the meaning of this superposition from the biological viewpoint? Can a gene really be in superposition of a few different epimutations?

Although our model is operational and in principle we are not interested in such questions, we make a comment to clarify the coupling of operational and biological descriptions of this situation. A cell by itself "knows its epigenome" at each instant of time; so it is well aware which epimutations took place up to this instant of time. However, a biologist performing an experiment with cells does not know the situation inside an individual cell in such detail. Superposition is related to uncertainty of the *observer's information.*

If epimutations in different genes are independent of each other, then the QL-state of a cell's epigenome is represented as the tensor product of states $|\psi_g\rangle$:

$$|\psi_{\rm epi}\rangle = |\psi_{g_1}\rangle \otimes ... \otimes |\psi_{g_m}\rangle. \tag{4}$$

However, in living cells, most of the genes and proteins are correlated somehow forming a big network system. So one epimutation affects other genes usually. Hence, the assumption of independent epimutations is nonbiological. Therefore we have to consider more general states describing the consistent epimutations of all genes in the genome of a cell. These are so called *entangled states* which are widely used in quantum information theory:

$$|\psi_{\rm epi}\rangle = \sum_{j_1 \cdots j_m} c_{j_1 \ldots j_m} |j_{g_1} \ldots j_{g_m}\rangle, \tag{5}$$

where $|j_{g_1} ... j_{g_m}\rangle$ is just the short notation for the tensor product of states of superpositions in various genes, $|j_{g_1} ... j_{g_m}\rangle \equiv |j_{g_1}\rangle \otimes ... \otimes |j_{g_m}\rangle$ and the sum of all squared coefficients is equal to 1.

We remark that the notion of entanglement is at the very heart of quantum mechanics. However, although it is widely used in quantum information, the understanding of the physical essence of entanglement is far from complete, see, e.g. [12] for debates. Nevertheless, in the quantum community there is the complete consensus that entanglement implies *correlations* – in our epigenetic modeling these are correlations between epimutations in different genes.

Typically in quantum foundations experts emphasize that correlations corresponding to an entangled state are "superstrong", i.e., they have amplitudes exceeding amplitudes which are possible for classical correlations. However, for

[5] Depending on the biological context, it is always possible to select a few epimutations of the main importance. Hence, the number k_g need not be very large. We state again that our model is operational. It need not be very detailed.

our modeling of epigenetic mutations, the debate on nonclassicality and on superstrength of entanglement-correlations is not important.

The key point of the application of entanglement in modeling of epimutations is its *nonlocal feature.* We remark that the notion of quantum nonlocality is often used vaguely. Typically, especially in relation to Bell's tests, nonlocality is treated (following D. Bohm and J. Bell) as nonlocality of hidden variables, i.e. those deterministic variables the existence of which is so actively debated in quantum foundations, since the very beginning of quantum mechanics. In our cell biological studies we cannot assume such "real physical nonlocality" of variables describing gene expression. We do not appeal to physical quantum effects in a cell. We state again that the origin of QL-representation is uncertainty in information about the cell's behavior in an experiment. We shall appeal to another sort of nonlocality which may be called intrinsic quantum nonlocality.

The form of the tensor space representation (5) of potential epimutations in the cell's genome implies that epimutation in one gene will immediately imply consistent epimutations in other genes. If the state (5) is not factorized, then by acting, i.e., through change in the environment, to one gene, say g_1, and inducing some epimutation in it, we can induce consistent epimutations in other genes.

This quantum nonlocality is the main source of the speedup of quantum computers. However, we do not advertise a rather common viewpoint that biological quantum computing plays some role in genetics and the brain's functioning. Quantum algorithms are based on *unitary dynamics* described by the Schrödinger's equation. In our opinion such dynamics cannot survive on the biological scales of space, time and temperature. In our QL-model a cell is an open QL-system; its dynamics is described by the quantum master equation; it is nonunitary. In particular, quantum entropy is not preserved, cf. Asano et al. [5] for QL cognitive modeling.

In our QL-model we also explore the intrinsic quantum nonlocality to speed up the epigenetic evolution in a living cell. Otherwise, i.e. by using a purely neo-Darwinian approach[6], we would be not able to explain the high speed of the epigenetic evolution. Evolution in the case of epimutations in a large number genes as the reaction to the environment would be too slow if epimutations inducing new levels of gene expressions would be randomly and independently generated and then selected.

Let an environment act on genes $g_1, ..., g_m$. Suppose that, for, e.g. g_1, as an individual gene, some epimutation, say M_{g_1} can be useful in this environment.

[6] Nowadays the term neo-Darwinism is used for theory of evolution, driven by natural selection acting on variation produced by *genetic mutation and genetic recombination.* Thus the Darwinian model of evolution liberated from ideas of Lamarckism was combined with genetics. Purely random genetic variations are subjected to natural selection under the environmental pressure. Neo-Darwinism is based on the main postulate of molecular biology: in a cell the information flow is possible only in one direction from DNA (RNA) to proteins, from genotype to phenotype; so, never backward: from phenotype to genotype.

However, this epimutation may disturb the functioning of other genes in a negative way. Hence, epimutations $M_{g_1}, ..., M_{g_n}$ induced by the environment have to be consistent. How can they become consistent? Either via iterations, first the state of epimutations $(M_{g_1}, ..., M_{g_n})$ is created, but the cell "feels" disagreement between levels of gene expression corresponding to these epimutations. New epimutations are induced by this inconsistency and so on. This process is similar to Darwinian natural selection and approaching consistency in gene expression would take too long a period (for the time scale of one living cell). Our proposal is that the dynamics are entangled and at one step all genes epimutate consistently. We state again that the main difference from quantum computing is using nonunitary evolution described by the quantum master equation, instead of the unitary (Schrödinger) evolution. Hence, we use entanglement, but without unitary evolution.

