On Canonical Embeddings of Residuated Groupoids

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Abstract. We prove that every symmetric residuated groupoid is embeddable in a boolean double residuated groupoid. Analogous results are obtained for other classes of algebras, e.g. (commutative) symmetric residuated semigroups, symmetric residuated unital groupoids, cyclic bilinear algebras. We also show that powerset algebras constructed in the paper preserve some Grishin axioms.

Keywords: representation theorems, residuated groupoids, symmetric residuated groupoids, boolean double residuated groupoid.

1 Introduction

Residuated groupoids with operations $\otimes, \backslash, /$ are models of Nonassociative Lambek Calculus (NL) [11] and other weak substructural logics [7]. Symmetric residuated groupoids with operations $\otimes, \backslash, /$ and dual operations $\oplus, \obslash, \oslash$ are models of certain symmetric substructural logics, as e.g. Grishin's extensions of the Lambek calculus [8]. In particular, Moortgat [16] studies a nonassociative symmetric substructural logic, called Lambek-Grishin calculus (LG), as a type logic for Type-Logical Grammar. Let us recall the calculus LG. Types are formed out of atomic types $p, q, r, \ldots$ by means of the following formation rule: if A, B are types, then also $A \otimes B$, $A \backslash B$, A/B, $A \oplus B$, $A \obslash B$, $A \oslash B$ are types. The minimal LG is given by the preorder axioms:

$$A \to A; \text{ if } A \to B \text{ and } B \to C \text{ then } A \to C,$$

together with the residuation and dual residuation laws:

$$A \to C/B \quad \text{iff} \quad A \otimes B \to C \quad \text{iff} \quad B \to A \backslash C,$$
$$C \oslash B \to A \quad \text{iff} \quad C \to A \oplus B \quad \text{iff} \quad A \obslash C \to B.$$

Algebraic models of this calculus are symmetric residuated groupoids.

Interesting extensions of this calculus can be obtained by adding Grishin axioms (see Section 5). Other well-known logics of that kind are Multiplicative Linear Logics, corresponding to commutative involutive symmetric residuated semigroups, and their noncommutative and nonassociative variants, e.g. InFL, InGL (see e.g. [1,7]).

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There are many natural constructions of multiple residuated groupoids, i.e. residuated groupoids with several residuation triples (see e.g. [5,9]). Dual residuated groupoids (satisfying the residuation law with respect to dual ordering $\geq$) can be constructed by using an involutive negation, e.g. set complementation $^{\sim}$, and defining the dual residuation triple:

$$X \oplus Y = (X^{\sim} \otimes Y^{\sim})^{\sim}, X \oslash Y = (X^{\sim} \backslash Y^{\sim})^{\sim}, X \oslash Y = (X^{\sim} / Y^{\sim})^{\sim}.$$

Białynicki-Birula and Rasiowa [2] show that every quasi-boolean algebra (i.e. a distributive lattice with an involutive negation, satisfying Double Negation and Transposition, or equivalently: Double Negation and one of De Morgan laws) is embeddable into a quasi-field of sets (i.e. a family of sets, closed under $\cup$, $\cap$ and a quasi-complement $\sim_g X = g\,[X]^{\sim}$, where g is an involutive mapping).

In this paper we prove similar results for symmetric residuated groupoids and related algebras. Our embedding preserves the residuated groupoid operations and negation(s). The target algebra is a field or a quasi-field of sets with additional operations of a symmetric residuated groupoid.

We prove that every symmetric residuated groupoid is a subalgebra of an algebra of the above form. As in [5], by a boolean residuated algebra one means a residuated algebra with additional boolean operations $^{\sim}, \cup, \cap$, and similarly for a quasi-boolean residuated algebra. More precisely, we show that every symmetric residuated groupoid can be embedded in a boolean double residuated groupoid, which is a field of sets with additional operations $\otimes_1$, $\backslash_1$, $/_1$, $\otimes_2$, $\backslash_2$, $/_2$ and $^{\sim}$ (dual operations are defined from $\otimes_2$, $\backslash_2$, $/_2$ as above). Analogous results are obtained for (commutative) symmetric residuated semigroups and other algebras of this kind. Furthermore, the target algebra always consists of subsets of some set, and the involutive negation is set complementation. The results elaborate final remarks of Buszkowski [5] who considers general residuated algebras, but does not provide any details of the representation. Let us notice that in [5] symmetric residuated algebras are called double residuated algebras.

We also show that the target algebra is a (commutative) semigroup, if the source algebra is so. Units 1 and 0 (for $\otimes$ and $\oplus$, respectively) are preserved, if the target algebra is restricted to a family of upsets. The latter is a quasi-field of sets, if the source algebra admits an involutive negation '$-$', and the embedding sends '$-$' to a quasi-complement. The target algebra is a cyclic bilinear algebra, if the source algebra is so.

Some ideas of our proofs are similar to those of Kurtonina and Moortgat [10] in their proof of the completeness of the minimal LG with respect to Kripke semantics. Our algebraic approach, however, reveals more uniformity of the whole construction, i.e. its two-level form where the first level is related to the ground level in the same way as the second one to the first one.

The paper is organized as follows. In Section 2 we discuss some basic notions. Section 3 contains some powerset constructions of residuated groupoids, dual residuated groupoids and symmetric residuated groupoids. The main result, a representation theorem for symmetric residuated groupoids, is proved in Section 4. In Section 5 we provide similar representation theorems for symmetric residuated semigroups, symmetric residuated unital groupoids and cyclic bilinear algebras. At the end of this section we consider some Grishin axioms.

2 Preliminaries

We begin this section with the definitions of some basic notions.

Let us recall that a structure $(M, \leq, \otimes)$ is *a partially ordered groupoid* (*p.o. groupoid*), if $\leq$ is a partial order and $\otimes$ is a binary operation monotone in both arguments i.e. $a \leq b$ implies $a \otimes c \leq b \otimes c$ and $c \otimes a \leq c \otimes b$, for $a, b, c \in M$.

A residuated groupoid is a structure $(M, \leq, \otimes, \backslash, /)$ such that $(M, \leq)$ is a poset, $(M, \otimes)$ is a groupoid, and $\otimes, \backslash, /$ satisfy the residuation law:

$$a \leq c/b \quad \text{iff} \quad a \otimes b \leq c \quad \text{iff} \quad b \leq a \backslash c,$$

for all $a, b, c \in M$. It is easy to show that if $(M, \leq, \otimes, \backslash, /)$ is a residuated groupoid, then $(M, \leq, \otimes)$ is a p.o. groupoid.

A dual residuated groupoid is defined as a structure $(M, \leq, \oplus, \oslash, \oslash)$ such that $(M, \leq)$ is a poset, $(M, \oplus)$ is a groupoid, and $\oplus, \oslash, \oslash$ satisfy the dual residuation law:

$$c \oslash b \leq a \quad \text{iff} \quad c \leq a \oplus b \quad \text{iff} \quad a \obslash c \leq b$$

for all $a, b, c \in M$. Again $(M, \leq, \oplus)$ is a p.o. groupoid.

A structure $\boldsymbol{M} = (M, \leq, \otimes, \backslash, /, \oplus, \obslash, \oslash)$ is called *a symmetric residuated groupoid* iff the $(\leq, \otimes, \backslash, /)$-reduct of $\boldsymbol{M}$ and the $(\leq, \oplus, \obslash, \oslash)$-reduct of $\boldsymbol{M}$ are a residuated groupoid and a dual residuated groupoid, respectively.

