Type Similarity for the Lambek-Grishin Calculus Revisited

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1 Introduction

The topic of this paper concerns a particular extension of Lambek's syntactic calculus [5] that was proposed by Grishin [4]. Roughly, the usual residuated family $(\otimes, /, \backslash)$ is extended by a *co*residuated triple $(\oplus, \oslash, \obslash)$ mirroring its behavior in the inequality sign:

$$A \otimes B \leq C \text{ iff } A \leq C/B \text{ iff } B \leq A\backslash C$$
$$C \leq A \oplus B \text{ iff } C \oslash B \leq A \text{ iff } A \obslash C \leq B$$

A survey of the various possible structural extensions reveals that besides same-sort associativity and/or commutativity of $\otimes$ and $\oplus$ independently, there exist as well interaction laws mixing the two vocabularies. We may categorize them along two dimensions, depending on whether they encode mixed associativity or -commutativity, and on whether they involve the tensor $\otimes$ and par $\oplus$ (type I, in Grishin's terminology) or the (co)implications (type IV):

	Type I	Type IV
Mixed associativity	$(A \oplus B) \otimes C \leq A \oplus (B \otimes C)$	$(A\backslash B) \oslash C \leq A\backslash(B \oslash C)$
	$A \otimes (B \oplus C) \leq (A \otimes B) \oplus C$	$A \obslash (B/C) \leq (A \obslash B)/C$
Mixed commutativity	$A \otimes (B \oplus C) \leq B \oplus (A \otimes C)$	$A \obslash (B\backslash C) \leq B\backslash(A \obslash C)$
	$(A \oplus B) \otimes C \leq (A \otimes C) \oplus B$	$(A/B) \oslash C \leq (A \oslash C)/B$

While our motivation for the classification into types I and IV may seem rather ad-hoc, one finds that the combined strength of these two groups allows for the either partial or whole collapse (depending on the presence of identity elements, or *units*) into same-sort commutativity and -associativity. Given that this result is hardly desirable from the linguistic standpoint, there is sufficient ground for making the distinction. Moortgat [8] thus proposed a number of calculi, jointly referred to by *Lambek-Grishin* (**LG**), which he considered of particular interest to linguistics. While all reject same-sort associativity and -commutativity, they adopt either one of the type I or IV groups of postulates, the results denoted $\mathbf{LG}_I$ and $\mathbf{LG}_{IV}$ respectively. On occasion, one speaks as well of $\mathbf{LG}_\varnothing$, in reference to the minimal base logic with no structural assumptions.

* This paper was written while the author was working on his thesis. See Bastenhof, *Categorial Symmetry*, PhD Thesis, Utrecht University, 2013. [Editors' note].

C. Casadio et al. (Eds.): Lambek Festschrift, LNCS 8222, pp. 28–50, 2014.
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Having explained the Lambek-Grishin calculi, we next discuss the concept of type-similarity. Besides model-theoretic investigations into derivability, people have sought to similarly characterize its symmetric-transitive closure under the absence of additives. Thus, we consider A and B *type similar*, written $\vdash A \sim B$, iff there exists a sequence of formulae $C_1 \ldots C_n$ s.t. $C_1 = A$, $C_n = B$, and either $C_i \leq C_{i+1}$ or $C_{i+1} \leq C_i$ for each $1 \leq i < n$.[1] For the traditional Lambek hierarchy, one finds their level of resource sensitivity reflected in the algebraic models for the corresponding notions of $\sim$, as summarized in the following table:

Calculus	Models	Reference
NL	quasigroup	Foret [3]
L	group	Pentus [10]
LP	Abelian group	Pentus [10]

With $\mathbf{LG}_{IV}$, however, Moortgat and Pentus (henceforth M&P, [9]) found that, while same-sort associativity and -commutativity remain underivable, the latter principles do hold at the level of type similarity. More specifically, we find there exist formulas serving as common ancestors or descendants (in terms of $\leq$):

$$\begin{tikzcd}
& D \arrow[dl, dashed] \arrow[dr, dashed] & \\
A \otimes B \arrow[dr, dashed] & & B \otimes A \arrow[dl, dashed] \\
& C &
\end{tikzcd}
\qquad
\begin{tikzcd}
& D \arrow[dl, dashed] \arrow[dr, dashed] & \\
(A \otimes B) \otimes C \arrow[dr, dashed] & & A \otimes (B \otimes C) \arrow[dl, dashed] \\
& C &
\end{tikzcd}$$

In general, we may prove (cf. [10]) that type similarity coincides with the existence of such *meets* D or *joins* C, as referred to by M&P (though not to be confused with terminology from lattice theory). With respect to linguistic applications, these findings suggest the possibility of tapping in on the flexibility of **L** and **LP** without compromising overall resource-sensitivity, simply by assigning the relevant joins or meets when specifying one's lexicon.

The current article defines a class of models w.r.t. which we prove soundness and completeness of type similarity in $\mathbf{LG}_{\varnothing}$ extended by type I or IV interactions, both with and without units. While M&P already provided analogous results of $\mathbf{LG}_{IV}$ inside Abelian groups, we here consider a notion of model better reflecting the presence of dual (co)residuated families of connectives, allowing for simpler proofs overall. Such results still leave open, however, the matter of deciding the existence of joins or meets. We first solve this problem for the specific case of $\mathbf{LG}_{I}$ together with units 0 and 1, taking a hint from M&P's Abelian group interpretation for $\mathbf{LG}_{IV}$. Decidability for type similarity in the remaining incarnations of **LG** is derived as a corollary.

We proceed as follows. First, §2 covers a general introduction to **LG** and to our formalism for describing the corresponding concept of derivation. We next illustrate type similarity in §3 with some typical instances in $\mathbf{LG}_{I}$ and $\mathbf{LG}_{IV}$.

[1] Our terminology is adapted from [9], revising that of Pentus [10], who previously spoke of *type conjoinability.*

Models for ~ in the presence of type I or IV interactions are defined in §4, along with proofs of soundness and completeness. Finally, an algorithm for generating joins inside $\mathbf{LG}_I$ in the precense of units is detailed in §5.

2 Lambek-Grishin Calculus

Lambek's non-associative syntactic calculus ([5], **NL**) combines linguistic inquiry with the mathematical rigour of proof theory. Corresponding to multiplicative, non-associative, non-commutative intuitionistic linear logic, its logical vocabulary includes a multiplicative conjunction (*tensor*) $\otimes$ with unit 1, along with direction-sensitive implications $\backslash$ and $/$. Grishin [4] first proposed adding DeMorgan-like duals (cf. remark 1 below), including multiplicative disjunction $\oplus$ (the *par*) with unit 0, as well as *subtractions* $\obslash$ and $\oslash$.

Definition 1. *Given a set of atoms $p, q, r, \ldots$, formulas are defined thus:*

$$\begin{array}{lll} A..E ::= p & & \textit{Atoms} \\ \quad \mid \ (A \otimes B) \mid (A \oplus B) & & \textit{Tensor vs. par} \\ \quad \mid \ (A/B) \mid (B \obslash A) & & \textit{Right division vs. left subtraction} \\ \quad \mid \ (B\backslash A) \mid (A \oslash B) & & \textit{Left division vs. right subtraction} \\ \quad \mid \ 1 \mid 0 & & \textit{Units} \end{array}$$

The associated concept of duality $^\infty$ is defined as follows:

$$\begin{array}{lll lll lll} p^\infty & := & p & (A \otimes B)^\infty & := & B^\infty \oplus A^\infty & (A \oplus B)^\infty & := & B^\infty \otimes A^\infty \\ 1^\infty & := & 0 & (A/B)^\infty & := & B^\infty \obslash A^\infty & (B \obslash A)^\infty & := & A^\infty / B^\infty \\ 0^\infty & := & 1 & (B\backslash A)^\infty & := & A^\infty \oslash B^\infty & (A \oslash B)^\infty & := & B^\infty \backslash A^\infty \end{array}$$

Remark 1. Note that if $^\infty$ is interpreted as negation, its defining clauses for the binary connectives read as De Morgan laws. That said, while $^\infty$ is indeed involutive, it is not, like negation, fixpoint-free, seeing as $p^\infty = p$.

While derivability may be characterized algebraically using inequalities $A \leq B$, we here instead use a *labelled deductive system* [6], adding an extra label f to further discriminate between different deductions.

Definition 2. *Fig.1 defines judgements $f : A \rightarrow B$, referred to by arrows.*

While we shall use arrows merely as a means of encoding derivations, one should note that, similar to λ-terms for intuitionistic logic, they may as well be considered amendable to computation (cf. [7]). The following is an easy observation:

Lemma 1. *If $f : A \rightarrow B$, then there exists $g : B^\infty \rightarrow A^\infty$.*

In practice, we use a more compact representation of derivations, compiling away monotonicity and composition. To this end, we first require the notions of positive and negative (formula) contexts.

Preorder laws

$$i_A : A \to A \qquad\qquad \frac{f : A \to B \quad g : B \to C}{(g \circ f) : A \to C}$$

Monotonicity

$$\frac{f : A \to B}{\begin{array}{c}(f \otimes C) : A \otimes C \to B \otimes C\\ (f/C) : A/C \to B/C\\ (C\backslash f) : C\backslash A \to C\backslash B\end{array}} \qquad \frac{f : A \to B}{\begin{array}{c}(f \oplus C) : A \oplus C \to B \oplus C\\ (f \oslash C) : A \oslash C \to B \oslash C\\ (C \obslash f) : C \obslash A \to C \obslash B\end{array}}$$

$$\frac{f : A \to B}{\begin{array}{c}(C \otimes f) : C \otimes A \to C \otimes B\\ (C/f) : C/B \to C/A\\ (f\backslash C) : B\backslash C \to A\backslash C\end{array}} \qquad \frac{f : A \to B}{\begin{array}{c}(C \oplus f) : C \oplus A \to C \oplus B\\ (C \oslash f) : C \oslash B \to C \oslash A\\ (f \obslash C) : B \obslash C \to A \obslash C\end{array}}$$

(Co)evaluation

$$\begin{array}{ll}
e^{/}_{A,B} : (A/B) \otimes B \to A & \qquad e^{\oslash}_{A,B} : A \to (A \oslash B) \oplus B\\
e^{\backslash}_{A,B} : B \otimes (B\backslash A) \to A & \qquad e^{\obslash}_{A,B} : A \to B \oplus (B \obslash A)\\
h^{/}_{A,B} : A \to (A \otimes B)/B & \qquad h^{\oslash}_{A,B} : (A \oplus B) \oslash B \to A\\
h^{\backslash}_{A,B} : A \to B\backslash(B \otimes A) & \qquad h^{\obslash}_{A,B} : B \obslash (B \oplus A) \to A
\end{array}$$

Units

$$\begin{array}{ll}
1_{A\otimes} : A \to A \otimes 1 & \qquad 0_{A\oplus} : A \oplus 0 \to A\\
1_{\otimes A} : A \to 1 \otimes A & \qquad 0_{\oplus A} : 0 \oplus A \to A\\
1^{*}_{A\otimes} : A \otimes 1 \to A & \qquad 0^{*}_{A\oplus} : A \to A \oplus 0\\
1^{*}_{\otimes A} : 1 \otimes A \to A & \qquad 0^{*}_{\oplus A} : A \to 0 \oplus A
\end{array}$$

Type I interactions — Type IV interactions

$$\begin{array}{ll}
a^{\oplus\otimes}_{A,B,C} : (A \oplus B) \otimes C \to A \oplus (B \otimes C) & \qquad \alpha^{\backslash\oslash}_{A,B,C} : (A\backslash B) \oslash C \to A\backslash(B \oslash C)\\
a^{\otimes\oplus}_{A,B,C} : A \otimes (B \oplus C) \to (A \otimes B) \oplus C & \qquad \alpha^{\obslash/}_{A,B,C} : A \obslash (B/C) \to (A \obslash B)/C\\
c^{\otimes\oplus}_{A,B,C} : A \otimes (B \oplus C) \to B \oplus (A \otimes C) & \qquad \gamma^{\obslash\backslash}_{A,B,C} : A \obslash (B\backslash C) \to B\backslash(A \obslash C)\\
c^{\oplus\otimes}_{A,B,C} : (A \oplus B) \otimes C \to (A \otimes C) \oplus B & \qquad \gamma^{/\oslash}_{A,B,C} : (A/B) \oslash C \to (A \oslash C)/B
\end{array}$$