We now proceed by operating with epigenetic markers of any origin as information quantities, i.e. without coupling each of them with a special form of cellular material. We enumerate all possible epigenetic markers which are involved in the process of evolution under the pressure of some fixed environment, $j = 1, ..., n$. Each marker can be quantified by the classical random variable $\xi_j = 1$, if this marker is created and then inherited, and $\xi_j = 0$, in the opposite case. These are observables which can be measured in experiments. The space of all classical states of the epigenome consists of vectors corresponding to fixing the values of all epigenetic markers: $\alpha = (\alpha_1, ..., \alpha_n)$, where $\alpha_j = 0, 1$. This classical state space consists of 2^n points. This space is the basis of the classical information description of the process of epigenetic evolution. However, we move to the quantum information description by assuming that classical states can form superpositions. To match with the Dirac ket-vector notation which is used in quantum physics, we denote the classical state α as $|\alpha\rangle$. Then the QL state space of (possible) epigenetic mutations, H, consists of superpositions of the form

$$|\psi\rangle = \sum_\alpha c_\alpha |\alpha\rangle,$$

where $\sum_\alpha |c_\alpha|^2 = 1$. This is the complex Hilbert space of the dimension 2^n. Now we repeat our previous considerations for epimutations of the chromatin-marking type. The QL adaptive dynamics described by the quantum master equation can be considered as a mixture of neo-Darwinian, neo-Lamarckian, and Wrightean evolution. This cocktail of stochasticity and determinism is consistently represented in the QL operational framework. The final steady state gives experimenters the classical probability distribution of the inherited epigenetic markers.

Entanglement may play an important role in the speedup of the epigenetic evolution. Since epimutations of the chromatin-marking type can be coupled to physical carriers, it was easy to use the standard notion of entanglement (as entanglement of systems) in the epigenetic framework. In general epigenetic markers are merely information structures in a cell such as, e.g. self-sustaining regulatory loops. However, we are lucky, since recently a new general viewpoint on entanglement was elaborated in the quantum information community. Entanglement can be considered not from the system viewpoint, but from the observer viewpoint. One considers a family of algebras of observables, say $\{\mathcal{A}_i\}$, on the total state space, in our case on H. Under some restrictions on these algebras the state space can be represented as the tensor product of subspaces corresponding to these algebras. In our case we consider algebras of observables corresponding to different epigenetic markers; corresponding subspaces are two dimensional qubit spaces, $H = \otimes_{j=1}^{n} H_{j;\mathrm{qubit}}$. Now we can use the notion of entanglement corresponding to this tensor product decomposition of the state space and repeat the speedup argument.

5 Concluding Discussion

As was pointed by one reviewer, "in standard quantum mechanics, the coefficients appearing in the decomposition of a superposed pure state (in a given basis) determine probabilities that do not admit an epistemic interpretation: they are ontic and cannot be interpreted in terms of uncertainty. This means that a GKSL transition changes the interpretation of the probabilities (ontic→epistemic). It seems instead that an uncertainty interpretation of the coefficients of a superposed pure epigenetic state is assumed here which plays a basic role. I think that the authors should clarify this point. My personal opinion is that an ontic interpretation of such coefficients would be compatible with the fact that epigenetic mutations are indeed potential, and the interaction with environment only actualizes some of them."

This is a fundamental interpretational issue of quantum theory; in fact, the issue of interpretation of the quantum state. This problem can be characterized by a huge diversity of opinions. The interpretation presented by this reviewer [18] is quite natural, but not the only one possible. In this paper we are not able to discuss this interpretational problem in more detail, see [1,2,18]. We keep to the subjective probability interpretation proposed recently by C. Fuchs. In this interpretation the quantum formalism is about the experimenter's knowledge and not about reality itself.

As was pointed out by another reviewer, "for Lamarckian evolution to proceed each cell must "know its epigenome" and the overall changes that have occurred in it through a process that is "non-local". This claim seems to me to be too strong for two reasons. First, Kimura postulated and won the Nobel Prize for his 'neutral current of evolution', holding that cellular evolution only occurs in non-functional areas that have not already been solved by evolution; e.g. not in heart muscle; not in eye tissues; etc. Thus, evolution should occur in non-functional areas of the DNA known as "junk" DNA. Second, authors claim that somehow the cell is not isolated from the environment, that it collects information from it as "environmental pressure" sufficiently well-enough to contradict neo-Darwinian evolution which proceeds with information flow in only one direction, i.e. "from "genotype to phenotype". Authors claim that cells know what all other cells perceive and how they change "instantly", but how does the cell know what is happening outside its immediate environment that causes cellular change from its "environmental pressure?"

This is an important comment which enlightens the most important aspects of our approach. First of all we state again that the present QL model was elaborated only for *epigenetic evolution*, i.e. we do not model changes in the genome, but only in the epigenome. Thus our epigenetic model of evolution does not contradict Kimura's model of neutral evolution on the level of the genome.[7] Kimura did not deny completely the importance of the environment pressure (on the genetic level). Kimura argued that molecular evolution is dominated by selectively neutral evolution, but at the phenotypic level, changes in characters were probably dominated by natural selection rather than sampling drift. Our model is also about phenotypic changes, but on the epigenetic level, and natural selection on the cellular level cannot contribute so much to the process of evolution, because the epigenetic changes have to happen inside a single generation. Nevertheless, our QL open system dynamics can be treated as a kind of natural selection, but on the molecular level in each single cell.

Our QL-model does not contradict the orthodox neo-Darwinian evolution which proceeds with information flow in only one direction, i.e. from "genotype to phenotype" since we do not claim that the genome is changed under the pressure of the environment (although such experimental data are intensively discussed in biological publications; see also [13]).

Finally, we point to cells' signaling as one of the basic mechanisms of epigenetic evolution and also cells' differentiation (the latter can be considered as a special case of epigenetic evolution). Following quantum tradition we use the terminology "instantly" for changes in cells, but this is (as well as in physics) just a mathematical metaphor[8] encrypting considerations of two time scales, one fine (for dynamical changes in epigenome) and another rough (for evolutionary-output changes through stabilization to the steady state).