An involutive residuated groupoid is a structure which arises from a residuated groupoid by adding a unary operation $-$ (we call it *an involutive negation*) which satisfies the following two conditions:

$$\begin{array}{ll} - - a = a & \text{(Double Negation)} \\ a \leq b \Rightarrow -b \leq -a & \text{(Transposition)} \end{array}$$

for all elements a, b. In a similar way we define involutive dual residuated groupoids, involutive symmetric residuated groupoids etc. Given lattice operations $\vee$, $\wedge$, the second condition is equivalent to $-(a \vee b) = (-a) \wedge (-b)$. Hence our involutive negation corresponds to a quasi-complement in the sense of [2] and a De Morgan negation (assuming Double Negation) in the sense of [6]. It is also called a cyclic negation in the literature on substructural logics (cf. [7]).

A multiple p.o. groupoid is an ordered algebra $\boldsymbol{M} = (M, \leq, \{\otimes\}_{i \in I})$ such that, for any $i \in I$, $(M, \leq, \otimes_i)$ is a p.o. groupoid. By *a multiple residuated groupoid* we mean a structure $\boldsymbol{M} = (M, \leq, \{\odot_i, \backslash_i, /_i\}_{i=1,\ldots,n})$ such that the $(\leq, \odot_i, \backslash_i, /_i)$ -reducts of $\boldsymbol{M}$ for $i = 1, 2, \ldots n$ are residuated groupoids.

In this paper we only consider *double residuated groupoids*, i.e. multiple residuated groupoids for $i = 1, 2$.

Let $\boldsymbol{M}$ be an involutive residuated groupoid. We define the structure $\boldsymbol{M}^{\ominus} = (M, \leq, \oplus, \oslash, \owedge, -)$ such that

$$a \oplus b = -((-a) \otimes (-b)),$$
$$a \oslash b = -((-a)/(-b)),$$
$$a \owedge b = -((-a)\backslash(-b)).$$

Now, let $\boldsymbol{M}$ denote a dual involutive residuated groupoid. We define the structure $\boldsymbol{M}^{-} = (M, \leq, \otimes, /, \backslash, -)$ such that

$$a \otimes b = -((-a) \oplus (-b)),$$
$$a/b = -((-a) \oslash (-b)),$$
$$a\backslash b = -((-a) \owedge (-b)).$$

Lemma 1. *If $\boldsymbol{M}$ is an involutive residuated groupoid, then $\boldsymbol{M}^{\ominus}$ is an involutive dual residuated groupoid. If $\boldsymbol{M}$ is an involutive dual residuated groupoid, then $\boldsymbol{M}^{-}$ is an involutive residuated groupoid.*

Proof. Assume that $\boldsymbol{M}$ is an involutive residuated groupoid. We show

$$c \oslash b \leq a \text{ iff } c \leq a \oplus b \text{ iff } a \owedge c \leq b.$$

We prove the first equivalence:
$c \leq a \oplus b$ iff $c \leq -((-a) \otimes (-b))$ iff $(-a) \otimes (-b) \leq -c$ iff
$-a \leq (-c)/(-b)$ iff $-((-c)/(-b)) \leq a$ iff $c \oslash b \leq a$
The second equivalence can be proved in an analogous way.

Assuming that $\boldsymbol{M}$ is an involutive dual residuated groupoid, the equivalences $a \leq c/b$ iff $a \otimes b \leq c$ iff $b \leq a\backslash c$ can be proved in an analogous way to the one above. □

Observe that $\boldsymbol{M}^{\ominus -} = \boldsymbol{M}$ and $\boldsymbol{M}^{-\ominus} = \boldsymbol{M}$.

It is easy to show that for symmetric residuated groupoids the following conditions hold:

$$a \otimes (a\backslash b) \leq b, \quad (b/a) \otimes a \leq b,$$
$$b \leq a \oplus (a \owedge b), \quad b \leq (b \oslash a) \oplus a,$$
$$a \leq b \;\Rightarrow\; c\backslash a \leq c\backslash b, \;\; a/c \leq b/c, \;\; b\backslash c \leq a\backslash c, \;\; c/b \leq c/a;$$
$$a \leq b \;\Rightarrow\; c \owedge a \leq c \owedge b, \;\; a \oslash c \leq b \oslash c, \;\; b \owedge c \leq a \owedge c, \;\; c \oslash b \leq c \oslash a.$$

3 A Powerset Construction

Concrete residuated groupoids can be constructed in various ways. A basic construction is the powerset residuated groupoid.

Given a groupoid $\boldsymbol{M} = (M, \otimes)$, we consider the powerset $\mathcal{P}(M)$ with operations defined as follows:

$$\begin{aligned} X \otimes Y &= \{a \otimes b : \; a \in X, \; b \in Y\}, \\ X\backslash Z &= \{c \in M : \; \forall a \in X \; a \otimes c \in Z\}, \\ Z/Y &= \{c \in M : \; \forall b \in Y \; c \otimes b \in Z\}. \end{aligned}$$

Then, $(\mathcal{P}(M), \subset, \otimes, \backslash, /)$ is a residuated groupoid; we denote this algebra by $\mathcal{P}(\boldsymbol{M})$. Every residuated groupoid can be embedded in a structure of this form as shown in [9]. In this paper, we apply a more general construction.

Starting from a p.o. groupoid $\boldsymbol{M} = (M, \leq, \otimes)$ one can define another powerset algebra which will be denoted by $\mathcal{P}^{\leq}(\boldsymbol{M})$. For $X, Y, Z \subset M$, we define operations:

$$\begin{aligned}
X \widehat{\otimes} Y &= \{c \in M : \exists a \in X\ \exists b \in Y\ a \otimes b \leq c\},\\
X \widehat{\backslash} Z &= \{b \in M : \forall a \in X\ \forall c \in M\ (a \otimes b \leq c \Rightarrow c \in Z)\},\\
Z \widehat{/} Y &= \{a \in M : \forall b \in Y\ \forall c \in M\ (a \otimes b \leq c \Rightarrow c \in Z)\}.
\end{aligned}$$

The following lemma holds.

Lemma 2. $\mathcal{P}^{\leq}(\boldsymbol{M}) = (\mathcal{P}(M), \subset, \widehat{\otimes}, \widehat{\backslash}, \widehat{/})$ *is a residuated groupoid.*

Proof. We prove that the residuation law holds, i.e.

$$Y \subset X \widehat{\backslash} Z \quad \text{iff} \quad X \widehat{\otimes} Y \subset Z \quad \text{iff} \quad X \subset Z \widehat{/} Y$$

for every $X, Y, Z \in \mathcal{P}(M)$.

Assume $Y \subset X \widehat{\backslash} Z$. Let $c \in X \widehat{\otimes} Y$. By the definition of operation $\widehat{\otimes}$, there exist $a \in X$ and $b \in Y$ such that $a \otimes b \leq c$. Since $b \in Y$, then $b \in X \widehat{\backslash} Z$. Hence, by the definition of operation $\widehat{\backslash}$, $c \in Z$.

Assume $X \widehat{\otimes} Y \subset Z$. Let $b \in Y$. Let $a \in X$, $c \in M$ and $a \otimes b \leq c$. By the definition of operation $\widehat{\otimes}$, we have $c \in X \widehat{\otimes} Y$, so $c \in Z$. Finally, by the definition of operation $\widehat{\backslash}$, $b \in X \widehat{\backslash} Z$.

