On Residuation

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Abstract. In this paper we explore the residuation laws that are at the basis of the Lambek calculus, and more generally of categorial grammar. We intend to show how such laws are characterized in the framework of a purely non-commutative fragment of linear logic, known as Cyclic Multiplicative Linear Logic.

Keywords: Lambek calculus, categorial grammar, linear logic, proof-net, cyclicity.

Introduction

In this paper we consider the *residuation laws*, that are at the basis of categorial grammar, and particularly, of the Lambek calculus, in the framework of the *cyclic multiplicative proof-nets* (CyM-PN).

In section 1 we show several presentations of residuation laws: under the most usual presentation these rules are treated as equivalences between statements concerning operations of categorial grammar, and under another presentation as equivalences between sequents of a sequent calculus for categorial grammar.

In section 2 we deal with the concept of proof-net and cyclic multiplicative proof-net. Proof-nets are proofs represented in a geometrical way, and indeed they represent proofs in Linear Logic. Cyclic multiplicative proof-net (CyM-PNs) represents proofs in Cyclic Multiplicative Linear Logic, a purely non-commutative fragment of linear logic.

In section 3, we show how the conclusions of a CyM-PN may be described in different ways which correspond to different sequents of CyMLL. In particular, there are 15 possibile ways to read the conclusions of an arbitrary CyM-proof-net with three conclusions.

In section 4 we consider a particular point of view on CyM-PNs and on the format of the sequents which describe the conclusions of CyM-PNs, the regular intuitionistic point of view. We shall show that the sequents considered equivalent in the presentation of the residuation laws in the sequent calculus style are exactly all the possible different ways to describe the conclusions of the same CyM-proof-net with three conclusions, assuming a regular intuitionistic point of view.

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1 Residuation Laws

Residuation laws are basic laws of categorial grammar, in particular the Lambek Calculus and its variant called Non Associative Lambek Calculus [10,11,6,7,13].

Residuation laws may be presented in a pure algebraic style (and this is the most usual presentation) and in a sequent calculus style.

In a pure algebraic style, the residuation laws involve

- a binary operation on a set M: $\cdot$ (the *residuated* operation, called *product*);
- two binary *residual* operations on the same set M: $\backslash$ (the *left residual* operation of the product) and $/$ (the *right residual* operation of the product);
- a partial ordering on the same set M: $\leq$.

In this algebraic style, the residuation laws state the following equivalences for every $a, b, c \in M$ (where M is equiped with the binary operations $\cdot$, $\backslash$, $/$, and with a binary relation $\leq$):

$$\text{(RES)} \quad a \cdot b \leq c \;\text{ iff }\; b \leq a \backslash c \;\text{ iff }\; a \leq c/b$$

An ordered algebra $(M, \leq, \cdot, /, \backslash)$ such that $(M, \leq)$ is a poset and $\cdot$, $/$, $\backslash$ are binary operations on M satisfying (RES) is called a *residuated groupoid* or, in the case the product $\cdot$ is associative, a *residuated semigroup* (see e. g. [6, p. 17], [12, pp. 670-71]) .

In a sequent calculus style, the residuation laws concern sequents of the form $E \vdash F$ where E and F are formulas of a formal language where the following binary connectives occur:

- the *residuated* connective, the *conjuntion*, denoted by $\cdot$ or by $\otimes$ in linear logic;
- the *left residual* connective, the left implication, denoted by $\backslash$ or by $\multimap$ in linear logic;
- the *right residual* connective, the right implication, denoted by $/$ or by $\circ\!\!-$ in linear logic.

