Studies of Numerical Scale Pedigree in Correspondence with Verbal Scale

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Abstract The analytic hierarchy process (AHP) allows decision makers (DMs) to entered judgments in the verbal mode. Obviously, a strict reliance on the corresponding Saaty 1–9 numeric scale can induce some inconsistency. Hence, it may be appropriate to calibrate the verbal scale in certain situations. Actually, various DMs have various numeric values even for the same verbal judgment. And for one and the same DM, the corresponding numeric values of verbal scales vary with judgment objects or object properties. We obtain a pedigree of numeric ratio scales and a transitive calibration, which marginally outperform the Saaty 1–9 scale in some instances. So, they may be useful substitutes for the Saaty 1–9 scale. And the pedigree can serve as a guide for DMs' confirming the numeric values of their verbal judgments.

Keywords Analytic hierarchy process · Expert choice · Numerical scale · Verbal responses

1 Introduction

According to Stevens' categorization [1], there are four levels of measurement. The levels, ranging from the lowest to the highest are Nominal, Ordinal, Interval, and Ratio. Each level has all of the meaning of the levels below plus additional

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Z. Wen and T. Li (eds.), *Foundations of Intelligent Systems*,
Advances in Intelligent Systems and Computing 277,
DOI: 10.1007/978-3-642-54924-3_68,

meaning. Nominal numbers, the lowest level in terms of the meaning conveyed, are just numerical representations for names. Ordinal numbers, as the name suggests, implies an order or ranking among elements. Interval scale data possess the meaning of Nominal and Ordinal data, as well as having meaning about the intervals between objects. Corresponding intervals on different parts of an interval scale have the same meaning. Interval level data can be used in arithmetic operations such as addition and multiplication. A mathematical definition of interval is: can be subjected to a linear transformation, or is invariant under the transformation $Y = aX + b$. Ratio level data (sometimes called ratio scale) is the highest level, having Nominal, Ordinal, and Interval properties, as well as the property of ratios. Corresponding ratios on different parts of a ratio scale have the same meaning. A mathematical definition of ratio is: admits multiplication by a constant, or is invariant under the transformation $Y = aX$. A ratio scale is said to have a true 'zero', however, the true zero can be conceptual and need not be observable. A ratio scale is often defined as one having a true zero point. However, for our purposes, it is easier to think of a ratio scale as one for which equivalent ratios are considered equal. Now, we discuss the ratio scale.

The analytic hierarchy process (AHP) developed by Saaty [2] has been widely applied to decision-making problems. AHP uses hierarchic structures to represent a decision problem and then develops priorities for the factors based on expert's judgment. The most remarkable advantages of AHP are that it produces numerical priorities from the subjective knowledge expressed in the estimates of pairwise comparison matrices (PCMs). Therefore, it can be employed to solve complicated, unstructured decision problems in many fields, such as the economic analysis, urban or regional planning and forecasting, etc [3–8].

Expert Choice (EC) allows you to enter judgments in either numerical, graphical, or verbal modes. Each judgment expresses the ratio of one element compared to another element. One of the strengths of AHP is that it allows decision-makers (DMs) to specify their preferences using a verbal scale. In the case where an AHP expert is helping a group or individual make a fuzzy decision, this verbal scale can be very useful. The verbal scale is essentially an ordinal scale. The question discussed in this paper is: given that a DM has entered judgments in the verbal mode, what numeric scale should be applied? Representatively, the Saaty 1–9 numeric scale is used. The question of scale has infrequently mentioned. Actually, the choice of numeric scale is an open research issue (see [9]). And it must be pointed out that any ratio scale can be used in this method. However, the 1–9 scale has proven to be an acceptable scale and is recommended for use in the AHP [10−12].

