Correlation Modeling of the Gravity Field in Classical Geodesy

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Contents

1 Introduction 1808
2 Correlation Functions 1810
2.1 Functions on the Sphere 1811
2.2 Functions on the Plane 1813
2.3 From the Sphere to the Plane 1816
2.4 Properties of Correlation Functions and PSDs 1817
3 Stochastic Processes and Covariance Functions 1821
3.1 Earth's Anomalous Gravitational Field 1824
3.2 The Disturbing Potential as a Stochastic Process 1826
4 Covariance Models 1830
4.1 Analytic Models 1831
4.2 The Reciprocal Distance Model 1832
4.3 Parameter Determination 1833
5 Summary and Future Directions 1835
Appendix A 1837
Appendix B 1840
References 1843

Abstract

The spatial correlation of the Earth's gravity field is well known and widely utilized in applications of geophysics and physical geodesy. This paper develops the mathematical theory of correlation functions, as well as covariance functions under a statistical interpretation of the field, for functions and processes on the sphere and plane, with formulation of the corresponding power spectral densities in the respective frequency domains and with extensions into the third

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W. Freeden et al. (eds.), *Handbook of Geomathematics*,
DOI 10.1007/978-3-642-54551-1_28

dimension for harmonic functions. The theory is applied, in particular, to the disturbing gravity potential with consistent relationships of the covariance and power spectral density to any of its spatial derivatives. An analytic model for the covariance function of the disturbing potential is developed for both spherical and planar application, which has analytic forms also for all derivatives in both the spatial and the frequency domains (including the along-track frequency domain). Finally, a method is demonstrated to determine the parameters of this model from empirical regional power spectral densities of the gravity anomaly.

1 Introduction

The Earth's *gravitational field* plays major roles in geodesy, geophysics, and geodynamics and is also a significant factor in specific applications such as precision navigation and satellite orbit analysis. With the advance of instrumentation technology over the last several decades, we now have gravitational models of high spatial resolution over most of the land areas, thanks to extensive ground and expanding airborne survey campaigns and over the oceans owing to satellite radar altimetry, which measures essentially a level surface. Recent satellite gravity missions (e.g., the Gravity Field and Steady-State Ocean Circulation Explorer (GOCE), Rummel et al. 2011) also have vastly improved the longer-wavelength parts of the model with globally distributed in situ measurements. Despite these improvements, there remain deficiencies in resolution, including a lack of uniformity and accuracy in some land areas, such as Antarctica and significant parts of Africa, South America, and Asia (Pavlis et al. 2012a). These gaps will be filled with continued measurement, mostly using airborne systems for efficient accessibility to remote regions.

Determining the required resolution and analyzing the effect or significance of the gravitational field at various scales for particular applications often rely on some a priori knowledge of the field. Also, the interpolation and extrapolation of the field from given discrete data and the prediction or estimation of field quantities other than those directly measured requires a weighting function based on the essential spatial correlative characteristics of the gravitational field. For these reasons, the study and development of correlation or covariance functions of the field have occupied geodesists and geophysicists in tandem with the advancements of measurement and instrument technology.

The rather slow attenuation of the field as a function of resolution gives it at regional scales a kind of random character, much like the Earth's topography. Indeed, the shorter spatial wavelengths of the gravitational field are in many cases highly correlated with the topography; and, profiles of topography, like coastlines, are known to be *fractals*, which arise from certain random fluctuations, analogous to Brownian motion (Mandelbrot 1983). Thus, we may argue that the Earth's gravitational field at fine scales also exhibits a stochastic nature (Jekeli 1991). This randomness in the field has been argued and counterargued for decades, but it does form the basis for one of the more successful estimation methods in physical geodesy, called least-squares collocation (Moritz 1980). In addition, the correlative description of the field is advantageous in more general error analyses of the problem of field modeling; and, it is particularly useful in generating synthetic fields for deterministic simulations of the field for Monte Carlo types of analyses.

The stochastic nature of the gravitational field, besides assumed primarily for the shorter wavelengths, is also limited to the horizontal dimensions. The variation in the vertical (above the Earth's surface) is constrained deterministically by the attenuation of the gravitational potential with distance from its source, as governed by the solution to Laplace's differential equation in free space. However, this constraint also extends the stochastic interpretation in estimation theory, since it analytically establishes mutually consistent correlations for vertical derivatives of the potential, or between its horizontal and vertical derivatives, or between the potential (and any of its derivatives) at different vertical levels. Thus, with the help of the corresponding covariance functions, one is able to estimate, for example, the geoid undulation from gravity anomaly data in a purely operational approach using no other physical models, which is the essence of the method of least-squares collocation.

It is necessary to distinguish and relate correlation and covariance functions as used in this text. The *covariance* function refers to random or stochastic processes and is the statistical expectation of the product of the centralized process at two points of the process (i.e., of two random variables with their means removed). The *correlation* function has more than one definition. As a natural extension of the Pearson correlation coefficient, it is the covariance function normalized by the square roots of the variances of the process at the two points (Priestley 1981). An alternative definition is the statistical expectation of the non-centralized product of the process at two points (Maybeck 1979). A third definition characterizes the correlation of deterministic (nonrandom) functions on the basis of averages of products over the domain of the function (de Coulon 1986). Ultimately, the covariance function and the correlation function, in its various incarnations, are related, but there is an advantage to distinguish between the stochastic and the non-stochastic versions. Minimum error variance estimation requires a stochastic interpretation, and the gravitational field is characterized stochastically in terms of covariance functions. If interpolation or filtering or simulation through arbitrary synthesis is the principal application, then it may be sufficient to dispense with the stochastic interpretation. If the stochastic process is ergodic then the average-based correlation function of its realization is the same as the its covariance function if the means are known and removed.