Fig. 1. Lambek-Grishin calculi presented using labelled deduction

Definition 3. *Define, by mutual induction, positive and negative contexts:*

$$\begin{array}{rl}
X^{+}[], Y^{+}[] ::= & [] \mid (X^{+}[] \otimes B) \mid (A \otimes Y^{+}[]) \mid (X^{+}[] \oplus B) \mid (A \oplus Y^{+}[])\\
\mid & (X^{+}[]/B) \mid (B\backslash X^{+}[]) \mid (X^{+}[] \oslash B) \mid (B \obslash X^{+}[])\\
\mid & (A/Y^{-}[]) \mid (Y^{-}[]\backslash A) \mid (A \oslash Y^{-}[]) \mid (Y^{-}[] \obslash A)
\end{array}$$

$$\begin{array}{rl}
X^{-}[], Y^{-}[] ::= & (X^{-}[] \otimes B) \mid (A \otimes Y^{-}[]) \mid (X^{-}[] \oplus B) \mid (A \oplus Y^{-}[])\\
\mid & (X^{-}[]/B) \mid (B\backslash X^{-}[]) \mid (X^{-}[] \oslash B) \mid (B \obslash X^{-}[])\\
\mid & (A/Y^{+}[]) \mid (Y^{+}[]\backslash A) \mid (A \oslash Y^{+}[]) \mid (Y^{+}[] \obslash A)
\end{array}$$

Evidently, given some $X^+[]$, $Y^-[]$ and $f : A \to B$, we have $X^+[f] : X^+[A] \to X^+[B]$ and $Y^-[f] : Y^-[B] \to Y^-[A]$. In practice, we often depict said arrows as an inference step, using the following shorthand notation:

$$\frac{X^+[\boxed{A}]}{X^+[A']}\,f \qquad \frac{Y^-[\boxed{A'}]}{Y^-[A]}\,f$$

having avoided writing $X^+[f]$ ($Y^-[f]$) by informally singling out the source of f using a box, thus unambiguously identifying the surrounding context $X^+[]$ ($Y^-[]$). Composition of arrows is compiled away by chaining inference steps. In practice, we often leave out subscripts in f, being easily inferable from context.

Our notation comes close to Brünnler and McKinley's use of *deep inference* for intuitionistic logic [1]. Contrary to their concept of derivability, however, our inference steps need not be restricted to primitive arrows. We next survey several definable arrows, proving useful in what is to follow.

Definition 4. *Lifting is defined*

$$\begin{array}{lcll}
l^{/}_{A,B} & := & ((e^{\backslash}_{B,A}/i_{A\backslash B}) \circ h^{/}_{A,A\backslash B}) & : A \to B/(A\backslash B) \\
l^{\backslash}_{A,B} & := & ((i_{B/A}\backslash e^{/}_{B,A}) \circ h^{\backslash}_{A,B/A}) & : A \to (B/A)\backslash A \\
l^{\oslash}_{A,B} & := & (h^{\oslash}_{A,A\obslash B} \circ (e^{\obslash}_{B,A} \oslash i_{A\obslash B})) & : B \oslash (A \obslash B) \to A \\
l^{\obslash}_{A,B} & := & (h^{\obslash}_{A,B\oslash A} \circ (i_{B\oslash A} \obslash e^{\oslash}_{B,A})) & : (B \oslash A) \obslash B \to A
\end{array}$$

Using the notation introduced in our previous discussion:

$$\cfrac{\cfrac{A}{\boxed{(A \otimes (A\backslash B))}/(A\backslash B)}\,h^{/}}{B/(A\backslash B)}\,e^{\backslash} \qquad \cfrac{\cfrac{A}{(B/A)\backslash\boxed{((B/A) \otimes A)}}\,h^{\backslash}}{(B/A)\backslash B}\,e^{/}$$

$$\cfrac{\cfrac{(B \oslash A) \obslash \boxed{B}}{(B \oslash A) \obslash ((B \oslash A) \oplus A)}\,e^{\oslash}}{A}\,h^{\oslash} \qquad \cfrac{\cfrac{\boxed{B} \oslash (A \obslash B)}{(A \oplus (A \obslash B)) \oslash (A \obslash B)}\,e^{\obslash}}{A}\,h^{\oslash}$$

Definition 5. *Grishin type I and IV interactions can alternatively be rendered*

$$\begin{array}{ll}
a^{\oslash}_{A,B,C} : (A \otimes B) \oslash C \to A \otimes (B \oslash C) & \alpha^{\oslash}_{A,B,C} : A \otimes (B \oslash C) \to (A \otimes B) \oslash C \\
a^{\obslash}_{A,B,C} : A \obslash (B \otimes C) \to (A \obslash B) \otimes C & \alpha^{\obslash}_{A,B,C} : (A \obslash B) \otimes C \to A \obslash (B \otimes C) \\
c^{\oslash}_{A,B,C} : (A \otimes B) \oslash C \to (A \oslash C) \otimes B & \gamma^{\oslash}_{A,B,C} : (A \oslash B) \otimes C \to (A \otimes C) \oslash B \\
c^{\obslash}_{A,B,C} : A \obslash (B \otimes C) \to B \otimes (A \obslash C) & \gamma^{\obslash}_{A,B,C} : A \otimes (B \obslash C) \to B \obslash (A \otimes C) \\
a^{/}_{A,B,C} : A \oplus (B/C) \to (A \oplus B)/C & \alpha^{/}_{A,B,C} : (A \oplus B)/C \to A \oplus (B/C) \\
a^{\backslash}_{A,B,C} : (A\backslash B) \oplus C \to A\backslash(B \oplus C) & \alpha^{\backslash}_{A,B,C} : A\backslash(B \oplus C) \to (A\backslash B) \oplus C \\
c^{/}_{A,B,C} : (A/B) \oplus C \to (A \oplus C)/B & \gamma^{/}_{A,B,C} : (A \oplus B)/C \to (A/C) \oplus B \\
c^{\backslash}_{A,B,C} : A \oplus (B\backslash C) \to B\backslash(A \oplus C) & \gamma^{\backslash}_{A,B,C} : A\backslash(B \oplus C) \to B \oplus (A\backslash C)
\end{array}$$

For Type I, we have the following definitions:

$$
\begin{array}{lcl}
a^{\oslash}_{A,B,C} & := & h^{\oslash}_{A\otimes(B\oslash C),C} \circ ((a^{\otimes\oplus}_{A,B\oslash C,C} \circ (i_A \otimes e^{\oslash}_{B,C})) \oslash i_C) \\
a^{\obslash}_{A,B,C} & := & h^{\obslash}_{(A\obslash B)\otimes C,A} \circ (i_A \obslash (a^{\oplus\otimes}_{A,A\obslash B,C} \circ (e^{\obslash}_{B,A} \otimes i_C))) \\
c^{\oslash}_{A,B,C} & := & h^{\oslash}_{(A\oslash C)\otimes B,C} \circ ((c^{\oplus\otimes}_{A\oslash C,C,B} \circ (e^{\oslash}_{A,C} \otimes i_B)) \oslash i_C) \\
c^{\obslash}_{A,B,C} & := & h^{\obslash}_{B\otimes(A\obslash C),A} \circ (i_A \obslash (c^{\otimes\oplus}_{B,A,A\obslash C} \circ (i_B \otimes e^{\obslash}_{C,A}))) \\
a^{/}_{A,B,C} & := & (((i_A \oplus e^{/}_{B,C}) \circ a^{\oplus\otimes}_{A,BC,C})/i_C) \circ h^{/}_{A\oplus(B/C),C} \\
a^{\backslash}_{A,B,C} & := & (i_A\backslash((e^{\backslash}_{B,A} \oplus i_C) \circ a^{\otimes\oplus}_{A,A\backslash B,C})) \circ h^{\backslash}_{(A\backslash B)\oplus C,A} \\
c^{/}_{A,B,C} & := & (((e^{/}_{A,B} \oplus i_C) \circ c^{\oplus\otimes}_{A/B,C,B})/i_B) \circ h^{/}_{(A/B)\oplus C,B} \\
c^{\backslash}_{A,B,C} & := & (i_B\backslash((i_A \oplus e^{\backslash}_{C,B}) \circ c^{\otimes\oplus}_{B,A,B\backslash C})) \circ h^{\backslash}_{A\oplus(B\backslash C),B}
\end{array}
$$

While for $\boldsymbol{LG}_{IV}$, we have:

$$
\begin{array}{lcl}
\alpha^{\oslash}_{A,B,C} & := & e^{\backslash}_{(A\otimes B)\oslash C,A} \circ (i_A \otimes (\alpha^{\backslash\oslash}_{A,A\otimes B,C} \circ (h^{\backslash}_{B,A} \oslash i_C))) \\
\alpha^{\obslash}_{A,B,C} & := & e^{/}_{A\obslash(B\otimes C),C} \circ ((\alpha^{\obslash/}_{A,B\otimes C,C} \circ (i_A \obslash h^{/}_{B,C})) \otimes i_C) \\
\gamma^{\oslash}_{A,B,C} & := & e^{/}_{(A\otimes C)\oslash B,C} \circ ((\gamma^{/\oslash}_{A\otimes C,C,B} \circ (h^{/}_{A,C} \oslash i_B)) \otimes i_C) \\
\gamma^{\obslash}_{A,B,C} & := & e^{\backslash}_{B\obslash(A\otimes C),A} \circ (i_A \otimes (\gamma^{\obslash\backslash}_{B,A,A\otimes C} \circ (i_B \obslash h^{\backslash}_{C,A}))) \\
\alpha^{/}_{A,B,C} & := & (i_A \oplus ((h^{\obslash}_{B,A}/i_C) \circ \alpha^{\obslash/}_{A,A\oplus B,C})) \circ e^{\obslash}_{(A\oplus B)/C,A} \\
\alpha^{\backslash}_{A,B,C} & := & (((i_A\backslash h^{\oslash}_{B,C}) \circ \alpha^{\backslash\oslash}_{A,B\oplus C,C}) \oplus i_C) \circ e^{\oslash}_{A\backslash(B\oplus C),C} \\
\gamma^{/}_{A,B,C} & := & (((h^{\oslash}_{A,B}/i_C) \circ \gamma^{/\oslash}_{A\oplus B,C,B}) \oplus i_B) \circ e^{\oslash}_{(A\oplus B)/C,B} \\
\gamma^{\backslash}_{A,B,C} & := & (i_B \oplus ((i_A\backslash h^{\obslash}_{C,B}) \circ \gamma^{\obslash\backslash}_{B,A,B\oplus C})) \circ e^{\obslash}_{A\backslash(B\oplus C),B}
\end{array}
$$

In practice, use of both type I and IV interactions may prove undesirable, given that their combined strength licenses same-sort associativity and -commutativity. To illustrate, we have the following arrow from $B\backslash(A\backslash C)$ to $A\backslash(B\backslash C)$:

$$(B \obslash 0_{\oplus(A\backslash C)}) \circ a^{\backslash}_{B,0,A\backslash C} \circ \gamma^{\backslash}_{A,B\backslash 0,C} \circ (A\backslash(\alpha^{\backslash}_{B,0,C} \circ (B\backslash 0^{*}_{\oplus C})))$$

with similar problems arising in the absence of units as well. In addition, units themselves are also suspect, sometimes inducing overgeneration where linguistic applications are concerned. In light of these remarks, we shall restrict our discussion to the following four calculi of interest:

	GrI	GrIV	Units
$\mathbf{LG}_I$	✓		
$\mathbf{LG}_{IV}$		✓	
$\mathbf{LG}_I^{0,1}$	✓		✓
$\mathbf{LG}_{IV}^{0,1}$		✓	✓

We use the following notational convention:

$$\mathbf{T} \vdash A \to B \text{ iff } \exists f, f : A \to B \text{ in } \mathbf{T}, \text{ where } \mathbf{T} \in \{\mathbf{LG}_I, \mathbf{LG}_{IV}, \mathbf{LG}_I^{0,1}, \mathbf{LG}_{IV}^{0,1}\}$$

In the case of statements valid for arbitrary choice of $\mathbf{T}$, or when the latter is clear from context, we simply write $\vdash A \to B$.