We discuss a sort of ratio scales, and produce a pedigree of numeric scales. And then we also obtain a transitive calibration based on the experimental results of Queen's University [13] and some assumptions. We then use these scales to evaluate the verbal assessments for a standard AHP problem: the relative distance from Philadelphia to six other world cities. Based on this example, the scales on

Table 1 The Saaty 1–9 scale

Numerical value	Verbal scale	Explanation
1.0	Equal importance of both elements	Two elements contribute equally
3.0	Moderate importance of one element over another	Experience and judgment favor one element over another
5.0	Strong importance of one element over another	An element is strongly favored
7.0	Very strong importance of one element over another	An element is very strongly dominant
9.0	Extreme importance of one element over another	An element is favored by at least an order of magnitude
2.0, 4.0, 6.0, 8.0	Intermediate values	Used to compromise between two judgments

the pedigree outperform the Saaty scale over a wide range of parameter values. And the transitive calibration also does very well for this problem. Of course, this is not to say that the other scales, including the Saaty 1–9 scale, perform badly. But they may be useful substitutes for the Saaty 1–9 scale in some instances.

As we all know, Law of Weber deals with the ability of individuals to differentiate small changes in measurable stimuli. In his discussion of Weber's Law, Saaty [2] offers one of Weber's examples where subjects are trying to determine which of two balls is heavier. Weber found that people while holding in their hands different weights, could distinguish them only if the margin of the weights increases with the weights. The margin is termed the "just noticeable difference." Note that the first derivative of scale value function should be an increment function. Our work is in accord with this quality.

2 Derivations of Ratio Scales

Given that $S_{b_1,b_2}(k)$ is a numeric ratio scale. Let $S_{b_1,b_2}(k) \in [1, M]$, $M \in R^+$. Now, we discuss a sort of ratio scales as follows.

$$S_{b_1,b_2}(k) = \frac{a_1 + b_1 k}{a_2 - b_2 k} \tag{1}$$

where a_1, a_2 are constants and $a_1, a_2 \geq 0$; b_1, b_2 are alterable coefficients; k can be any real number and represents the grade. For the sake of accord with Saaty 1–9 scale, we can assume that k is a natural number and $k \geq 1$.

Obviously, $S_{b_1,b_2}(k)$ is a function relating to k and b_1, b_2.

Let $S_{b_1,b_2}(1) = 1$. We have $a_1 + b_1 = a_2 - b_2$.

Let $S_{b_1,b_2}(M) = M$. We have $a_1 + Mb_1 = M(a_2 - Mb_2)$.

Let $t = \frac{b_1}{b_2}$. So,

$$S_t(k) = \frac{M + tk}{M + 1 + t - k} \tag{2}$$

Because $S_t(k) \geq 1$ and the equal sign is only for $k = 1$, we have $t > -1$.

1. If t and M are constants, $S_t(k)$ varies with k.

Formula (2) leads to

$$\frac{\mathrm{d}S_t(k)}{\mathrm{d}k} = \frac{(t+1)M + t + t^2}{(M + 1 + t - k)^2} \geq 0 \tag{3}$$

$$\frac{\mathrm{d}^2 S_t(k)}{\mathrm{d}k^2} = 2\frac{(t+1)M + t + t^2}{(M + 1 + t - k)^3} \geq 0 \tag{4}$$

If k is a natural number, we have

$$\Delta S = S_t(k) - S_t(k-1) = \frac{M + tM + t + t^2}{(M + 1 + t - k)(M + 2 + t - k)} \tag{5}$$

Apparently, $S_t(k)$ is an increment function and the first derivative of $S_t(k)$ is also an increment function if t is limited. That is to say, given a limited value of t, ΔS increases with k. The quality is in accord with Law of Weber.

2. If k and M are constants, $S_t(k)$ varies with t.

Formula (2) leads to

$$S_t(k) = k - \frac{(k-1)M - k^2 + k}{M + 1 + t - k} \tag{6}$$

$$\frac{\mathrm{d}S_t(k)}{\mathrm{d}t} = \frac{(M-k)(k-1)}{(M + 1 + t - k)^2} \geq 0 \tag{7}$$

$$\frac{\mathrm{d}^2 S_t(k)}{\mathrm{d}t^2} = -\frac{2(M-k)(k-1)}{(M + 1 + t - k)^3} \leq 0 \tag{8}$$

Actually, t can be any real number meeting $t > -1$. If $t \to +\infty$, $\frac{\mathrm{d}S_t(k)}{\mathrm{d}k} = 0$, and $S_t(k) = k$. Apparently, the Saaty 1–9 numeric scale is a special case of $S_t(k)$. That is to say, if $t \to +\infty$ and $k \in [1, 9], S_{+\infty}(k)$ is the 1–9 numeric scale. Formula (6) shows that $S_t(k)$ is actually the calibration of Saaty 1–9 scale. According to (8), we know that the increment of $S_t(K)$ is on the decrease when t is on the increase.