In a sequent calculus style, the residuation laws state the following equivalences between *contest-free* sequents of such a formal language: for every formula A, B, C

$$\text{(RES)} \quad A \cdot B \vdash C \;\text{ iff }\; B \vdash A \backslash C \;\text{ iff }\; A \vdash C/B$$

or (using the linear logic symbols):

$$\text{(RES)} \quad A \otimes B \vdash C \;\text{ iff }\; B \vdash A \multimap C \;\text{ iff }\; A \vdash C \circ\!\!- B$$

2 Cyclic Multiplicative Proof-Nets, CyM-PN

2.1 Multiplicative Proof-Nets and CyM-PN

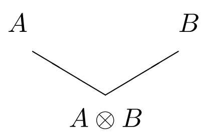
A multiplicative proof-net is a graph such that:

– the nodes are decorated by formulas of the fragment of Linear Logic which is called *multiplicative linear logic* (without units), i.e. the nodes are decorated by formulas constructed by starting with atoms by means of the binary connectives $\otimes$ (*multiplicative conjunction*) and $\parr$ (*multiplicative disjunction*), where
  - for each atom X there is another atom which is the dual of X and is denoted by $X^{\perp}$, in a way such that, for every atom X, $X^{\perp\perp} = X$;
  - for each formula A the linear negation $A^{\perp}$ is defined as follows, in order to satisfy the principle $A^{\perp\perp} = A$:
    - if A is an atom, $A^{\perp}$ is the atom which is the dual of A,
    - $(B \otimes C)^{\perp} = C^{\perp} \parr B^{\perp}$
    - $(B \parr C)^{\perp} = C^{\perp} \otimes B^{\perp}$;

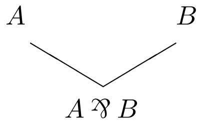
– edges are grouped by *links* and the links are:
  - the *axiom-link*, a binary link (i.e. a link with two nodes and one edge) with no premise, in which the two nodes are conclusions and each node is decorated by the linear negation of the formula decorating the other one; i.e. the conclusions of an axiom link are decorated by two formulas A, $A^{\perp}$

  - the *cut-link*, another binary link (i.e. a link with two nodes and one edge) where there is no conclusion and the two nodes are premises: each node is decorated by the linear negation of the formula decorating the other one, i.e. the premises of an axiom link are decorated by two formulas A, $A^{\perp}$

  - the $\otimes$-*link*, a ternary link (i.e. a link with three nodes and two edges), where two nodes are premises (the first premise and the second premise) and the other node is the conclusion, there is an edge between the first premise and the conclusion and another edge between the second premise and the conclusion, and the conclusion is decorated by a formula $A \otimes B$, where A is the formula decorating the first premise and B is the formula decorating the second premise



- the $\parr$-*link*, another ternary link (i.e. a link with three nodes and two edges), where two nodes are premises (the first premise and the second premise) and the other node is the conclusion, there is an edge between the first premise and the conclusion and another edge between the second premise and the conclusion, and the conclusion is decorated by a formula $A \parr B$, where A is the formula decorating the first premise and B is the formula decorating the second premise



– each node is the premise of at most one link, and is the conclusion of exactly one link; the nodes which are not premises of links are called the *conclusions of the proof-net*;
– for each "switching" the graph is acyclic and connected, where a "switching" of the graph is the removal of one edge in each $\parr$-link of the graph.

We point out that *left* and *right* residual connectives may be defined as follows, by means of the linear negation and the multiplicative disjunction:

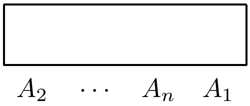
$$A \multimap C = A^{\perp} \parr C \qquad C \multimapinv A = C \parr A^{\perp}$$

A *cyclic multiplicative proof-net* (CyM-PN) is a multiplicative proof-net s. t.

– the graph is *planar*, i. e. the graph may be drawn on the plane with no crossing of edges,
– the conclusions are in a *cyclic order*, induced by the "trips" inside the proof-net (as defined in [1]; trips are possibile ways to visit the graph); this cyclic order of the conclusions corresponds to the order of the conclusions (from left to right, when the graph is written on the plane as a planar graph, i.e. with no crossing of edges) by adding that the "rightmost" conclusion is before the "leftmost" one.

