L-Completeness of the Lambek Calculus with the Reversal Operation Allowing Empty Antecedents

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Abstract. In this paper we prove that the Lambek calculus allowing empty antecedents and enriched with a unary connective corresponding to language reversal is complete with respect to the class of models on subsets of free monoids (L-models).

1 The Lambek Calculus with the Reversal Operation

We consider the calculus L, introduced in [4]. The set $\mathrm{Pr} = \{p_1, p_2, p_3, \dots\}$ is called the set of *primitive types*. *Types* of L are built from primitive types using three binary connectives: $\backslash$ *(left division)*, $/$ *(right division)*, and $\cdot$ *(multiplication)*; we shall denote the set of all types by Tp. Capital letters $(A, B, \dots)$ range over types. Capital Greek letters (except Σ) range over finite (possibly empty) sequences of types; Λ stands for the empty sequence. Expressions of the form $\Gamma \to C$, where $\Gamma \neq \Lambda$, are called *sequents* of L.

Axioms: $A \to A$.

Rules:

$$\frac{A\Pi \to B}{\Pi \to A \setminus B}\ (\to \setminus),\ \Pi \neq \Lambda \qquad \frac{\Pi \to A \quad \Gamma B \Delta \to C}{\Gamma \Pi (A \setminus B) \Delta \to C}\ (\setminus \to)$$

$$\frac{\Pi A \to B}{\Pi \to B \,/\, A}\ (\to /),\ \Pi \neq \Lambda \qquad \frac{\Pi \to A \quad \Gamma B \Delta \to C}{\Gamma (B \,/\, A) \Pi \Delta \to C}\ (/ \to)$$

$$\frac{\Pi \to A \quad \Delta \to B}{\Pi \Delta \to A \cdot B}\ (\to \cdot) \qquad \frac{\Gamma A B \Delta \to C}{\Gamma (A \cdot B) \Delta \to C}\ (\cdot \to)$$

$$\frac{\Pi \to A \quad \Gamma A \Delta \to C}{\Gamma \Pi \Delta \to C}\ (\mathrm{cut})$$

The (cut) rule is eliminable [4].

We also consider an extra unary connective $^{\mathrm{R}}$ (written in the postfix form, A^{R}). The extended set of types is denoted by Tp^{R}. For a sequence of types $\Gamma = A_1 A_2 \dots A_n$ let $\Gamma^{\mathrm{R}} \leftrightharpoons A_n^{\mathrm{R}} \dots A_2^{\mathrm{R}} A_1^{\mathrm{R}}$ ("$\leftrightharpoons$" here and further means "equal by definition").

The calculus L^{R} is obtained from L by adding three rules for $^{\mathrm{R}}$:

$$\frac{\Gamma \to C}{\Gamma^{\mathrm{R}} \to C^{\mathrm{R}}}\ (^{\mathrm{R}} \to {}^{\mathrm{R}}) \qquad \frac{\Gamma A^{\mathrm{RR}} \Delta \to C}{\Gamma A \Delta \to C}\ (^{\mathrm{RR}} \to)_{\mathrm{E}} \qquad \frac{\Gamma \to C^{\mathrm{RR}}}{\Gamma \to C}\ (\to {}^{\mathrm{RR}})_{\mathrm{E}}$$

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Dropping the $\Pi \neq \Lambda$ restriction on the $(\to \backslash)$ and $(\to /)$ rules of L leads to *the Lambek calculus allowing empty antecedents* called L^*. The calculus $\mathrm{L}^{*\mathrm{R}}$ is obtained from L^* by changing the type set from Tp to Tp^{R} and adding the $({}^{\mathrm{R}} \to {}^{\mathrm{R}})$, $({}^{\mathrm{RR}} \to)_{\mathrm{E}}$, and $(\to {}^{\mathrm{RR}})_{\mathrm{E}}$ rules.

Unfortunately, no cut elimination theorem is known for L^{R} and $\mathrm{L}^{*\mathrm{R}}$. Nevertheless, L^{R} is a conservative extension of L, and $\mathrm{L}^{*\mathrm{R}}$ is a conservative extension of L^*:

Lemma 1. *A sequent formed of types from* Tp *is provable in* L^{R} *(*$\mathrm{L}^{*\mathrm{R}}$*) if and only if it is provable in* L *(resp.,* L^**).*

This lemma will be proved later via a semantic argument.

2 Normal Form for Types

The ${}^{\mathrm{R}}$ connective in the Lambek calculus and linear logic was first considered in [5] (there it is denoted by $\breve{}$). In [5], this connective is axiomatised using Hilbert-style axioms:

$$A^{\mathrm{RR}} \leftrightarrow A \qquad \text{and} \qquad (A \cdot B)^{\mathrm{R}} \leftrightarrow B^{\mathrm{R}} \cdot A^{\mathrm{R}}.$$

Here $F \leftrightarrow G$ ("F is *equivalent* to G") is a shortcut for two sequents: $F \to G$ and $G \to F$. The relation $\leftrightarrow$ is reflexive, symmetric, and transitive (due to the rule (cut)). Using (cut) one can prove that if $\mathrm{L}^{\mathrm{R}} \vdash F_1 \to G_1$, $F_1 \leftrightarrow F_2$, and $G_1 \leftrightarrow G_2$, then $\mathrm{L}^{\mathrm{R}} \vdash F_2 \to G_2$. Also, $\leftrightarrow$ is a *congruence relation,* in the following sense: if $A_1 \leftrightarrow A_2$ and $B_1 \leftrightarrow B_2$, then $A_1 \cdot B_1 \leftrightarrow A_2 \cdot B_2$, $A_1 \backslash B_1 \leftrightarrow A_2 \backslash B_2$, $B_1 \,/\, A_1 \leftrightarrow B_2 \,/\, A_2$, $A_1^{\mathrm{R}} \leftrightarrow A_2^{\mathrm{R}}$.