The proof of the second equivalence is analogous. $\square$

The same construction can be performed with the reverse ordering $\geq$. Starting from a p.o. groupoid $\boldsymbol{M} = (M, \leq, \oplus)$, we define a dual powerset algebra $\mathcal{P}^{\geq}(\boldsymbol{M})$. For $X, Y, Z \subset M$, we define operations:

$$\begin{aligned}
X \bar{\oplus} Y &= \{c \in M : \exists a \in X\ \exists b \in Y\ \ c \leq a \oplus b\},\\
X \bar{\obslash} Z &= \{b \in M : \forall a \in X\ \forall c \in M\ \ (c \leq a \oplus b \Rightarrow c \in Z)\},\\
Z \bar{\oslash} Y &= \{a \in M : \forall b \in Y\ \forall c \in M\ \ (c \leq a \oplus b \Rightarrow c \in Z)\}.
\end{aligned}$$

Lemma 3. $\mathcal{P}^{\geq}(\boldsymbol{M}) = (\mathcal{P}(M), \subset, \bar{\oplus}, \bar{\obslash}, \bar{\oslash})$ *is a residuated groupoid.*

Proof. Observe that $\boldsymbol{M}' = (M, \geq, \oplus)$ is a p.o. groupoid. $\mathcal{P}^{\geq}(\boldsymbol{M})$ is exactly the algebra considered in Lemma 2 for $\boldsymbol{M}'$. $\square$

In all cases we obtained some powerset residuated groupoids. Dual residuated groupoids can be constructed from them in the way described in Lemma 1. Of course, $\mathcal{P}^{\leq}(\boldsymbol{M})$ and $\mathcal{P}^{\geq}(\boldsymbol{M})$ can be expanded by the set complementation:

$$X^{\sim} = \{a \in M :\ a \notin X\}.$$

Clearly, $^{\sim}$ is an involutive negation on $\mathcal{P}(M)$. We can define dual operations on $\mathcal{P}^{\geq}(\boldsymbol{M})$ as follows:

$$X \widehat{\oplus} Y = (X^{\sim} \bar{\oplus}\, Y^{\sim})^{\sim},$$
$$X \widehat{\oslash} Z = (X^{\sim} \bar{\oslash}\, Z^{\sim})^{\sim},$$
$$Z \widehat{\oslash} Y = (Z^{\sim} \bar{\oslash}\, Y^{\sim})^{\sim}.$$

Lemma 4. *$\mathcal{P}_d{}^{\geq}(\boldsymbol{M}) = (\mathcal{P}(M), \subset, \widehat{\oplus}, \widehat{\oslash}, \widehat{\oslash})$ is a dual residuated groupoid.*

Proof. It is an easy consequence of Lemma 1 and Lemma 3. $\square$

The next lemma shows an alternative way of defining operations $\widehat{\oplus}, \widehat{\oslash}, \widehat{\oslash}$.

Lemma 5. *The operations $\widehat{\oplus}, \widehat{\oslash}, \widehat{\oslash}$ can also be defined as follows:*

$$X \widehat{\oplus} Y = \{c \in M : \forall a, b \in M\ (c \leq a \oplus b \Rightarrow (a \in X \vee b \in Y))\},$$
$$X \widehat{\oslash} Z = \{b \in M : \exists a \notin X\ \exists c \in Z\ \ c \leq a \oplus b\},$$
$$Z \widehat{\oslash} Y = \{a \in M : \exists b \notin Y\ \exists c \in Z\ \ c \leq a \oplus b\}.$$

Let $\boldsymbol{M} = (M, \leq, \otimes, \backslash, /, \oplus, \oslash, \oslash)$ be a symmetric residuated groupoid. By $\mathcal{P}_{\leq}(\boldsymbol{M})$ we denote the algebra $(\mathcal{P}(M), \subset, \widehat{\otimes}, \widehat{\backslash}, \widehat{/}, \widehat{\oplus}, \widehat{\oslash}, \widehat{\oslash})$, where $\widehat{\otimes}$, $\widehat{\backslash}$, $\widehat{/}$ and $\widehat{\oplus}$, $\widehat{\oslash}$, $\widehat{\oslash}$ are defined as for $\mathcal{P}^{\leq}(\boldsymbol{M})$ and for $\mathcal{P}_d{}^{\geq}(\boldsymbol{M})$, respectively.

Lemma 6. *For any symmetric residuated groupoid $\boldsymbol{M}$, $\mathcal{P}_{\leq}(\boldsymbol{M})$ is a symmetric residuated groupoid.*

Proof. It is an immediate consequence of Lemma 2 and Lemma 4. $\square$

Let $(M, \leq)$ be a poset. *An upset* is a set $X \subset M$ such that, if $x \in X$ and $x \leq y$, then $y \in X$, for all $x, y \in M$. *A downset* is a set $X \subset M$ such that, if $x \in X$ and $y \leq x$, then $y \in X$, for all $x, y \in M$.

By *a principal upset* (*downset*) *generated by* $a \in M$ we mean the set of all $b \in M$ such that $a \leq b$ ($b \leq a$). We denote it $\lceil a \rceil$ ($\lfloor a \rfloor$).

Observe that for any $X, Y \subset M$, $X \widehat{\otimes} Y$, $X \widehat{\backslash} Y$, $Y \widehat{/} X$ are upsets on $(M, \leq)$. Similarly, $X \bar{\oplus} Y$, $X \bar{\oslash} Y$, $Y \bar{\oslash} X$ are downsets. Consequently, $X \widehat{\oplus} Y$, $X \widehat{\oslash} Y$, $Y \widehat{\oslash} X$ are upsets.

Let us denote by $U_{\boldsymbol{M}}$ the set $\{X \subset M :\ X \text{ is an upset}\}$. Let us denote by $\boldsymbol{U_M}$ the partially ordered algebra $(U_{\boldsymbol{M}}, \subset, \widehat{\otimes}, \widehat{\backslash}, \widehat{/}, \widehat{\oplus}, \widehat{\oslash}, \widehat{\oslash})$. Observe that $\boldsymbol{U_M}$ is a subalgebra of $\mathcal{P}_{\leq}(\boldsymbol{M})$. Clearly, $\boldsymbol{U_M}$ is a symmetric residuated groupoid. Let us denote $D_{\boldsymbol{M}} = \{X \subset M : X \text{ is a downset}\}$ and $\boldsymbol{D_M} = (D_{\boldsymbol{M}}, \subset, \bar{\oplus}, \bar{\oslash}, \bar{\oslash})$. Observe that $\boldsymbol{D_M}$ is a subalgebra of $\mathcal{P}^{\geq}(\boldsymbol{M})$, where $\boldsymbol{M} = (M, \leq, \oplus)$ is a p.o. groupoid.

Unfortunately, we know no embedding of the symmetric residuated groupoid $\boldsymbol{M}$ into $\mathcal{P}_{\leq}(\boldsymbol{M})$. The values of such an embedding should be upsets. Neither $h(a) = \lceil a \rceil$, nor $h(a) = \lfloor a \rfloor^{\sim}$ satisfies the homomorphism conditions for all operations $\otimes, \backslash, /, \oplus, \oslash, \oslash$. For instance, the first does not satisfy $h(a \backslash b) = h(a) \widehat{\backslash} h(b)$.