For example, consider *E. coli diauxie.* Because E. coli receives the environmental condition (glucose/lactose), it changes its gene expression, adapts to its environment, and finally the genotype/phenotype is transfered to the next daughter cells. But we call this phenomenon adaptation not epimutation. And even in narrower sense, the epimutation at chromatin modification level occurs in a similar way as the diauxie of E. coli. By receiving environmental information, the signal changes the cellular metabolism and also the modification system of DNA and finally such modification changes give the Lamarckian evolution (=adaptation) of the cell itself and sometimes the changes can be transmitted to the next generation (=epimutation = Lamarckian evolution). These adaptation can be possible with *the signal transduction network in the cell.* And we demonstrated (by a mathematical QL model) that really living systems having such network interacting systems of every composite element can behave as a quantum computer with entangled state of any kinds of responses finally giving decoherence of the final answer to the environmental pressure (observation). So every cell knows its environment at any time and this knowledge is "nonlocal" (at the time scale of cellular generations).

[7] In our opinion, Kimura proposed that evolution or DNA changes occur neutral, but he did not propose that evolution occurs at non-functional areas. Even at functional areas DNA changes occur at any time, but when the phenotype of functional change appears (for example at heart muscle), we really have evolution making the birth of new species.

[8] For example, in quantum formalism one speaks about collapse of the wave function which happens instantly. However, by proceeding more carefully one talks about change of the quantum state as the result of measurement or more generally interaction with an environment. And such a process is not instantaneous.

Acknowledgments. This paper was finished during the visit of A. Khrennikov to the Center of Quantum BioInformatics of Tokyo University of Science, February-March 2013.

References

1. Khrennikov, A.: Ubiquitous Quantum Structures: From Psychology to Finances. Springer, Berlin (2010)
2. Haven, E., Khrennikov, A.: Quantum Social Science. Cambridge Press, Cambridge (2013)
3. Asano, M., Ohya, M., Khrennikov, A.: Quantum-like model for decision making process in two players game. Found. Phys. **41**(3), 538–548 (2010)
4. Asano, M., Ohya, M., Tanaka, Yu., Khrennikov, A., Basieva, I.: On application of Gorini-Kossakowski-Sudarshan-Lindblad equation in cognitive psychology. Open. Syst. Inf. Dyn. **17**, 1–15 (2010)
5. Asano, M., Ohya, M., Tanaka, Yu., Khrennikov, A., Basieva, I.: Dynamics of entropy in quantum-like model of decision making. J. Theor. Biol. **281**, 56–64 (2011)
6. Asano, M., Basieva, I., Khrennikov, A., Ohya, M., Tanaka, Yu., Yamato, I.: Quantum-like model for the adaptive dynamics of the genetic regulation of E. coli's metabolism of glucose/lactose. Syst. Synth. Biol. **6**, 1–7 (2012)
7. Asano, M., Basieva, I., Khrennikov, A., Ohya, M., Tanaka, Yu., Yamato, I.: Quantum-like model of diauxie in Escherichia coli: operational description of precultivation effect. J. Theor. Biol. **314**, 130–137 (2012)
8. Dzhafarov, E.N., Kujala, J.V.: Quantum entanglement and the issue of selective influences in psychology: an overview. In: Busemeyer, J.R., Dubois, F., Lambert-Mogiliansky, A., Melucci, M. (eds.) QI 2012. LNCS, vol. 7620, pp. 184–195. Springer, Heidelberg (2012)
9. Busemeyer, J.R., Bruza, P.: Quantum Models of Cognition and Decision. Cambridge Press, Cambridge (2012)
10. Ohya, M., Volovich, I.: Mathematical Foundations of Quantum Information and Computation and Its Applications to Nano- and Bio-systems. Springer, New York (2011)
11. Jablonka, E.J., Raz, G.: Transgenerational epigenetic inheritance: prevalence, mechanisms, and implications for the study of heredity and evolution. Q. Rev. Biol. **84**, 131–176 (2009)
12. Khrennikov, A.: Contextual Approach to Quantum Formalism. Springer, Berlin (2009)
13. Ogryzko, V.V.: A quantum-theoretical approach to the phenomenon of directed mutations in bacteria (hypothesis). Biosystems **43**, 83–95 (1997)
14. Ogryzko, V.V.: On two quantum approaches to adaptive mutations in bacteria. http://arxiv.org/ftp/arxiv/papers/0805/0805.4316.pdf
15. McFadden, J.J., Al-Khalili, J.V.: A quantum mechanical model of adaptive mutation. Biosystems **50**, 203–211 (1999)
16. McFadden, J.: Quantum Evolution. Harper Collins, London (2000)
17. Donald, M.J.: A Review of Quantum Evolution. quant-ph/0101019
18. Atmanspacher, H., Primas, H.: Epistemic and ontic quantum realities. In: Adenier, G., Khrennikov, A.Yu. (eds.) Foundations of Probability and Physics-3. Conference Proceedings, vol. 750, pp. 49–62. American Institute of Physics, Melville (2005)

Quantum Interactions as Niche-Structure

Michael J. Pocklington(✉)

Department of Genetics, University of Leicester, Leicester LE1 7RH, UK
mjp7@leicester.ac.uk

Abstract. In the field of systems biology, the molecular interactions constituting the life of organisms may be represented in interaction maps. The genetic interaction map is unique because it concerns only the interactions of an organism, occupying its niche, producing biological fitness. The procedure for obtaining this map can be idealised. Underlying the map for any occupied niche there must exist a temporally closed (developmental) information-processing structure, or causal plexus. This must be located within a temporally open (evolutionary) niche-structure, or causal nexus. The framework allows simplification naturally, identifying molecules and sub-molecular particles also as occupied niches. Quantum interactions are then rendered as the merging and splitting of niches, the flux determined by the entropy gradient. Across all levels of the self-similar niche-structure continuum (the causal nexus), occupied niches determine the construction of empty niches, and empty niches provide the potential for the evolution of the occupied niche-structure.