These axioms are provable in L^{R} and, vice versa, adding them to L yields a calculus equivalent to L^{R}. The same is true for $\mathrm{L}^{*\mathrm{R}}$ and L^* respectively.

Furthermore, the following two equivalences hold in L^{R} and $\mathrm{L}^{*\mathrm{R}}$:

$$(A \backslash B)^{\mathrm{R}} \leftrightarrow B^{\mathrm{R}} \,/\, A^{\mathrm{R}} \qquad \text{and} \qquad (B \,/\, A)^{\mathrm{R}} \leftrightarrow A^{\mathrm{R}} \backslash B^{\mathrm{R}}.$$

Using the four equivalences above one can prove by induction that any type $A \in \mathrm{Tp}^{\mathrm{R}}$ is equivalent to its *normal form* $tr(A)$, defined as follows:

1. $tr(p_i) \leftrightharpoons p_i$;
2. $tr(p_i^{\mathrm{R}}) \leftrightharpoons p_i^{\mathrm{R}}$;
3. $tr(A \cdot B) \leftrightharpoons tr(A) \cdot tr(B)$;
4. $tr(A \backslash B) \leftrightharpoons tr(A) \backslash tr(B)$;
5. $tr(B \,/\, A) \leftrightharpoons tr(B) \,/\, tr(A)$;
6. $tr((A \cdot B)^{\mathrm{R}}) \leftrightharpoons tr(B^{\mathrm{R}}) \cdot tr(A^{\mathrm{R}})$;
7. $tr((A \backslash B)^{\mathrm{R}}) \leftrightharpoons tr(B^{\mathrm{R}}) \,/\, tr(A^{\mathrm{R}})$;
8. $tr((B \,/\, A)^{\mathrm{R}}) \leftrightharpoons tr(A^{\mathrm{R}}) \backslash tr(B^{\mathrm{R}})$;
9. $tr(A^{\mathrm{RR}}) \leftrightharpoons tr(A)$.

In the normal form, the ${}^{\mathrm{R}}$ connective can appear only on occurrences of primitive types. Obviously, $tr(tr(A)) = tr(A)$ for every type A.

We also consider variants of L and L^* with $\mathrm{Tp} \cup \{p^{\mathrm{R}} \mid p \in \mathrm{Tp}\}$ instead of Tp as the set of primitive types. These calculi will be called L' and $L^{*\prime}$ respectively. Obviously, if a sequent is provable in L', then all its types are in normal form and this sequent is provable in L^{R} (and the same for $L^{*\prime}$ and $L^{*\mathrm{R}}$). Later we shall prove the converse statement:

Lemma 2. *A sequent $F_1 \dots F_n \to G$ is provable in L^{R} (resp., $L^{*\mathrm{R}}$) if and only if the sequent $tr(F_1) \dots tr(F_n) \to tr(G)$ is provable in L' (resp., $L^{*\prime}$).*

3 L-Models

Now let Σ be an alphabet (an arbitrary nonempty set, finite or countable). By Σ^+ we denote the set of all nonempty words over Σ; the set of all words over Σ, including the empty word, is denoted by Σ^*. The set Σ^* with the operation of word concatenation is the *free monoid* generated by Σ; the empty word ϵ is the unit of this monoid. Subsets of Σ^* are called *languages* over Σ. The set Σ^+ with the same operation is the *free semigroup* generated by Σ. Its subsets are *languages without the empty word.*

The set $\mathcal{P}(\Sigma^*)$ of all languages is also a monoid: if $M, N \subseteq \Sigma^*$, then let $M \cdot N$ be $\{uv \mid u \in M, v \in N\}$; the singleton $\{\epsilon\}$ is the unit. Likewise, the set $\mathcal{P}(\Sigma^+)$ is a semigroup with the same multiplication operation.

On these two structures one can also define two *division* operations: $M \setminus N \risingdotseq \{u \in \Sigma^* \mid (\forall v \in M)\, vu \in N\}$, $N \,/\, M \risingdotseq \{u \in \Sigma^* \mid (\forall v \in M)\, uv \in N\}$ for $\mathcal{P}(\Sigma^*)$, and $M \setminus N \risingdotseq \{u \in \Sigma^* \mid (\forall v \in M)\, vu \in N\}$, $N \,/\, M \risingdotseq \{u \in \Sigma^+ \mid (\forall v \in M)\, uv \in N\}$ for $\mathcal{P}(\Sigma^+)$. Note that, unlike multiplication, the $\mathcal{P}(\Sigma^*)$ version of division operations does not coincide with the $\mathcal{P}(\Sigma^+)$ one even for languages without the empty word. For example, if $M = N = \{a\}$ ($a \in \Sigma$), then $M \setminus N$ is $\{\epsilon\}$ in $\mathcal{P}(\Sigma^*)$ and empty in $\mathcal{P}(\Sigma^+)$.

These three operations on languages naturally correspond to three connectives of the Lambek calculus, thus giving an interpretation for Lambek types and sequents. An *L-model* is a pair $\mathcal{M} = \langle \Sigma, w \rangle$, where Σ is an alphabet and w is a function that maps Lambek calculus types to languages over Σ, such that $w(A \cdot B) = w(A) \cdot w(B)$, $w(A \setminus B) = w(A) \setminus w(B)$, and $w(B \,/\, A) = w(B) \,/\, w(A)$ for all $A, B \in \mathrm{Tp}$. One can consider models either with or without the empty word, depending on what set of languages ($\mathcal{P}(\Sigma^*)$ or $\mathcal{P}(\Sigma^+)$), and, more importantly, what version of the division operations is used. Models with and without the empty word are similar but different (in particular, models with the empty word are not a generalisation of models without it). Obviously, w can be defined on primitive types in an arbitrary way, and then it is uniquely propagated to all types.