3 Diamond Property and Examples

Definition 6. *Given $\boldsymbol{T} \in \{\boldsymbol{LG}_I, \boldsymbol{LG}_{IV}, \boldsymbol{LG}_I^{0,1}, \boldsymbol{LG}_{IV}^{0,1}\}$, we say A, B are type similar in $\boldsymbol{T}$, written $\boldsymbol{T} \vdash A \sim B$, iff $\exists C, \boldsymbol{T} \vdash A \to C$ and $\boldsymbol{T} \vdash B \to C$.*

Following [10] and [9], we say that the C witnessing $\mathbf{T} \vdash A \sim B$ is a *join* for A, B, not to be confused with the notion of joins familiar from lattice theory. Keeping with tradition, we write $\vdash A \sim B$ in case a statement is independent of the particular choice of $\mathbf{T}$. We have the following equivalent definition.

Lemma 2. *Formulas A, B are type similar in $\boldsymbol{T}$ iff there exists D s.t. $\boldsymbol{T} \vdash D \to A$ and $\boldsymbol{T} \vdash D \to B$.*

Proof. The following table provides for each choice of $\mathbf{T}$ the solution for D in case the join C is known, and conversely. Note q refers to an arbitrary atom.

$\mathbf{T}$	Solution for C	Solution for D
$\mathbf{LG}_I$	$((B \otimes B) \oplus (B \obslash A))/D$	$C \obslash ((A/B) \otimes (B \oplus B))$
$\mathbf{LG}_{IV}$	$((D/B)\backslash q) \oplus ((D/A)\backslash(q \obslash D))$	$((C/q) \oslash (A \obslash C)) \otimes (q \oslash (B \obslash C))$
$\mathbf{LG}_I^{0,1}$	$(1 \oslash D) \oplus (A \otimes B)$	$(B \oplus A) \otimes (C\backslash 0)$
$\mathbf{LG}_{IV}^{0,1}$	$(A \otimes B)/(1 \oslash (D\backslash 0))$	$((1 \oslash C)\backslash 0) \obslash (B \oplus A)$

Fig.2 shows the derivations for the joins, assuming $f : D \to A$ and $g : D \to B$, those concerning the solutions for D being essentially dual under $^\infty$.

Lem.2 is commonly referred to by the *diamond property*, in reference to the following equivalent diagrammatic representation:

$$\begin{tikzcd}[column sep=small] & \exists D \arrow[dl, dashed] \arrow[dr, dashed] & \\ A \arrow[dr] & & B \arrow[dl] \\ & C & \end{tikzcd} \quad \text{iff} \quad \begin{tikzcd}[column sep=small] & D \arrow[dl] \arrow[dr] & \\ A \arrow[dr, dashed] & & B \arrow[dl, dashed] \\ & \exists C & \end{tikzcd}$$

The formula D is also referred to as a *meet* for A, B. If C is known, we write $A \sqcap_C B$ for the meet constructed in Lem.2, while conversely we write $A \sqcup_D B$ for the join obtained from D. Clearly, if $\vdash A \sqcup_D B \to E$ ($\vdash E \to A \sqcap_C B$), then also $\vdash A \to E$ ($\vdash E \to A$), $\vdash B \to E$ ($\vdash E \to B$) and $\vdash D \to E$ ($\vdash E \to C$).

Remark 2. M&P provide an alternative solution for $\mathbf{LG}_{IV}$, defining $A \sqcap_C B = (A/C) \otimes (C \oslash (B \obslash C))$ and $A \sqcup_D B = ((D/B)\backslash D) \otimes (D\backslash A)$. Though smaller in size compared to ours, the latter allows for easier generalization. For example, in the following event, suppose we wish to find a meet for A_1 and A_4:

$$\begin{tikzcd}[column sep=small] A_1 \arrow[dr] & & A_2 \arrow[dl] \arrow[dr] & & A_3 \arrow[dl] \arrow[dr] & & A_4 \arrow[dl] \\ & C_1 & & C_2 & & C_3 & \end{tikzcd}$$

$$\begin{array}{cl} A & \\ \hline (\boxed{A}\otimes B)/B & h^{/} \\ \hline ((B\oplus(B\obslash A))\otimes B)/\boxed{B} & e^{\obslash} \\ \hline \boxed{((B\oplus(B\obslash A))\otimes B)}/D & g \\ \hline ((B\otimes B)\oplus(B\obslash A))/D & a^{\otimes\oplus} \end{array} \qquad \begin{array}{cl} B & \\ \hline (B\otimes\boxed{A})/A & h^{/} \\ \hline (B\otimes(B\oplus(B\obslash A)))/\boxed{A} & e^{\obslash} \\ \hline \boxed{(B\otimes(B\oplus(B\obslash A)))}/D & f \\ \hline ((B\otimes B)\oplus(B\obslash A))/D & a^{\otimes\oplus} \end{array}$$

$$\begin{array}{cl} A & \\ \hline (\boxed{B}/A)\backslash B & l^{\backslash} \\ \hline (D/A)\backslash\boxed{B} & g \\ \hline (D/A)\backslash((D/B)\backslash\boxed{D}) & l^{\backslash} \\ \hline (D/A)\backslash\boxed{((D/B)\backslash(q\oplus(q\obslash D)))} & e^{\obslash} \\ \hline (D/A)\backslash(((D/B)\backslash q)\oplus(q\obslash D)) & \alpha^{\backslash} \\ \hline ((D/B)\backslash q)\oplus((D/A)\backslash(q\obslash D)) & \gamma^{\backslash} \end{array} \qquad \begin{array}{cl} B & \\ \hline (\boxed{A}/B)\backslash A & l^{\backslash} \\ \hline (D/B)\backslash\boxed{A} & f \\ \hline (D/B)\backslash\boxed{((D/A)\backslash D)} & l^{\backslash} \\ \hline (D/B)\backslash(q\oplus(q\obslash((D/A)\backslash D))) & e^{\obslash} \\ \hline ((D/B)\backslash q)\oplus\boxed{(q\obslash((D/A)\backslash D))} & \alpha^{\backslash} \\ \hline ((D/B)\backslash q)\oplus((D/A)\backslash(q\obslash D)) & \gamma^{\obslash\backslash} \end{array}$$

$$\begin{array}{cl} B & \\ \hline \boxed{1}\otimes B & 1 \\ \hline ((1\oslash D)\oplus D)\otimes B & e^{\oslash} \\ \hline (1\oslash D)\oplus(\boxed{D}\otimes B) & a^{\oplus\otimes} \\ \hline (1\oslash D)\oplus(A\otimes B) & g \end{array} \qquad \begin{array}{cl} A & \\ \hline A\otimes\boxed{1} & 1 \\ \hline A\otimes((1\oslash D)\oplus D) & e^{\oslash} \\ \hline (1\oslash D)\oplus(A\otimes\boxed{D}) & c^{\otimes\otimes} \\ \hline (1\oslash D)\oplus(A\otimes B) & f \end{array}$$

$$\begin{array}{cl} B & \\ \hline \boxed{(B\otimes(1\oslash(D\backslash 0)))}/(1\oslash(D\backslash 0)) & h^{/} \\ \hline (\boxed{(B\otimes 1)}\oslash(D\backslash 0))/(1\oslash(D\backslash 0)) & \alpha^{\oslash} \\ \hline (\boxed{B}\oslash(D\backslash 0))/(1\oslash(D\backslash 0)) & 1 \\ \hline ((A\backslash\boxed{(A\otimes B)})\oslash(D\backslash 0))/(1\oslash(D\backslash 0)) & h^{\backslash} \\ \hline ((\boxed{A}\backslash((A\otimes B)\oplus 0))\oslash(D\backslash 0))/(1\oslash(D\backslash 0)) & 0 \\ \hline (\boxed{(D\backslash((A\otimes B)\oplus 0))}\oslash(D\backslash 0))/(1\oslash(D\backslash 0)) & f \\ \hline \boxed{(((A\otimes B)\oplus(D\backslash 0))\oslash(D\backslash 0))}/(1\oslash(D\backslash 0)) & \gamma^{\backslash} \\ \hline (A\otimes B)/(1\oslash(D\backslash 0)) & h^{\oslash} \end{array}$$

$$\begin{array}{cl} A & \\ \hline (A\otimes B)/\boxed{B} & h^{/} \\ \hline (A\otimes B)/(1\oslash(B\obslash\boxed{1})) & l^{\oslash} \\ \hline (A\otimes B)/(1\oslash\boxed{(B\obslash(D\backslash(D\otimes 1)))}) & h^{\backslash} \\ \hline (A\otimes B)/(1\oslash(D\backslash(B\obslash\boxed{(D\otimes 1)}))) & \gamma^{\obslash\backslash} \\ \hline (A\otimes B)/(1\oslash(D\backslash(B\obslash\boxed{D}))) & 1^{*} \\ \hline (A\otimes B)/(1\oslash(D\backslash(B\obslash\boxed{B}))) & g \\ \hline (A\otimes B)/(1\oslash(D\backslash\boxed{(B\obslash(B\oplus 0))})) & 0 \\ \hline (A\otimes B)/(1\oslash(D\backslash 0)) & h^{\obslash} \end{array}$$

Fig. 2. Derivations for the joins constructed in Lem.2

Normally, we would suffice by repeated applications of the diamond property:

$$\begin{tikzcd}[column sep=small]
 & & & D_6 \arrow[dl, dashed] \arrow[dr, dashed] & & & \\
 & & D_4 \arrow[dl, dashed] \arrow[dr, dashed] & & D_5 \arrow[dl, dashed] \arrow[dr, dashed] & & \\
 & D_1 \arrow[dl, dashed] \arrow[dr, dashed] & & D_2 \arrow[dl, dashed] \arrow[dr, dashed] & & D_3 \arrow[dl, dashed] \arrow[dr, dashed] & \\
A_1 \arrow[dr] & & A_2 \arrow[dl] \arrow[dr] & & A_3 \arrow[dl] \arrow[dr] & & A_4 \arrow[dl] \\
 & C_1 & & C_2 & & C_3 &
\end{tikzcd}$$

Note D_6 derives each of A_1, A_2, A_3, A_4. Restricting to A_1, A_4, we have a shorter solution obviously generalizing $A \sqcap_C B = ((C/q) \oslash (A \obslash C)) \otimes (q \oslash (B \obslash C))$:

$$((C_2/q) \oslash (A_1 \obslash C_3)) \otimes (((q/q) \oslash (A_4 \obslash C_1)) \otimes (((q/q) \oslash (A_2 \obslash C_2)) \otimes (q \oslash (A_3 \obslash C_2))))$$

Lemma 3. *[9] Already in the base logic $\mathbf{LG}_\varnothing$, type similarity satisfies*

1. *Reflexivity.* $\vdash A \sim A$
2. *Transitivity.* $\vdash A \sim B$ *and* $\vdash B \sim C$ *imply* $\vdash A \sim C$
3. *Symmetry.* $\vdash A \sim B$ *implies* $\vdash B \sim A$
4. *Congruence.* $\vdash A_1 \sim A_2$ *and* $\vdash B_1 \sim B_2$ *imply* $\vdash A_1\ \delta\ B_1 \sim A_2\ \delta\ B_2$ *for any* $\delta \in \{\otimes, /, \backslash, \oplus, \obslash, \oslash\}$.

We next illustrate $\sim$'s expressivity. While some examples were already considered by M&P for $\mathbf{LG}_{IV}$, we here provide alternative (often shorter) solutions. For reasons of space, we often omit the derivations witnessing our claims.

Lemma 4. *Neutrals.* $\vdash C\backslash C \sim D/D$

Proof. We have a join $(((C\backslash C) \oslash D) \oplus ((D \oslash C) \oplus C))/(C\backslash C)$ for $\mathbf{LG}_I$, as well as a meet $(C \oslash (C \obslash (C \otimes D))) \otimes (D \oslash (D \obslash (C \otimes D)))$ for $\mathbf{LG}_{IV}$.

The next few lemmas detail associativity and commutativity properties; underivable, but still valid at the level of type similarity.