Keywords: Quantum interaction · Niche-structure · Natural interaction map · Imaginary chemostat · Causal plexus

1 Introduction

In this paper I wish to draw attention to an increasingly popular visualisation technique in the field of systems biology, the interaction map. There are many different types of interaction map, but the genetic interaction map is special. It visualises the network of interactions constituting organismic function using information gathered from genetic mutations, using only one kind of measurement: the measurement of fitness. Organisms may be broadly construed as entities defined by their ability to occupy a niche, allowing generalisation beyond biology; and the map illuminates exactly those interactions yielding entity survival. In other words, they are *the interactions that matter.*

In its experimental production, the genetic interaction map emerges from pairwise combinations of mutations in genetically tractable micro-organisms. When they are introduced into the same cell, two mutations are seen to be functionally independent if the double mutant has a fitness deficit that is the expected product of the mutations acting alone. A genetic interaction refers to the case giving an unexpected fitness measurement against the aforementioned

H. Atmanspacher et al. (Eds.): QI 2013, LNCS 8369, pp. 335–347, 2014.
DOI: 10.1007/978-3-642-54943-4_31,

background. For example, two minor mutations, placed together in the same gene, could unexpectedly cause the gene product to catastrophically fail. On the other hand, two serious mutations placed together in a gene could unexpectedly compensate for each other. The interaction shows that the mutations have some functional relationship to each other; in this example, physical proximity in a gene product such as a protein, which could be mis-folded.

Genetic interactions between pairs of mutations are not limited to co-location in the same gene. This is because the molecular activity of the cell is embodied in space; it is organised in pathways and physical structures which are encoded by separate genes. Mutations affecting adjacent products in pathways, or contained in structures in physical contact, will reveal interactions linking separate genes. This is both the theory, and the observation. Very large numbers of pairwise interactions of mutations, combined into a map, may usefully depict every genetically-determined function of an organism, organised into a functionally coherent whole (see Fig. 1; [1]). It is not to be confused with other types of interaction map, or with genetic regulatory networks.

The earliest interaction maps were drawn by [2], when they were called genetic complementation maps. There was some early discussion concerning the philosophy underlying the interaction map itself; but then, as now, interest was directed at the content of the map. We shall not be interested in the content of the map. We are interested in what sorts of entities can yield such a map, and what the map actually means, i.e. the type of territory that the map depicts.

In order to address such questions, we will employ a thought-experiment. We will use existing knowledge to recursively define and idealise the procedure for obtaining the map. This gives the natural genetic interaction map, which is unique and perfect for every organism. Because it is perfect, it can never be completely achieved other than by the idealised method outlined here. The natural genetic interaction map may be likened to the square root of 2, which can not be expressed as a precise numerical value, but can be defined by an exact procedure.

In the laboratory, an artificial environment known as a chemostat can be used to obtain the fitness data for the map. In this paper, an imaginary chemostat is the idealised method for determining fitness. Properly constructed, the imaginary chemostat must produce a fitness measurement that is a probability per unit time that the mutant organism will be lost from the chemostat. Significantly, this measurement of fitness is not constrained by ideas about what constitutes an *organism*. Entities other than classical organisms could maintain themselves in a chemostat. Similarly, the idea of a *gene* is here superseded by an operational definition. Any defect, heritably transmitted by any entity, could have its fitness consequences measured, and would yield a mark on a map. In biological organisms, this includes epigenetic information. Heritable transmission outside of canonical genetics is found in realms other than organisms, such as systems of human organisation and production, containing such things as programs, propositions, procedures, protocols, plans, concepts, and customs. Heritably transmitted information is also contained in realms lacking conspicuous symbolic content, such as those described by chemical or electronic feedback loops and networks. Examples of the latter are communication networks and brains.

This method of visualisation suggests a single extended framework for understanding nature that is not merely a unity of representation. Not only are niche-filling entities conspicuous by their abundance in nature, but they are linked in ecosystems continuously across different realms. Here we extend the framework from biological organisms downwards to entities in the quantum realm. The framework does not suffer from a classical provenance, and has much in common with the quantum theory.

2 Construction of the Genetic Interaction Map

Recent advances in automated high-throughput technology and software techniques have allowed the construction of interaction maps for free-living organisms (Fig. 1; [1]). Construction of genetic interaction maps relies on the expectation that if two mutations are functionally independent, the fitness phenotype of the double mutant would be expected to be the product of the measured fitness phenotypes of the two mutants. Suppose we have two functionally independent mutations, each with a fitness of 0.5 relative to the strain without a mutation (the wild-type). Then if the two functionally independent mutations are introduced into the same genetic background, the relative fitness of the double mutant is expected to be 0.25. Deviations from this expectation imply the existence of functional associations that embody the life of the organism. Here is not the place to enumerate the various ways in which mutations can interfere with each other to produce the deviations seen, but the interested reader may find useful [1,4] and the references contained therein. Be aware that the classical study of epistasis, which holds genetic interactions as its subject of interest, tolerates many mathematically courageous formulations that are not relevant to our current analysis.

The archetypal measurement of fitness with the greatest sensitivity is to grow each mutant together with the wild-type strain under competition in a chemostat. A chemostat allows the continuous culture of organisms under essentially constant conditions. It comprises a culture vessel through which there is a continuous defined flow of nutrient medium. Provided the flow rate of the medium does not exceed the growth rate of the organism, the organism will survive in the chemostat, and essentially constant conditions are obtained. *Essentially constant* means that averaged over the duration of many inhabited chemostats of the same type, overall conditions will have a constant distribution i.e. there exists a definite state space in the sense of dynamical systems theory.

With a few well-known and rare exceptions, the wild-type strain will eventually displace the mutant strain in the chemostat, and the time taken for it to do this is the measure of the (lack of) relative fitness in the mutant. Precision in the measurement of fitness can be increased by repeated measurements. There is no value in combining together more than two mutations. This is because we wish to obtain a map, for which pairwise interactions are sufficient. The organism under study needs to be genetically pure except for the mutations in question.

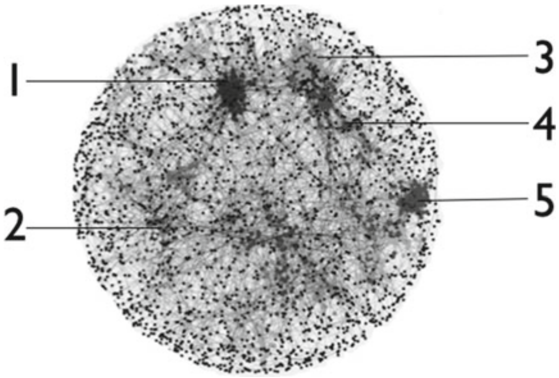


Fig. 1. The genetic interaction map for yeast *Saccharomyces cerevisiae*, redrawn from [1]. Each node (dot) represents a gene, but each gene could in turn be comprised of thousands of nodes, each representing a single mutation. Clusters of nodes show interacting genes, participating in identifiable cellular functions; here there are shown five labelled examples: 1, secretion; 2, ribosomal functions; 3, glycosylation and cell wall biosynthesis; 4, cellular morphogenesis; 5, DNA replication and repair.