Thus, one may start with the formulation of the physical correlation of the gravitational field without the stochastic underpinning and introduce the stochastic interpretation as needed. Since one of the main applications is the popular least-squares collocation in physical geodesy, the terminology of covariance functions dominates the later chapters. Whether from the more general or the stochastic viewpoint, the correlative methods can be extended to other fields on the Earth's surface and to fields that are harmonic in free space. For example, the anomalous magnetic potential (due to the magnetization of the crust material induced by the main, outer-core-generated field of the Earth) also satisfies Laplace's differential equation.

Thus, it shares basic similarities to the anomalous gravitational field. Under certain, albeit rather restrictive assumptions, one field may even be represented in terms of the other (Poisson's relationship; Baranov 1957). Although this relationship has not been studied in detail from the stochastic or more general correlative viewpoint, it does open numerous possibilities in estimation and error analysis.

Finally, it is noted that spatial data analyses in geophysics, specifically the optimal prediction and interpolation of geophysical signals, known as *kriging* (Olea 1999), rely as does collocation in geodesy on a correlative interpretation of the signals. Semi-variograms, instead of *correlation functions*, are used in kriging, but they are closely related. Therefore, a study of modeling one (correlations or covariances, in the present case) immediately carries over to the other.

The following chapters review *correlation functions* on the sphere and plane, as well as the transforms into their respective spatial frequency domains. For the stochastic understanding of the geopotential field, the covariance function is introduced, under the assumption of *ergodicity* (hence, *stationarity*). Again, the frequency domain formulation, that is, the power spectral density of the field, is of particular importance. The method of covariance propagation, which is indispensable in such estimation techniques as least-squares collocation, naturally motivates the analytic modeling of covariance functions. Models have occupied physical geodesists since the utility of least-squares collocation first became evident, and myriad types of models and approaches exist. In this paper, a single yet comprehensive, adaptable, and flexible model is developed that offers consistency among all derivatives of the potential, whether in spherical or planar coordinates, and in the space or frequency domains. Methods to derive appropriate parameters for this model conclude the essential discussions of this paper.

2 Correlation Functions

We start with functions on the sphere and develop the concept of the correlation function without the need for a stochastic foundation. The statistical interpretation may be imposed at a later time when it is convenient or necessary to do so. As it happens, the infinite plane as functional domain offers more than one option for developing correlation functions, depending on the class of functions, and, therefore, will be treated after considering the unit sphere, σ. Other types of surfaces that approximate the Earth's surface more accurately (ellipsoid, geoid, topographic surface) could also be contemplated. However, the extension of the correlation function into space according to potential theory and the development of a useful duality in the spatial frequency domain then become more problematic, if not impossible. In essence, we require surfaces on which functions have a spectral decomposition and such that convolutions transform into the frequency domain as products of spectra. The latter requirement is tied to the analogy between convolutions and correlations. Furthermore, the surface should be sufficiently simple as a boundary in the solution to Laplace's equation for the gravitational potential. To satisfy all these requirements and with a view toward practical applications, the present discussion is restricted to the plane and the sphere.

Although data on the surface are always discrete, we do not consider discrete functions. Rather, it is always assumed that the data are samples of a continuous function. Then, the correlation functions to be defined are also continuous, and correlations among the data are interpreted as samples of the correlation function.

2.1 Functions on the Sphere

Let g and h be continuous, square-integrable functions on the unit sphere, σ, i.e.,

$$\frac{1}{4\pi}\iint_{\sigma} g^2 d\sigma < \infty, \quad \frac{1}{4\pi}\iint_{\sigma} h^2 d\sigma < \infty, \tag{1}$$

and suppose they depend on the spherical polar coordinates, $\{(\theta,\lambda)\,|0 \le \theta \le \pi,\ 0 \le \lambda < 2\pi\}$. Each function may be represented in terms of its *Legendre transform* as an infinite series of spherical harmonics,

$$g(\theta,\lambda) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} G_{n,m}\bar{Y}_{n,m}(\theta,\lambda), \tag{2}$$

where the Legendre transform, or the *Legendre spectrum* of g, is

$$G_{n,m} = \frac{1}{4\pi}\iint_{\sigma} g(\theta,\lambda)\,\bar{Y}_{n,m}(\theta,\lambda)\,d\sigma, \tag{3}$$

and where the functions, $\bar{Y}_{n,m}(\theta,\lambda)$, are *surface spherical harmonics* defined by

$$\bar{Y}_{n,m}(\theta,\lambda) = \bar{P}_{n,|m|}(\cos\theta)\begin{cases}\cos m\lambda, & m \ge 0\\ \sin|m|\,\lambda, & m < 0\end{cases} \tag{4}$$

The functions, $\bar{P}_{n,m}$, are *associated Legendre functions* of the first kind, fully normalized so that

$$\frac{1}{4\pi}\iint_{\sigma} \bar{Y}_{n',m'}(\theta,\lambda)\,\bar{Y}_{n,m}(\theta,\lambda)\,d\sigma = \begin{cases}1, & n' = n \text{ and } m' = m\\ 0, & n' \ne n \text{ or } m' \ne m\end{cases} \tag{5}$$

A similar relationship exists between h and its Legendre transform, $H_{n,m}$. The degree and order, (n, m), are wave numbers belonging to the frequency domain. The unit sphere is used here only for convenience, and any sphere (radius, R) may be used. The Legendre spectrum then refers to this sphere.

We define the correlation function of g and h as

$$\phi_{gh}(\psi,\alpha)=\frac{1}{4\pi}\iint_{\sigma} g(\theta,\lambda)\,h\left(\theta',\lambda'\right)\sin\theta d\theta d\lambda, \tag{6}$$

where the points (θ,λ) and (θ',λ') are related by

$$\cos\psi=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\left(\lambda-\lambda'\right), \tag{7}$$

$$\tan\alpha=\frac{-\sin\theta'\sin(\lambda-\lambda')}{\sin\theta\cos\theta'-\cos\theta\sin\theta'\cos(\lambda-\lambda')}, \tag{8}$$

and where the integration is performed over all pairs of points, (θ,λ) and (θ',λ'), separated by the fixed spherical distance, ψ, and oriented by the fixed azimuth, α.