We construct the higher-level algebra $\mathcal{P}_{\leq}(\boldsymbol{U_M})$. In this algebra the operations are denoted by $\otimes, \backslash, /, \oplus, \obslash, \oslash$. They can explicitly be defined as follows:

$$
\begin{aligned}
\mathcal{X} \otimes \mathcal{Y} &= \{Z \in U_{\boldsymbol{M}} : \exists X \in \mathcal{X}\ \exists Y \in \mathcal{Y}\ \ X \widehat{\otimes} Y \subset Z\},\\
\mathcal{X} \backslash \mathcal{Z} &= \{Y \in U_{\boldsymbol{M}} : \forall X \in \mathcal{X}\ \forall Z \in U_{\boldsymbol{M}}\ (X \widehat{\otimes} Y \subset Z \Rightarrow Z \in \mathcal{Z})\},\\
\mathcal{Z} / \mathcal{Y} &= \{X \in U_{\boldsymbol{M}} : \forall Y \in \mathcal{Y}\ \forall Z \in U_{\boldsymbol{M}}\ (X \widehat{\otimes} Y \subset Z \Rightarrow Z \in \mathcal{Z})\},\\
\mathcal{X} \oplus \mathcal{Y} &= \{Z \in U_{\boldsymbol{M}} : \forall X \in U_{\boldsymbol{M}}\ \forall Y \in U_{\boldsymbol{M}}\ (Z \subset X \widehat{\oplus} Y \Rightarrow (X \in \mathcal{X} \vee Y \in \mathcal{Y}))\},\\
\mathcal{X} \obslash \mathcal{Z} &= \{Y \in U_{\boldsymbol{M}} : \exists X \notin \mathcal{X}\ \exists Z \in \mathcal{Z}\ \ Z \subset X \widehat{\oplus} Y\},\\
\mathcal{Z} \oslash \mathcal{Y} &= \{X \in U_{\boldsymbol{M}} : \exists Y \notin \mathcal{Y}\ \exists Z \in \mathcal{Z}\ \ Z \subset X \widehat{\oplus} Y\},
\end{aligned}
$$

for all $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subset U_{\boldsymbol{M}}$.

The following lemma holds.

Lemma 7. $\mathcal{P}_{\leq}(\boldsymbol{U_M}) = (\mathcal{P}(U_{\boldsymbol{M}}), \subset, \otimes, \backslash, /, \oplus, \obslash, \oslash)$ *is a symmetric residuated groupoid.*

Proof. It is an immediate consequence of Lemma 6. $\square$

Clearly, $\mathcal{P}_{\leq}(\boldsymbol{U_M})$ with complementation $^\sim$ is an involutive symmetric residuated groupoid. Further, $\mathcal{P}_{\leq}(\boldsymbol{U_M})$ is a boolean symmetric residuated groupoid, since $\mathcal{P}(U_{\boldsymbol{M}})$ is a boolean algebra (a field of all subsets of a set).

4 Main Theorem

In this section, we prove the main result of the paper.

Theorem 1. *Every symmetric residuated groupoid $\boldsymbol{M}$ is embeddable into the boolean symmetric residuated groupoid $\mathcal{P}_{\leq}(\boldsymbol{U_M})$.*

Proof. We define a function $h : M \to \mathcal{P}(U_{\boldsymbol{M}})$ by setting: $h(a) = \{X \in U_{\boldsymbol{M}} : a \in X\}$.

First, we show that h preserves the order, i.e.

$$a \leq b \quad \text{iff} \quad h(a) \subset h(b), \quad \text{for all } a, b \in M.$$

($\Rightarrow$) Suppose $a \leq b$. Let $X \in h(a)$. By the definition of h, $a \in X$. X is an upset, hence $a \in X$ and $a \leq b$ imply $b \in X$. Thus $X \in h(b)$.

($\Leftarrow$) Suppose $h(a) \subset h(b)$. We have $a \in \lceil a \rceil \in h(a)$. Hence, $\lceil a \rceil \in h(b)$. By the definition of h, $b \in \lceil a \rceil$, it means that $a \leq b$.

We show that h preserves all operations.

First, we show that $h(a \otimes b) = h(a) \otimes h(b)$.

($\subseteq$) Let $Z \in h(a \otimes b)$. We have then $a \otimes b \in Z$. Since $Z \in U_{\boldsymbol{M}}$, then by the definition of operation $\widehat{\otimes}$, $\lceil a \rceil \widehat{\otimes} \lceil b \rceil \subset Z$. We have $\lceil a \rceil \in h(a)$, $\lceil b \rceil \in h(b)$. Then, by the definition of operation $\otimes$, we obtain $Z \in h(a) \otimes h(b)$.

($\supseteq$) Let $Z \in h(a) \otimes h(b)$. By the definition of operation $\otimes$, there exist $X \in h(a)$ and $Y \in h(b)$ such that $X \widehat{\otimes} Y \subset Z$. By the definition of h, $a \in X$ and $b \in Y$. Hence by the definition of operation $\widehat{\otimes}$, $a \otimes b \in X \widehat{\otimes} Y$. Thus, $a \otimes b \in Z$, and finally $Z \in h(a \otimes b)$.

Now, we show that $h(a\backslash b) = h(a)\backslash h(b)$.

($\subseteq$) Let $Y \in h(a\backslash b)$. We have then $a\backslash b \in Y$. Take $X \in h(a)$, $Z \in U_{\boldsymbol{M}}$ such that $X \widehat{\otimes} Y \subset Z$. Since $a \in X$, $a \otimes (a\backslash b) \leq b$, so $b \in X \widehat{\otimes} Y$. Hence $b \in Z$. Thus $Z \in h(b)$ and $Y \in h(a)\backslash h(b)$.

($\supseteq$) Let $Y \in h(a)\backslash h(b)$. We have $\lceil a \rceil \in h(a)$. By the definition of operation $\backslash$, for all $Z \in U_{\boldsymbol{M}}$ the following implication holds: if $\lceil a \rceil \widehat{\otimes} Y \subset Z$, then $Z \in h(b)$. We have then $\lceil a \rceil \widehat{\otimes} Y \in h(b)$, and hence $b \in \lceil a \rceil \widehat{\otimes} Y$. By the definition of operation $\widehat{\otimes}$, there exist $a' \in \lceil a \rceil$ and $y \in Y$ such that $a' \otimes y \leq b$. Hence $y \leq a'\backslash b \leq a\backslash b$, so $a\backslash b \in Y$. Thus $Y \in h(a\backslash b)$.

One proves $h(a/b) = h(a)/h(b)$ in an analogous way.

Now, we show that $h(a \oplus b) = h(a) \oplus h(b)$.

($\subseteq$) Let $Z \in h(a \oplus b)$. We have then $a \oplus b \in Z$. Let $X \in U_{\boldsymbol{M}}$, $Y \in U_{\boldsymbol{M}}$ be such that $Z \subset X \widehat{\oplus} Y$. Then $a \oplus b \in X \widehat{\oplus} Y$. By the definition of operation $\widehat{\oplus}$, we have $a \in X$ or $b \in Y$, so $X \in h(a)$ or $Y \in h(b)$. By the definition of operation $\oplus$, we obtain $Z \in h(a) \oplus h(b)$.

($\supseteq$) Let $Z \in h(a) \oplus h(b)$. By the definition of operation $\oplus$, for all $X \in U_{\boldsymbol{M}}$, $Y \in U_{\boldsymbol{M}}$, if $Z \subset X \widehat{\oplus} Y$ and $X \notin h(a)$, then $Y \in h(b)$. Let X be $\lfloor a \rfloor^{\sim}$ and let Y be $\lfloor a \rfloor^{\sim} \widehat{\oslash} Z$. We have then $Z \subset \lfloor a \rfloor^{\sim} \widehat{\oplus} (\lfloor a \rfloor^{\sim} \widehat{\oslash} Z)$. Since $\lfloor a \rfloor^{\sim} \notin h(a)$, therefore $\lfloor a \rfloor^{\sim} \widehat{\oslash} Z \in h(b)$, so $b \in \lfloor a \rfloor^{\sim} \widehat{\oslash} Z$. By the definition of operation $\widehat{\oslash}$, there exist $a' \notin \lfloor a \rfloor^{\sim}$ and $c \in Z$ such that $c \leq a' \oplus b$. Since $a' \leq a$, so $c \leq a \oplus b$. Hence $a \oplus b \in Z$ and $Z \in h(a \oplus b)$.

Finally, we show that $h(a \oslash b) = h(a) \oslash h(b)$.