A sequent $F_1 \dots F_n \to G$ is considered *true* in a model $\mathcal{M}$ ($\mathcal{M} \vDash F_1 \dots F_n \to G$) if $w(F_1) \cdot \ldots \cdot w(F_n) \subseteq w(G)$. If the sequent has an empty antecedent ($n = 0$), i. e., is of the form $\to G$, then it is considered true if $\epsilon \in w(G)$. This implies that such sequents are never true in L-models without the empty word. L-models give sound and complete semantics for L and L^*, due to the following theorem:

Theorem 1. *A sequent is provable in* L *if and only if it is true in all L-models without the empty word. A sequent is provable in* L^* *if and only if it is true in all L-models with the empty word.*

This theorem is proved in [8] for L and in [9] for L^*; its special case for the product-free fragment (where we keep only types without multiplication) is much easier and appears in [1].

Note that for L and L-models without the empty word it is sufficient to consider only sequents with one type in the antecedent, since $L \vdash F_1 F_2 \ldots F_n \to G$ if and only if $L \vdash F_1 \cdot F_2 \cdot \ldots \cdot F_n \to G$. For L^* and L-models with the empty word it is sufficient to consider only sequents with empty antecedent, since $L^* \vdash F_1 \ldots F_{n-1} F_n \to G$ if and only if $L^* \vdash \to F_n \setminus (F_{n-1} \setminus \ldots \setminus (F_1 \setminus G) \ldots))$.

4 L-Models with the Reversal Operation

The new $^{\mathrm{R}}$ connective corresponds to the *language reversal* operation. For $u = a_1 a_2 \ldots a_n$ ($a_1, \ldots, a_n \in \Sigma$, $n \geq 1$) let $u^{\mathrm{R}} \leftrightharpoons a_n \ldots a_2 a_1$; $\epsilon^{\mathrm{R}} \leftrightharpoons \epsilon$. For a language M let $M^{\mathrm{R}} \leftrightharpoons \{u^{\mathrm{R}} \mid u \in M\}$. The notion of L-model is easily modified to deal with the new connective by adding additional constraints on w: $w(A^{\mathrm{R}}) = w(A)^{\mathrm{R}}$ for every type A.

One can easily show that the calculi L^{R} and $L^{*\mathrm{R}}$ are sound with respect to L-models with the reversal operation (without and with the empty word respectively). Now, using this soundness statement and Pentus' completeness theorem (Theorem 1), we can prove Lemma 1 (conservativity of L^{R} over L and $L^{*\mathrm{R}}$ over L^*): if a sequent is provable in L^{R} (resp., $L^{*\mathrm{R}}$) and does not contain the $^{\mathrm{R}}$ connective, then it is true in all L-models without the empty word (resp., with the empty word). Moreover, in these L-models the language reversal operation is never used. Therefore, the sequent involved is provable in L (resp., L^*) due to the completeness theorem.

The completeness theorem for L^{R} is proved in [3] (the product-free case is again easy and is handled in [6] using Buszkowski's argument [1]):

Theorem 2. *A sequent is provable in* L^{R} *if and only if it is true in all L-models with the reversal operation and without the empty word.*

In this paper we present a proof for the $L^{*\mathrm{R}}$ version of this theorem:

Theorem 3. *A sequent is provable in* $L^{*\mathrm{R}}$ *if and only if it is true in all L-models with the reversal operation and without the empty word.*

The proof basically duplicates the proof of Theorem 2 from [3]; changes are made to handle the empty word cases.

The main idea is as follows: if a sequent in normal form is not provable in $L^{*\mathrm{R}}$, then it is not provable in $L^{*\prime}$. Therefore, by Theorem 1, there exists a model in which this sequent is not true, but this model does not necessarily satisfy all of the conditions $w(A^{\mathrm{R}}) = w(A)^{\mathrm{R}}$. We want to modify our model by adding $w(A^{\mathrm{R}})^{\mathrm{R}}$ to $w(A)$. For L^{R} [3], we can first make the sets $w(A^{\mathrm{R}})^{\mathrm{R}}$ and $w(A)$

disjoint by replacing every letter $a \in \Sigma$ by a long word $a^{(1)} \dots a^{(N)}$ ($a^{(i)}$ are symbols from a new alphabet); then the new interpretation for A is going to be $w(A) \cup w(A^{\mathrm{R}})^{\mathrm{R}} \cup T$ with an appropriate "trash heap" set T. For $\mathrm{L}^{*\mathrm{R}}$, we cannot do this directly, because ϵ will still remain the same word after the substitution of long words for letters. Fortunately, the model given by Theorem 1 enjoys a sort of weak universal property: if a type A is a subtype of our sequent, then $\epsilon \in w(A)$ if *and only if* $\mathrm{L}^{*\prime} \vdash \to A$. Hence, if $\epsilon \in w(A)$, then $\epsilon \in w(A^{\mathrm{R}})$, and vice versa, so the empty word does not do any harm here.

Note that essentially here we need only the fact that our sequent is not derivable in $\mathrm{L}^{*\prime}$, but not $\mathrm{L}^{*\mathrm{R}}$, and from this assumption we prove the existence of a model falsifying it. Hence, the sequent is not provable in $\mathrm{L}^{*\mathrm{R}}$. Therefore, we have proved Lemma 2.