If the spherical harmonic series, Eq. (2), for g and h are substituted into Eq. (6), we find that, due to the special geometry of the sphere, no simple analytic expression results unless we further average over all azimuths, α, thus imposing *isotropy* on the correlation function. Therefore, we redefine the correlation function of g and h (on the sphere) as follows:

$$\phi_{gh}(\psi)=\frac{1}{8\pi^2}\int_0^{2\pi}\iint_{\sigma} g(\theta,\lambda)\,h\left(\theta',\lambda'\right)\sin\theta d\theta d\lambda d\alpha. \tag{9}$$

More precisely, this is the cross-correlation function of g and h. The *autocorrelation function* of g is simply $\phi_{gg}(\psi)$. The prefixes, cross- and auto-, are used mostly to emphasize a particular application and may be dropped when no confusion arises.

Because of its sole dependence on ψ, ϕ_{gh} can be expressed as an infinite series of Legendre polynomials:

$$\phi_{gh}(\psi)=\sum_{n=0}^{\infty}(2n+1)\left(\Phi_{gh}\right)_n P_n(\cos\psi), \tag{10}$$

where the coefficients, $\left(\Phi_{gh}\right)_n$, constitute the Legendre transform of ϕ_{gh}:

$$\left(\Phi_{gh}\right)_n=\frac{1}{2}\int_0^{\pi}\phi_{gh}(\psi)\,P_n(\cos\psi)\sin\psi d\psi. \tag{11}$$

Substituting the decomposition formula for the Legendre polynomial,

$$P_n(\cos\psi)=\frac{1}{2n+1}\sum_{m=-n}^{n}\bar{Y}_{n,m}(\theta,\lambda)\,\bar{Y}_{n,m}\left(\theta',\lambda'\right), \tag{12}$$

and Eq. (9) into Eq. (11) and then simplifying using the orthogonality, Eq. (5), and the definition of the Legendre spectrum, Eq. (3), we find:

$$\left(\Phi_{gh}\right)_n = \frac{1}{2n+1}\sum_{m=-n}^{n}\frac{1}{4\pi}\iint_{\sigma} g\left(\theta,\lambda\right)\bar{Y}_{n,m}\left(\theta,\lambda\right)\left(\frac{1}{4\pi}\iint_{\sigma} h\left(\theta',\lambda'\right)\bar{Y}_{n,m}\left(\theta',\lambda'\right)\sin\psi\, d\psi\, d\alpha\right)\sin\theta\, d\theta\, d\lambda$$
$$= \frac{1}{2n+1}\sum_{m=-n}^{n} G_{n,m}H_{n,m} \tag{13}$$

where (θ, λ) is constant in the inner integral. The quantities, $\left(\Phi_{gh}\right)_n$, constituting the Legendre transform of the correlation function, may be called the (cross-) *power spectral density (PSD)* of g and h. They are determined directly from the Legendre spectra of g and h. The (auto-) PSD of g is simply

$$\left(\Phi_{gg}\right)_n = \frac{1}{2n+1}\sum_{m=-n}^{n} G_{n,m}^2. \tag{14}$$

The terminology that refers the correlation function to "power" is appropriate since it is an integral divided by the solid angle of the sphere. For functions on the plane, we make a distinction between energy and power, depending on the class of functions.

2.2 Functions on the Plane

On the infinite plane with Cartesian coordinates, $\{(x_1, x_2)\,|\,-\infty < x_1 < \infty, -\infty < x_2 < \infty\}$, we consider several possibilities for the functions. The situation is straightforward if the functions are periodic and square integrable over the domain of a period or are square integrable over the plane. Anticipating no confusion, these functions again are denoted, g and h. For the periodic case, with periods, Q_1 and Q_2, in the respective coordinates,

$$\frac{1}{Q_1Q_2}\int_0^{Q_1}\int_0^{Q_2} g^2 dx_1 dx_2 < \infty, \quad \frac{1}{Q_1Q_2}\int_0^{Q_1}\int_0^{Q_2} h^2 dx_1 dx_2 < \infty; \tag{15}$$

and each function may be represented in terms of its *Fourier transform* as an infinite series of sines and cosines, conveniently combined using the complex exponential:

$$g\left(x_1, x_2\right) = \frac{1}{Q_1Q_2}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} G_{k_1,k_2} e^{i2\pi\left(\frac{k_1x_1}{Q_1}+\frac{k_2x_2}{Q_2}\right)}, \tag{16}$$

where the Fourier transform, or the *Fourier spectrum* of g, is given by

$$G_{k_1,k_2} = \int_0^{Q_1}\int_0^{Q_2} g\left(x_1, x_2\right) e^{-i2\pi\left(\frac{k_1 x_1}{Q_1}+\frac{k_2 x_2}{Q_2}\right)} dx_1 dx_2, \tag{17}$$

and a similar relationship exists between h and its transform, H_{k_1,k_2}. Again, the integers, k_1, k_2, are the wave numbers in the frequency domain.

Assuming both functions have the same periods, the *correlation function* of g and h is defined by

$$\phi_{gh}\left(s_1, s_2\right) = \frac{1}{Q_1 Q_2}\int_{-Q_1/2}^{Q_1/2}\int_{-Q_2/2}^{Q_2/2} g^*\left(x_1', x_2'\right) h\left(x_1' + s_1, x_2' + s_2\right) dx_1' dx_2', \tag{18}$$

where g^* is the complex conjugate of g (we deal only with real functions but need this formal definition). The independent variables are the differences between points of evaluation of h at (x_1, x_2) and g^* at $\left(x_1', x_2'\right)$, respectively, as follows:

$$s_1 = x_1 - x_1', \quad s_2 = x_2 - x_2'. \tag{19}$$

The integration is performed with s_1 and s_2 fixed and requires recognition of the fact that g and h are periodic.