($\subseteq$) Let $Y \in h(a \oslash b)$. We have then $a \oslash b \in Y$. We know that $\lfloor a \rfloor^{\sim} \notin h(a)$ and $\lceil b \rceil \in h(b)$. We show $\lceil b \rceil \subset \lfloor a \rfloor^{\sim} \widehat{\oplus} Y$. Let $d \in \lceil b \rceil$, so $b \leq d$. Let $d \leq x \oplus y$ and $x \notin \lfloor a \rfloor^{\sim}$. So $x \leq a$, and then $d \leq a \oplus y$. We obtain $a \oslash d \leq y$, so $a \oslash b \leq y$. Hence $y \in Y$. Consequently, $d \in \lfloor a \rfloor^{\sim} \widehat{\oplus} Y$. Therefore, by the definition of operation $\oslash$, $Y \in h(a) \oslash h(b)$.

($\supseteq$) Let $Y \in h(a) \oslash h(b)$. By the definition of operation $\oslash$, there exist $X \notin h(a)$ and $Z \in h(b)$ such that $Z \subset X \widehat{\oplus} Y$. We have then $a \notin X$ and $b \in Z$. Since $X \in U_{\boldsymbol{M}}$, $X \subset \lfloor a \rfloor^{\sim}$, so $Z \subset \lfloor a \rfloor^{\sim} \widehat{\oplus} Y$, and hence $b \in \lfloor a \rfloor^{\sim} \widehat{\oplus} Y$. Since $b \leq a \oplus (a \oslash b)$ and $a \notin \lfloor a \rfloor^{\sim}$, then $a \oslash b \in Y$. Thus $Y \in h(a \oslash b)$.

One proves $h(a \oslash b) = h(a) \oslash h(b)$ in an analogous way. $\square$

It is easy to deduce from Theorem 1 that the Lambek-Grishin calculus is a conservative fragment of the Boolean Generalized Lambek Calculus from [5].

Representation theorems are studied by many authors. Bimbó and Dunn in [3] prove representation theorems for some types of generalized Galois logics (gaggles) such as boolean, distributive and partial (multi-)gaggles. To preserve operations, the set of upsets $U_{\boldsymbol{M}}$ in our case is replaced by the set of ultrafilters on M for boolean gaggles and by the set of prime filters on M for distributive lattices in [3].

5 Variants

In this section, based on the main result of the paper, we discuss certain variants of the representation theorem.

Let $\boldsymbol{M}$ be a symmetric residuated groupoid.

Fact 1. *If operation $\otimes$ (resp. $\oplus$) is associative in $\boldsymbol{M}$, then operation $\widehat{\otimes}$ (resp. $\widehat{\oplus}$) is associative in $\mathcal{P}_{\leq}(\boldsymbol{M})$.*

Proof. We show $(X\widehat{\otimes}Y)\widehat{\otimes}Z \subset X\widehat{\otimes}(Y\widehat{\otimes}Z)$. Let $x \in (X\widehat{\otimes}Y)\widehat{\otimes}Z$. Then there exist $a \in X\widehat{\otimes}Y$ and $b \in Z$ such that $a \otimes b \leq x$, and next, there exist $c \in X$, $d \in Y$ such that $c \otimes d \leq a$. Hence $(c \otimes d) \otimes b \leq x$. By the associativity of $\otimes$ in $\boldsymbol{M}$, $c \otimes (d \otimes b) \leq x$. Consequently, $x \in X\widehat{\otimes}(Y\widehat{\otimes}Z)$. The reverse inclusion can be proved in an analogous way.

In order to prove the associativity of operation $\widehat{\oplus}$, let us observe that operation $\bar{\oplus}$ is associative in the residuated groupoid $\mathcal{P}^{\geq}(\boldsymbol{M})$, where $\boldsymbol{M} = (M, \leq, \oplus)$. The latter fact can be proved in a similar way as above. Thus, $(X\widehat{\oplus}Y)\widehat{\oplus}Z =$ $= ((X\widehat{\oplus}Y)^{\sim}\bar{\oplus}Z^{\sim})^{\sim} = ((X^{\sim}\bar{\oplus}Y^{\sim})\bar{\oplus}Z^{\sim})^{\sim} = (X^{\sim}\bar{\oplus}(Y^{\sim}\bar{\oplus}Z^{\sim}))^{\sim} = X\widehat{\oplus}(Y\widehat{\oplus}Z)$. □

Observe that the associativity of operation $\widehat{\otimes}$ (resp. $\widehat{\oplus}$) implies the associativity of operation $\otimes$ (resp. $\oplus$) in $\mathcal{P}_{\leq}(\boldsymbol{U_M})$.

Fact 2. *If operation $\otimes$ (resp. $\oplus$) is commutative in $\boldsymbol{M}$, then operation $\widehat{\otimes}$ (resp. $\widehat{\oplus}$) is commutative in $\mathcal{P}_{\leq}(\boldsymbol{M})$.*

Proof. Assume that operation $\otimes$ is commutative in $\boldsymbol{M}$. Then $X\widehat{\otimes}Y = \{c \in M : \exists a \in X\ \exists b \in Y\ a \otimes b \leq c\} = \{c \in M : \exists b \in Y\ \exists a \in X\ b \otimes a \leq c\} = Y\widehat{\otimes}X$.

Assuming the commutativity of operation $\oplus$ in $\boldsymbol{M}$, we can show in a similar way that $X\bar{\oplus}Y = Y\bar{\oplus}X$. Thus, $X\widehat{\oplus}Y = (X^{\sim}\bar{\oplus}\, Y^{\sim})^{\sim} = (Y^{\sim}\bar{\oplus}\, X^{\sim})^{\sim} = Y\widehat{\oplus}X$. □

Observe that the commutativity of operation $\widehat{\otimes}$ (resp. $\widehat{\oplus}$) implies the commutativity of operation $\otimes$ (resp. $\oplus$) in $\mathcal{P}_{\leq}(\boldsymbol{U_M})$.

The above facts and observations allow us to state the following representation theorem for semigroups and commutative semigroups.

Theorem 2. *Every (commutative) symmetric residuated semigroup can be embedded into the (commutative) boolean symmetric residuated semigroup.*

A unital groupoid is an algebra $(M, \otimes, 1)$ such that $(M, \otimes)$ is a groupoid and 1 is a unit element for $\otimes$. *A symmetric residuated unital groupoid* is a structure $\boldsymbol{M} = (M, \leq, \otimes, \backslash, /, 1, \oplus, \oslash, \varnothing, 0)$ such that the $(\leq, \otimes, \backslash, /, \oplus, \oslash, \varnothing)$-reduct of $\boldsymbol{M}$ is a symmetric residuated groupoid, 1 is a unit element for $\otimes$ and 0 is a unit element for $\oplus$. A monoid is a unital semigroup and a symmetric residuated monoid is a symmetric residuated unital semigroup.

Let $\boldsymbol{M}$ be a symmetric residuated unital groupoid. In $\boldsymbol{U_M}$ there exists a unit element $\mathbb{1}$ satisfying $X\widehat{\otimes}\, \mathbb{1} = X = \mathbb{1}\, \widehat{\otimes}X$, namely $\mathbb{1} = \lceil 1 \rceil$. If X is an upset, then

$X\widehat{\otimes}\lceil 1\rceil = \{c \in M : \exists a \in X\ \exists b \in \lceil 1\rceil\ a \otimes b \le c\} = \{c \in M : \exists a \in X\ a \le c\} = X = \lceil 1\rceil\,\widehat{\otimes} X$. In $\boldsymbol{D_M}$ there exists a zero element $\underline{0}$ satisfying $X\bar{\oplus}\,\underline{0} = X = \underline{0}\,\bar{\oplus} X$, namely $\underline{0} = \lfloor 0\rfloor$. If X is a downset, then $X\bar{\oplus}\lfloor 0\rfloor = \{c \in M : \exists a \in X\ \exists b \in \lfloor 0\rfloor\ c \le a \oplus b\} = \{c \in M : \exists a \in X\ c \le a\} = X = \lfloor 0\rfloor\,\bar{\oplus}\, X$.