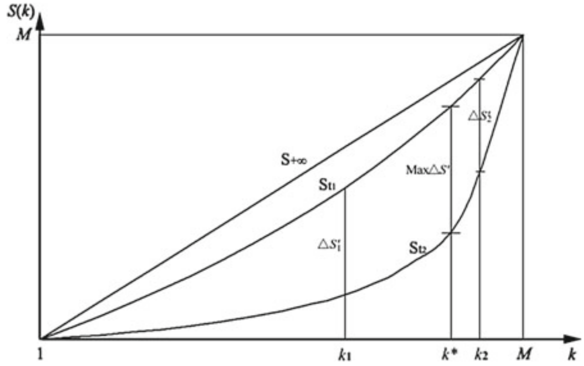


Fig. 1 The changing process of difference between $S_{t_1}(k)$ and $S_{t_2}(k)$

Let $t_1 > t_2 > -1$. We have

$$\Delta S' = S_{t_1}(k) - S_{t_2}(k) = \frac{(t_1 - t_2)(M - k)(k - 1)}{(M + 1 + t_1 - k)(M + 1 + t_2 - k)} \tag{9}$$

Let $\frac{\partial \Delta S'}{\partial k} = 0$. We can always find a value of k, which leads to a maximum value of $\Delta S'$ and ranges between 1 and M. The changing process of difference between $S_{t_1}(k)$ and $S_{t_2}(k)$ is described as the following graph. Apparently, $\Delta S'$ has a maximum value when $k = k^*$, whereas $\Delta S'$ is lesser when other values are assigned to k (Fig. 1).

3. About M.

Given that t is a constant. Let $M' > M$. Suppose that the scale $S_t(k)$ should increase the maximum calibration from M to M'. The variation is described as Fig. 2.

The scale is newly denoted by $S_t(k, M)$. We have

$$S_t(k, M) = \frac{M + tk}{M + 1 + t - k} \tag{10}$$

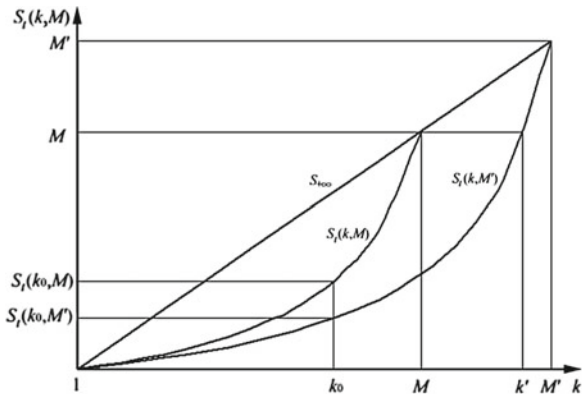
This leads to

$$\frac{\partial S_t(k, M)}{\partial M} = 1 + t - k - tk \le 0 \tag{11}$$

Obviously, $S_t(k_0, M) \ge S_t(k_0, M')$, where $k_0 \in [1, M]$.

Let $S_t(k', M') = \frac{M' + tk'}{M' + 1 + t - k'} = M$. We have

Fig. 2 The variation of curves relating to various values of M



$$k' = M + \frac{(M-1)(M'-M)}{t+M} \geq M \tag{12}$$

That's to say, $M' \geq S_t(k,M) \geq M$, where $M' \geq k \geq k'$.