5 L-Completeness of $\mathbf{L}^{*\mathbf{R}}$ (Proof)

Let $\mathrm{L}^{*\mathrm{R}} \nvdash \to G$ (as mentioned earlier, it is sufficient to consider sequents with empty antecedent). Also let G be in normal form (otherwise replace it by $tr(G)$).

Since $\mathrm{L}^{*\mathrm{R}} \nvdash \to G$, $\mathrm{L}^{*\prime} \nvdash \to G$. The calculus $\mathrm{L}^{*\prime}$ is essentially the same as L^*, therefore Theorem 1 gives us a structure $\mathcal{M} = \langle \Sigma, w \rangle$ such that $\epsilon \notin w(G)$. The structure $\mathcal{M}$ indeed falsifies $\to G$, but it is not a model in the sense of our new language: some of the conditions $w(p_i^{\mathrm{R}}) = w(p_i)^{\mathrm{R}}$ might be not satisfied.

Let Φ be the set of all subtypes of G (including G itself; the notion of subtype is understood in the sense of L^{R}).

The construction of $\mathcal{M}$ (see [9]) guarantees that the following two statements hold for every $A \in \Phi$:

1. $w(A) \neq \varnothing$;
2. $\epsilon \in w(A) \iff \mathrm{L}^{*\prime} \vdash \to A$.

We introduce an inductively defined counter $f(A)$, $A \in \Phi$: $f(p_i) \leftrightharpoons 1$, $f(p_i^{\mathrm{R}}) \leftrightharpoons 1$, $f(A \cdot B) \leftrightharpoons f(A) + f(B) + 10$, $f(A \setminus B) \leftrightharpoons f(B)$, $f(B \,/\, A) \leftrightharpoons f(B)$. Let $K \leftrightharpoons \max\{f(A) \mid A \in \Phi\}$, $N \leftrightharpoons 2K + 25$ (N should be odd, greater than K, and big enough itself).

Let $\Sigma_1 \leftrightharpoons \Sigma \times \{1, \dots, N\}$. We shall denote the pair $\langle a, j \rangle \in \Sigma_1$ by $a^{(j)}$. Elements of Σ and Σ_1 will be called *letters* and *symbols* respectively. A symbol can be *even* or *odd* depending on the parity of the superscript. Consider a homomorphism $h \colon \Sigma^* \to \Sigma_1^*$, defined as follows: $h(a) \leftrightharpoons a^{(1)} a^{(2)} \dots a^{(N)}$ ($a \in \Sigma$), $h(a_1 \dots a_n) \leftrightharpoons h(a_1) \dots h(a_n)$, $h(\epsilon) = \epsilon$. Let $P \leftrightharpoons h(\Sigma^+)$. Note that h is a bijection between Σ^* and $P \cup \{\epsilon\}$ and between Σ^+ and P.

Lemma 3. *For all $M, N \subseteq \Sigma^*$ we have*

1. $h(M \cdot N) = h(M) \cdot h(N)$;
2. *if $M \neq \varnothing$, then $h(M \setminus N) = h(M) \setminus h(N)$ and $h(N \,/\, M) = h(N) \,/\, h(M)$.*

Proof

1. By the definition of a homomorphism.

2. $\boxed{\subseteq}$ Let $u \in h(M \setminus N)$. Then $u = h(u')$ for some $u' \in M \setminus N$. For all $v' \in M$ we have $v'u' \in N$. Take an arbitrary $v \in h(M)$, $v = h(v')$ for some $v' \in M$. Since $u' \in M \setminus N$, $v'u' \in N$, whence $vu = h(v')h(u') = h(v'u') \in h(N)$. Therefore $u \in h(M) \setminus h(N)$.
   $\boxed{\supseteq}$ Let $u \in h(M) \setminus h(N)$. First we claim that $u \in P \cup \{\epsilon\}$. Suppose the contrary: $u \notin P \cup \{\epsilon\}$. Take $v' \in M$ (M is nonempty by assumption). Since $v = h(v') \in P \cup \{\epsilon\}$, $vu \notin P \cup \{\epsilon\}$. On the other hand, $vu \in h(N) \subseteq P \cup \{\epsilon\}$. Contradiction. Now, since $u \in P \cup \{\epsilon\}$, $u = h(u')$ for some $u' \in \Sigma^+$. For an arbitrary $v' \in M$ and $v \leftrightharpoons h(v')$ we have $h(v'u') = vu \in h(N)$, whence $v'u' \in N$, whence $u' \in M \setminus N$. Therefore, $u = h(u') \in h(M \setminus N)$.
   The / case is handled symmetrically.

We construct a new model $\mathcal{M}_1 = \langle \Sigma_1, w_1 \rangle$, where $w_1(z) \leftrightharpoons h(w(z))$ ($z \in \mathrm{Pr}'$). Due to Lemma 3, $w_1(A) = h(w_1(A))$ for all $A \in \Phi$, whence $w_1(F) = h(w(F)) \not\subseteq h(w(G)) = w_1(G)$ ($\mathcal{M}_1$ is also a countermodel in the language without $^{\mathrm{R}}$). Note that $w_1(A) \subseteq P \cup \{\epsilon\}$ for any type A; moreover, if $A \in \Phi$, then $\epsilon \in w_1(A)$ if and only if $\mathrm{L}^{*\prime} \vdash \to A$.