Graphical techniques can incorporate fitness data for all the pairwise combinations of mutants represented as nodes in a static graphical plot. This can be achieved in a variety of ways in practice, (see for example Refs. [1,4]); but for our purposes we need only imagine one simple method. For each pairwise mutation-combination, the deviation from expected fitness is a precisely measured quantity. This quantity can be used to establish an edge length between each pair of nodes, by taking the mean edge length between all the nodes and adjusting it for each pair of nodes, bringing two nodes closer together if they show a deviation from expected fitness (which indicates an interaction). Pairs of mutations showing a large deviation in fitness will assume a position in close proximity to each other, provided other mutation-combinations allow it; whereas pairs of mutations showing no deviation in expected fitness will come to occupy positions on opposite sides of the map, were it not for connections with other mutations. The genetic interaction map in (Fig. 1), redrawn from Reference [1] illustrates a similar visualisation method to the one described above. The resulting map of interactions points to the underlying information processing structure, exemplified in biochemical pathways and signal transduction pathways, and constituting the computational activity of the organism (see, for example [3]).

Specific environmental perturbations (changes in the nutrient medium or chemostat parameters) can be contrived, producing nodes on the map representing environmental conditions that will connect to the nodes representing genetic elements involved in dealing with those conditions. The network of interactions therefore reaches out into the environment, to include nutrients, toxins, and physical conditions.

3 The Imaginary Chemostat

Recall the measurement of fitness, in which two types of organism are grown together in the laboratory chemostat until only type one remains. The procedure can be repeated until a statistically relevant result is obtained. Since we know the map depends for accuracy on an arbitrarily large number of runs of the chemostat, let us imagine being able to make a very large number of runs (say 10^{100}) for each mutant. The chemostat conditions for the runs would have to be standardised, which we can imagine without being specific. We would expect the interaction map to sharpen up indefinitely; but we could expand the view of the map and continuously inspect it for greater detail. Since we would now be making a practically impossible number of measurements, we might just as well free the chemostat from any other practical limitation, such as size and shape of containing vessel, and constituents of the medium flowing through the chemostat.

For the measurement of fitness it is necessary that an organism must not mutate, in other words, we must be able to stabilise (or freeze) genetic information. This is a requirement for holding constant the object of inquiry.

We need not assume that genetic information must be carried by DNA, because we could collect every different mutant that could ever occur in our organism by waiting until such a mutation were to occur; and then stabilise it. A mutation identifies an inherited trait. If an inherited trait could be carried by self-replicating RNA, or a methylation pattern independent of the DNA base sequence, or a prion, or indeed anything at all, we could imagine stabilising it, isolating the strain of organism carrying it, and measuring its fitness in the same way as for any other kind of mutant. Since we have fully defined the environment in which the organism survives, any mutation, and hence functional genetic information, is operationally defined by its effect on fitness. If it has an effect on the life of the organism, it will appear as a node on the map.

4 What the Genetic Interaction Map Means

As is well known, the map and the territory are not the same thing. In its construction, the genetic interaction map is a static graphical plot of failures in fitness due to mutant gene combinations; so what, exactly, is the territory? It appears intuitively obvious that the map refers to a network structure describing all the molecular activities of the organism; indeed, the integrity of the entire field of molecular biology rests on the existence of such a structure. An underlying functional network may be held in the mind of the biologist (for examples, see Refs. [3,5,6]) but only within the context of experiments and contrived observations, or in some other way disconnected from the selecting environment. An equivalent interpretation of this functional network is that it corresponds to naturally-selected biological computation in the organism. Here, we identify it as the whole functional network corresponding to the survival-strategy of the real organism, naturally selected in its real environment. It must be a unitary object because, insofar as we can freeze the genetic information that defines the organism, the map is singular and precise, and it must therefore be temporally closed. In other words, we must have accounted for all the possibilities for the developmental trajectory of the network. Since we understand that the network depicts, for a genetically-defined organism, the causal flow of information, energy and materials, we may call it a *causal plexus*. This distinguishes it from the wider, temporally open, *causal nexus*.

5 Conceptual Simplification

Here we aim to conceptually simplify the system, perhaps to reveal its fundamental nature. This process of conceptual reduction I will call *reductio ad extremum*. Starting with an organism such as yeast, we could grow it in increasingly permissive (that is, less demanding) imaginary chemostat environments. Then we could delete the unnecessary genes, giving us a simpler type of organism, and a simpler map. We could delete the genes encoding metabolic enzymes, because we could supply their products in the chemostat medium. Functions such as sporulation, sexual reproduction, stationary phase, aerobic respiration, and much regulatory activity would be redundant in a suitable chemostat, and the genes encoding these functions could be deleted. We would still have an organism, a niche, a map, and an underlying causal plexus.

Continued survival of an entity in the chemostat need not rely on cellular replication, since continuous occupation of the chemostat vessel could be achieved by the growth of hyphae, that is, without cellular separation. A cellular structure involving membranes is not required, since a structure such as a polymeric molecule could propagate in the chemostat. We could delete the genes encoding the system for energy metabolism, but then we would have to supply (for example) ATP in the chemostat medium, otherwise the individual entities would not be able to grow. However, if the entity was simple enough, we would not need an energy currency such as ATP; instead, we could rely on a supply of energetically activated precursors for building the structure. Expressed more generally, an entropy gradient would, at this point, to be required.

After many rounds of *reductio ad extremum* we would be left with a simplified organism, and a simplified interaction map (Fig. 2). Eventually, we would arrive at either of two possibilities.