Given that l is a natural number. Considering the following two scales for a certain t, both of which have $(l + 2)$ levels.

$$S_t(k,M)\text{: } 1, C_1, C_2, \ldots C_i, \ldots, C_l, M$$
$$S_t(k,M')\text{: } 1, C'_1, C'_2, \ldots, C'_i, \ldots, C'_l, M'.$$

Suppose that C_i and C'_i, for all $i = 1, 2, \ldots, l$, correspond to one and the same verbal judgment and meet $\frac{M-C_i}{C_i-1} = \frac{M'-C'_i}{C'_i-1}$.

This leads to $C'_i - C_i = \frac{M'-M}{M-1}(C_i - 1) \geq 0$.

In other words, for the same verbal judgment except *equality* and *extremity*, the numeric values of $S_t(k,M')$ are more than those of $S_t(k,M)$.

Now, we assign various values to t, and then we can get various numeric scales. Let $M = 9$. We can derive the following table and graph from the above.

The above is a pedigree of numeric scales. In the Table 2, there are various scales to choose from. In fact, t varies with DMs' opinions or follows one's inclinations. That's to say, various DMs have various numeric ratios even for the same verbal judgment. And for one and the same DM, t varies with judgment objects or object properties (Fig. 3).

3 Consistency of the Verbal Scale

Consider the following scale denoted by $S(k)$ (Table 3):

Table 2 Pedigree of numeric ratio scales

k	$-1 < t < -0.9$	$S_{-0.9}(k)$	$S_{-0.2}(k)$	$S_{-0.1}(k)$
1	–	$1.000 = 0.500/0.500$	$1.000 = 0.500/0.500$	$1.000 = 0.500/0.500$
3	–	$1.033 = 0.508/0.492$	$1.235 = 0.553/0.447$	$1.261 = 0.558/0.442$
5	–	$1.098 = 0.523/0.477$	$1.667 = 0.625/0.375$	$1.735 = 0.634/0.366$
7	–	$1.286 = 0.563/0.437$	$2.714 = 0.731/0.269$	$2.862 = 0.741/0.259$
9	–	$9.000 = 0.900/0.100$	$9.000 = 0.900/0.100$	$9.000 = 0.900/0.100$
2, 4, 6, 8	–	1.014, 1.059, 1.161, 1.636	1.103, 1.414, 2.053, 4.111	1.114, 1.458, 2.154, 4.316

k	$S_0(k)$	$S_{1/2}(k)$	$S_1(k)$	$S_2(k)$
1	$1.000 = 0.500/0.500$	$1.000 = 0.500/0.500$	$1.000 = 0.500/0.500$	$1.000 = 0.500/0.500$
3	$1.286 = 0.563/0.437$	$1.400 = 0.583/0.417$	$1.500 = 0.600/0.400$	$1.667 = 0.625/0.375$
5	$1.800 = 0.643/0.357$	$2.091 = 0.676/0.324$	$2.333 = 0.700/0.300$	$2.714 = 0.731/0.269$
7	$3.000 = 0.750/0.250$	$3.571 = 0.781/0.219$	$4.000 = 0.800/0.200$	$4.600 = 0.821/0.179$
9	$9.000 = 0.900/0.100$	$9.000 = 0.900/0.100$	$9.000 = 0.900/0.100$	$9.000 = 0.900/0.100$
2, 4, 6, 8	1.125, 1.500, 2.250, 4.500	1.176, 1.692, 2.667, 5.200	1.222, 1.857, 3.000, 5.667	1.300, 2.125, 3.500, 6.250

k	$S_{10}(k)$	$S_{100}(k)$	$S_t(k), t > 100$	$S_{+\infty}(k)$
1	$1.000 = 0.500/0.500$	$1.000 = 0.500/0.500$	–	$1.000 = 0.500/0.500$
3	$2.294 = 0.696/0.304$	$2.888 = 0.743/0.257$	–	$3.000 = 0.750/0.250$
5	$3.933 = 0.797/0.203$	$4.848 = 0.829/0.171$	–	$5.000 = 0.833/0.167$
7	$6.077 = 0.859/0.141$	$6.883 = 0.873/0.127$	–	$7.000 = 0.875/0.125$
9	$9.000 = 0.900/0.100$	$9.000 = 0.900/0.100$	–	$9.000 = 0.900/0.100$
2, 4, 6, 8	1.611, 3.063, 4.929, 7.417	1.935, 3.858, 5.856, 7.931	–	2.000, 4.000, 6.000, 8.000