The zero element $\mathbb{O} \in \mathcal{P}_{\le}(\boldsymbol{M})$ satisfying $X\widehat{\oplus}\ \mathbb{O} = X = \mathbb{O}\ \widehat{\oplus} X$ is $\lfloor 0\rfloor^{\sim}$. We have $X\widehat{\oplus}\lfloor 0\rfloor^{\sim} = (X^{\sim}\bar{\oplus}\lfloor 0\rfloor)^{\sim} = (X^{\sim})^{\sim} = X = \lfloor 0\rfloor^{\sim}\widehat{\oplus}X$.

Now, we pass to $\mathcal{P}_{\le}(\boldsymbol{U_M})$. Notice that, for any $\mathcal{X}, \mathcal{Y} \subset U_{\boldsymbol{M}}$, the sets $\mathcal{X} \otimes \mathcal{Y}$, $\mathcal{X}\backslash\mathcal{Y}$, $\mathcal{Y}/\mathcal{X}$, $\mathcal{X} \oplus \mathcal{Y}$, $\mathcal{X} \oslash \mathcal{Y}$, $\mathcal{Y} \oslash \mathcal{X}$ are upsets with respect to $\subset$ on $\mathcal{P}(U_{\boldsymbol{M}})$. Consequently, the set $\mathcal{U}_{\mathcal{P}(U_M)}$, of all upsets on $\mathcal{P}(U_{\boldsymbol{M}})$, is a subalgebra of $\mathcal{P}_{\le}(\boldsymbol{U_M})$. The unit element and the zero element can be defined as follows:

$\mathbf{1} = \{X \in U_{\boldsymbol{M}} : 1 \in X\} = h(1)$,

$\mathbf{0} = \{X \in U_{\boldsymbol{M}} : 0 \in X\} = h(0)$.

We have $\mathcal{X} \otimes \mathbf{1} = \{Z \in U_{\boldsymbol{M}} : \exists X \in \mathcal{X}\ \exists Y \in \mathbf{1}\quad X\widehat{\otimes}Y \subset Z\} = \{Z \in U_{\boldsymbol{M}} : \exists X \in \mathcal{X}\ \ X\widehat{\otimes}\lceil 1\rceil \subset Z\} = \{Z \in U_{\boldsymbol{M}} : \exists X \in \mathcal{X}\ \ X \subset Z\} = \mathcal{X} = \mathcal{X} \otimes \mathbf{1}$.

We have for all $Y \in U_{\boldsymbol{M}}, 0 \notin Y$ if, and only if, $Y \subset \lfloor 0\rfloor^{\sim}$. In other words, $\lfloor 0\rfloor^{\sim}$ is the greatest upset Y such that $0 \notin Y$. We prove $\mathcal{X} = \mathcal{X} \oplus \mathbf{0}$ for any $\mathcal{X} \in \mathcal{U}_{\mathcal{P}(U_M)}$. We show $\mathcal{X} \subset \mathcal{X} \oplus \mathbf{0}$. Assume $Z \in \mathcal{X}$. Let $Z \subset X\widehat{\oplus}Y$. Hence $X\widehat{\oplus}Y \in \mathcal{X}$. Assume $Y \notin \mathbf{0}$, hence $0 \notin Y$. Since $Y \subset \lfloor 0\rfloor^{\sim}$, then $X\widehat{\oplus}Y \subset X\widehat{\oplus}\lfloor 0\rfloor^{\sim} = X$. Consequently, $X \in \mathcal{X}$, which yields $Z \in \mathcal{X} \oplus \mathbf{0}$. Now, we show $\mathcal{X} \oplus \mathbf{0} \subset \mathcal{X}$. Assume $Z \in \mathcal{X} \oplus \mathbf{0}$. We have $Z \subset Z\widehat{\oplus}\lfloor 0\rfloor^{\sim}$ and $0 \notin \lfloor 0\rfloor^{\sim}$, so $\lfloor 0\rfloor^{\sim} \notin \mathbf{0}$. It yields $Z \in \mathcal{X}$. $\mathcal{X} = \mathbf{0} \oplus \mathcal{X}$, for $\mathcal{X} \in \mathcal{U}_{\mathcal{P}(U_M)}$, can be proved in a similar way. We have then, $\mathcal{X} \oplus \mathbf{0} = \mathcal{X} = \mathbf{0} \oplus \mathcal{X}$.

$\mathcal{U}_{\mathcal{P}(U_M)}$ is a subalgebra of $\mathcal{P}_{\le}(\boldsymbol{U_M})$. We have shown above that h embeds $\boldsymbol{M} = (M, \le, \otimes, \backslash, /, 1, \oplus, \oslash, \oslash, 0)$ into the algebra $\mathcal{U}_{\mathcal{P}(U_M)}$ and $h(1) = \mathbf{1}$, $h(0) = \mathbf{0}$. Notice that $\mathcal{U}_{\mathcal{P}(U_M)}$ is not closed under $^{\sim}$, in general (similarly, $U_{\boldsymbol{M}}$ is not closed under $^{\sim}$).

If $\boldsymbol{M}$ is an involutive symmetric residuated groupoid, then $U_{\boldsymbol{M}}$ (resp. $\mathcal{U}_{\mathcal{P}(U_M)}$) is closed under an involutive negation (a quasi-complement in the sense of [2]). We define $g : \mathcal{P}(M) \mapsto \mathcal{P}(M)$ as follows:

$$g(X) = (-X)^{\sim},$$

where $-X = \{-a : a \in X\}$. Clearly, $(-X)^{\sim} = -(X^{\sim})$, hence $g(g(X)) = X$, and $X \subset Y$ entails $g(Y) \subset g(X)$. Consequently, g is an involutive negation on $\mathcal{P}(M)$. Further, $U_{\boldsymbol{M}}$ is closed under g, so $(U_{\boldsymbol{M}}, \subset, \widehat{\otimes}, \widehat{\backslash}, \widehat{/}, \widehat{\oplus}, \widehat{\oslash}, \widehat{\oslash}, g)$ is an involutive symmetric residuated groupoid.

We define an involutive negation $\sim_g$ on $\mathcal{P}(U_{\boldsymbol{M}})$ as follows:

$$\sim_g (\mathcal{X}) = g\,[\mathcal{X}]^{\sim}\ .$$

Clearly, $\sim_g$ arises from g in the same way as g arises from $-$. Consequently, $\sim_g$ is an involutive negation on $\mathcal{P}(U_{\boldsymbol{M}})$, and $\mathcal{U}_{\mathcal{P}(U_M)}$ is closed under $\sim_g$. We show that $h(-a) = \sim_g h(a)$ for all $a \in X$, for the mapping h defined above.

We have to show that $X \in \sim_g h(a)$ iff $X \in h(-a)$. The following equivalences hold: $X \in \sim_g h(a)$ iff $X \in g\,[h(a)]^{\sim}$ iff $X \notin g\,[h(a)]$ iff $g(X) \notin h(a)$ iff $(-X)^{\sim} \notin h(a)$ iff $a \notin (-X)^{\sim}$ iff $a \in -X$ iff $-a \in X$ iff $X \in h(-a)$.

Buszkowski [4] proved that each residuated semigroup with De Morgan negation is isomorphically embeddable into some residuated semigroup of cones with quasi-boolean complement. The following theorem yields a related result.