Lemma 5. *Symmetry.* $\vdash A\backslash B \sim B/A$

Proof. For $\mathbf{LG}_I$ we have a join $(((A\backslash B) \oslash A) \oplus B)/(A\backslash B)$,

$$\dfrac{\dfrac{\dfrac{A\backslash B}{((A\backslash B) \oslash A) \oplus \boxed{A}}\ e^{\oslash}}{((A\backslash B) \oslash A) \oplus (B/(A\backslash B))}\ h^{/}}{(((A\backslash B) \oslash A) \oplus B)/(A\backslash B)}\ a^{/} \qquad \dfrac{\dfrac{\dfrac{\dfrac{B/A}{((B/A) \otimes \boxed{(A\backslash B)})/(A\backslash B)}\ h^{/}}{\boxed{((B/A) \otimes (((A\backslash B) \oslash A) \oplus A))}/(A\backslash B)}\ e^{\oslash}}{(((A\backslash B) \oslash A) \oplus \boxed{((B/A) \otimes A)})/(A\backslash B)}\ c^{\otimes\oplus}}{(((A\backslash B) \oslash A) \oplus B)/(A\backslash B)}\ e^{/}$$

while for $\mathbf{LG}_{IV}$, we have a meet $A \oslash (B \obslash (A \otimes A))$.

$$\begin{array}{cl} \boxed{A} \oslash (B \obslash (A \otimes A)) & \\ \hline (A\backslash(A \otimes A)) \oslash (B \obslash (A \otimes A)) & h^{\backslash} \\ \hline A\backslash\boxed{((A \otimes A) \oslash (B \obslash (A \otimes A)))} & \alpha^{\backslash\oslash} \\ \hline A\backslash B & l^{\oslash} \end{array} \qquad \begin{array}{cl} \boxed{A} \oslash (B \obslash (A \otimes A)) & \\ \hline ((A \otimes A/A) \oslash (B \obslash (A \otimes A)) & h^{/} \\ \hline \boxed{((A \otimes A) \oslash (B \obslash (A \otimes A)))}/A & \gamma^{/\oslash} \\ \hline B/A & l^{\oslash} \end{array}$$

Lemma 6. *Rotations.* $\vdash A\backslash(C/B) \sim (A\backslash C)/B$ *and* $\vdash A\backslash(B\backslash C) \sim B\backslash(A\backslash C)$

Proof. In $\mathbf{LG}_I$, we have $A\backslash(B\backslash((B \oslash A) \oplus ((A \oslash B) \oplus C)))$ as a join for $A\backslash(B\backslash C)$ and $B\backslash(A\backslash C)$. To derive $\vdash A\backslash(C/B) \sim (A\backslash C)/B$, we proceed as follows:

1. $\mathbf{LG}_I \vdash A\backslash(C/B) \sim (C/B)/A$ (Lem.5)
2. $\mathbf{LG}_I \vdash (C/B)/A \sim (C/A)/B$ (shown above)
3. $\mathbf{LG}_I \vdash (C/A)/B \sim (A\backslash C)/B$ (Lem.5 and L.3(4))
4. $\mathbf{LG}_I \vdash A\backslash(C/B) \sim (A\backslash C)/B$ (Lem.3(2), 1,2,3)

For $\mathbf{LG}_{IV}$, we have a meet $((C \oslash (C/B)) \obslash q) \otimes ((C \oslash (A\backslash C)) \obslash (q\backslash C))$ witnessing $\mathbf{LG}_{IV} \vdash A\backslash(C/B) \sim (A\backslash C)/B$, as well as $\mathbf{LG}_{IV} \vdash A\backslash(B\backslash C) \sim B\backslash(A\backslash C)$ with meet $((C \oslash (A\backslash C)) \obslash q) \otimes ((C \oslash (B\backslash C)) \obslash (q\backslash C))$.

Lemma 7. *Distributivity.* $\vdash A \otimes (B/C) \sim (A \otimes B)/C$

Proof. For $\mathbf{LG}_I$, note $\vdash A \otimes (B/C) \sim A \otimes (C\backslash B)$ and $\vdash (A \otimes B)/C \sim C\backslash(A \otimes B)$ by Lem.5 and Lem.3(4). Thus, it suffices to show $\mathbf{LG}_I \vdash A \otimes (C\backslash B) \sim C\backslash(A \otimes B)$, fow which we have a join $C\backslash((A \oslash C) \oplus (C \otimes B))$.

$$\begin{array}{cl} A \otimes (C\backslash B) & \\ \hline C\backslash(C \otimes (\boxed{A} \otimes (C\backslash B))) & h^{\backslash} \\ \hline C\backslash(C \otimes \boxed{(((A \oslash C) \oplus C) \otimes (C\backslash B))}) & e^{\oslash} \\ \hline C\backslash\boxed{(C \otimes ((A \oslash C) \oplus (C \otimes (C\backslash B))))} & a^{\oplus\otimes} \\ \hline C\backslash((A \oslash C) \oplus (C \otimes \boxed{(C \otimes (C\backslash B))})) & c^{\otimes\oplus} \\ \hline C\backslash((A \oslash C) \oplus (C \otimes B)) & e^{\backslash} \end{array} \qquad \begin{array}{cl} C\backslash(\boxed{A} \otimes B) & \\ \hline C\backslash\boxed{(((A \oslash C) \oplus C) \otimes B)} & e^{\oslash} \\ \hline C\backslash((A \oslash C) \oplus (C \otimes B)) & a^{\oplus\otimes} \end{array}$$

In $\mathbf{LG}_{IV}$, we have meet $A \otimes ((A \oslash (B \obslash (A \otimes B))) \otimes (B \oslash (((A \otimes B)/C) \obslash (A \otimes B))))$.

Lemma 8. *Commutativity.* $\vdash A \otimes B \sim B \otimes A$

Proof. We have a join $(A \oslash B) \oplus (B \otimes B)$ for $\mathbf{LG}_I$,

$$\begin{array}{cl} \boxed{A} \otimes B & \\ \hline ((A \oslash B) \oplus B) \otimes B & e^{\oslash} \\ \hline (A \oslash B) \oplus (B \otimes B) & a^{\oplus\otimes} \end{array} \qquad \begin{array}{cl} B \otimes \boxed{A} & \\ \hline B \otimes ((A \oslash B) \oplus B) & e^{\oslash} \\ \hline (A \oslash B) \oplus (B \otimes B) & c^{\otimes\oplus} \end{array}$$

as well as a meet $(B \otimes B) \obslash (B \otimes ((B \oplus A) \otimes B))$. For $\mathbf{LG}_{IV}$, we have a meet $(((A/B) \oslash ((A \otimes B) \obslash (A \otimes A))) \otimes (B \oslash (B \obslash (A\backslash(A \otimes B))))) \otimes A$.

Lemma 9. *Associativity.* $\vdash (A \otimes B) \otimes C \sim A \otimes (B \otimes C)$

Proof. For $\mathbf{LG}_I$, we have meet $(A \otimes A) \obslash (A \otimes ((A \otimes (A \oplus B)) \otimes C))$, and join $(B \oslash ((A \backslash q) \oplus (Q/C))) \oplus (q \oplus q)$. In $\mathbf{LG}_{IV}$, use the diamond property after getting a meet D_1 from Lem.7 for $((A \otimes (B \otimes C))/C) \otimes C$ and $(A \otimes ((B \otimes C)/C)) \otimes C$:

$$\begin{tikzcd}
 & & D_2 \arrow[dl, dashed] \arrow[dr, dashed] & \\
 & D_1 \arrow[dl, dashed] \arrow[dr, dashed] & & (A \otimes B) \otimes C \arrow[dl] \\
((A \otimes (B \otimes C))/C) \otimes C \arrow[d] & & (A \otimes ((B \otimes C)/C)) \otimes C & \\
A \otimes (B \otimes C) & & &
\end{tikzcd}$$

Remark 3. While the above lemmas immediately extend to $\mathbf{LG}_I^{0,1}$ and $\mathbf{LG}_{IV}^{0,1}$, the presence of units often allows for simpler joins and meets. For example, we have the following joins (J) and meets (M) in $\mathbf{LG}_I^{0,1}$ and $\mathbf{LG}_{IV}^{0,1}$:

LEMMA	$\mathbf{LG}_I^{0,1}$	$\mathbf{LG}_{IV}^{0,1}$
Neutrals	J. $C \backslash ((1 \oslash D) \oplus (C \otimes D))$	J. $((1 \oslash C) \backslash 0) \backslash ((C \otimes D)/D)$
Symmetry	J. $(1 \oslash A) \oplus B$	M. $1 \oslash (B \obslash A)$
Commutativity	J. $(1 \oslash (1/A)) \oplus B$	M. $((1 \oslash A) \backslash 0) \otimes B$
Associativity	J. $(1 \oslash (1/A)) \oplus (B \otimes C)$	M. $A \otimes (((1 \oslash B) \backslash 0) \otimes C)$

4 Completeness Results

We consider models built upon algebraic structures featuring two binary operations $\times$ and $+$, related by linear distributivity. Their definition derives from the linear distributive categories of [2] by turning their arrows into equivalences.

Definition 7. *A linearly distributive algebra is a 6-tuple* $\mathscr{A} = \langle A, \times, +, ^\perp, \top, \perp \rangle$ *of type* $\langle 2, 2, 1, 0, 0 \rangle$ *satisfying*

1. *Associativity.* $(A \times B) \times C = A \times (B \times C)$; $(A + B) + C = A + (B + C)$
2. *Commutativity.* $A \times B = B \times A$; $A + B = B + A$
3. *Units.* $A \times \top = A$; $A + \perp = A$
4. *Inverses.* $A^\perp \times A = \perp$; $A^\perp + A = \top$
5. *Linear distributivity.* $A \times (B + C) = (A \times B) + C$

Definition 8. *A model* $\mathscr{M}$ *for* $\sim$ *is a pair* $\langle \mathscr{A}, v \rangle$ *extending* $\mathscr{A}$ *with a valuation* v *mapping atoms into* $\mathscr{A}$*, extended inductively to an interpretation* $[\![\cdot]\!]$*:*

$$\begin{array}{lll}
[\![p]\!] := v(p) & [\![A \otimes B]\!] := [\![A]\!] \times [\![B]\!] & [\![A \oplus B]\!] := [\![A]\!] + [\![B]\!] \\
[\![1]\!] := \top & [\![A/B]\!] := [\![A]\!] + [\![B]\!]^\perp & [\![B \obslash A]\!] := [\![B]\!]^\perp \times [\![A]\!] \\
[\![0]\!] := \perp & [\![B \backslash A]\!] := [\![B]\!]^\perp + [\![A]\!] & [\![A \oslash B]\!] := [\![A]\!] \times [\![B]\!]^\perp
\end{array}$$

Note that, for arbitrary A, $[\![{}^1 A]\!] = [\![A^1]\!] = [\![{}^0 A]\!] = [\![A^0]\!] = [\![A]\!]^\perp$. E.g., $[\![A^1]\!] = [\![A \obslash 1]\!] = [\![A]\!]^\perp \times \top = [\![A]\!]^\perp$, and $[\![{}^0 A]\!] = [\![0/A]\!] = \perp + [\![A]\!]^\perp = [\![A]\!]^\perp$. M&P as well conducted model-theoretic investigations into type similarity for $\mathbf{LG}_{IV}$.

Their interpretation, however, takes as target the *free* Abelian group generated by the atomic formulae and an additional element $\obar$,

$$\begin{array}{llll} [\![p]\!]' := p & [\![A \otimes B]\!]' := [\![A]\!]' \cdot [\![B]\!]' & [\![A \oplus B]\!]' := [\![A]\!]' \cdot \obar^{-1} \cdot [\![B]\!]' \\ & [\![A/B]\!]' := [\![A]\!]' \cdot [\![B]\!]'^{-1} & [\![B \obslash A]\!]' := [\![B]\!]'^{-1} \cdot \obar \cdot [\![A]\!]' \\ & [\![B\backslash A]\!]' := [\![B]\!]'^{-1} \cdot [\![A]\!]' & [\![A \oslash B]\!]' := [\![A]\!]' \cdot \obar \cdot [\![B]\!]'^{-1} \end{array}$$

writing 1 for unit and $^{-1}$ for inverse. While not reconcilable with Def.8 in that it does not offer a concrete instance of a linearly distributive algebra, the decidability of the word problem in free Abelian groups implies the decidability of type similarity as a corollary of completeness. The current investigation rather aims at a concept of model that better reflects the coexistence of residuated and coresiduated triples in the source language. While we can still prove type similarity complete w.r.t. the freely generated such model, as shown in Lem.14, the inference of decidability requires additional steps. Specifically, we will use Moortgat and Pentus' models in as inspiration in §5 to define, for each formula, a 'normal form', possibly involving units, w.r.t. which it is found type similar. We then decide type similarity at the level of such normal forms by providing an algorithm for generating joins, settling the word problem in the freely generated linear distributive algebra as a corollary, ensuring, in turn, the desired result.

Lemma 10. *If* $\vdash A \to B$, *then* $[\![A]\!] = [\![B]\!]$ *in every model.*

Proof. By induction on the arrow witnessing $\vdash A \to B$.