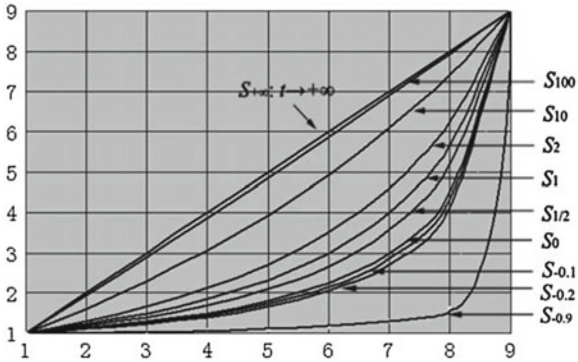


Fig. 3 Pedigree of numeric ratio scales

It is considered that a strict reliance on the Saaty verbal scale can induce some inconsistency. Suppose a DM has specified that:

1. Object A is *moderately more important* than Object B, and
2. Object B is *moderately more important* than Object C.

If the DM is to be numerically consistent, he would have to respond that Object A is *extremely more important* than Object C. However, according to an experiment of Queen's University [13], the answer should be: Object A is *strongly more important* than Object C. That is to say, $S(3)^2 = S(5)$. The corresponding numeric scale of the verbal scale is related to the meaning of the words in language. According to the experimental result, we might as well suppose the linguistic logical relations of the verbal scale are as follows…

Let k, l are natural numbers, $k, l \in [1, 9]$, and $k \geq l$.

$$S(k) \leq S(k)S(l) \leq S(k+2) \tag{13}$$

for all k, l. And

$$S^2(k) = S(k+2) \tag{14}$$

where $k \geq 3$.

Let $S(3) = \eta$, $k' \in N$. According to formula (14), we have $S(2k'-1) = \eta^{2^{k'-2}}$ and $S(2k') = S(2)^{2^{k'-1}}$, where k' is natural number, $k' \geq 2$. Let $S(9) = 9$. We have $\eta = 1.316$. Suppose $S(2) = \sqrt{S(1)S(3)} = \eta^{1/2}$. So, we have

$$S(k) = \eta^{2^{k'-2}} \tag{15}$$

where $k = \begin{cases} 2k'-1, & k' \geq 2 \\ 2k', & k' \geq 1 \end{cases}$

Obviously,

Table 3 The correspondence between the verbal and numeric scale

Value of numeric scale	Meaning of verbal scale	Abbreviation
$S(1) = 1$	Equal importance of both elements	E
$S(3)$	Moderate importance of one element over another	M
$S(5)$	Strong importance of one element over another	S
$S(7)$	Very strong importance of one element over another	V
$S(9)$	Extreme importance of one element over another	Ex
$S(2)$, $S(4)$, $S(6)$, $S(8)$	Intermediate values	I(−)

Table 4 The numeric values and expressions of the scale

Abbreviation	Value of numeric scale	Expression	Meaning of verbal scale
E	1.000 = 0.500/0.500	1	Equal importance of both elements
M	1.316 = 0.568/0.432	η	Moderate importance of one element over another
S	1.732 = 0.634/0.366	η^2	Strong importance of one element over another
V	3.000 = 0.750/0.250	η^4	Very strong importance of one element over another
Ex	9.000 = 0.900/0.100	η^8	Extreme importance of one element over another
I(−)	1.147, 1.510, 2.280, 5.196	$\eta^{1/2}, \eta^{3/2}, \eta^3, \eta^6$	Intermediate values

$$S(2k') = \sqrt{S(2k'-1)S(2k'+1)} \tag{16}$$

And the numbers and ratios of the scale are as follows, which meet consistency perfectly on instinct.

We calculated the correlations between $S(k)$ and the scales listed in Table 4. These correlations are presented in the following table.

According to Table 5, when t ranges between −0.2 and 1, the correlations are higher than 0.99. That's to say, for all values of t in this range, $S_t(k)$ is highly approaching to the above scale and maybe accords with the linguistic logical relations of verbal scale in a higher degree. On the other hand, when $t > 1$ or $t < -0.2, S_t(k)$ differs greatly $S(k)$.