In the first possibility, we could follow the process of *reductio ad extremum* to the point where the entity in the imaginary chemostat would assemble spontaneously from the precursors supplied in the chemostat medium. Inoculation of an existing entity would not be necessary. We could separately make defined changes ("mutations") to the medium entering the chemostat, and likewise to the entity in the inoculum. This would yield a genetic interaction map in which all the mutations would map to the chemostat itself (the medium, or the vessel walls). Mutations made to the entity in the inoculum would make no difference to the presence of the entity in the chemostat; they would not be represented on

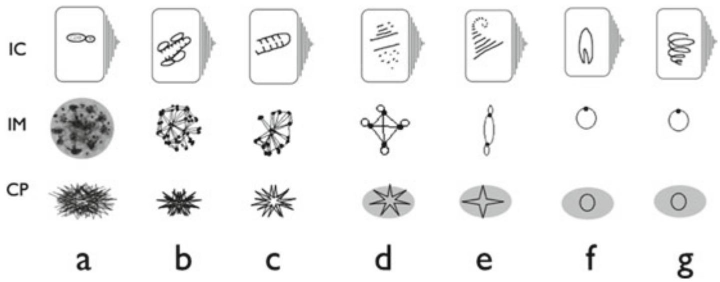


Fig. 2. Examples of entities in the conceptual simplification. Left to right, increasingly permissive medium. IC, imaginary chemostat. IM, imagined genetic interaction map. CP, imagined causal plexus. a, yeast. b, ribonucleoprotein complex. c, nucleic acid. d, self-templating polymer or crystal. e, non-templating polymer growing by terminal addition of subunits. f, flame. g, vortex.

the map. We could call the entity a *nonsicle* (non-cycle), and we could call the highly contrived medium an *assertive medium*.

In the second possibility, occupation of the chemostat would depend upon an initial inoculation with a niche-filling entity. In this case, it would indeed be possible to produce an interaction map for the inoculated entity. We could call the chemostat environment a *permissive medium*. A familiar example is to be found in the commonly-used polymerase chain reaction (PCR). Here, the permissive medium supplies energetically-activated monomers, polymerase enzymes, pulses of heat to allow separation of the hydrogen-bonded strands, and a suitable aqueous environment. We could usefully hold a generalised conception of a PCR-like reaction for open-ended propagation, while recognising that self-templating molecules need not be limited to nucleic acids. After this fashion, a large variety of simple self-reproducing physical entities could be propagated in an imaginary chemostat, since the medium, physical activity, and vessel walls of the imaginary chemostat are not limited in any way.

We could obtain interaction maps for crystals, micelles, hydrophobic complexes, or propagating molecular agglomerations generally (agitation or heat pulses will be assumed). All that is necessary is that the entities will fill a chemostat by growth from an inoculum, thereafter to propagate and survive. We can imagine other well-known self-maintaining entities, such as those generally described as dissipative structures: a non-exhaustive list would include flames, corrosions, vortexes, avalanches, and dust-bunnies. We expect to be able to construct imaginary chemostats, and derive interaction maps for all of the aforementioned entities.

The functional networks underlying the very simplest physical entities in the examples above could be described as limit-stable positive feedback loops, or state-cycles (Fig. 2). These state-cycle "organisms" will have few-node interaction maps, perhaps corresponding to the sites for addition of subunits, or sites of physical fragility.

6 The Molecule as an Occupied Niche

Consider the case of a population of entities such as crystals or polymeric molecules, as described above, propagating by cycles of breakage and growth within a permissive medium, with the potential to indefinitely occupy the imaginary chemostat. We could ask how many individual molecules there would have to be in the chemostat to qualify as a replicating entity. Measurement of survival in the imaginary chemostat requires only that an organism or entity can be reliably observed. We are forced to conclude that to qualify as an entity occupying the chemostat, we do not need a population larger than one (or two, depending on the stage of replication).

Thus the entity occupying the niche could be a single polymeric molecule, replicating by a cycle of growth and fragmentation, in a suitable *permissive medium*. Let us now ask if we can simplify it further. Observing a cycle of polymerisation and breakage, we can reduce this conceptually to a dimer. In other words, the entity inhabiting the imaginary chemostat is now a dimeric molecule, dissociating into monomers under the conditions provided by the permissive medium, such that each monomer is then available for attachment to an activated monomer, again provided by the permissive medium, eventually forming a new dimer. The cycle of reproduction would continue, and the chemostat would be filled. We continue to see an entity in a niche; we could obtain a "mutant", and construct an interaction map.

This can be simplified even further without departing from the conceptual framework. The entity occupying the chemostat does not have to be a homodimeric molecule; it could be a heterodimer. How large does the smaller subunit of the heterodimer have to be, to qualify for restoring the entity to a mature form? It could be a single atom, say of oxygen; and thus our entity could be maintained indefinitely through cycles of oxidization-reduction in a suitable chemostat. But why restrict our notion of dynamic identity to oxidation-reduction cycles? Why not maintain identity through addition and removal of a single electron? Or a photon?

If it was a photon, we would now have reduced the entity inhabiting the imaginary chemostat, originally undeniably an organism within a niche, to the size and complexity of a single excitable molecule propagating through time in a permissive medium that supplies an energy gradient.

Surely, niche-filling does not reduce to excitability? After all, a single stable molecule in the ground state is hardly any different from its dynamically excited counterpart, and it, too, propagates through time, by virtue of its internal sub-molecular interactions. We should therefore consider the dynamically excited molecule and its ground-state counterpart as separated by a step in appearance, but not a jump in realm.

In either case, the imaginary chemostat is not now a container; instead, it is a localized arrangement of energy (a container of sorts) where the positional information in the trajectories of the sub-molecular particles in the molecule itself constitute self-maintaining structural information. Note that the imaginary chemostat continues to function as before: as a niche-filling entity. The molecule does have a measurable fitness, since it could (and inevitably would) disintegrate with some non-zero probability, were we to "mutate" the positional information carried by atomic structure. We could draw an interaction map, by "mutating" the molecular structure. Now we would be interfering with the energy states of electrons, atoms, and perhaps other particles, thereby changing the fitness (stability) of the molecule. If we could interfere with these vital molecular components one at a time, and in pairwise combination as we did for a biological organism, we could obtain a "genetic interaction map" for the molecule.