Theorem 1. *If* $A \sim B$, *then* $[\![A]\!] = [\![B]\!]$ *in every model.*

Proof. If $A \sim B$, we have a join C for A and B. By Lem.10, $\vdash A \to C$ and $\vdash B \to C$ imply $[\![A]\!] = [\![C]\!]$ and $[\![B]\!] = [\![C]\!]$, and hence $[\![A]\!] = [\![B]\!]$.

To prove completeness, we define a syntactic model wherein the interpretations of formulae are (constructively) shown to coincide with their equivalence classes under $\sim$. In defining said model, we use the following lemmas.

Lemma 11. *We have* $\vdash (A\backslash A) \oslash A \sim (A \obslash A)/A$.

Proof. Lem.5 gives meets D_1, D_2 for $\vdash (A\backslash A)\oslash A \sim A \obslash (A\backslash A)$ and $\vdash (A\obslash A)/A \sim A\backslash(A \obslash A)$. As such, we have a join C witnessing $\vdash A \obslash (A\backslash A) \sim A\backslash(A \obslash A)$, so that another use of the diamond property provides the desired meet D_3:

$$\begin{tikzcd}
 & & & D_3 \arrow[dll, dashed] \arrow[drr, dashed] & & & \\
 & D_1 \arrow[dl, dashed] \arrow[dr, dashed] & & & & D_2 \arrow[dl, dashed] \arrow[dr, dashed] & \\
(A\backslash A) \oslash A & & A \obslash (A\backslash A) \arrow[dr, dashed] & & A\backslash(A \obslash A) \arrow[dl, dashed] & & A\backslash(A \oslash A) \\
 & & & C & & &
\end{tikzcd}$$

Lemma 12. $1 \oslash A \sim A \obslash 1 \sim 0/A \sim A\backslash 0$ *in* $\boldsymbol{LG}_I^{0,1}$ *and* $\boldsymbol{LG}_{IV}^{0,1}$.

Proof. That $\vdash 1 \oslash A \sim A \obslash 1$ and $\vdash 0/A \sim A\backslash 0$ are immediate consequences of Lem.5. Furthermore, $\mathbf{LG}^{0,1}_{IV} \vdash 1 \oslash A \rightarrow A\backslash 0$, as shown on the left, while for $\mathbf{LG}_I$ we apply the diamond property, as shown on the right,

$$
\begin{array}{cl}
1 \oslash A & \\ \hline
A\backslash\boxed{(A \otimes (1 \oslash A))} & h\backslash \\ \hline
A\backslash(\boxed{(A \otimes 1)} \oslash A) & \alpha^{\oslash} \\ \hline
A\backslash(\boxed{A} \oslash A) & 1^* \\ \hline
A\backslash\boxed{((0 \oplus A) \oslash A)} & 0^* \\ \hline
A\backslash 0 & h^{\oslash}
\end{array}
\qquad
\begin{tikzcd}
 & D \arrow[dl, dashed] \arrow[dr, dashed] & \\
A\backslash((A \otimes 1) \oslash A) \arrow[d] \arrow[dr] & & 1 \oslash A \arrow[dl] \\
A\backslash 0 & A\backslash(A \otimes (1 \oslash A)) &
\end{tikzcd}
$$

Definition 9. *We construct a syntactic model by building a linearly distributive algebra upon the set of equivalence classes* $[A]_\sim := \{B \mid \vdash A \sim B\}$ *of formulae w.r.t.* $\sim$*. The various operations of the algebra are defined as follows:*

$$
\begin{array}{lll}
[A]_\sim \times [B]_\sim := [A \otimes B]_\sim & \top := [A\backslash A]_\sim & [A]^\perp_\sim := [(A\backslash A) \oslash A]_\sim \\
[A]_\sim + [B]_\sim := [A \oplus B]_\sim & \perp := [A \obslash A]_\sim & \qquad\; = [(A \obslash A)/A]_\sim
\end{array}
$$

For $\boldsymbol{LG}^{0,1}_I$ *and* $\boldsymbol{LG}^{0,1}_{IV}$*, the following simpler definitions suffice:*

$$
\begin{array}{lll}
[A]_\sim \times [B]_\sim := [A \otimes B]_\sim & \top := [1]_\sim & [A]^\perp_\sim := [1 \oslash A]_\sim = [A \obslash 1]_\sim \\
[A]_\sim + [B]_\sim := [A \oplus B]_\sim & \perp := [0]_\sim & \qquad\; = [0/A]_\sim = [A\backslash 0]_\sim
\end{array}
$$

Finally, we define the valuation by $v(p) := [p]_\sim$ *for arbitrary atom* p*.*

Lemma 13. *The syntactic model is well-defined.*

Proof. We check the equations of Def.7. Definition unfolding reduces (1) to showing $\vdash (A \otimes B) \otimes C \sim A \otimes (B \otimes C)$ and $\vdash (A \oplus B) \oplus C \sim A \oplus (B \oplus C)$. Both follow from Lem.9, noting that for the latter we can take the dual of a meet (join) for $\vdash C^\infty \otimes (B^\infty \otimes A^\infty) \sim (C^\infty \otimes B^\infty) \otimes A^\infty$ under $^\infty$. Similarly, (2) and (4) are immediate consequences of Lem.8 and Lem.11 (Lem.12 in the presence of units), while (3) is equally straightforward. This leaves (5), demanding $\vdash A \otimes (B \oplus C) \sim (A \otimes B) \oplus C$. We have $\mathbf{LG}_I \vdash A \otimes (B \oplus C) \rightarrow (A \otimes B) \oplus C$, while for $\mathbf{LG}_{IV}$ we use the diamond property:

$$
\begin{tikzcd}
 & D \arrow[dl, dashed] \arrow[dr, dashed] & \\
A \otimes (B \oplus C) \arrow[dr] & & A \otimes (A\backslash((A \otimes B) \oplus C)) \arrow[dl] \arrow[d] \\
 & A \otimes ((A\backslash(A \otimes B)) \oplus C) & (A \otimes B) \oplus C
\end{tikzcd}
$$

While we could proceed to prove $[\![A]\!] = [A]_\sim$ in the syntactic model for arbitrary A, we prove a slightly more involved statement, the increase in complexity paying off when proving decidability of type similarity in Thm.4. Write $\mathscr{A}(Atom)$ for the linear distributive algebra freely generated by the atoms.

Lemma 14. *If* $[\![A]\!] = [\![B]\!]$ *in* $\mathscr{A}(Atom)$*, then also* $\vdash A \sim B$*.*

Proof. We follow the strategy pioneered by Pentus [10]. Consider the homomorphic extension h of $p \mapsto [p]_\sim$ (cf. Def.9). We prove, for arbitrary A, that $h([\![A]\!]) = [A]_\sim$, taking $[\![A]\!]$ to be the interpretation of A in $\mathscr{A}(Atom)$. Hence, if $[\![A]\!] = [\![B]\!]$ in $\mathscr{A}(Atom)$, then also $h([\![A]\!]) = h([\![B]\!])$, so that $[A]_\sim = [B]_\sim$, and thus $\vdash A \sim B$. Proceeding by induction, the cases $A = p$, $A = 1$ and $A = 0$ follow by definition, while simple definitional unfolding suffices if $A = A_1 \otimes A_2$ or $A = A_1 \oplus A_2$. The cases $A = A_1 / A_2$, $A = A_2 \backslash A_1$, $A = A_1 \oslash A_2$ and $A = A_2 \obslash A_1$ are all alike, differing primarily in the number of applications of Lem.5. We demonstrate with $A = A_1 / A_2$. In $\mathbf{LG}_I$ and $\mathbf{LG}_{IV}$, we have

$$h([\![A_1/A_2]\!]) = h([\![A_1]\!]) + h([\![A_2]\!])^\perp = [A_1]_\sim + [A_2]_\sim^\perp = [A_1 \oplus ((A_2 \obslash A_2)/A_2)]_\sim$$

Thus, we have to show $\vdash A_1 \oplus ((A_2 \obslash A_2)/A_2) \sim A_1/A_2$:

1. $\vdash A_2 \obslash A_2 \sim A_1 \oslash A_1$ (Lem.4)
2. $\vdash A_1 \oslash A_1 \sim A_1 \obslash A_1$ (Lem.5)
3. $\vdash A_2 \obslash A_2 \sim A_1 \obslash A_1$ (Transitivity, 1, 2)
4. $\vdash A_1 \oplus ((A_2 \obslash A_2)/A_2) \sim A_1 \oplus ((A_1 \obslash A_1)/A_2)$ (Congruence, 3)
5. $\vdash A_1 \oplus ((A_1 \obslash A_1)/A_2) \sim (A_1 \oplus (A_1 \obslash A_1))/A_2$
6. $\vdash (A_1 \oplus (A_1 \obslash A_1))/A_2 \leftarrow A_1/A_2$
7. $\vdash A_1 \oplus ((A_2 \obslash A_2)/A_2) \sim A_1/A_2$ (Transitivity, 4, 5, 6)

In the presence of units, we have to show instead $\vdash A_1 \oplus (0/A_2) \sim A_1/A_2$, the desired proof being essentially a simplification of that found above.

Theorem 2. *If* $[\![A]\!] = [\![B]\!]$ *in every model, then* $\vdash A \sim B$.

Proof. If $[\![A]\!] = [\![B]\!]$ in every model, then in particular in $\mathscr{A}(Atom)$, and hence $\vdash A \sim B$ by Lem.14.

5 Generating Joins

We next present an algorithm for generating joins and meets in $\mathbf{LG}_I^{0,1}$, deriving decidability for the remaining incarnations of $\mathbf{LG}$ as a corollary. We proceed in two steps. First, we define for each formula A a 'normal form' $\|A\|^\circ$ w.r.t. which it is shown type similar by some join C_A. Whether or not any A and B are type similar is then decided for $\|A\|^\circ$ and $\|B\|^\circ$, an affirmative answer, witnessed by some meet D, implying the existence of a join C for A and B by the diamond property. The following figure summarizes the previous discussion.

$$\begin{tikzcd}
 & & & D \arrow[dl, dashed] \arrow[dr, dashed] & & & \\
A \arrow[dr, dashed] & & \|A\|^\circ \arrow[dl, dashed] & & \|B\|^\circ \arrow[dr, dashed] & & B \arrow[dl, dashed] \\
 & C_A \arrow[drr, dashed] & & & & C_B \arrow[dll, dashed] & \\
 & & & C & & &
\end{tikzcd}$$

Definition 10. *We define the maps $\|\cdot\|^\circ$ and $\|\cdot\|^\bullet$ by mutual induction:*

$$
\begin{array}{rcl}
\|p\|^\circ & := & p \\
\|1\|^\circ & := & 1 \\
\|0\|^\circ & := & 0 \\
\|A \otimes B\|^\circ & := & \|A\|^\circ \otimes \|B\|^\circ \\
\|A/B\|^\circ & := & \|A\|^\circ \otimes \|B\|^\bullet \\
\|B\backslash A\|^\circ & := & \|A/B\|^\circ \\
\|A \oplus B\|^\circ & := & \|A\|^\circ \otimes ((1/0) \otimes \|B\|^\circ) \\
\|A \oslash B\|^\circ & := & \|A\|^\circ \otimes (0 \otimes \|B\|^\bullet) \\
\|B \obslash A\|^\circ & := & \|A \oslash B\|^\circ
\end{array}
\qquad
\begin{array}{rcl}
\|p\|^\bullet & := & 1/p \\
\|1\|^\bullet & := & 1 \\
\|0\|^\bullet & := & 1/0 \\
\|A \otimes B\|^\bullet & := & \|A\|^\bullet \otimes \|B\|^\bullet \\
\|A/B\|^\bullet & := & \|A\|^\bullet \otimes \|B\|^\circ \\
\|B\backslash A\|^\bullet & := & \|A/B\|^\bullet \\
\|A \oplus B\|^\bullet & := & \|A\|^\bullet \otimes (\|B\|^\bullet \otimes 0) \\
\|A \oslash B\|^\bullet & := & \|A\|^\bullet \otimes ((1/0) \otimes \|B\|^\circ) \\
\|B \obslash A\|^\bullet & := & \|A \oslash B\|^\bullet
\end{array}
$$

Compare the above definition to the Abelian group interpretation of M&P: multiplications $A \cdot B$ and inverses A^{-1} are rendered as $A \otimes B$ and $1/A$, while 0 replaces the special atom $\oplus$. We now need only solve the problem of generating joins for the formulas in the images of $\|\cdot\|^\circ$ and $\|\cdot\|^\bullet$, relying on the result, proved presently, that $\vdash A \sim \|A\|^\circ$ and $\vdash 1/A \sim \|A\|^\bullet$.