Now we introduce several auxiliary subsets of Σ_1^+ (by $\mathrm{Subw}(M)$ we denote the set of all nonempty subwords of words from M, i.e. $\mathrm{Subw}(M) \leftrightharpoons \{u \in \Sigma_1^+ \mid (\exists v_1, v_2 \in \Sigma_1^*)\, v_1 u v_2 \in M\}$):
$T_1 \leftrightharpoons \{u \in \Sigma_1^+ \mid u \notin \mathrm{Subw}(P \cup P^{\mathrm{R}})\}$;
$T_2 \leftrightharpoons \{u \in \mathrm{Subw}(P \cup P^{\mathrm{R}}) \mid$ the first or the last symbol of u is even$\}$;
$E \leftrightharpoons \{u \in \mathrm{Subw}(P \cup P^{\mathrm{R}}) - (P \cup P^{\mathrm{R}}) \mid$ both the first symbol and the last symbol of u are odd$\}$.

The sets P, P^{R}, T_1, T_2, and E form a partition of Σ_1^+ into nonintersecting parts. The set Σ_1^* is now split into six disjoint subsets: P, P^{R}, T_1, T_2, E, and $\{\epsilon\}$. For example, $a^{(1)}b^{(10)}a^{(2)} \in T_1$, $a^{(N)}b^{(1)} \ldots b^{(N-1)} \in T_2$, $a^{(7)}a^{(6)}a^{(5)} \in E$ ($a, b \in \Sigma$). Let $T \leftrightharpoons T_1 \cup T_2$, $T_i(k) \leftrightharpoons \{u \in T_i \mid |u| \geq k\}$ ($i = 1, 2$, $|u|$ is the length of u), $T(k) \leftrightharpoons T_1(k) \cup T_2(k) = \{u \in T \mid |u| \geq k\}$. Note that if the first or the last symbol of u is even, then $u \in T$, no matter whether it belongs to $\mathrm{Subw}(P \cup P^{\mathrm{R}})$. The index k (possibly with subscripts) here and further ranges from 1 to K. For all k we have $T(k) \supseteq T(K)$.

Lemma 4

1. *$P \cdot P \subseteq P$, $P^{\mathrm{R}} \cdot P^{\mathrm{R}} \subseteq P^{\mathrm{R}}$;*
2. *$T^{\mathrm{R}} = T$, $T(k)^{\mathrm{R}} = T(k)$;*
3. *$P \cdot P^{\mathrm{R}} \subseteq T(K)$, $P^{\mathrm{R}} \cdot P \subseteq T(K)$;*
4. *$P \cdot T \subseteq T(K)$, $T \cdot P \subseteq T(K)$;*
5. *$P^{\mathrm{R}} \cdot T \subseteq T(K)$, $T \cdot P^{\mathrm{R}} \subseteq T(K)$;*
6. *$T \cdot T \subseteq T$.*

Proof

1. Obvious.
2. Directly follows from our definitions.
3. Any element of $P \cdot P^{\mathrm{R}}$ or $P^{\mathrm{R}} \cdot P$ does not belong to $\mathrm{Subw}(P \cup P^{\mathrm{R}})$ and its length is at least $2N > K$. Therefore it belongs to $T_1(K) \subseteq T(K)$.

4. Let $u \in P$ and $v \in T$. If $v \in T_1$, then uv is also in T_1. Let $v \in T_2$. If the last symbol of v is even, then $uv \in T$. If the last symbol of v is odd, then $uv \notin \mathrm{Subw}(P \cup P^{\mathrm{R}})$, whence $uv \in T_1 \subseteq T$. Since $|uv| > |u| \geq N > K$, $uv \in T(K)$.
   The claim $T \cdot P \subseteq T$ is handled symmetrically.
5. $P^{\mathrm{R}} \cdot T = P^{\mathrm{R}} \cdot T^{\mathrm{R}} = (T \cdot P)^{\mathrm{R}} \subseteq T(K)^{\mathrm{R}} = T(K)$. $T \cdot P^{\mathrm{R}} = T^{\mathrm{R}} \cdot P^{\mathrm{R}} = (P \cdot T)^{\mathrm{R}} \subseteq T(K)^{\mathrm{R}} = T(K)$.
6. Let $u, v \in T$. If at least one of these two words belongs to T_1, then $uv \in T_1$. Let $u, v \in T_2$. If the first symbol of u or the last symbol of v is even, then $uv \in T$. In the other case u ends with an even symbol, and v starts with an even symbol. But then we have two consecutive even symbols in uv, therefore $uv \in T_1$.

Let us call words of the form $a^{(i)}a^{(i+1)}a^{(i+2)}$, $a^{(N-1)}a^{(N)}b^{(1)}$, and $a^{(N)}b^{(1)}b^{(2)}$ ($a, b \in \Sigma$, $1 \leq i \leq N-2$) *valid triples of type I* and their reversals (namely, $a^{(i+2)}a^{(i+1)}a^{(i)}$, $b^{(1)}a^{(N)}a^{(N-1)}$, and $b^{(2)}b^{(1)}a^{(N)}$) *valid triples of type II.* Note that valid triples of type I (resp., of type II) are the only possible three-symbol subwords of words from P (resp., P^{R}).

Lemma 5. *A word u of length at least three is a subword of a word from $P \cup P^{\mathrm{R}}$ if and only if any three-symbol subword of u is a valid triple of type I or II.*

Proof. The nontrivial part is "if". We proceed by induction on $|u|$. Induction base ($|u| = 3$) is trivial. Let u be a word of length $m+1$ satisfying the condition and let $u = u'x$ ($x \in \Sigma_1$). By induction hypothesis ($|u'| = m$), $u' \in \mathrm{Subw}(P \cup P^{\mathrm{R}})$. Let $u' \in \mathrm{Subw}(P)$ (the other case is handled symmetrically); u' is a subword of some word $v \in P$. Consider the last three symbols of u. Since the first two of them also belong to u', this three-symbol word is a valid triple of type I, not type II. If it is of the form $a^{(i)}a^{(i+1)}a^{(i+2)}$ or $a^{(N)}b^{(1)}b^{(2)}$, then x coincides with the symbol next to the occurrence of u' in v, and therefore $u = u'x$ is also a subword of v. If it is of the form $a^{(N-1)}a^{(N)}b^{(1)}$, then, provided $v = v_1u'v_2$, v_1u' is also an element of P, and so is the word $v_1u'b^{(1)}b^{(2)}\ldots b^{(N)}$, which contains $u = u'b^{(1)}$ as a subword. Thus, in all cases $u \in \mathrm{Subw}(P)$.