The correlation function is periodic with the same periods as for g and h, and its Fourier transform, that is, the power spectral density (PSD), is discrete and given by

$$\left(\Phi_{gh}\right)_{k_1,k_2} = \int_0^{Q_1}\int_0^{Q_2} \phi_{gh}\left(s_1, s_2\right) e^{-i2\pi\left(\frac{k_1 s_1}{Q_1}+\frac{k_2 s_2}{Q_2}\right)} ds_1 ds_2. \tag{20}$$

Substituting the correlation function, defined by Eq. (18) into Eq. (20), yields after some straightforward manipulations (making use of Eq. (17) and the periodicity of its integrand):

$$\left(\Phi_{gh}\right)_{k_1,k_2} = \frac{1}{Q_1 Q_2} G_{k_1,k_2}^* H_{k_1,k_2}. \tag{21}$$

Analogous to the spherical case, Eq. (13), the PSD of periodic functions on the plane can be determined directly from their Fourier series coefficients.

A very similar situation arises for nonperiodic functions that are nevertheless square integrable on the plane:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^2 dx_1 dx_2 < \infty, \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^2 dx_1 dx_2 < \infty. \tag{22}$$

In this case, the *Fourier transform* relationships for the function are given by

$$g\left(x_1, x_2\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G\left(f_1, f_2\right) e^{i2\pi\left(f_1 x_1 + f_2 x_2\right)} df_1 df_2, \tag{23}$$

$$G\left(f_1, f_2\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(x_1, x_2\right) e^{-i2\pi\left(f_1 x_1 + f_2 x_2\right)} dx_1 dx_2$$

where f_1 and f_2 are corresponding spatial (cyclical) frequencies. The correlation function is given by

$$\phi_{gh}\left(s_1, s_2\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^*\left(x_1', x_2'\right) h\left(x_1' + s_1, x_2' + s_2\right) dx_1' dx_2'; \tag{24}$$

and its Fourier transform is easily shown to be

$$\Phi_{gh}\left(f_1, f_2\right) = G^*\left(f_1, f_2\right) H\left(f_1, f_2\right). \tag{25}$$

This Fourier transform of the correlation function is called more properly the *energy spectral density*, since the correlation function is simply the integral of the product of function. The square integrability of the functions implies that they have finite energy.

Later we consider stochastic processes on the plane that are stationary, which means that they are not square integrable. For this case, one may relax the integrability condition to

$$\lim_{E_1\to\infty}\lim_{E_2\to\infty}\frac{1}{E_1 E_2}\int_{-E_1/2}^{E_1/2}\int_{-E_2/2}^{E_2/2} |g|^2 dx_1 dx_2 < \infty, \tag{26}$$

and we say that g has finite power (energy per domain unit). Analogously, the *correlation function* is given by

$$\phi_{gh}(s_1, s_2) = \lim_{E_1\to\infty} \lim_{E_2\to\infty} \frac{1}{E_1 E_2} \int_{-E_1/2}^{E_1/2} \int_{-E_2/2}^{E_2/2} g^*\left(x_1', x_2'\right) h\left(x_1' + s_1, x_2' + s_2\right) \times dx_1' dx_2', \tag{27}$$

but the Fourier transforms of the functions, g and h, do not exist in the usual way (as in Eq. (23)). On the other hand, the correlation function is square integrable and, therefore, possesses a Fourier transform, that is, the PSD of g and h:

$$\Phi_{gh}(f_1, f_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{gh}(s_1, s_2)\, e^{-i2\pi(f_1 s_1 + f_2 s_2)} ds_1 ds_2. \tag{28}$$

Consider truncated functions defined on a finite domain:

$$\bar{g}(x_1, x_2) = \begin{cases} g(x_1, x_2), \; x_1 \in [-E_1/2, E_1/2] \text{ and } x_2 \in [-E_2/2, E_2/2] \\ 0 \qquad\qquad\qquad\qquad \text{otherwise} \end{cases} \tag{29}$$

and similarly for $\bar{h}$. Then $\bar{g}$ and $\bar{h}$ are square integrable on the plane and have Fourier transforms, $\bar{G}$ and $\bar{H}$, respectively; e.g.,

$$\bar{G}(f_1, f_2) = \int_{-E_1/2}^{E_1/2} \int_{-E_2/2}^{E_2/2} g(x_1, x_2)\, e^{-i2\pi(f_1 x_1 + f_2 x_2)} dx_1 dx_2. \tag{30}$$

It is now straightforward to show that in this case, the Fourier transform of ϕ_{gh} is given by

$$\Phi_{gh}(f_1, f_2) = \lim_{E_1\to\infty} \lim_{E_2\to\infty} \frac{1}{E_1 E_2} \bar{G}^*(f_1, f_2)\, \bar{H}(f_1, f_2). \tag{31}$$

In practice, this *power spectral density* can only be approximated due to the required limit operators. However, the essential relationship between (truncated) function spectra and the PSD is once more evident.

2.3 From the Sphere to the Plane

For each class of functions on the plane, we did not need to impose isotropy on the correlation function. However, isotropy proves useful in comparisons to the spherical correlation function at high spatial frequencies. In the case of the nonperiodic functions on the plane, a simple averaging over azimuth changes the Fourier transform of the correlation function to its Hankel transform:

$$\Phi_{gh}(f) = 2\pi \int_0^\infty \phi_{gh}(s) s\, J_0(2\pi f s)\, ds, \;\; \phi_{gh}(s) = 2\pi \int_0^\infty \Phi_{gh}(f) f\, J_0(2\pi f s)\, df, \tag{32}$$

where $s = \sqrt{s_1^2 + s_2^2}$ and $f = \sqrt{f_1^2 + f_2^2}$, and J_0 is the zero-order Bessel function of the first kind.