Lemma 15. *There exist maps $f(\cdot)$ and $g(\cdot)$ mapping any given A to joins witnessing $\vdash A \sim \|A\|^\circ$ and $\vdash 1/A \sim \|A\|^\bullet$ respectively.*

Proof. The (mutual) inductive definition of the desired maps is presented in parallel with the proof of their correctness. In the base cases, we set

$$
\begin{array}{lll}
f(p) := p & f(1) := 1 & f(0) := 0 \\
g(p) := 1/p & g(1) := 1 & g(0) := 1/0
\end{array}
$$

Correctness is nigh immediate, noting $\vdash 1/1 \to 1$ for $g(1)$. The diamond property is used for most of the inductive cases. To illustrate, consider $A = A_1 \oslash A_2$ and $A = A_2 \obslash A_1$, handled similarly. Starting with $f(\cdot)$, we have, by induction hypothesis, joins $f(A_1)$ and $g(A_2)$ for $\vdash A_1 \sim \|A_1\|^\circ$ and $\vdash (1/A_2) \sim \|A_2\|^\bullet$. Hence, by Lem.3(4), we have a join $f(A_1) \otimes (0 \otimes g(A_2))$ for $\vdash (A_1 \otimes (0 \otimes (1/A_2)) \sim (\|A_1\|^\circ \otimes (0 \otimes \|A_2\|^\bullet))$. In addition, we have joins

$$
\begin{tikzcd}[column sep=tiny]
A_1 \oslash A_2 \arrow[dr] & & A_1 \otimes (0 \otimes (1/A_2)) \arrow[dl] \\
& ((A_1 \oslash A_2) \oplus A_2) \oplus ((1 \oplus A_2) \obslash 1) &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}[column sep=tiny]
A_2 \obslash A_1 \arrow[dr] & & A_1 \otimes (0 \otimes (1/A_2)) \arrow[dl] \\
& (A_2 \oplus (A_2 \obslash A_1)) \oplus ((A_2 \oplus 1) \obslash 1) &
\end{tikzcd}
$$

We demonstrate the derivability claims found in the left diagram in Fig.3, those found in the right diagram being shown similarly. With these findings, we may now define $f(A_1 \oslash A_2)$ through the diamond property:

$$
\begin{tikzcd}[column sep=tiny]
A_1 \oslash A_2 \arrow[dr] & & A_1 \otimes (0 \otimes (1/A_2)) \arrow[dl] \arrow[dr] & & \|A_1\|^\circ \otimes (0 \otimes \|A_2\|^\bullet) \arrow[dl] \\
& ((A_1 \oslash A_2) \oplus A_2) \oplus ((1 \oplus A_2) \obslash 1) \arrow[dr, dashed] & & f(A_1) \otimes (0 \otimes g(A_2)) \arrow[dl, dashed] & \\
& & f(A_1 \oslash A_2) & &
\end{tikzcd}
$$

$$
\begin{array}{cl}
A_1 \oslash A_2 & \\ \hline
(A_1 \oslash A_2) \otimes \boxed{1} & 1 \\ \hline
(A_1 \oslash A_2) \otimes ((1 \oplus A_2) \oplus ((1 \oplus A_2) \obslash 1)) & e^{\obslash} \\ \hline
\boxed{((A_1 \oslash A_2) \otimes (1 \oplus A_2))} \oplus ((1 \oplus A_2) \obslash 1) & a^{\otimes\oplus} \\ \hline
(\boxed{((A_1 \oslash A_2) \otimes 1)} \oplus A_2) \oplus ((1 \oplus A_2) \obslash 1) & a^{\otimes\oplus} \\ \hline
((A_1 \oslash A_2) \oplus A_2) \oplus ((1 \oplus A_2) \obslash 1) & 1^*
\end{array}
$$

$$
\begin{array}{cl}
A_1 \otimes (0 \otimes \boxed{(1/A_2)}) & \\ \hline
A_1 \otimes (0 \otimes ((1/A_2) \otimes \boxed{1})) & 1 \\ \hline
A_1 \otimes (0 \otimes \boxed{((1/A_2) \otimes ((1 \oplus A_2) \oplus ((1 \oplus A_2) \obslash 1)))}) & e^{\obslash} \\ \hline
A_1 \otimes (0 \otimes (\boxed{((1/A_2) \otimes (1 \oplus A_2))} \oplus ((1 \oplus A_2) \obslash 1))) & a^{\otimes\oplus} \\ \hline
A_1 \otimes (0 \otimes ((1 \oplus \boxed{((1/A_2) \otimes A_2)}) \oplus ((1 \oplus A_2) \obslash 1))) & c^{\otimes\oplus} \\ \hline
A_1 \otimes \boxed{(0 \otimes ((1 \oplus 1) \oplus ((1 \oplus A_2) \obslash 1)))} & e^{/} \\ \hline
A_1 \otimes (\boxed{(0 \otimes (1 \oplus 1))} \oplus ((1 \oplus A_2) \obslash 1)) & a^{\otimes\oplus} \\ \hline
A_1 \otimes ((\boxed{(0 \otimes 1)} \oplus 1) \oplus ((1 \oplus A_2) \obslash 1)) & a^{\otimes\oplus} \\ \hline
A_1 \otimes (\boxed{(0 \oplus 1)} \oplus ((1 \oplus A_2) \obslash 1)) & 1^* \\ \hline
A_1 \otimes (1 \oplus ((1 \oplus A_2) \obslash 1)) & 0 \\ \hline
\boxed{(A \otimes 1)} \oplus ((1 \oplus A_2) \obslash 1) & a^{\otimes\oplus} \\ \hline
\boxed{A_1} \oplus ((1 \oplus A_2) \obslash 1) & 1^* \\ \hline
((A_1 \oslash A_2) \oplus A_2) \oplus ((1 \oplus A_2) \obslash 1) & e^{\oslash}
\end{array}
$$

Fig. 3. Showing $\mathbf{LG}_I \vdash A_1 \oslash A_2 \to ((A_1 \oslash A_2) \oplus A_2) \oplus ((1 \oplus A_2) \obslash 1)$ and $\mathbf{LG}_I \vdash A_1 \otimes (0 \otimes (1/A_2)) \to ((A_1 \oslash A_2) \oplus A_2) \oplus ((1 \oplus A_2) \obslash 1)$

while $f(A_2 \obslash A_1)$ is similarly defined

$$(A_2 \oplus (A_2 \obslash A_1)) \oplus ((A_2 \oplus 1) \obslash 1) \sqcup_{A \otimes (0 \otimes (1/A_2))} (f(A_1) \otimes (0 \otimes g(A_2)))$$

The same strategy is used to define $g(A_1 \oslash A_2)$ and $g(A_2 \obslash A_1)$, this time employing joins $(1 \oslash A_1) \oplus (1 \oplus (0 \obslash A_2))$ and $(1 \oslash A_1) \oplus ((A_2 \oslash 0) \oplus 1)$ witnessing $\vdash 1/(A_1 \oslash A_2) \sim (1/A_1) \otimes ((1/0) \otimes A_2)$ and $\vdash 1/(A_2 \obslash A_1) \sim (1/A_1) \otimes ((1/0) \otimes A_2)$. In the same vein, we can handle a significant portion of the remaining cases:

$$
\begin{aligned}
g(A_1 \otimes A_2) &:= ((1 \oplus (A_2 \obslash 1)) \oplus (A_1 \obslash 1)) \sqcup_{(1/A_1) \otimes (1/A_2)} (g(A_1) \otimes g(A_2)) \\
g(A_1/A_2) &:= (1 \oplus ((A_1/A_2) \obslash 1)) \sqcup_{(1/A_1) \otimes A_2} (g(A_1) \otimes f(A_2)) \\
g(A_2 \backslash A_1) &:= (1 \oplus ((A_2 \backslash A_1) \obslash 1)) \sqcup_{(1/A_2) \otimes A_2} (g(A_1) \otimes f(A_2)) \\
f(A_1 \oplus A_2) &:= (A_1 \oplus (0 \obslash A_2)) \sqcup_{A_1 \otimes ((1/0) \otimes A_2)} f(A_1) \otimes ((1/0) \otimes f(A_2)) \\
g(A_1 \oplus A_2) &:= (1 \oplus ((A_1 \oplus A_2) \obslash 1)) \sqcup_{(1/A_1) \otimes ((1/A_2) \otimes 0)} (g(A_1) \otimes (g(A_2) \otimes 0))
\end{aligned}
$$

To show $\vdash (1/A_1) \otimes ((1/A_2) \otimes 0) \to 1 \oplus ((A_1 \oplus A_2) \obslash 1)$ for the definition of $g(A_1 \oplus A_2)$ can be a bit tricky, so we give the derivation in Fig.4. We are left with the following cases, handled without use of the diamond property:

$$
\begin{array}{cl}
(1/A_1) \otimes (\boxed{(1/A_2)} \otimes 0) & \\ \hline
(1/A_1) \otimes (((1/A_2) \otimes \boxed{1}) \otimes 0) & 1 \\ \hline
(1/A_1) \otimes (\boxed{((1/A_2) \otimes ((A_1 \oplus A_2) \oplus ((A_1 \oplus A_2) \obslash 1)))} \otimes 0) & e^{\obslash} \\ \hline
(1/A_1) \otimes ((\boxed{((1/A_2) \otimes (A_1 \oplus A_2))} \oplus ((A_1 \oplus A_2) \obslash 1)) \otimes 0) & a^{\otimes\oplus} \\ \hline
(1/A_1) \otimes (((A_1 \oplus \boxed{((1/A_2) \otimes A_2)}) \oplus ((A_1 \oplus A_2) \obslash 1)) \otimes 0) & a^{\otimes\oplus} \\ \hline
(1/A_1) \otimes \boxed{(((A_1 \oplus 1) \oplus ((A_1 \oplus A_2) \obslash 1)) \otimes 0)} & e^{/} \\ \hline
(1/A_1) \otimes (\boxed{((A_1 \oplus 1) \otimes 0)} \oplus ((A_1 \oplus A_2) \obslash 1)) & c^{\oplus\otimes} \\ \hline
(1/A_1) \otimes ((A_1 \oplus \boxed{(1 \otimes 0)}) \oplus ((A_1 \oplus A_2) \obslash 1)) & a^{\oplus\otimes} \\ \hline
(1/A_1) \otimes (\boxed{(A_1 \oplus 0)} \oplus ((A_1 \oplus A_2) \obslash 1)) & 1^{*} \\ \hline
(1/A_1) \otimes (A_1 \oplus ((A_1 \oplus A_2) \obslash 1)) & 0 \\ \hline
\boxed{((1/A_1) \otimes A_1)} \oplus ((A_1 \oplus A_2) \obslash 1) & a^{\otimes\oplus} \\ \hline
1 \oplus ((A_1 \oplus A_2) \obslash 1) & e^{/}
\end{array}
$$

Fig. 4. Showing $\mathbf{LG}_I \vdash (1/A_1) \otimes ((1/A_2) \otimes 0)$

$$
\begin{array}{rcl}
f(A_1 \otimes A_2) & := & f(A_1) \otimes f(A_2) \\
f(A_1/A_2) & := & (f(A_1) \oplus (A_2 \obslash g(A_2))) \oplus ((1/A_2) \obslash 1) \\
f(A_2\backslash A_1) & := & (f(A_1) \oplus (A_2 \obslash g(A_2))) \oplus ((1/A_2) \obslash 1)
\end{array}
$$

Fig.5 shows well-definedness of $f(A_1/A_2)$, with $f(A_2\backslash A_1)$ handled similarly.

We shall decide type similarity by reference to the following invariants.