As shown in [1], we may represent a CyM-proof-net π as a planar graph as follows:

where $A_1, \ldots, A_n$ are the conclusions of π in their cyclic order (A_2 is the immediate successor of A_1, ..., A_n is the immediate successor of A_{n-1}, A_1 is the immediate successor of A_n). There are other representations of the same CyM-proof-net π as a planar graph, i.e. for each conclusion A of π, we may represent π as a planar graph in such a way that A is the first conclusion going from left to right. For example, we may reprensent π in such a way that the first conclusion (from the left to the right) is A_2 and the last conclusion is A_1, i.e. :



A CyM-PN is *cut-free* iff it contains no cut-link.

An important theorem (*cut-elimination theorem* or *normalization theorem* for proof-nets) states that every CyM-PN can be transformed in a cut-free CyM-PN with the same conclusions. We may therefore restrict our attention to cut-free CyM-PN.

2.2 Terminal Links in CyM-PN. Irreducible CyM-PN

A ternary link of a CyM-PN π is *terminal* iff the conclusion of the link is also a conclusion of π.

It is immediate, from the definition of CyM-PN, to prove the following propositions on terminal $\parr$-links (see also [15]):

- if π is a CyM-PN and we remove from π a terminal $\parr$-link with conclusion $A \parr B$, by keeping the premises A and B which become conclusions of the graph, then we obtain a CyM-PN where, in the cyclic order of its conclusions, the conclusion $A \parr B$ is replaced by the two conclusions A, B, with B the immediate successor of A;
- if π is a CyM-proof-net, A and B are two conclusions of π and A is the immediate predecessor of B, in the cyclic order of the conclusions of π, then by adding to π a terminal $\parr$-link with first premise A and second premise B, we obtain a CyM-proof-net where, in the cyclic order of its conclusions, the pair of consecutive conclusions A, B is replaced by the conclusion $A \parr B$.

Remark that this proposition does not hold for terminal $\otimes$-links, so that there is a very strong geometrical difference between $\otimes$-links and $\parr$-links in CyM-PN.

Therefore we may remove one, more than one, or all the terminal $\parr$-links from a CyM-PN π and we still obtain a CyM-PN ψ, and from ψ we may return back to π. Similarly we may add to a CyM-PN ψ a new terminal $\parr$-link (where the premises are two conclusions, and the conclusion which is the first premise is the immediate predecessor of the conclusion which is the second premise), and we still obtain a CyM-PN π, and from π we may return back to ψ.

It is important to realize that the act of adding a terminal $\parr$-link to a CyM-PN, when the second premise is the immediate successor of the first one, and the act of removing a terminal $\parr$-link, do not fundamentally *modify* the proof-net.

On the basis of these properties, we may introduce the following definitions.

- Two CyM-PN π and ψ are *equivalent* iff each CyM-PN can be obtained from the other one by removing or by adding terminal $\parr$-links in the way indicated above.
- A CyM-PN is *irreducible*, iff no terminal link is a $\parr$-link.

We may then express the properties introduced above in the following way:

– *every CyM-PN is equivalent to a unique irreducible CyM-PN.*

Remark that if a CyM-PN π is equivalent to an irreducible CyM-PN ψ, then the conclusions of π differ from the conclusions of ψ as follows: some consecutive conclusions of ψ are replaced - as a conclusion of π - by the formula constructed from these conclusions by using the connective $\parr$ and by preserving the cyclic order of these conclusions. E.g. a CyM-PN π with conclusions $A, B \parr (C \parr D), E \parr F, G$, when A, B, C, D, E, F, G are formulas in which the main connective is not $\parr$, is equivalent to the irreducible CyM-PN ψ with conclusions A, B, C, D, E, F, G.

We may limit ourself to only dealing with *irreducible* CyM-PN's, considering every CyM-PN π as a different way to read the conclusions of the unique irreducible CyM-PN ψ equivalent to π. In the above example, the addition of terminal $\parr$-links to the irreducible CyM-PN ψ in order to get the CyM-PN π, may be considered as a way of reading the conclusions A, B, C, D, E, F, G of ψ in the form $A, B \parr (C \parr D), E \parr F, G$.