An approximate relation between the transforms of the planar and spherical isotropic correlation functions follows from the asymptotic relationship between Legendre polynomials and Bessel functions:

$$\lim_{n\to\infty} P_n\left(\cos\frac{x}{n}\right) = J_0(x)\,, \quad \text{for } x > 0. \tag{33}$$

If we let $x = 2\pi f\, s$, where $s = R\psi$, and R is the radius of the sphere, then with $2\pi f \approx n/R$, we have $x/n = \psi$. Hence, for large n (or small ψ),

$$P_n(\cos\psi) \approx J_0(2\pi f\, s)\,. \tag{34}$$

Now, discretizing the second of Eqs. (32) (with $df = 1/(2\pi R)$) and substituting Eq. (33) yields (again, with $2\pi f \approx n/R$)

$$\phi_{gh}(s) \approx \sum_{n=0}^{\infty} \frac{n}{2\pi R^2} \Phi_{gh}\left(\frac{n}{2\pi R}\right) P_n\left(\cos\frac{s}{R}\right). \tag{35}$$

Comparing this with the spherical correlation function, Eq. (10), we see that

$$(2n+1)\left(\Phi_{gh}\right)_n \approx \frac{n}{2\pi R^2}\Phi_{gh}(f)\,, \quad \text{where } f \approx \frac{n}{2\pi R}. \tag{36}$$

This relationship between planar and spherical PSDs holds only for *isotropic correlation functions* and for large n or f.

2.4 Properties of Correlation Functions and PSDs

Correlation functions satisfy certain properties that should then also hold for corresponding models and may aid in their development. The autocorrelation is a *positive definite* function, since its eigenvalues defined by its spectrum, the PSD, are positive; e.g., see Eq. (14) or from Eq. (31):

$$\Phi_{gg}(f_1, f_2) = \lim_{E_1 \to \infty} \lim_{E_2 \to \infty} \frac{1}{E_1 E_2} \left| \bar{G}(f_1, f_2) \right|^2 \geq 0. \tag{37}$$

The values of the autocorrelation function for nonzero argument are not greater than at the origin:

$$\phi_{gg}(\psi) \leq \phi_{gg}(0), \quad \psi > 0; \qquad \phi_{gg}(s_1, s_2) \leq \phi_{gg}(0, 0), \quad \sqrt{s_1^2 + s_2^2} > 0; \tag{38}$$

where equality would imply a perfectly correlated function (a constant). The inequalities (38) are proved using Schwartz's inequality applied to the Eqs. (6) and (24), respectively. Note that cross correlations may be larger in absolute value than their values at the origin (e.g., if they vanish there).

Because of the imposed *isotropy*, spherical correlation functions are not defined for $\psi < 0$. On the other hand, planar correlation functions may be formulated for all quadrants; and, they satisfy:

$$\phi_{gh}(-s_1, s_2) = \phi_{hg}^{*}(s_1, -s_2), \tag{39}$$

which follows readily from their definition, given by Eqs. (24) or (27). Clearly, the autocorrelation function of a real function is symmetric with respect to the origin, even if not isotropic.

The correlation function of a derivative is the *derivative of the correlation*. For finite energy functions, we find immediately from Eq. (24) that

$$\begin{aligned} \frac{\partial}{\partial s_k} \phi_{gh}(s_1, s_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^{*}\left(x_1', x_2'\right) \frac{\partial h}{\partial s_k} \left(x_1' + s_1, x_2' + s_2\right) dx_1' dx_2' \\ &= \phi_{g, \frac{\partial h}{\partial x_k}}(s_1, s_2), \quad k = 1, 2. \end{aligned} \tag{40}$$

From this and Eq. (39), we also have

$$\begin{aligned} \frac{\partial}{\partial s_k} \phi_{gh}(s_1, s_2) = \frac{\partial}{\partial s_k} \phi_{hg}^{*}(-s_1, -s_2) = -\phi_{h, \frac{\partial g}{\partial x_k}}^{*}(-s_1, -s_2) = -\phi_{\frac{\partial g}{\partial x_k}, h}(s_1, s_2), \\ k = 1, 2. \end{aligned} \tag{41}$$

The minus sign may be eliminated with the definition of s_k, Eqs. (19). We have $\partial / \partial s_k = \partial / \partial x_k^{(h)} = -\partial / \partial x_k^{(g)}$, where $x_k^{(g)}$ and $x_k^{(h)}$ refer, respectively, to the coordinates of g and h. Therefore,

$$\begin{aligned} \phi_{\frac{\partial g}{\partial x_k^{(g)}}, h}(s_1, s_2) &= \frac{\partial}{\partial x_k^{(g)}} \phi_{gh}(s_1, s_2), \\ \phi_{g, \frac{\partial h}{\partial x_k^{(h)}}}(s_1, s_2) &= \frac{\partial}{\partial x_k^{(h)}} \phi_{gh}(s_1, s_2). \end{aligned} \tag{42}$$

The same results may be shown for correlation functions of other types of functions on the plane (where the derivations in the case of the limit operators require a bit more care).

Higher-order derivatives follow naturally, and indeed, we see that the correlation function of any linear operators on functions, $L^{(g)}g$ and $L^{(h)}h$, is the combination of these linear operators applied to the correlation function:

$$\phi_{L^{(g)}g,L^{(h)}h} = L^{(g)}\left(L^{(h)}\phi_{gh}\right). \tag{43}$$

Independent variables are omitted since this property, known as the *law of propagation of correlations*, holds also for the spherical case.

The PSDs of derivatives of functions on the plane follow directly from the inverse transform of the correlation function:

$$\phi_{gh}(s_1,s_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{gh}(f_1,f_2)\, e^{i2\pi(f_1s_1+f_2s_2)}df_1df_2. \tag{44}$$

With Eqs. (42), we find

$$\phi_{\frac{\partial g}{\partial x_k^{(g)}},\frac{\partial h}{\partial x_k^{(h)}}}(s_1,s_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{gh}(f_1,f_2)\,\frac{\partial^2}{\partial x_k^{(g)}\partial x_k^{(h)}} e^{i2\pi(f_1s_1+f_2s_2)}df_1df_2. \tag{45}$$

From this (and Eqs. (19)) one may readily infer the following general formula for the PSD of the derivatives of g and h of any order:

$$\Phi_{g_{p_1p_2},h_{q_1q_2}}(f_1,f_2) = (-1)^{p_1+p_2}(i2\pi f_1)^{p_1+q_1}(i2\pi f_2)^{p_2+q_2}\,\Phi_{gh}(f_1,f_2), \tag{46}$$

where $g_{p_1p_2} = \partial^{p_1+p_2}g\big/\left(\partial x_1^{p_1}\partial x_2^{p_2}\right)$ and $h_{q_1q_2} = \partial^{q_1+q_2}h\big/\left(\partial x_1^{q_1}\partial x_2^{q_2}\right)$. These expressions could be obtained also through Eqs. (21), (25), or (31), from the spectra of the function derivatives, which have a corresponding relationship to the spectra of the functions.