Actually, if k is a continuous variable, we can derive $S(k)$ from any scale function on the pedigree. For example, when $t = 0$, the corresponding values of k are: 1, 2.1534, 3.1611, 4.0397, 4.8037, 6.0526, 7, 8.2679, 9. In other words, for $t = 0$, when $k = 3.1611$, $S_t(k) = S(3)$ and the corresponding meaning of the scale value is *moderate importance of one element over another*, and so on at higher levels.

Table 5 The correlations between $S(k)$ and $S_t(k)$

Scale	$S_{-0.9}(k)$	$S_{-0.2}(k)$	$S_{-0.1}(k)$	$S_0(k)$	$S_{1/2}(k)$	$S_1(k)$	$S_2(k)$	$S_{10}(k)$	$S_{100}(k)$	$S_{+\infty}(k)$
Correlation	0.904	0.992	0.994	0.996	0.997	0.991	0.976	0.907	0.848	0.838

4 An Example

In order to demonstrate the effects of this pedigree, we consider Saaty's example of distance measurement [2], which had been also discussed by Harker and Vargas [9], Finan and Hurley [13]. The following verbal assessments were made of the relative distance between Philadelphia and six other cities (Table 6).

Harker and Vargas [9] evaluated this matrix using the Saaty 1–9 scale and four other reasonable alternatives. They measured the correlation between each scale's weights and the true weights. The Saaty 1–9 scale produced the highest correlation at 0.979.

We evaluated the weights for the same matrix using all the scales on the pedigree for various values of t and calculated the correlations between the true weights and the weights obtained for each value of t. These correlations are presented in the following table.

We also calculate the correlations between the true weights and the weights obtained by using $S(k)$. The correlation is 0.992.

In our view, reasonable values for t range between -0.1 and 1. For all values of t in this range, the correlations of the numeric scales, not only in Table 7 but also in Table 5, are higher than 0.99. Of course, this is not to say that the other scales, including the Saaty 1–9 scale, perform badly. But they do very well for this relative distance problem, and, on instinct, meets consistency or the linguistic logical relations of verbal scale in a higher degree. And $S(k)$ also does very well for this problem, which is a transitive calibration and meets consistency perfectly in appearance.

5 Conclusion

Apparently, various DMs have various numeric values even for the same verbal judgment, and, for one and the same DM, the corresponding numeric values of verbal scales vary with judgment objects or object properties. The pedigree can be used to serve as a guide for DMs' confirming the numeric values of their verbal judgments. The scales on the pedigree do outperform the Saaty scale over a wide range of parameter values. And the transitive calibration also does very well for this problem. Of course, the improved performance is marginal. So, this is not to say that the other scales, including the Saaty 1–9 scale, perform badly. But they may be useful substitutes for the Saaty 1–9 scale in some instances.

Table 6 The verbal assessments of distance measurement

	Cairo	Tokyo	Chicago	San Fran	London	Montreal
Cairo	1	–	I(V–Ex)	M	M	V
Tokyo	M	1	Ex	M	M	Ex
Chicago	–	–	1	–	–	I(E–M)
San Fran	–	–	I(S–V)	1	–	I(S–V)
London	–	–	S	M	1	I(S–V)
Montreal	–	–	–	–	–	1

Table 7 The correlations between the true weights and the weights obtained for each t

Scale	$S_{-0.9}(k)$	$S_{-0.2}(k)$	$S_{-0.1}(k)$	$S_0(k)$	$S_{1/2}(k)$	$S_1(k)$	$S_2(k)$	$S_{10}(k)$	$S_{100}(k)$	$S_{+\infty}(k)$
Correlation	0.911	0.988	0.991	0.992	0.995	0.995	0.994	0.988	0.980	0.979

Acknowledgments This work was partially by the National Natural Science Foundation of China (NSFC) under grant No. 71101076 and by the Social Science Foundation of Ministry of Education of the People's Republic of China under grant No. 10YJA630147.

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