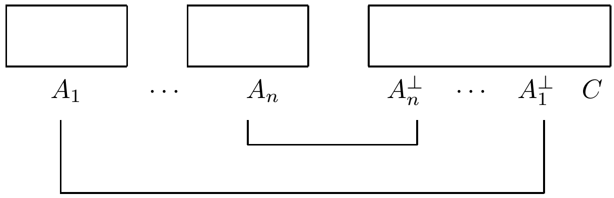
2.3 Focusing on Conclusions: Outputs and Inputs

When π is a CyM-PN, we may focus on one of the conclusions of π and consider it as the *output* of π, whereas the other conclusions play the role of *inputs*; i.e. we may say that they are nodes waiting for something (waiting for some formulas) in order to get the focused conclusion of π.

Let us denote by $A^\perp$ the conclusion of a CyM-PN π, when this conclusion is considered as waiting for the formula A. Remark that each conclusion of a CyM-PN may be considered as waiting for a formula: this possibility is given by the cut-rule that establishes the communication between two formulas, one of which is the dual of the other, where each formula is waiting for its dual.

Except in the case of a CyM-PN with only one conclusion - the choice to focus on a conclusion C is arbitrary and may be revised: i.e. each conclusion may be focused! Indeed, if we focus on a conclusion C of a CyM-PN π, this conclusion may be read as the output of π and, as a consequence, all the other conclusions have to be considered as inputs of π. But the nature of a CyM-PN allows to change the focus, i.e. to change the choice of the conclusion which is considered as an output. Every conclusion of a CyM-proof-net may be considered as an output, and the choice may be changed. This possibility corresponds also to the logical nature of a proof. A proof of B from the hypothesis A is a proof with conclusions B and $A^\perp$: a proof with output B, waiting for an input A, or a proof with ouput $A^\perp$ (the negation of A), waiting for an input $B^\perp$ (the negation of B).

Moreover, when π is a CyM-PN and C is a conclusion of π, we get a CyM-PN with just the unique conclusion C in the case in which, for each other conclusion $A^\perp$, there is a corresponding CyM-proof-net with conclusion A: it is enough to apply the cut rule n times, where $n + 1$ is the number of the conclusions of π:



Considering this example remark that, in the planar representation of π from left to right, if the conclusion $A_i^\perp$ occurs before the conclusion $A_j^\perp$, then the proof-net with conclusion A_j occurs before the proof-net with conclusion A_i.

Given a CyM-PN π, we may also focus on more than one conclusion, in particular on more than one consecutive conclusions; in this way the focused conclusions of π are considered as *outputs* and the other conclusions of π as *inputs*.

The focus on one conclusion or on more than one consecutive conclusions of a CyM-PN does not modify the graph, but it is simply a way to consider the graph, a way to describe the graph, in terms of some inputs and some outputs (e.g. in the represntation of a CyM-PN by an intuitionistic sequent, as we shall show in section 4).

2.4 Schematic CyM-PN

The *schema* of a cut-free CyM-PN π, with conclusions occurring in the cyclic order A_1, …, A_n, is what we get from π by removing all the decorations of the nodes and by denoting the conclusions (in their cyclic order) by the integers $1, \ldots, n$.

A *schematic CyM-PN* is the schema of a cut-free CyM-PN.

Remark that - if π is a CyM-PN - then the decoration of the nodes is induced from the decoration of the conclusions of the axiom links in the schema of π.

A schematic CyM-PN with n conclusions will be represented as

where the order of the conclusions is the cyclic order $1, \cdots, n$ (i.e. the conclusion $i+1$ is the immediate successor of the conclusion i for $i \neq n$ and the conclusion 1 is the immediate successor of the conclusion n). Every decoration of the axiom links in such a schematic cut-free CyM-PN produces a CyM-PN with conclusions decorated by formulas, i.e.

3 Sequents of CyMLL: Ways of Describing the Conclusions of CyM-PNs

3.1 Sequents of CyMLL and Conclusions of CyM-PN

A sequent of CyMLL is an expression of the form

$$\vdash \Gamma$$

where Γ is a finite sequence of formulas of the language of CyMLL.

A sequent $\vdash \Gamma$ of CyMLL is *irreducible* iff no formula in Γ is of the form $A \parr B$.