Now we construct one more model $\mathcal{M}_2 = \langle \Sigma_1, w_2 \rangle$, where $w_2(p_i) \leftrightharpoons w_1(p_i) \cup w_1(p_i^{\mathrm{R}})^{\mathrm{R}} \cup T$, $w_2(p_i^{\mathrm{R}}) \leftrightharpoons w_1(p_i)^{\mathrm{R}} \cup w_1(p_i^{\mathrm{R}}) \cup T$. This model is a model even in the sense of the enriched language. To finish the proof, we need to check that $\mathcal{M}_2 \not\models \to G$, e.g. $w_2(G) \not\ni \epsilon$.

Lemma 6. *For any $A \in \Phi$ the following holds:*

1. $w_2(A) \subseteq P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T$;
2. $w_2(A) \supseteq T(f(A))$;
3. $w_2(A) \cap (P \cup \{\epsilon\}) = w_1(A)$ *(in particular, $w_2(A) \cap (P \cup \{\epsilon\}) \neq \varnothing$);*
4. $w_2(A) \cap (P^{\mathrm{R}} \cup \{\epsilon\}) = w_1(tr(A^{\mathrm{R}}))^{\mathrm{R}}$ *(in particular, $w_2(A) \cap (P^{\mathrm{R}} \cup \{\epsilon\}) \neq \varnothing$);*
5. $\epsilon \in w_2(A) \iff \mathrm{L}^{*\prime} \vdash \to A$.

Proof. We prove statements 1–4 simultaneously by induction on type A.

The induction base is trivial. Further we shall refer to the i-th statement of the induction hypothesis ($i = 1, 2, 3, 4$) as "IH-i".

1. Consider three possible cases.

a) $A = B \cdot C$. Then $w_2(A) = w_2(B) \cdot w_2(C) \subseteq (P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T) \cdot (P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T) \subseteq P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T$ (Lemma 4).

b) $A = B \setminus C$. Suppose the contrary: in $w_2(A)$ there exists an element $u \in E$. Then $vu \in w_2(C)$ for any $v \in w_2(B)$. We consider several subcases and show that each of those leads to a contradiction.

i) $u \in \mathrm{Subw}(P)$, and the superscript of the first symbol of u (as $\epsilon \notin E$, u contains at least one symbol) is not 1. Let the first symbol of u be $a^{(i)}$. Note that i is odd and $i > 2$. Take $v = a^{(3)} \ldots a^{(N)} a^{(1)} \ldots a^{(i-1)}$. The word v has length at least $N \geq K$ and ends with an even symbol, therefore $v \in T(K) \subseteq T(f(B)) \subseteq w_2(B)$ (IH-2). On the other hand, $vu \in \mathrm{Subw}(P)$ and the first symbol and the last symbol of vu are odd. Therefore, $vu \in E$ and $vu \in w_2(C)$, but $w_2(C) \cap E = \varnothing$ (IH-1). Contradiction.

ii) $u \in \mathrm{Subw}(P)$, and the first symbol of u is $a^{(1)}$ (then the superscript of the last symbol of u is not N, because otherwise $u \in P$). Take $v \in w_2(B) \cap (P \cup \{\epsilon\})$ (this set is nonempty due to IH-3). If $v = \epsilon$, then $vu = u \in E$. Otherwise the first and the last symbol of vu are odd, and $vu \in \mathrm{Subw}(P) - P$, and again we have $vu \in E$. Contradiction.

iii) $u \in \mathrm{Subw}(P^{\mathrm{R}})$, and the superscript of the first symbol of u is not N (the first symbol of u is $a^{(i)}$, i is odd). Take $v = a^{(N-2)} \ldots a^{(1)} a^{(N)} \ldots a^{(i+1)} \in T(K) \subseteq w_2(B)$. Again, $vu \in E$.

iv) $u \in \mathrm{Subw}(P^{\mathrm{R}})$, and the first symbol of u is $a^{(N)}$. Take $v \in w_2(B) \cap (P^{\mathrm{R}} \cup \{\epsilon\})$ (nonempty due to IH-4). $vu \in E$.

c) $A = C \,/\, B$. Proceed symmetrically.

2. Consider three possible cases.

a) $A = B \cdot C$. Let $k_1 \leftrightharpoons f(B)$, $k_2 \leftrightharpoons f(C)$, $k \leftrightharpoons k_1 + k_2 + 10 = f(A)$. Due to IH-2, $w_2(B) \supseteq T(k_1)$ and $w_2(C) \supseteq T(k_2)$. Take $u \in T(k)$. We have to prove that $u \in w_2(A)$. Consider several subcases.

i) $u \in T_1(k)$. By Lemma 5 ($|u| \geq k > 3$ and $u \notin \mathrm{Subw}(P \cup P^{\mathrm{R}})$) in u there is a three-symbol subword xyz that is not a valid triple of type I or II. Divide the word u into two parts, $u = u_1 u_2$, such that $|u_1| \geq k_1 + 5$, $|u_2| \geq k_2 + 5$. If needed, shift the border between parts by one symbol to the left or to the right, so that the subword xyz lies entirely in one part. Let this part be u_2 (the other case is handled symmetrically). Then $u_2 \in T_1(k_2)$. If u_1 is also in T_1, then the proof is finished. Consider the other case. Note that in any word from $\mathrm{Subw}(P \cup P^{\mathrm{R}})$ among any three consecutive symbols at least one is even. Shift the border to the left by at most 2 symbols to make the last symbol of u_1 even. Then $u_1 \in T(k_1)$, and u_2 remains in $T_1(k_2)$. Thus $u = u_1 u_2 \in T(k_1) \cdot T(k_2) \subseteq w_2(B) \cdot w_2(C) = w_2(A)$.