For functions on the sphere, the situation is hardly as simple. Indeed, this writer is unaware of formulas for the PSDs of horizontal derivatives, with the exception of an approximation for the average horizontal derivative,

$$d_Hg(\theta,\lambda) = \sqrt{\left(\frac{\partial g}{\partial\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{\partial g}{\partial\lambda}\right)^2}. \tag{47}$$

Making use of an orthogonality proved by Jeffreys (1955):

$$\frac{1}{4\pi}\iint_{\sigma}\left(\frac{\partial \bar{Y}_{n,m}}{\partial \theta}\frac{\partial \bar{Y}_{p,q}}{\partial \theta}+\frac{1}{\sin^2\theta}\frac{\partial \bar{Y}_{n,m}}{\partial \lambda}\frac{\partial \bar{Y}_{p,q}}{\partial \lambda}\right)d\sigma=\begin{cases} n\,(n+1)\,, & n=p \text{ and } m=q \\ 0, & n\neq p \text{ or } m\neq q \end{cases} \tag{48}$$

the autocorrelation function of $d_H g$ at $\psi = 0$ from Eq. (9) becomes

$$\begin{aligned}\phi_{d_H g, d_H g}(0) &= \frac{1}{4\pi}\iint_{\sigma}\left(\left(\frac{\partial g}{\partial \theta}\right)^2+\left(\frac{1}{\sin\theta}\frac{\partial g}{\partial \lambda}\right)^2\right)\sin\theta\, d\theta\, d\lambda \\ &= \sum_{n=0}^{\infty} n\,(n+1)\sum_{m=-n}^{n} G_{nm}^2. \end{aligned} \tag{49}$$

It is tempting to identify the PSD by comparing this result to Eq. (10), but Eq. (49) proves this form of the correlation function only for $\psi = 0$. The error in this approximation of the PSD of $d_H g$ is an open question.

For functions, $\widehat{g}\,(x_1, x_2; z)$, that satisfy *Laplace's equation*, $\nabla^2 \widehat{g} = 0$, in the space exterior to the plane (i.e., they are harmonic for $z > 0$) and satisfy the boundary condition, $\widehat{g}\,(x_1, x_2; 0) = g\,(x_1, x_2)$, the Fourier spectrum on any plane with $z = z_0 > 0$ is related to the spectrum of g:

$$\widehat{G}\,(f_1, f_2; z_0) = G\,(f_1, f_2)\, e^{-2\pi f z_0}, \tag{50}$$

where $f^2 = f_1^2 + f_2^2$. Similarly, for functions, $\widehat{g}\,(\theta, \lambda; r)$, harmonic outside the sphere $(r > R)$ that satisfy $\widehat{g}\,(\theta, \lambda; R) = g\,(\theta, \lambda)$, the Legendre spectrum on any sphere with $r = r_0 > R$ is related to the spectrum of g according to

$$\widehat{G}_{n,m}\,(r_0) = \left(\frac{R}{r_0}\right)^{n+1} G_{n,m}. \tag{51}$$

Therefore, the corresponding spectral densities are analogously related. In general, the cross PSD of g at level, $z = z_g$, and h at level, $z = z_h$, is given (e.g., substituting Eq. (50) for $\widehat{g}$ and $\widehat{h}$ into Eq. (31)) by

$$\Phi_{\widehat{g}\widehat{h}}\left(f_1, f_2; z_g, z_h\right) = e^{-2\pi f \left(z_g + z_h\right)}\, \Phi_{gh}\,(f_1, f_2)\,. \tag{52}$$

Note that the altitudes add in the exponent. Similarly, for cross PSDs of functions on spheres, $r = r_g$ and $r = r_h$, we have

$$\left(\Phi_{\widehat{g}\widehat{h}}\left(r_g, r_h\right)\right)_n = \left(\frac{R^2}{r_g r_h}\right)^{n+1} \left(\Phi_{gh}\right)_n. \tag{53}$$

Although the altitude variables were treated strictly as parameters in these PSDs, one may consider briefly the corresponding correlation functions as "functions" of z and r, respectively, for the sole purpose of deriving the correlation functions of vertical (radial) derivatives. Indeed, it is readily seen from the definitions, Eqs. (9) and (27), for the cross correlation of $\widehat{g}\left(\theta,\lambda;r_g\right)$ and $\widehat{h}\left(\theta,\lambda;r_h\right)$ that

$$\phi_{\frac{\partial\widehat{g}}{\partial r_g},\frac{\partial\widehat{h}}{\partial r_h}}\left(\psi;r_g,r_h\right)=\frac{\partial^2}{\partial r_g\partial r_h}\phi_{\widehat{g}\widehat{h}}\left(\psi;r_g,r_h\right),$$
$$\phi_{\frac{\partial\widehat{g}}{\partial z_g},\frac{\partial\widehat{h}}{\partial z_h}}\left(s_1,s_2;z_g,z_h\right)=\frac{\partial^2}{\partial z_g\partial z_h}\phi_{\widehat{g}\widehat{h}}\left(s_1,s_2;z_g,z_h\right),\tag{54}$$

and the law of propagation of correlations, Eq. (43), holds also for this linear operator. It should be stressed, however, that the *correlation function* is essentially a function of variables on the plane or sphere; no integration of products of functions takes place in the third dimension.