An important theorem (the *sequentialisation theorem*, [1]) states: in the sequent calculus for CyMLL one is able to prove a sequent $\vdash \Gamma$ iff there is a CyM-PN where the conclusions are in the cyclic order induced by Γ, i.e. by taking the first formula in Γ as the immediate successor of the last formula in Γ.

The above considerations are summarized in the following statement: each sequent may be considered as the list of all the conclusions of a possible CyM-PN, by starting with one of the conclusions and by listing all the conclusions on the basis of their cyclic order, and each provable sequent is the list of all the conclusions of a real CyM-PN.

A sequent $\vdash \Gamma$ is *derivable* from a sequent $\vdash \Delta$ in CyMLL iff in the sequent calculus for CyMLL one is able to derive $\vdash \Delta$ from $\vdash \Gamma$, i.e. iff from every CyM-PN with conclusions in the cyclic order induced by Γ one gets also a CyM-PN with conclusions in the cyclic order induced by Δ.

If Γ is a finite sequence of formulas, then we denote by $\Gamma^\perp$ the finite sequence of the linear negations of each formula of Γ in the reverse order; i.e., if Γ is the sequence $A_1, \ldots, A_n$, then $\Gamma^\perp$ is the sequence $A_n^\perp, \ldots, A_1^\perp$.

When π is a CyM-PN and we focus on the conclusion C of π as an output so that all the other conclusions are expressed by the linear negations $A^\perp$ of formulas A belonging to a finite sequence of formulas of CyMLL, the derivable sequent corresponding to π is of the form $\vdash \Gamma^\perp, C$. It is usual to write this sequent also as $\Gamma \vdash C$, i.e. by putting the inputs before $\vdash$ and the output after $\vdash$.

This means that, if a CyM-PN has two conclusions, then we may focus on a conclusion B and consider the other conclusion as an input, i.e. as $A^\perp$, therefore writing $A \vdash B$; or we may focus on the conclusion $A^\perp$ and consider B as waiting for $B^\perp$, i.e. reading B as $B^{\perp\perp}$, therefore writing $B^\perp \vdash A^\perp$.

The above considerations are summarized in the following statement: each sequent of the form $\Gamma \vdash C$ may be considered as the reading of a possible CyM-PN modulo the *focalization* on one of the conclusions (the conclusion C on the right side of the sequent).

3.2 Equivalent Sequents

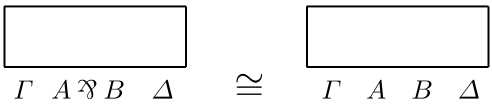
Two sequents $\vdash \Gamma$ and $\vdash \Delta$ are *equivalent* in CyMLL iff $\vdash \Gamma$ is derivable from $\vdash \Delta$, and viceversa, in the sequent calculus for CyMLL.

In other terms, two sequents $\vdash \Gamma$ and $\vdash \Delta$ are *equivalent* iff

- from every CyM-PN with conclusions in the cyclic order induced by Γ, we get a CyM-PN with conclusions in the cyclic order induced by Δ;
- from every CyM-PN with conclusions in the cyclic order induced by Δ, we get also a CyM-PN with conclusions in the cyclic order induced by Γ.

It is easy to verify that

- each sequent of the form $\vdash \Gamma, A \parr B, \Delta$ is equivalent in CyMLL to the sequent $\vdash \Gamma, A, B, \Delta$; therefore each sequent of CyMLL is equivalent to an *irreducible* sequent;



- the elements of each equivalence class of sequents, under the equivalence relation defined above, are
  - *irreducible sequents* which induce the same cyclic order,
  - all the sequents which are derivable from one of the irreducible sequents of the same class, by replacing two or more consecutive conclusions by a single conclusion which is obtained putting the connective $\parr$ between these formulas - according to their order - under an arbitrary use of brackets.