Theorem 3. *Every involutive symmetric residuated* (*unital*) *groupoid is embeddable into a quasi-boolean symmetric residuated* (*unital*) *groupoid, and similarly for involutive residuated* (*commutative*) *semigroups and monoids.*

A bilinear algebra can be defined as a symmetric residuated monoid with two negations $\sim$, $-$, satisfying:

$$\begin{aligned} \sim -a = a &= - \sim a, \\ \sim (a \otimes b) &= (\sim b) \oplus (\sim a), \\ -(a \otimes b) &= (-b) \oplus (-a), \\ \sim a &= a \backslash 0, \\ -a &= 0/a, \end{aligned}$$

for all elements a, b. An equivalent notion was defined in Lambek [14,15] as an algebra corresponding to Bilinear Logic. Bilinear Logic is equivalent to the multiplicative fragment of Noncommutative MALL of Abrusci [1]. Some lattice models of this logic are discussed by Lambek in [13]. Cyclic Noncommutative MALL of Yetter [17] gives rise to cyclic bilinear algebras.

A cyclic bilinear algebra is a bilinear algebra $\boldsymbol{M}$ such that $\sim a = -a$; equivalently $\boldsymbol{M}$ is an involutive symmetric residuated monoid, satisfying:

$$\begin{aligned} -(a \otimes b) &= (-b) \oplus (-a), \\ -a &= a \backslash 0 = 0/a, \end{aligned}$$

for all $a, b \in M$.

Let $\boldsymbol{M} = (M, \leq, \otimes, \backslash, /, 1, \oplus, \oslash, \varobslash, 0, -)$ be a cyclic bilinear algebra. We show that the involutive function g defined above satisfies:

$$\begin{aligned} g(X \widehat{\otimes} Y) &= g(Y) \widehat{\oplus}\, g(X), \\ g(X) = X \widehat{\backslash} \lfloor 0 \rfloor^{\sim} &= \lfloor 0 \rfloor^{\sim} \widehat{/} X, \end{aligned}$$

for all $X, Y \in U_{\boldsymbol{M}}$.

We show the first equation. Assume $-(a \otimes b) = (-b) \oplus (-a)$. We have $g(X \widehat{\otimes} Y) = g(Y) \widehat{\oplus}\, g(X)$ iff $-(X \widehat{\otimes} Y)^{\sim} = (-Y)^{\sim} \widehat{\oplus} (-X)^{\sim}$ iff $-(X \widehat{\otimes} Y) = (-Y) \bar{\oplus} (-X)$. The following equivalences hold: $c \in -(X \widehat{\otimes} Y)$ iff $-c \in X \widehat{\otimes} Y$ iff there exist $a \in X$ and $b \in Y$ such that $a \otimes b \leq -c$ iff there exist $a \in X$ and $b \in Y$ such that $c \leq -(a \otimes b) = (-b) \oplus (-a)$ iff there exist $a' \in -X$ and $b' \in -Y$ such that $c \leq b' \oplus a'$ iff $c \in (-Y) \bar{\oplus} (-X)$. So, we have shown $g(X \widehat{\otimes} Y) = g(Y) \widehat{\oplus}\, g(X)$.

Now, we prove $g(X) = X \widehat{\backslash} \lfloor 0 \rfloor^{\sim}$ i.e. $(-X)^{\sim} = X \widehat{\backslash} \lfloor 0 \rfloor^{\sim}$, or equivalently $-X = (X \widehat{\backslash} \lfloor 0 \rfloor^{\sim})^{\sim}$. We have $b \in (X \widehat{\backslash} \lfloor 0 \rfloor^{\sim})^{\sim}$ iff $b \notin X \widehat{\backslash} \lfloor 0 \rfloor^{\sim}$ iff there exist $a \in X$ and $c \in M$ such that $a \otimes b \leq c$ and $c \notin \lfloor 0 \rfloor^{\sim}$ (i.e. $c \leq 0$) iff there exists $a \in X$ such that $a \otimes b \leq 0$ iff there exists $a \in X$ such that $a \leq 0/b = -b$ iff $-b \in X$ iff $b \in -X$.

One proves $g(X) = \lfloor 0 \rfloor^{\sim} \widehat{/} X$ in an analogous way.

Since $\sim_g$ arises from g in the same way as g arises from $-$, one can analogously show that the involutive negation $\sim_g$ satisfies:

$$\sim_g(\mathcal{X} \otimes \mathcal{Y}) = \sim_g(\mathcal{Y}) \oplus \sim_g(\mathcal{X}),$$
$$\sim_g(\mathcal{X}) = \mathcal{X} \backslash \mathbf{0} = \mathbf{0} / \mathcal{X},$$

for all $\mathcal{X}, \mathcal{Y} \in \mathcal{U}_{\mathcal{P}(U_M)}$. We obtain the following theorem:

Theorem 4. *Every cyclic bilinear algebra $\boldsymbol{M}$ is embeddable into the quasi-boolean cyclic bilinear algebra $\mathcal{U}_{\mathcal{P}(U_M)}$, which is a quasi-field of sets.*

An analogous result can be proved for bilinear algebras, but then the target algebra $\mathcal{U}_{\mathcal{P}(U_M)}$ is a weak quasi-field of sets with two weak quasi-complements $\sim_g \mathcal{X} = g[\mathcal{X}]^{\sim}$, $-_f \mathcal{X} = f[\mathcal{X}]^{\sim}$, where $g(X) = (\sim X)^{\sim}$, $f(X) = (-X)^{\sim}$. We also have $h(\sim a) = \sim_g h(a)$, $h(-a) = -_f h(a)$. If the associativity of $\otimes$ and $\oplus$ is not assumed, then similar results can be obtained for cyclic InGL-algebras (without lattice operations) in the sense of [7].

Grishin [8] considered a formal system whose algebraic models are symmetric residuated monoids with 0 additionally satisfying the laws of mixed associativity:

1. $a \otimes (b \oplus c) \leq (a \otimes b) \oplus c$
2. $(a \oplus b) \otimes c \leq a \oplus (b \otimes c)$

We propose to call such structures *associative Lambek-Grishin algebras* (associative LG-algebras). Omitting the associativity of $\otimes, \oplus$, one obtains a more general class of LG-algebras.

Moortgat [16] and other authors consider systems admitting so-called Grishin axioms. Some axioms of that kind are listed below.

	Associativity	Commutativity
Group I	$1_a.\ a \otimes (b \oplus c) \leq (a \otimes b) \oplus c$	$1_c.\ a \otimes (b \oplus c) \leq b \oplus (a \otimes c)$
	$2_a.\ (a \oplus b) \otimes c \leq a \oplus (b \otimes c)$	$2_c.\ (a \oplus b) \otimes c \leq (a \otimes c) \oplus b$
Group II	$1_a.\ (a \otimes b) \otimes c \leq a \otimes (b \otimes c)$	$1_c.\ a \otimes (b \otimes c) \leq b \otimes (a \otimes c)$
	$2_a.\ a \otimes (b \otimes c) \leq (a \otimes b) \otimes c$	$2_c.\ (a \otimes b) \otimes c \leq (a \otimes c) \otimes b$
Group III	$1_a.\ (a \oplus b) \oplus c \leq a \oplus (b \oplus c)$	$1_c.\ a \oplus (b \oplus c) \leq b \oplus (a \oplus c)$
	$2_a.\ a \oplus (b \oplus c) \leq (a \oplus b) \oplus c$	$2_c.\ (a \oplus b) \oplus c \leq (a \oplus c) \oplus b$
Group IV	$1_a.\ (a \backslash b) \oslash c \leq a \backslash (b \oslash c)$	$1_c.\ a \obslash (b \backslash c) \leq b \backslash (a \obslash c)$
	$2_a.\ a \obslash (b / c) \leq (a \obslash b) / c$	$2_c.\ (a / b) \oslash c \leq (a \oslash c) / b$

Some axiomatization of a bilinear algebra obtained by adding selected Grishin axioms was described by Lambek in [12].