ii) $u \in T_2(k)$. Let u end with an even symbol (the other case is symmetric). Divide the word u into two parts, $u = u_1 u_2$, $|u_1| \geq k_1 + 5$, $u_2 \geq k_2 + 5$, and shift the border (if needed), so that the last symbol of u_1 is even. Then both u_1 and u_2 end with an even symbol, and therefore $u_1 \in T(k_1)$ and $u_2 \in T(k_2)$.

b) $A = B \setminus C$. Let $k \leftrightharpoons f(C) = f(A)$. By IH-2, $w_2(C) \supseteq T(k)$. Take $u \in T(k)$ and an arbitrary $v \in w_2(B) \subseteq P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T$. By Lemma 4, statements 4–6, $vu \in (P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T) \cdot T \subseteq T$, and since $|vu| \geq |u| \geq k$, $vu \in T(k) \subseteq w_2(C)$. Thus $u \in w_2(A)$.

c) $A = C \,/\, B$. Symmetrically.

3. Consider three possible cases.

a) $A = B \cdot C$.

$\boxed{\supseteq}$ $u \in w_1(A) = w_1(B) \cdot w_1(C) \subseteq w_2(B) \cdot w_2(C) = w_2(A)$ (IH-3); $u \in P \cup \{\epsilon\}$.

$\boxed{\subseteq}$ Suppose $u \in P$ and $u \in w_2(A) = w_2(B) \cdot w_2(C)$. Then $u = u_1 u_2$, where $u_1 \in w_2(B)$ and $u_2 \in w_2(C)$. First we claim that $u_1 \in P \cup \{\epsilon\}$. Suppose the contrary. By IH-1, $u_1 \in P^{\mathrm{R}} \cup T$, $u_2 \in P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T$, and therefore $u = u_1 u_2 \in (P^{\mathrm{R}} \cup T) \cdot (P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T) \subseteq P^{\mathrm{R}} \cup T$ (Lemma 4, statements 1, 3–6). Hence $u \notin P \cup \{\epsilon\}$. Contradiction. Thus, $u_1 \in P \cup \{\epsilon\}$. Similarly, $u_2 \in P \cup \{\epsilon\}$, and by IH-3 we obtain $u_1 \in w_1(B)$ and $u_2 \in w_1(C)$, whence $u = u_1 u_2 \in w_1(A)$.

b) $A = B \setminus C$.

$\boxed{\supseteq}$ Take $u \in w_1(B \setminus C) \subseteq P \cup \{\epsilon\}$. First we consider the case where $u = \epsilon$. Then we have $\mathrm{L}^{*\prime} \vdash \; \rightarrow B \setminus C$, whence $u = \epsilon \in w_2(B \setminus C)$. Now let $u \in P$. For any $v \in w_1(B)$ we have $vu \in w_1(C)$. We claim that $u \in w_2(B \setminus C)$. Take $v \in w_2(B) \subseteq P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T$ (IH-1). If $v \in P \cup \{\epsilon\}$, then $v \in w_1(B)$ (IH-3), and $vu \in w_1(C) \subseteq w_2(C)$ (IH-3). If $v \in P^{\mathrm{R}} \cup T$, then $vu \in (P^{\mathrm{R}} \cup T) \cdot P \subseteq T(K) \subseteq w_2(C)$ (Lemma 4, statements 3 and 4, and IH-2). Therefore, $u \in w_2(B) \setminus w_2(C) = w_2(B \setminus C)$.

$\boxed{\subseteq}$ If $u \in w_2(B \setminus C)$ and $u \in P \cup \{\epsilon\}$, then for any $v \in w_1(B) \subseteq w_2(B)$ we have $vu \in w_2(C)$. Since $v, u \in P \cup \{\epsilon\}$, $vu \in P \cup \{\epsilon\}$. By IH-3, $vu \in w_1(C)$. Thus $u \in w_1(B \setminus C)$.

c) $A = C \,/\, B$. Symmetrically.

4. Consider three cases.

a) $A = B \cdot C$. Then $tr(A^{\mathrm{R}}) = tr(C^{\mathrm{R}}) \cdot tr(B^{\mathrm{R}})$.

$\boxed{\supseteq}$ $u \in w_1(tr(A^{\mathrm{R}}))^{\mathrm{R}} = w_1(tr(C^{\mathrm{R}}) \cdot tr(B^{\mathrm{R}}))^{\mathrm{R}} = \big(w_1(tr(C^{R})) \cdot w_1(tr(B^{\mathrm{R}}))\big)^{\mathrm{R}} = w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \cdot w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}} \subseteq w_2(B) \cdot w_2(C) = w_2(A)$ (IH-4); $u \in P^{\mathrm{R}}$.