Let us consider a CyM-PN π. We may describe the cyclic order of its conclusions by means of a sequent $\vdash \Gamma$, where Γ is a sequence of formulas which contains exactly the conclusions of π and induces the cyclic order of the conclusions of π; i.e. Γ is the sequence of the conclusions of π in a planar representation of π (from left to right). Moreover, if Δ induces the same cyclic order as Γ, then $\vdash \Gamma$ and $\vdash \Delta$ are both descriptions of the cyclic order of the conclusions of π, the difference between $\vdash \Gamma$ and $\vdash \Delta$ being only a different way to consider (to see) the conclusions of π, and no modification of π is performed when we prefer the description $\vdash \Delta$ instead of $\vdash \Gamma$.

Therefore, the cyclic order of the conclusions of a CyM-PN may be described in several ways which include all the sequents $\vdash \Gamma$ such that Γ induces the cyclic order of the conclusions of π. So, if a CyM-PN π has n conclusions, there are at least n sequents which are descriptions of the cyclic order of the conclusions of π, and all these sequents are *equivalent*.

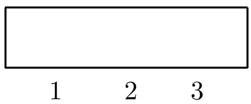
But there are other ways to describe the conclusions of a CyM-PN π: these ways are all the other sequents which are equivalent to the sequents $\vdash \Gamma$ where Γ induces the cyclic order of the conclusions of π. They are exactly all the sequents obtained by putting the connective $\parr$ between these conclusions - according to their order - under an arbitrary use of brackets.

Therefore, two sequents $\vdash \Gamma$ and $\vdash \Delta$ are *equivalent* in CyMLL iff $\vdash \Gamma$ and $\vdash \Delta$ are two different ways to describe the conclusions of the same possible CyM-PN. Two different ways to describe the conclusions of a possible CyM-PN by means of two different but equivalent sequents may have one of the following features:

- both the sequents describe the same cyclic order of the conclusions, but in two different ways, i.e. by starting with two different conclusions;
- two or more consecutive conclusions of one sequent are replaced in the other sequent by a single conclusion which is obtained by putting the connective $\parr$ between these formulas - according to their order - under an arbitrary use of brackets.

3.3 The Case of CyM-PNs with Three Conclusions

Let us consider a schematic CyM-PN π with three conclusions denoted by $1, 2, 3$ and let us suppose that the cyclic order of the conclusions is that 2 comes after 1, 3 comes after 2, and 1 comes after 3, i.e.



Let us consider the equivalent sequents which are different ways to describe the conclusions of such a schematic CyM-PN π.

- The following equivalent and irreducible sequents are descriptions of the cyclic order of the conclusions of π:

$$\vdash 1, 2, 3 \qquad \vdash 3, 1, 2 \qquad \vdash 2, 3, 1$$

- On this basis, all the possible descriptions of the conclusions of π are the following equivalent sequents:

$\vdash 1, 2, 3$	$\vdash 1 \parr 2, 3$	$\vdash 1, 2 \parr 3$	$\vdash (1 \parr 2) \parr 3$	$\vdash 1 \parr (2 \parr 3)$
$\vdash 3, 1, 2$	$\vdash 3 \parr 1, 2$	$\vdash 3, 1 \parr 2$	$\vdash (3 \parr 1) \parr 2$	$\vdash 3 \parr (1 \parr 2)$
$\vdash 2, 3, 1$	$\vdash 2 \parr 3, 1$	$\vdash 2, 3 \parr 1$	$\vdash (2 \parr 3) \parr 1$	$\vdash 2 \parr (3 \parr 1)$

where:

- the sequents in the first columm are irreducible and induce the same cyclic order of the conclusions $1, 2, 3$
- in each row there are the sequents obtained from the first sequent (an irreducible sequent) by adding a $\parr$ between the first two conclusions (second column), between the last two conclusions (third column), between the two conclusions of the second sequent, and between the two conclusions of the third sequent;
- for each sequent of the second column there is a sequent in the third column such that both the sequents induce the same cyclic order;

- the sequents in the fourth anf fifth colums are all the sequents which allow to express the cyclic order of the conclusions of π by means of an unique expression constructed by using twice the operation $\parr$.