The cross PSDs of vertical derivatives, therefore, are easily derived by applying Eqs. (54) to the inverse transforms of the correlation functions, Eqs. (44) and (10), with extended expressions for the PSDs, Eqs. (52) and (53). The result is

$$\Phi_{\widehat{g}_{z_g^j}\widehat{h}_{z_h^k}}\left(f_1,f_2;z_g,z_h\right)=\frac{\partial^{j+k}}{\partial z_g^j\partial z_h^k}\left(e^{-2\pi f\left(z_g+z_h\right)}\right)\Phi_{gh}\left(f_1,f_2\right)$$
$$=\left(-2\pi f\right)^{j+k}e^{-2\pi f\left(z_g+z_h\right)}\Phi_{gh}\left(f_1,f_2\right),\tag{55}$$

$$\left(\Phi_{\widehat{g}_{r_g^j}\widehat{h}_{r_h^k}}\left(r_g,r_h\right)\right)_n=\frac{\partial^{j+k}}{\partial r_g^j\partial r_h^k}\left(\frac{R^2}{r_gr_h}\right)^{n+1}\left(\Phi_{gh}\right)_n,\tag{56}$$

where $\widehat{g}_{z_g^j}=\partial^j\widehat{g}/\partial z_g^j$, $\widehat{h}_{z_h^k}=\partial^k\widehat{h}/\partial z_h^k$, $\widehat{g}_{r_g^j}=\partial^j\widehat{g}/\partial r_g^j$, and $\widehat{h}_{r_h^k}=\partial^k\widehat{h}/\partial r_h^k$. Thus, the PSD for any combination of horizontal and vertical derivatives of g and h on horizontal planes in Cartesian space may be obtained by appending the appropriate factors to Φ_{gh}. The same holds for any combination of vertical derivatives of g and h on concentric spheres.

3 Stochastic Processes and Covariance Functions

A *stochastic* (or *random*) *process* is a collection, discrete or continuous, of random variables that are associated with a deterministic variable, in our case, a point on the plane or sphere. At each point, the process is random with an underlying probability distribution. A probability domain or sample space for each random variable is implied but omitted in the following simplified notation; in fact, the distribution may be unknown. If each random variable takes on a specific value from the corresponding sample spaces, then the process is said to be realized, and this realization is a function of the point coordinates. Thus, we continue to use the notation, g, to represent a continuous stochastic process, with the understanding that for any fixed point, it is a random variable.

We assume that the process is wide-sense stationary, meaning that all statistics up to second order are invariant with respect to the origin of the space variable. Then, the expectation of g at all points is the same constant, and the covariance between g at any two points depends only on the displacement (vector) of one point relative to the other.

Typically, besides not knowing the probability distribution, we have access only to a single realization of the stochastic process, which makes the estimation of essential statistics such as the mean and covariance problematic, unless we invoke an additional powerful condition characteristic of many processes – *ergodicity*. For ergodic processes the statistics associated with the underlying probability law, based on the statistical (*ensemble*) expectation, are equivalent to the statistics derived from *space-based* averages of a single realization of the process. *Stationarity* is necessary but not sufficient for ergodicity. Also, we consider only wide-sense ergodicity. It can be shown that stationary stochastic processes whose underlying probability distribution is Gaussian is also ergodic (Moritz 1980; Jekeli 1991). We do not need this result since the probability distribution is not needed in our developments; and, indeed, ergodic processes on the sphere cannot be Gaussian (Lauritzen 1973).

For stochastic processes on the sphere, we define the space average as

$$M\left(\cdot\right)=\frac{1}{4\pi}\iint\limits_{\sigma}\left(\cdot\right)d\sigma. \tag{57}$$

Let g and h be two such processes that are ergodic (hence, also stationary) and let their means, according to Eq. (57), be denoted μ_g and μ_h. Then, the covariance function of g and h is given by

$$M\left(\left(g-\mu_g\right)\left(h-\mu_h\right)\right)=\frac{1}{4\pi}\iint\limits_{\sigma}\left(g\left(\theta,\lambda\right)-\mu_g\right)\left(h\left(\theta',\lambda'\right)-\mu_h\right)\sin\theta d\theta d\lambda, \tag{58}$$

which, by the stationarity, depends only on the relative location of g and h, that is, on (ψ,α), as given by Eqs. (7) and (8). We will assume without loss in generality that the means of the processes are zero (if not, redefine the process with its constant mean removed). Then, clearly, the covariance function is like the correlation function, Eq. (6), except the interpretation is for stochastic processes. We continue to use the same notation, however, and further redefine the covariance function to be isotropic by including an average in azimuth, α, as in Eq. (9). The Legendre transform of the covariance function is also the (cross) PSD of g and h and is given by Eq. (13). The quantities,

$$\left(c_{gh}\right)_n = (2n+1)\left(\Phi_{gh}\right)_n, \tag{59}$$

are known as *degree variances*, or variances per degree, on account of the total variance being, from Eq. (10),

$$\phi_{gh}(0) = \sum_{n=0}^{\infty}\left(c_{gh}\right)_n. \tag{60}$$

Ergodic processes on the plane are not square integrable since they are also stationary, and we define the average operator as

$$M(\cdot) = \lim_{E_1\to\infty}\lim_{E_2\to\infty}\frac{1}{E_1E_2}\int_{-E_1/2}^{E_1/2}\int_{-E_2/2}^{E_2/2}(\cdot)\,dx_1dx_2. \tag{61}$$

The *covariance function* under the assumption of zero means for g and h is, again, the correlation function given by Eq. (27). However, the PSD requires some additional derivation since the truncated stochastic processes, $\bar{g}$ and $\bar{h}$, defined as in Eq. (29), are not stationary and, therefore, not ergodic.