Definition 11. *For arbitrary p, we define by mutual inductions the functions* $|\cdot|_p^+$ *and* $|\cdot|_p^-$ *counting, respectively, the numbers of positive and negative occurrences of p inside their arguments. First, the positive count:*

$$
\begin{array}{lll}
|r|_p^+ := 1 \text{ iff } r = p & |A \otimes B|_p^+ := |A|_p^+ + |B|_p^+ & |A \oplus B|_p^+ := |A|_p^+ + |B|_p^+ \\
|1|_p^+ := 0 & |A/B|_p^+ := |A|_p^+ + |B|_p^- & |B \obslash A|_p^+ := |A|_p^+ + |B|_p^- \\
|0|_p^+ := 0 & |B\backslash A|_p^+ := |A|_p^+ + |B|_p^- & |A \oslash B|_p^+ := |A|_p^+ + |B|_p^-
\end{array}
$$

and similarly, the negative count:

$$
\begin{array}{lll}
|r|_p := 0 & |A \otimes B|_p^- := |A|_p^- + |B|_p^- & |A \oplus B|_p^- := |A|_p^- + |B|_p^- \\
|1|_p := 0 & |A/B|_p^- := |A|_p^- + |B|_p^+ & |B \obslash A|_p^- := |A|_p^- + |B|_p^+ \\
|0|_p := 0 & |B\backslash A|_p^- := |A|_p^- + |B|_p^+ & |A \oslash B|_p^- := |A|_p^- + |B|_p^+
\end{array}
$$

The atomic count $|A|_p$ *for p is defined* $|A|_p^+ - |A|_p^-$*. In a similar fashion, we define positive and negative counts* $|A|_0^+$ *and* $|A|_0^-$ *for occurrences of the unit* 0 *inside A, and set* $|A|_0 := |A|_0^+ - |A|_0^-$*.*

In practice, the previously defined counts shall prove only of interest with arguments of the form $\|A\|^\circ$. In the case of arbitrary formulas, we therefore define (by mutual induction) the positive and negative *operator counts* $|A|_\oplus^+$ and $|A|_\oplus^-$ (resembling, though slightly differing from, a concept of [9] bearing the same name), recording the values of $|\|A\|^\circ|_0^+$ and $|\|A\|^\circ|_0^-$ respectively.

$$
\begin{array}{cl}
\|A_1\|^\circ \otimes \|A_2\|^\bullet & \\ \hline
\|A_1\|^\circ \otimes (\boxed{1} \otimes \|A_2\|^\bullet) & 1 \\ \hline
\|A_1\|^\circ \otimes \boxed{(((1/A_2) \oplus ((1/A_2) \obslash 1)) \otimes \|A_2\|^\bullet)} & e^{\obslash} \\ \hline
\|A_1\|^\circ \otimes (((1/A_2) \otimes \boxed{\|A_2\|^\bullet}) \oplus ((1/A_2) \obslash 1)) & c^{\oplus\otimes} \\ \hline
\|A_1\|^\circ \otimes (((1/A_2) \otimes \boxed{g(A_2)}) \oplus ((1/A_2) \obslash 1)) & IH \\ \hline
\|A_1\|^\circ \otimes (\boxed{((1/A_2) \otimes (A_2 \oplus (A_2 \obslash g(A_2))))} \oplus ((1/A_2) \obslash 1)) & e^{\obslash} \\ \hline
\|A_1\|^\circ \otimes ((\boxed{((1/A_2) \otimes A_2)} \oplus (A_2 \obslash g(A_2))) \oplus ((1/A_2) \obslash 1)) & a^{\otimes\oplus} \\ \hline
\|A_1\|^\circ \otimes ((1 \oplus (A_2 \obslash g(A_2))) \oplus ((1/A_2) \obslash 1)) & e^{/} \\ \hline
\boxed{(\|A_1\|^\circ \otimes (1 \oplus (A_2 \obslash g(A_2))))} \oplus ((1/A_2) \obslash 1) & a^{\otimes\oplus} \\ \hline
(\boxed{(\|A_1\|^\circ \otimes 1)} \oplus (A_2 \obslash g(A_2))) \oplus ((1/A_2) \obslash 1) & a^{\otimes\oplus} \\ \hline
(\boxed{\|A_1\|^\circ} \oplus (A_2 \obslash g(A_2))) \oplus ((1/A_2) \obslash 1) & 1^* \\ \hline
(f(A_1) \oplus (A_2 \obslash g(A_2))) \oplus ((1/A_2) \obslash 1) & IH
\end{array}
$$

$$
\begin{array}{cl}
A_1/A_2 & \\ \hline
(A_1/A_2) \otimes \boxed{1} & 1 \\ \hline
(A_1/A_2) \otimes ((1/A_2) \oplus ((1/A_2) \obslash 1)) & e^{\obslash} \\ \hline
((A_1/A_2) \otimes \boxed{(1/A_2)}) \oplus ((1/A_2) \obslash 1) & a^{\otimes\oplus} \\ \hline
((A_1/A_2) \otimes \boxed{g(A_2)}) \oplus ((1/A_2) \obslash 1) & IH \\ \hline
\boxed{((A_1/A_2) \otimes (A_2 \oplus (A_2 \obslash g(A_2))))} \oplus ((1/A_2) \obslash 1) & e^{\obslash} \\ \hline
(\boxed{((A_1/A_2) \otimes A_2)} \oplus (A_2 \obslash g(A_2))) \oplus ((1/A_2) \obslash 1) & a^{\otimes\oplus} \\ \hline
(\boxed{A_1} \oplus (A_2 \obslash g(A_2))) \oplus ((1/A_2) \obslash 1) & e^{/} \\ \hline
(f(A_1) \oplus (A_2 \obslash g(A_2))) \oplus ((1/A_2) \obslash 1) & IH
\end{array}
$$

Fig. 5. Proving well-definedness of $f(A_1/A_2)$

Definition 12. *For arbitrary A, define the positive and negative operator counts $|A|^+_\oplus$ and $|A|^-_\oplus$ are defined by induction over A, as follows:*

$$
\begin{array}{lll}
|p|^+_\oplus := 0 & |A \otimes B|^+_\oplus := |A|^+_\oplus + |B|^+_\oplus & |A \oplus B|^+_\oplus := |A|^+_\oplus + |B|^+_\oplus \\
|1|^+_\oplus := 0 & |A/B|^+_\oplus := |A|^+_\oplus + |B|^-_\oplus & |B \obslash A|^+_\oplus := |A|^+_\oplus + |B|^-_\oplus + 1 \\
|0|^+_\oplus := 1 & |B\backslash A|^+_\oplus := |A|^+_\oplus + |B|^-_\oplus & |A \oslash B|^+_\oplus := |A|^+_\oplus + |B|^-_\oplus + 1
\end{array}
$$

and

$$
\begin{array}{lll}
|p|^-_\oplus := 0 & |A \otimes B|^-_\oplus := |A|^-_\oplus + |B|^-_\oplus & |A \oplus B|^-_\oplus := |A|^-_\oplus + |B|^-_\oplus + 1 \\
|1|^-_\oplus := 0 & |A/B|^-_\oplus := |A|^-_\oplus + |B|^+_\oplus & |B \obslash A|^-_\oplus := |A|^-_\oplus + |B|^+_\oplus \\
|0|^-_\oplus := 0 & |B\backslash A|^-_\oplus := |A|^-_\oplus + |B|^+_\oplus & |A \oslash B|^-_\oplus := |A|^-_\oplus + |B|^+_\oplus
\end{array}
$$

Finally, the operator count $|A|_\oplus$ is defined $|A|^+_\oplus - |A|^-_\oplus$.

Lemma 16. *For any A, $|A|^+_\oplus = |\|A\|^\circ|^+_0 = |\|A\|^\bullet|^-_0$ and $|A|^-_\oplus = |\|A\|^\bullet|^+_0 = |\|A\|^\circ|^-_0$.*

Lemma 17. *If $\vdash A \to B$, then $|A|_\oplus = |B|_\oplus$, and $|A|_p = |B|_p$ for all p_i.*

Corollary 1. *If* $\vdash A \sim B$, *then* $|A|_{\oplus} = |B|_{\oplus}$, *and* $|A|_p = |B|_p$ *for all* p_i.

We now prove the inverse of the above corollary. Our aim is to define a meet for $\|A\|^{\circ}$ and $\|B\|^{\circ}$, entering into the construction of a join for A and B through use of the diamond property along with $f(A)$ and $f(B)$. To this end, we first require a few more definitions and lemmas. The following is an easy observation.

Lemma 18. *Formulas* $\|C\|^{\circ}, \|C\|^{\bullet}$ *for any* C *are included in the proper subset of* $\mathscr{F}(Atom)$ *generated by the following grammar:*

$$\phi ::= 0 \mid p_i \mid (1/0) \mid (1/p_i)$$
$$A^{nf}, B^{nf} ::= 1 \mid \phi \mid (A^{nf} \otimes B^{nf})$$

Thus, positive and negative occurrences of 0 (p_i) take the forms 0 (p_i) and 1/0 ($1/p_i$), being glued together through $\otimes$ only. We next detail the corresponding notion of *context*. Through universal quantification over said concept in stating derivability of certain rules pertaining to the Grishin interactions (cf. Lem.19), we obtain the non-determinacy required for the construction of the desired meet.

Definition 13. *A (tensor) context* $A^{\otimes}[]$ *is a bracketing of a series of formulae connected through* $\otimes$, *containing a unique occurrence of a hole* $[]$*:*

$$A^{\otimes}[], B^{\otimes}[] ::= [] \mid (A^{\otimes}[] \otimes B) \mid (A \otimes B^{\otimes}[])$$

Given $A^{\otimes}[], B$, *let* $A^{\otimes}[B]$ *denote the substitution of* B *for* $[]$ *in* $A^{\otimes}[]$.

We next characterize (half of) the type I Grishin interaction using contexts.

Lemma 19. *If* $\vdash A^{\otimes}[B \oslash C] \to D$, *then* $\vdash B \oslash A^{\otimes}[C] \to D$.

Proof. Assuming $f : A^{\otimes}[B \oslash C] \to D$, we proceed by induction on $A^{\otimes}[]$. The base case being immediate, we check $A^{\otimes}[] = A_1^{\otimes}[] \otimes A_2$ and $A^{\otimes}[] = A_1 \otimes A_2^{\otimes}[]$:

$$\dfrac{\dfrac{\dfrac{B \oslash (A_1^{\otimes}[C] \otimes A_2)}{\boxed{(B \oslash A_1^{\otimes}[C])} \otimes A_2}\, a^{\oslash}}{A_1^{\otimes}[B \oslash C] \otimes A_2}\, IH}{D}\, f \qquad \dfrac{\dfrac{\dfrac{B \oslash (A_1 \otimes A_2^{\otimes}[C])}{A_1 \otimes \boxed{(B \oslash A_2^{\otimes}[C])}}\, IH}{A_1 \otimes A_2^{\otimes}[B \oslash C]}\, c^{\oslash}}{D}\, f$$

The nondeterminacy required for the construction of our desired meet is obtained through the liberty of choosing one's context in instantiating the above rules. In practice, we only require recourse to the following restricted form.

Corollary 2. *If* $\vdash A^{\otimes}[B] \to C$, *then* $\vdash (1 \oslash B) \oslash A^{\otimes}[1] \to C$.

Proof. Suppose $f : (1 \oslash B) \oslash A^{\otimes}[1] \to C$. We then proceed as follows:

$$\dfrac{\dfrac{\dfrac{(1 \oslash B) \oslash A^{\otimes}[1]}{A^{\otimes}[\boxed{((1 \oslash B) \oslash 1)}]}\, Lem.19}{A^{\otimes}[B]}\, l^{\oslash}}{C}\, f$$

Theorem 3. *$\vdash A \sim B$ if $|A|_\oplus = |B|_\oplus$ and $|\|A\|^\circ|_p = |\|B\|^\circ|_p$ for all p.*

Proof. First, we require some notation. We shall write a (non-empty) *list* of formulas $[A_1, \ldots, A_n, B]$ to denote the right-associative bracketing of $A_1 \obslash \ldots A_n \obslash B$. Further, given $n \geq 0$, let A^n denote the list of n repetitions of A. Finally, we write ++ for list concatenation. Now let there be given an enumeration

$$p_1, p_2, \ldots p_n$$

of all the atoms occurring in A and B. Define

$$\begin{aligned} k &:= max(|\|A\|^\circ|_0^+, |\|B\|^\circ|_0^+) = max(|A|_\oplus^+, |B|_\oplus^+) \\ l &:= max(|\|A\|^\circ|_0^-, |\|B\|^\circ|_0^-) = max(|A|_\oplus^-, |B|_\oplus^-) \\ k(i) &:= max(|\|A\|^\circ|_{p_i}^+, |\|B\|^\circ|_{p_i}^+) \ (1 \leq i \leq n) \\ l(i) &:= max(|\|A\|^\circ|_{p_i}^-, |\|B\|^\circ|_{p_i}^-) \ (1 \leq i \leq n) \end{aligned}$$