Thus, there are 15 different ways of describing the conclusions of the graph π represented above. Remark that the schematic CyM-PN is not really modified when we prefer one of these ways, since the introduction of terminal $\parr$ links does not really modify a schematic CyM-PN π.

4 Residuation Laws as Regular Intuitionistic Descriptions of Conclusions of Intuitionistic CyM-PNs

4.1 Intuitionistic CyM-PNs

Let us call *intuitionistic* a CyM-PN π in which the focus is placed on only one conclusion of π.

The denomination *intuitionistic* is approriate, since in an intutionistic CyM-PN there is exactly one conclusion and an arbitrary finite number of inputs, as required by the intuitionistic point of view of programs and proofs.

Therefore, in each intuitionistic CyM-PN π:

- we cannot focus on more than one conclusion;
- the change of the focus is the change to another intuitionistic CyM-PN;
- there is exactly one conclusion which is considered as output - i.e. the focused conclusion - whereas all the other conclusions are considered as inputs.

Let us label with a formula C the unique focused conclusion of an intuitionistic CyM-PN π, whereas any other conclusion of π is waiting for something and is then labeled by $A^\perp$ where A is a a formula (a type).

We wish to emphasize that an intuitionistic CyM-PN is simply the addition of a fixed focus on one conclusion of a CyM-PN. As a result, each intuitionistic CyM-PN is also a CyM-PN, and each CyM-PN may be considered (when we add a focus on one conclusion) as an intuitionistic CyM-PN.

4.2 Regular Intuitionistic Description of CyM-PNs

Of course, the possible descriptions of the conclusions of an intuitionistic CyM-PN π are the descriptions of π by means of equivalent sequents of CyMLL.

But the specific character of an intuitionistic CyM-PN, i.e. the focus on exactly one conclusion, imposes to write under a special format, the *intuitionistic format*, the sequents which describe the conclusions of the CyM-PN.

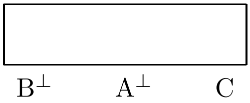
Let $\vdash \Gamma$ be a sequent which represents a way to describe the conclusions of a intuitionistic CyM-PN π with focused conclusion C: $\vdash \Gamma$ is of the form $\vdash \Delta^\perp, D, \Lambda^\perp$ where D is the formula C or a formula obtained from several conclusions of π including the focused conclusion C by means of the connective $\parr$. The *intuitionistic format* of $\vdash \Gamma$ is the expression $\Delta, \Lambda \vdash D$.

An *intuitionistic description* of the conclusions of an intuitionistic CyM-PN is the intuitionistic format of a sequent which is a description of the conclusions of π.

An intuitionistic description of the conclusions of an intuitionistic CyM-PN is *regular* iff it is of the form $E \vdash D$ where E and D are formulas.

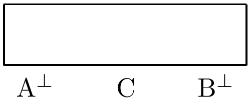
4.3 The Case of Intuitionistic CyM-PNs with Three Conclusions

Every intuitionistic CyM-PN π with three conclusions may be represented as

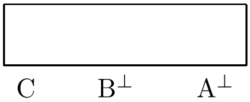


where the conclusion denoted by C is the one that is treated as the output of π and the other two conclusions are those considered as inputs of π.

Remark that we may represent π also as



or as



The 15 equivalent sequents which are the possible descriptions of the conclusions of π are the following:

$$\begin{array}{lllll}
\vdash B^\perp, A^\perp, C & \vdash B^\perp \parr A^\perp, C & \vdash B^\perp, A^\perp \parr C & \vdash (B^\perp \parr A^\perp) \parr C & \vdash B^\perp \parr (A^\perp \parr C) \\
\vdash C, B^\perp, A^\perp & \vdash C \parr B^\perp, A^\perp & \vdash C, B^\perp \parr A^\perp & \vdash (C \parr B^\perp) \parr A^\perp & \vdash C \parr (B^\perp \parr A^\perp) \\
\vdash A^\perp, C, B^\perp & \vdash A^\perp \parr C, B^\perp & \vdash A^\perp, C \parr B^\perp & \vdash (A^\perp \parr C) \parr B^\perp & \vdash A^\perp \parr (C \parr B^\perp)
\end{array}$$