It is likely that any or all of the above features of niche-filling entities could be encompassed in some kind of taxonomic structure. The current framework is not constrained by taxonomic considerations. Instead, it draws attention to a type of structure that demands a taxonomy. However, such a taxonomy, even if it was useful, could make no difference to the underlying reality.

Any suggestion that there is more than one type of natural entity in the chain from molecule to organism must suggest a stage in the proposed sequence of r*eductio ad extremum* at which one type of entity becomes another. For example, proposed niche-filling entities that have not yet been shown to arise in nature (such as replicating molecules) might be dismissed from constituting examples of *real* entities, thereby disallowing the *reductio ad extremum* to continue beyond this point, and imposing an early discontinuity in the type of entity. However, this suggestion would continually have to be updated to take into account the discovery of self-maintaining molecular systems on other planets, and is therefore arbitrary, and must be discounted.

Thus we find that the framework provided by the imaginary chemostat does not suffer from the need to draw a discontinuity in the spectrum of complexity from the organism down to the molecule. It would recognise a discontinuity if such a discontinuity was found (as in the example of the nonsicle described above, in which the entity occupying the niche has an empty interaction map); but all the entities described above, from complex organisms down to molecules, have an equivalent type. They are all self-maintaining, niche-filling entities for which we can, in principle, derive an interaction map, in turn pointing to an underlying functional network, here called a causal plexus.

It is reasonable to suggest that if we can arrive at a molecule from an organism by *reductio ad extremum*, then it is trivial to extend this framework further downwards to include atoms and subatomic particles; indeed, it would be unnatural not to proceed with such an extension. This is because not only molecules, but all the components that make up a molecule, ultimately fermions and bosons, propagate through time in the same way relative to the bottom level permissive medium. This universality of behaviour for all entities at the level of the molecule and below finds expression in the quantum theory. We may identify the permissive medium as the level at which Planck-scale interactions occur.

In ecological niche-structures, the process by which one niche is successfully invaded by organisms from another may be called natural selection. Biologists routinely visualise this as a series of connected niches in a niche-structure, such as an evolutionary series of yeast, which might evolve to exploit a novel type of sugar molecule. Here, change in the overall genetic makeup of the organism is an example of downward causation [7], caused by natural selection for survival of novel mutants in successive niches. I will suggest that the same process explains the selection of an empty niche-structure in the permissive medium, and that this in turn constrains the propagation of particles through the permissive medium.

7 Discussion

We have seen that the occupied niche is a fundamental entity, not limited to populations of biological organisms, deserving of a special status, and here called the *causal plexus*. The imaginary chemostat was the tool of thought used to arrive at this fundamental entity, and to extend a framework encompassing instances of it, down to the quantum realm. I will argue elsewhere that this framework must encompass all of reality, including such things as human organisations, minds, and abstract structures such as mathematics, rendered as niche structures.

The occupied niche revealed in this way is a singular, perfect object. This is because the growth of organisms demands the definition of a niche which completely defines all of the possible states of occupation resulting in an exact conception of fitness. The measurement of survival, expressed as fitness, repeated indefinitely, can become as precise as we like, thus yielding the singular object for the precisely defined occupied niche and no other. Note that it is a conceptual object. The real organisms we see in the world around us are particular instances of this object. They are all different from each other. However similar they may be in appearance to each other, in genetic definition and environmental provenance, they will always be distinguishable by the method described (remember that real organisms can not be genetically "frozen"). This means that different individuals within a species, for example, will not occupy exactly the same niche, but a *niche-cloud*. Species have found a way to exploit niche-clouds through the generation and maintenance of appropriate variability.

But let us stay with the singular, conceptual object, the occupied niche. As well as having a singular nature, it is *temporally closed*; it is also dynamically self-renewing, provided it does not interact with another system. If it were to interact outside of its definition as described, it would have been changed; we would no longer looking at the same object, and its behaviour would not be defined in the framework. Its envelope, or interface with the environment, is part of its (self-renewing) functional network; it has zero-order interaction kinetics; and it can form new stable structures by engaging with other niches, forming a niche-structure. Interactions take the form of binary fusion, or they can disintegrate into a product-pair, the directionality provided by natural selection. The foregoing is a description of niches occupied by biological organisms (Sects. 4 and 5), yet it could also be a description of quantum particles. The explanation for the similarity of description offered here is that they are, indeed, the same type of object.

All interactions between them must also be of a single type, as visualised within the current framework. Consider the constitution of *the organism* as an occupied niche. It is visualised (Sects. 4, 5, and 6 above) as a functional network of molecular reactions. But the framework reveals molecules themselves to be occupied niches, so the molecular reactions within an organism itself must constitute an *internal* niche-structure. Here, energy and structural information (internal niche structure) may be seen as alternative visualisations of the same thing. Thus the framework as a whole describes a single universal niche-structure. There are no absolute levels of organisation spanning the niche-structure, since organisms not only eat each other, they may also eat molecules (or, indeed, photons) directly. Thus a level of organisation may be a pedagogical convenience, but a mature viewpoint sees just one, big, self-similar niche-structure, or *causal nexus*.

According to this niche-filling conception of reality, a particle is a locally stable occupied niche, or niche-particle, a *causal plexus*. Niche-particles are stable because on their scale of existence there is nowhere else for energy or structural information to go to. We could acknowledge the notion of a Hamiltonian system, but here we need only care about a defined system of occupied and empty niches with a precisely allowed quantity of energy for each niche-particle type. Since niches are tightly defined in themselves, not only must occupied niches have a singular identity, but energy (equivalently, structural information) must be conserved when changes in niche occupancy occur. This parallels the ecological case in which the flow of information or energy must be completely accounted for, and at least in principle, all the occupied niches identified, and their sources of energy and information traced.