We now witness $\vdash \|A\|^\circ \sim \|B\|^\circ$ by a meet

$$\begin{aligned} D \;:=\; & (1 \oslash p_1)^{k(1)} \;{+}{+}\; (1 \oslash (1/p_1))^{l(1)} \\ & {+}{+} \ldots \\ & {+}{+}\; (1 \oslash p_n)^{k(n)} \;{+}{+}\; (1 \oslash (1/p_n))^{l(n)} \\ & {+}{+}\; (1 \oslash 0)^k \qquad {+}{+}\; (1 \oslash (1/0))^l \qquad {+}{+}\; [1] \end{aligned}$$

Since we know from Lem.15 that $\vdash A \sim \|A\|^\circ$ and $\vdash B \sim \|B\|^\circ$ with joins $f(A)$ and $f(B)$, we can construct a join $f(A) \sqcup_D f(B)$ witnessing $\vdash A \sim B$. Suffice it to show that D, as defined above, is indeed a meet for $\|A\|^\circ$ and $\|B\|^\circ$. W.l.o.g., we show $\vdash D \to \|A\|^\circ$, dividing our proof in three steps. We shall a running example for illustrating each step, considering the concrete case where $A = p_2 \otimes (p_1/p_2)$ and $B = p_3 \oslash (p_3 \oslash p_1)$. Then

$$\begin{aligned} \|A\|^\circ &= p_2 \otimes (p_1 \otimes (1/p_2)) \\ \|B\|^\circ &= p_3 \otimes (0 \otimes ((1/p_3) \otimes ((1/0) \otimes p_1))) \\ D &= [1 \oslash p_1, 1 \oslash p_2, 1 \oslash (1/p_2), 1 \oslash p_3, 1 \oslash (1/p_3), 1 \oslash 0, 1 \oslash (1/0), 1] \end{aligned}$$

$$\begin{array}{lllllll} k(1) = 1 & & k(2) = 1 & & k(3) = 1 & & k = 1 \\ l(1) = 0 & & l(2) = 1 & & l(3) = 1 & & l = 1 \end{array}$$

1. First, note that we have

   If $f : E \to F$, also $(f \circ (1^*_{\otimes E} \circ (e^{/}_{1,G} \otimes i_E))) : ((1/G) \otimes G) \otimes E \to F$ $(*)$

   Starting with $i_{\|A\|^\circ} : \|A\|^\circ \to \|A\|^\circ$, for $i = 1$ to n, recursively apply $(*)$ $k(i) - |\|A\|^\circ|_{p_i}^+$ ($= l(i) - |\|A\|^\circ|_{p_i}^-$, by $|A|_{p_i} = |B|_{p_i}$) times, instantiating G with p_i, followed by another $k - |\|A\|^\circ|_\oplus^+$ ($= l - |\|A\|^\circ|_\oplus^-$, since $|A|_\oplus = |B|_\oplus$) recursive applications, this time instantiating G by 0. In our example, we obtain the following arrows:

$$\dfrac{\dfrac{\dfrac{\dfrac{(\boxed{((1/0) \otimes 0)} \otimes (((1/p_3) \otimes p_3) \otimes \|A\|^\circ))}{(\boxed{1} \otimes (((1/p_3) \otimes p_3) \otimes \|A\|^\circ))}\, e^{/}}{\boxed{((1/p_3) \otimes p_3)} \otimes \|A\|^\circ}\, 1^*}{\boxed{1} \otimes \|A\|^\circ}\, e^{/}}{\|A\|^\circ}\, 1^* \qquad \dfrac{\dfrac{\boxed{((1/p_2) \otimes p_2)} \otimes \|B\|^\circ}{\boxed{1} \otimes \|B\|^\circ}\, e^{/}}{\|B\|^\circ}\, 1^*$$

$$
\begin{prooftree}
\AxiomC{$[1 \oslash p_1, 1 \oslash p_2, 1 \oslash (1/p_2), 1 \oslash p_3, 1 \oslash (1/p_3), 1 \oslash 0] +\!\!+ \boxed{[1 \oslash (1/0), ((1 \otimes 1) \otimes ((1 \otimes 1) \otimes (1 \otimes (1 \otimes 1))))]}$}
\RightLabel{$Cor.2$}
\UnaryInfC{$[1 \oslash p_1, 1 \oslash p_2, 1 \oslash (1/p_2), 1 \oslash p_3, 1 \oslash (1/p_3)] +\!\!+ \boxed{[1 \oslash 0, (((1/0) \otimes 1) \otimes ((1 \otimes 1) \otimes (1 \otimes (1 \otimes 1))))]}$}
\RightLabel{$Cor.2$}
\UnaryInfC{$[1 \oslash p_1, 1 \oslash p_2, 1 \oslash (1/p_2), 1 \oslash p_3] +\!\!+ \boxed{[1 \oslash (1/p_3), (((1/0) \otimes 0) \otimes ((1 \otimes 1) \otimes (1 \otimes (1 \otimes 1))))]}$}
\RightLabel{$Cor.2$}
\UnaryInfC{$[1 \oslash p_1, 1 \oslash p_2, 1 \oslash (1/p_2)] +\!\!+ \boxed{[1 \oslash p_3, (((1/0) \otimes 0) \otimes (((1/p_3) \otimes 1) \otimes (1 \otimes (1 \otimes 1))))]}$}
\RightLabel{$Cor.2$}
\UnaryInfC{$[1 \oslash p_1, 1 \oslash p_2] +\!\!+ \boxed{[1 \oslash (1/p_2), (((1/0) \otimes 0) \otimes (((1/p_3) \otimes p_3) \otimes (1 \otimes (1 \otimes 1))))]}$}
\RightLabel{$Cor.2$}
\UnaryInfC{$[1 \oslash p_1] +\!\!+ \boxed{[1 \oslash p_2, (((1/0) \otimes 0) \otimes (((1/p_3) \otimes p_3) \otimes (1 \otimes (1 \otimes (1/p_2)))))]}$}
\RightLabel{$Cor.2$}
\UnaryInfC{$[1 \oslash p_1, (((1/0) \otimes 0) \otimes (((1/p_3) \otimes p_3) \otimes (p_2 \otimes (1 \otimes (1/p_2)))))]$}
\RightLabel{$Cor.2$}
\UnaryInfC{$((1/0) \otimes 0) \otimes (((1/p_3) \otimes p_3) \otimes (p_2 \otimes (p_1 \otimes (1/p_2))))$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$[1 \oslash p_1, 1 \oslash p_2, 1 \oslash (1/p_2), 1 \oslash p_3, 1 \oslash (1/p_3), 1 \oslash 0] +\!\!+ \boxed{[1 \oslash (1/0), (1 \otimes 1) \otimes (1 \otimes (1 \otimes (1 \otimes (1 \otimes 1))))]}$}
\RightLabel{$Cor.2$}
\UnaryInfC{$[1 \oslash p_1, 1 \oslash p_2, 1 \oslash (1/p_2), 1 \oslash p_3, 1 \oslash (1/p_3)] +\!\!+ \boxed{[1 \oslash 0, (1 \otimes 1) \otimes (1 \otimes (1 \otimes (1 \otimes ((1/0) \otimes 1))))]}$}
\RightLabel{$Cor.2$}
\UnaryInfC{$[1 \oslash p_1, 1 \oslash p_2, 1 \oslash (1/p_2), 1 \oslash p_3] +\!\!+ \boxed{[1 \oslash (1/p_3), (1 \otimes 1) \otimes (1 \otimes (0 \otimes (1 \otimes ((1/0) \otimes 1))))]}$}
\RightLabel{$Cor.2$}
\UnaryInfC{$[1 \oslash p_1, 1 \oslash p_2, 1 \oslash (1/p_2)] +\!\!+ \boxed{[1 \oslash p_3, (1 \otimes 1) \otimes (1 \otimes (0 \otimes ((1/p_3) \otimes ((1/0) \otimes 1))))]}$}
\RightLabel{$Cor.2$}
\UnaryInfC{$[1 \oslash p_1, 1 \oslash p_2] +\!\!+ \boxed{[1 \oslash (1/p_2), (1 \otimes 1) \otimes (p_3 \otimes (0 \otimes ((1/p_3) \otimes ((1/0) \otimes 1))))]}$}
\RightLabel{$Cor.2$}
\UnaryInfC{$[1 \oslash p_1] +\!\!+ \boxed{[1 \oslash p_2, ((1/p_2) \otimes 1) \otimes (p_3 \otimes (0 \otimes ((1/p_3) \otimes ((1/0) \otimes 1))))]}$}
\RightLabel{$Cor.2$}
\UnaryInfC{$[1 \oslash p_1, ((1/p_2) \otimes p_2) \otimes (p_3 \otimes (0 \otimes ((1/p_3) \otimes ((1/0) \otimes 1))))]$}
\RightLabel{$Cor.2$}
\UnaryInfC{$((1/p_2) \otimes p_2) \otimes (p_3 \otimes (0 \otimes ((1/p_3) \otimes ((1/0) \otimes p_1))))$}
\end{prooftree}
$$

Fig. 6. Illustrating the proof of Thm. 3

Note that the antecedent of $\rightarrow$ now contains exactly $k(i)$ (k) and $l(i)$ (l) occurrences of p_i (0) and $1/p_i$ $(1/0)$ respectively.

2. For $i = 1$ to n, apply the following procedure. Starting with the arrow constructed in the previous step, recursively apply Cor.2 $k(i)$ times, instantiating B with p_i, followed by another $l(i)$ applications where B is instantiated with $1/p_i$. Finally, we repeat the above procedure one last time with the positive and negative occurrences of 0. We continue with our example in Fig.6.
3. From D, we can derive the antecedent of the arrow in the previous step through repeated applications of 1, thus obtaining the desired result.

As a corollary of the above theorem, we can prove the decidability of the word problem in $\mathscr{A}(Atom)$. Lem.14 in turn implies decidability of type similarity in each of the variants of the Lambek-Grishin calculus discussed in this chapter.

Lemma 20. *For any expression ϕ in $\mathscr{A}(Atom)$, there exists a formula A in $\boldsymbol{LG}_I^{0,1}$ s.t. $[\![A]\!] = \phi$.*

Proof. We define the map $[\![\cdot]\!]^{-1}$ taking ϕ to a formula, as follows:

$$\begin{array}{rlrl} [\![p]\!]^{-1} &:= p & [\![\phi^\perp]\!]^{-1} &:= 0/[\![\phi]\!]^{-1} \\ [\![\top]\!]^{-1} &:= 1 & [\![\perp]\!]^{-1} &:= 0 \\ [\![\phi \times \psi]\!]^{-1} &:= [\![\phi]\!]^{-1} \otimes [\![\psi]\!]^{-1} & [\![\phi + \psi]\!]^{-1} &:= [\![\phi]\!]^{-1} \oplus [\![\psi]\!]^{-1} \end{array}$$

An easy induction ensures $[\![[\![\phi]\!]^{-1}]\!] = \phi$. To illustrate, consider the case $\phi^\perp$: $[\![[\![\phi^\perp]\!]^{-1}]\!] = [\![0/[\![\phi]\!]^{-1}]\!] = \perp + [\![[\![\phi]\!]^{-1}]\!]^\perp = \perp + \phi^\perp = \phi^\perp$.

Lemma 21. *For any $\phi, \psi \in \mathscr{A}(Atom)$, we can decide whether or not $\phi = \psi$.*

Proof. By Thm.3, we can decide $\mathbf{LG}_I^{0,1} \vdash [\![\phi]\!]^{-1} \sim [\![\psi]\!]^{-1}$ through comparison of atomic- and operator counts. If affirmative, then also $[\![[\![\phi]\!]^{-1}]\!] = [\![[\![\psi]\!]^{-1}]\!]$ by Thm.1, i.e., $\phi = \psi$ by Lem.20. If instead $\mathbf{LG}_I^{0,1} \nvdash [\![\phi]\!]^{-1} \sim [\![\psi]\!]^{-1}$, then also $[\![[\![\phi]\!]^{-1}]\!] \neq [\![[\![\psi]\!]^{-1}]\!]$, i.e., $\phi \neq \psi$ by Thm.2.

Theorem 4. *For any A, B, it is decidable whether $\boldsymbol{T} \vdash A \sim B$ for any $\boldsymbol{T} \in \{\boldsymbol{LG}_I, \boldsymbol{LG}_{IV}, \boldsymbol{LG}_I^{0,1}, \boldsymbol{LG}_{IV}^{0,1}\}$.*

Proof. Use Lem.21 to decide whether or not $[\![A]\!] = [\![B]\!]$ in $\mathscr{A}(Atom)$. If so, then $\vdash A \sim B$ by Lem.14. Otherwise, $\nvdash A \sim B$ by Thm.1.

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