The intuitionistic format of these sequents is as follows, representing each formula $(E^\perp \parr F^\perp)$ with its dual formula $(F \otimes E)^\perp$:

$$\begin{array}{lllll}
A, B \vdash C & A \otimes B \vdash C & B \vdash A^\perp \parr C & \vdash (A \otimes B)^\perp \parr C & \vdash B^\perp \parr (A^\perp \parr C) \\
A, B \vdash C & A \vdash C \parr B^\perp & A \otimes B \vdash C & \vdash (C \parr B^\perp) \parr A^\perp & \vdash C \parr (A \otimes B)^\perp \\
A, B \vdash C & B \vdash A^\perp \parr C & A \vdash C \parr B^\perp & \vdash (A^\perp \parr C) \parr B^\perp & \vdash A^\perp \parr (C \parr B^\perp)
\end{array}$$

Observe that all the sequents of the first column receive the same intuitionistic format, and that the second and the third columns contain the same sequents in the intuitionistic format. Thus, all the intuitionistic descriptions of the conclusions of the intuitionistic CyM-PN π are 10 (one in the first column, 3 in the second and third column, 3 in the fourth column and 3 in the last column).

Among these 10 intuitionistic descriptions of the intuitionistic CyM-NP π the *regular* ones are the sequents occurring in the second column or, equivalently, in the third column, i.e. there are only 3 regular intuitionistic descriptions of the CyM-NP π :

$$A \otimes B \vdash C \quad B \vdash A^{\perp} \parr C \quad A \vdash C \parr B^{\perp}$$

If we replace every formula $E^{\perp} \parr F$ by $E \multimap F$ and every formula $E \parr F^{\perp}$ by $E \multimapinv F$, then we obtain that all the possible regular intuitionistic representations of an intuitionistic CyM-PN with three conclusions, in which the focus is on the conclusion C, are the following equivalent sequents:

$$A \otimes B \vdash C \quad B \vdash A \multimap C \quad A \vdash C \multimapinv B$$

i. e.

$$A \bullet B \vdash C \quad B \vdash A \backslash C \quad A \vdash C/B$$

i.e. the sequents considered equivalent when the residuation laws of categorial grammar are represented in the sequent calculus style.

Therefore, we may say that residuation laws - when presented in the sequent calculus style - express the equivalence between the 3 sequents which are all the possible *regular intuitionistic descriptions of the conclusions* of the same CyMLL-PN with three conclusions:

$$A \otimes B \vdash C \quad B \vdash A \multimap C \quad A \vdash C \multimapinv B$$

More generally, we may consider as *general residuation laws* the equivalence between the 10 sequents which are all the possible *intuitionistic descriptions of the conclusions* of the same CyMLL-PN with three conclusions:

$$A, B \vdash C$$

$$\begin{array}{lll} A \otimes B \vdash C & B \vdash A \multimap C & A \vdash C \multimapinv B \\ \vdash A \otimes B \multimap C & \vdash (A \multimap C) \multimapinv B & \vdash (C \multimapinv B) \multimapinv A \\ \vdash C \multimapinv A \otimes B & \vdash B \multimap (A \multimap C) & \vdash A \multimap (C \multimapinv B) \end{array}$$

Conclusion

As a conclusion of our work, we would like to present the lines for further investigations as a generalization of the results obtained in this paper.

Residuation laws are the most simple examples of a large class of laws which are considered in categorial grammars, the class containing e.g. the following rules: monotonicity rules, application rules, expansion rules, transitivity rules, composition rules [10,8,9].

It would be very interesting to extend the present investigation to the full set of categorial grammar rules by adopting the same methodology presented here:

one starts by representing these rules in a sequent calculus style, and then shows that they correspond to properties or transformations of proof-nets (under a particular point of view).

In this way, we will be able to discover and represent the geometrical properties of the set of categorial grammar rules, having been facilitated in the investigation of the logical properties of these rules by the techniques and results of the theory of proof-nets (and viceversa).

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