A particle has to take some definite identity, and some people find it difficult to accept that this might be of a distributed form. Yet we are familiar with niches being distributed in space, so we should have no difficulty in thinking of an occupied niche constituting a fundamental particle as a particle's -worth of energy (niche-particle), necessarily carrying with it positional information (structural information) giving it a type-identity. As with any occupied niche conventionally understood, all a niche-particle can do is invade neighbouring niches that allow maintenance of energy (information). Since it has a singular nature, all energy and information must be accounted for. It could enter a neighbouring empty niche, losing identity in the previous niche but maintaining identity in the new niche (propagation); or, if the neighbouring niche is already filled, it could merge its energy and structural information with it, creating a new type of niche-particle with a type-identity embracing both of the contributing niche-particles (interaction). Run the interaction in reverse to allow the niche-particle to disintegrate. The entropy gradient provides directionality by supplying or removing niche-particles at either end of the niche-structure.

Returning to the bottom-level permissive medium, let us suppose that the current state of reality comprises niche-particles propagating through time and space, where the permissive medium comprises only empty niches, like a template or skeleton upon which occupied niches evolve. Where might these empty niches come from? Recalling that all niches are like islands in time, it could be suggested that the empty niches in the permissive medium necessarily take the form of closed time-symmetrical causal loops; necessarily, because energy, or structural information, can not be supplied or removed without reason. Closed time-symmetrical causal loops would be equivalent to the structures visualised by Feynman diagrams. We could further suppose that these structures could (indeed, must) be chained together over arbitrary distances in space, provided that they never violate closed time-symmetrical causality. Such a structure of empty niches could terminate at occupied niches (niche-particles), providing an opportunity for the niche-particles to invade the empty niche structure. In this case, the empty niche-structure (alternatively understood as a template of *virtual particles*) could be seen as naturally selected by the current state of all real niche-particles. We could say that the empty niche-structure is downwardly caused [7] by the current occupied niche-structure.

Since time is entirely constituted by change at the niche-particle level, the rate of change of the underlying permissive medium in creating the empty niche structure has no meaning, i.e., it is extra-temporal. Seen from the vantage point of occupied niches, any change in the empty niche-structure provided by the permissive medium would be instantaneous, and no structure deeper than the permissive medium is called for. This idea seems to resonate with the idea of the quantum potential of [8], (although here without a constraining mathematical formalism) as it could be said to be a self-organizing process of the underlying field that lays down a guide or template. Since there is no reason why the empty niche structures of the type suggested could not communicate through the whole permissive medium, it would effectively involve the *whole field*, and carry information about any experimental arrangement in which a particle would find itself.

Thus the niche-particle is trapped in a niche-structure selected by current reality. This would be an expression of the current experimental setup. If the experimental setup were to be changed, (in particular, the orientation or presence of slits, arms or mirrors) then the empty niche-structure, which defines the route the particle will take, must adjust to this changed conformation. Since time arises out of changes in the occupied niche-structure, not the empty niche-structure *beneath* it, any change in the empty niche-structure would always be perceived from above as instantaneous. There would be no constraint on the distance over which an empty niche-structure could develop; only that it should be maximally economical in an energetic (informational) sense. Thus it would conform to the principle of least action.

An approach taken by many investigators in the field of Quantum Interactions is to take an understanding of the quantum formalism, and apply it to problems resistant to other techniques. The approach taken in this paper was different, extending from obligatory biological foundations only. Revealed in the genetic interaction map was an image of "the organism" as a naturally-selected functional network, and then as an occupied niche. Reduction allowed the development of a framework, with quantum particles visualised as occupied niches, evolving upon an empty niche-structure, in turn selected within the permissive medium constituting Planck-scale interactions. The framework is simple, has no algebraic content, and may help towards a better understanding of quantum interactions, and to further the interpretation of the quantum theory generally.

Acknowledgements. I am grateful for helpful discussions with Vic Norris, Dominic Widdows, Peter Wittek, Terry Robinson and Mike Halsall.

References

1. Costanzo, M., Baryshnikova, A., Bellay, J., Kim, Y., Spear, E.D., Sevier, C.S., Ding, H., Koh, J.L., Toufighi, K., Mostafavi, S., Prinz, J., St Onge, R.P., VanderSluis, B., Makhnevych, T., Vizeacoumar, F.J., Alizadeh, S., Bahr, S., Brost, R.L., Chen, Y., Cokol, M., Deshpande, R., Li, Z., Lin, Z.Y., Liang, W., Marback, M., Paw, J., San Luis, B.J., Shuteriqi, E., Tong, A.H., van Dyk, N., Wallace, I.M., Whitney, J.A., Weirauch, M.T., Zhong, G., Zhu, H., Houry, W.A., Brudno, M., Ragibizadeh, S., Papp, B., Pal, C., Roth, F.P., Giaever, G., Nislow, C., Troyanskaya, O.G., Bussey, H., Bader, G.D., Gingras, A.C., Morris, Q.D., Kim, P.M., Kaiser, C.A., Myers, C.L., Andrews, B.J., Boone, C.: The genetic landscape of a cell. Science **327**, 425–431 (2010)
2. Benzer, S.: On the topology of the genetic fine structure. Proc. Natl. Acad. Sci. USA **45**, 1607–1620 (1959)
3. Bray, D.: Wetware: A Computer in Every Living Cell. Yale University Press, New Haven (2009)
4. Costanzo, M., Baryshnikova, A., Myers, C.L., Andrews, B., Boone, C.: Charting the genetic interaction map of a cell. Curr. Opin. Biotechnol. **22**, 66–74 (2011)
5. Jeong, H., Tombor, B., Albert, R., Oltvai, Z.N., Barabasi, A.-L.: The large-scale organization of metabolic networks. Nature **407**, 651–654 (2000)
6. Peter, S., Veloz, T., Dittrich, P.: Feasibility of organizations - a refinement of chemical organization theory with application to P systems. In: Gheorghe, M., Hinze, T., Păun, G., Rozenberg, G., Salomaa, A. (eds.) CMC 2010. LNCS, vol. 6501, pp. 325–337. Springer, Heidelberg (2010)
7. Campbell, D.T.: 'Downward Causation' in hierarchically organised biological systems. In: Ayala, F.J., Dobzhansky, T.G. (eds.) Studies in the Philosophy of Biology: Reduction and Related Problems, pp. 179–186. University of California Press, Berkeley (1974)
8. Bohm, D., Hiley, B.J., Kaloyerou, P.N.: An ontological basis for the quantum theory. Phys. Rep. **144**, 321–375 (1987)

Author Index