$\boxed{\subseteq}$ Let $u \in P^{\mathrm{R}}$ and $u \in w_2(A) = w_2(B) \cdot w_2(C)$. Then $u = u_1 u_2$, where $u_1 \in w_2(B)$, $u_2 \in w_2(C)$. We claim that $u_1, u_2 \in P^{\mathrm{R}} \cup \{\epsilon\}$. Suppose the contrary. By IH-1, $u_1 \in P \cup T$, $u_2 \in P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T$, whence $u = u_1 u_2 \in (P \cup T) \cdot (P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T) \subseteq P \cup T$. Contradiction. Thus, $u_1 \in P^{\mathrm{R}} \cup \{\epsilon\}$, and therefore $u_2 \in P^{R} \cup \{\epsilon\}$, and, using IH-4, we obtain $u_1 \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}}$, $u_2 \in w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}}$. Hence $u = u_1 u_2 \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \cdot w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}} = \big(w_1(tr(C^{\mathrm{R}})) \cdot w_1(tr(B^{\mathrm{R}}))\big)^{\mathrm{R}} = w_1(tr(C^{\mathrm{R}}) \cdot tr(B^{\mathrm{R}}))^{\mathrm{R}} = w_1(tr(A^{\mathrm{R}}))^{\mathrm{R}}$.

b) $A = B \setminus C$. Then $tr(A^{\mathrm{R}}) = tr(C^{\mathrm{R}}) \,/\, tr(B^{\mathrm{R}})$.

$\boxed{\supseteq}$ Let $u \in w_1(tr(C^{\mathrm{R}}) \,/\, tr(B^{\mathrm{R}}))^{\mathrm{R}} = w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \setminus w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}}$, First we consider the case where $u = \epsilon$. Then $\mathrm{L}^{*\prime} \vdash \; \rightarrow tr(C^{\mathrm{R}}) \,/\, tr(B^{\mathrm{R}})$, whence $\epsilon \in w_2(tr(C^{\mathrm{R}}) \,/\, tr(B^{\mathrm{R}})) = w_2(tr(A^{\mathrm{R}}))$. Therefore, $u \in w_2(tr(A^{\mathrm{R}}))^{\mathrm{R}}$. Now let $u \in$

P^{R}. For every $v \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}}$ we have $vu \in w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}}$. We claim that $u \in w_2(B \setminus C)$. Take an arbitrary $v \in w_2(B) \subseteq P \cup P^{\mathrm{R}} \cup \{\epsilon\} \cup T$ (IH-1). If $v \in P^{\mathrm{R}} \cup \{\epsilon\}$, then $v \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}}$ (IH-4), whence $vu \in w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}} \subseteq w_2(C)$. If $v \in P \cup T$, then (since $u \in P^{\mathrm{R}}$) we have $vu \in (P \cup T) \cdot P^{\mathrm{R}} \subseteq T(K) \subseteq w_2(C)$ (Lemma 4 and IH-2).

$\boxed{\subseteq}$ If $u \in w_2(B \setminus C)$ and $u \in P^{\mathrm{R}} \cup \{\epsilon\}$, then for any $v \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \subseteq w_2(B)$ we have $vu \in w_2(C)$. Since $v, u \in P^R \cup \{\epsilon\}$, $vu \in P^{\mathrm{R}} \cup \{\epsilon\}$, therefore $vu \in w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}}$ (IH-4). Thus $u \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \setminus w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}} = w_1(A^{\mathrm{R}})^{\mathrm{R}}$.

c) $A = C \,/\, B$. Symmetrically.

This completes the proof of statements 1–4 of Lemma 6. Statement 5 follows from statement 3 and immediately yields Theorem 3 ($\mathrm{L}^{*\prime} \not\vdash \to G$, whence $\epsilon \notin w_2(G)$).

6 Grammars and Complexity

The Lambek calculus and its variants are used for describing formal languages via Lambek categorial grammars. An L^*-*grammar* is a triple $\mathcal{G} = \langle \Sigma, H, \triangleright \rangle$, where Σ is a finite alphabet, $H \in \mathrm{Tp}$, and $\triangleright$ is a finite correspondence between Tp and Σ ($\triangleright \subset \mathrm{Tp} \times \Sigma$). The *language generated by* $\mathcal{G}$ is the set of all nonempty words $a_1 \dots a_n$ over Σ for which there exist types $B_1, \dots, B_n$ such that $\mathrm{L}^* \vdash B_1 \dots B_n \to H$ and $B_i \triangleright a_i$ for all $i \le n$. We denote this language by $\mathfrak{L}(\mathcal{G})$. The notion of L-*grammar* is defined in a similar way. These class of grammars are *weakly equivalent* to the classes of context-free grammars with and without ϵ-rules in the following sense:

Theorem 4. *A formal language is context-free if and only if it is generated by some* L^**-grammar. A formal language without the empty word is context-free if and only if it is generated by some* L*-grammar.* [7] [2]

By modifying our definition in a natural way one can introduce the notion of $\mathrm{L}^{*\mathrm{R}}$-grammar and L^{R}-grammar. These grammars also generate precisely all context-free languages (resp., context-free languages without the empty word):

Theorem 5. *A formal language is context-free if and only if it is generated by some* $\mathrm{L}^{*\mathrm{R}}$*-grammar. A formal language without the empty word is context-free if and only if it is generated by some* L^{R}*-grammar.*

Proof. The "only if" part follows directly from Theorem 4 due to the conservativity of $\mathrm{L}^{*\mathrm{R}}$ over L^* and L^{R} over L (Lemma 1).

The "if" part is proved by replacing all types in an $\mathrm{L}^{*\mathrm{R}}$-grammar (L^*-grammar) by their normal forms and applying Lemma 2.

Since $A \,/\, B$ is equivalent in L^{R} and $\mathrm{L}^{*\mathrm{R}}$ to $(B^{\mathrm{R}} \setminus A^{\mathrm{R}})^{\mathrm{R}}$, and the derivability problem in Lambek calculus with two division operators is NP-complete [10] (this holds both for L and L^*), the derivability problem is NP-complete even for the fragment of L^{R} ($\mathrm{L}^{*\mathrm{R}}$) with one division.

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