We show that the powerset algebras, defined above, preserve axioms from Groups I-IV. We denote by I.1_a the first axiom from Group I of Associativity, and similarly for the other axioms.

Let $\boldsymbol{A}$ be a symmetric residuated groupoid.

Proposition 1. *If the axiom I.1$_a$ (resp. IV.1$_a$, I.2$_a$, IV.2$_a$, I.1$_c$, IV.1$_c$, I.2$_c$, IV.2$_c$) is valid in the basic algebra $\boldsymbol{A}$, then the corresponding axiom IV.1$_a$ (resp. I.1$_a$, IV.2$_a$, I.2$_a$, IV.1$_c$, I.1$_c$, IV.2$_c$, I.2$_c$) is valid in the algebra $\mathcal{P}_{\leq}(\boldsymbol{A})$.*

Proof. Assume that the mixed associativity law I.1$_a$ is valid in algebra $\boldsymbol{A}$. We show that the appropriate law IV.1$_a$ is valid in $\mathcal{P}_{\leq}(\boldsymbol{A})$: $(A\widehat{\backslash}B)\widehat{\oslash}C \subseteq A\widehat{\backslash}(B\widehat{\oslash}C)$.

Let $x \in (A\widehat{\backslash}B)\widehat{\oslash}C$. By the definition of operation $\widehat{\oslash}$ there exist $c \notin C$ and $b \in A\widehat{\backslash}B$ such that $b \leq x \oplus c$. We fix $a \in A$. By monotonicity of $\otimes$ we obtain $a \otimes b \leq a \otimes (x \oplus c)$. By assumption $a \otimes b \leq (a \otimes x) \oplus c$. Since $a \in A$ and $b \in A\widehat{\backslash}B$, then $a \otimes b \in B$ by the definition of operation $\widehat{\backslash}$. Since $c \notin C$ and $a \otimes b \in B$, so $a \otimes x \in B\widehat{\oslash}C$. Consequently, $x \in A\widehat{\backslash}(B\widehat{\oslash}C)$.

Assume now that the mixed associativity law IV.1$_a$ is valid in $\boldsymbol{A}$. We show that the appropriate law I.1$_a$ is valid in $\mathcal{P}_{\leq}(\boldsymbol{A})$: $A\widehat{\otimes}(B\widehat{\oplus}C) \subseteq (A\widehat{\otimes}B)\widehat{\oplus}C$.

Let $x \in A\widehat{\otimes}(B\widehat{\oplus}C)$. By the definition of operation $\widehat{\otimes}$ there exist $a \in A$ and $b \in B\widehat{\oplus}C$ such that $a \otimes b \leq x$. We claim that $x \in (A\widehat{\otimes}B)\widehat{\oplus}C$ i.e. for all u, v: if $x \leq u \oplus v$ then $u \in A\widehat{\otimes}B$ or $v \in C$. Assume that $x \leq u \oplus v$. Suppose that $v \notin C$. We show that $u \in A\widehat{\otimes}B$. By the residuation law we have $x \oslash v \leq u$. Take $a \in A$. By monotonicity of $\backslash$ we obtain $a\backslash(x \oslash v) \leq a\backslash u$ and by assumption $(a\backslash x) \oslash v \leq a\backslash u$. By the residuation we have $a\backslash x \leq (a\backslash u) \oplus v$. Since $b \leq a\backslash x$ then $b \leq (a\backslash u) \oplus v$. Since $b \in B\widehat{\oplus}C$ and $v \notin C$, so $a\backslash u \in (B\widehat{\oplus}C)\widehat{\oslash}C \subseteq B$. We have $a \in A$ and $a\backslash u \in B$. Consequently, $u \in A\widehat{\otimes}B$.

Assume that the mixed (weak)-commutativity law I.1$_c$ is valid in $\boldsymbol{A}$. We show that the appropriate law IV.1$_c$ is valid in $\mathcal{P}_{\leq}(\boldsymbol{A})$: $A\widehat{\obslash}(B\widehat{\backslash}C) \subseteq B\widehat{\backslash}(A\widehat{\obslash}C)$.

Let $x \in A\widehat{\obslash}(B\widehat{\backslash}C)$. There exist $a \notin A$ and $c \in B\widehat{\backslash}C$ such that $c \leq a \oplus x$. We fix $b \in B$. By monotonicity of $\otimes$ we obtain $b \otimes c \leq b \otimes (a \oplus x)$. By assumption $b \otimes c \leq a \oplus (b \otimes x)$. Since $b \in B$ and $c \in B\widehat{\backslash}C$, then $b \otimes c \in C$. We have $a \notin A$ and $b \otimes c \in C$, so $b \otimes x \in A\widehat{\obslash}C$. Consequently, $x \in B\widehat{\backslash}(A\widehat{\obslash}C)$.

Assume now that the mixed (weak)-commutativity law IV.1$_c$ is valid in $\boldsymbol{A}$. We show that the law I.1$_c$ is valid in $\mathcal{P}_{\leq}(\boldsymbol{A})$: $A\widehat{\otimes}(B\widehat{\oplus}C) \subseteq B\widehat{\oplus}(A\widehat{\otimes}C)$.

Let $x \in A\widehat{\otimes}(B\widehat{\oplus}C)$. There exist $a \in A$ and $b \in B\widehat{\oplus}C$ such that $a \otimes b \leq x$. We claim that $x \in B\widehat{\oplus}(A\widehat{\otimes}C)$ i.e. for all u, v: if $x \leq u \oplus v$ then $u \in B$ or $v \in A\widehat{\otimes}C$. Assume that $x \leq u \oplus v$. Suppose that $u \notin B$. We show that $v \in A\widehat{\otimes}C$. By the residuation law we have $u \obslash x \leq v$. Take $a \in A$. By monotonicity of $\backslash$ we obtain $a\backslash(u \obslash x) \leq a\backslash v$ and by assumption $u \obslash (a\backslash x) \leq a\backslash v$. By the residuation we have $a\backslash x \leq u \oplus (a\backslash v)$. Since $b \leq a\backslash x$ then $b \leq u \oplus (a\backslash v)$. Since $b \in B\widehat{\oplus}C$ and $u \notin B$, so $a\backslash v \in B\widehat{\obslash}(B\widehat{\oplus}C) \subseteq C$. We have $a \in A$ and $a\backslash v \in C$. Consequently, $v \in A\widehat{\otimes}C$.

The cases for axioms I.2$_a$, IV.2$_a$, I.2$_c$ and IV.2$_c$ are proved in a similar way. $\square$

Proposition 2. *If the axiom II.1$_a$ (resp. II.2$_a$, III.1$_a$, III.2$_a$, II.1$_c$, II.2$_c$, III.1$_c$, III.2$_c$) is valid in the basic algebra $\boldsymbol{A}$, then the corresponding axiom II.2$_a$ (resp. II.1$_a$, III.2$_a$, III.1$_a$, II.2$_c$, II.1$_c$, III.2$_c$, III.1$_c$) is valid in the algebra $\mathcal{P}_{\leq}(\boldsymbol{A})$.*

We omit an easy proof of this proposition.

Corollary 1. *If $\boldsymbol{A}$ satisfies an axiom from the above list, then $\mathcal{P}_{\leq}(\boldsymbol{U_A})$ satisfies the same axiom.*

This corollary yields the following proposition.

Proposition 3. *If $\boldsymbol{A}$ is an (resp. associative) LG-algebra, then $\mathcal{U}_{\mathcal{P}(\boldsymbol{U_A})}$ is an (resp. associative) LG-algebra.*

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