Since both $\bar{g}$ and $\bar{h}$ are random for each space variable, their Fourier transforms, $\bar{G}$ and $\bar{H}$, are also stochastic in nature. Consider first the ensemble expectation of the product of transforms, given by Eq. (30),

$$\mathcal{E}\left(\bar{G}^*(f_1,f_2)\,\bar{H}(f_1,f_2)\right) = \left(\int_{-E_1/2}^{E_1/2}\int_{-E_2/2}^{E_2/2}\int_{-E_1/2}^{E_1/2}\int_{-E_2/2}^{E_2/2}\mathcal{E}\left(g\left(x_1',x_2'\right)h(x_1,x_2)\right)\right.$$
$$\left. e^{i2\pi\left(f_1\left(x_1'-x_1\right)+f_2\left(x_2'-x_2\right)\right)}dx_1dx_2dx_1'dx_2'\right) \tag{62}$$

The expectation inside the integrals is the same as the space average and is the *covariance function* of g and h, as defined above, which because of their stationarity depends only on the coordinate differences, $s_1 = x_1 - x_1'$ and $s_2 = x_2 - x_2'$. It can be shown Brown (1983, p. 86) that the integrations reduce to

$$\mathcal{E}\left(\bar{G}^*(f_1,f_2)\,\bar{H}(f_1,f_2)\right) =$$
$$E_1E_2\left(\int_{-E_1}^{E_1}\int_{-E_2}^{E_2}\left(1-\frac{|s_1|}{2E_1}\right)\left(1-\frac{|s_2|}{2E_2}\right)\phi_{gh}(s_1,s_2)\,e^{-i2\pi(f_1s_1+f_2s_2)}ds_1ds_2\right). \tag{63}$$

In the limit, the integrals on the right side approach the Fourier transform of the covariance function, that is, the PSD, $\Phi_{gh}(f_1,f_2)$; and, we have

$$\Phi_{gh}(f_1, f_2) = \lim_{E_1 \to \infty} \lim_{E_2 \to \infty} \mathcal{E}\left(\frac{1}{E_1 E_2} \bar{G}^*(f_1, f_2)\, \bar{H}(f_1, f_2) \right). \tag{64}$$

Again, in practice, this PSD can only be approximated due to the limit and expectation operators.

We have shown that under appropriate assumptions (ergodicity), the covariance functions of stochastic processes on the sphere or plane are essentially identical to the corresponding correlation functions that were developed without a stochastic foundation. The only exception occurs in the relationship between Fourier spectra and the (Fourier) PSD (compare Eqs. (31) and (64)). Furthermore, from Eqs. (62) through (64) we have also shown that the covariance function of a stochastic process is the Fourier transform of the PSD, given by Eq. (64). This is a statement of the more general *Wiener-Khintchine theorem* (Priestley 1981).

Although there are opposing schools of thought as to the stochastic nature of a field like Earth's gravitational potential, we will argue (see below) that the stochastic interpretation is entirely legitimate. Moreover, the *stochastic interpretation* of the gravitational field is widely, if not uniformly, accepted in geodesy (e.g., Moritz 1978, 1980; Hofmann-Wellenhof and Moritz 2005), as is the covariance nomenclature. Moritz (1980) provided compelling justifications to view the gravitational field as a stochastic process on the plane or sphere. The use of covariance functions also emphasizes that the significance of correlations among functions lies in their variability irrespective of the means (which we will always assume to be zero). For these reasons, we will henceforth in our applications to the Earth's gravitational field refer only to covariance functions, use the same notation, and use all the properties and relationships derived for correlation functions.

3.1 Earth's Anomalous Gravitational Field

The masses of the Earth, including all material below its surface, as well as the atmosphere, generate the gravitational field, which in vacuum is harmonic and satisfies Laplace's differential equation. For present purposes we neglect the atmosphere (and usually its effect is removed from data) so that for points, x, above the surface, the gravitational potential, V, fulfills Laplace's equation,

$$\nabla^2 V(x) = 0. \tag{65}$$

Global solutions to this equation depend on boundary values of V or its derivatives on some mathematically convenient bounding surface. Typically this surface is a sphere with radius, a, and the solution is then expressed in spherical polar coordinates, (r, θ, λ), as an infinite series of solid spherical harmonic functions, $\bar{Y}_{n,m}(\theta, \lambda) \big/ r^{n+1}$, for points outside the sphere:

$$V(r,\theta,\lambda) = \frac{GM}{a}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(\frac{a}{r}\right)^{n+1} C_{n,m}\bar{Y}_{n,m}(\theta,\lambda), \tag{66}$$

where GM is Newton's gravitational constant times the total mass of the Earth (this scale factor is determined from satellite tracking data); and $C_{n,m}$ is a coefficient of degree, n, and order, m, determined from V and/or its radial derivatives on the bounding sphere (obtained, e.g., from measurements of gravity). Modern solutions also make use of satellite tracking data and in situ measurements of the field by satellite-borne instruments to determine these coefficients.

In a coordinate system fixed to the Earth, we define the *gravity potential* as the sum of the gravitational potential, V, due to mass attraction and the (nongravitational) potential, φ, whose gradient equals the centrifugal acceleration due to Earth's rotation:

$$W(x) = V(x) + \varphi(x). \tag{67}$$

If we define a normal (i.e., reference) gravity potential, $U = V^{ellip} + \phi$, associated with a corotating material ellipsoid, such that on this ellipsoid, $U|_{x\in\text{ellip}} = U_0$, then the difference, called the *disturbing potential*,

$$T(x) = W(x) - U(x), \tag{68}$$

is also a harmonic function in free space and may be represented as a spherical harmonic series:

$$T(r,\theta,\lambda) = \frac{GM}{a}\sum_{n=2}^{\infty}\sum_{m=-n}^{n}\left(\frac{a}{r}\right)^{n+1} \delta C_{n,m}\bar{Y}_{n,m}(\theta,\lambda), \tag{69}$$

where the $\delta C_{n,m}$ are coefficients associated with the difference, $V - V^{ellip}$. The total ellipsoid mass is set equal to the Earth's total mass, so that $\delta C_{0,0} = 0$; and, the coordinate origin is placed at the center of mass of the Earth (and ellipsoid), implying that the first moments of the mass distribution all vanish: $\delta C_{1,m} = 0$ for $m = -1, 0, 1$.

The set of spherical harmonic coefficients, $t_{n,m} = (GM/a)\,\delta C_{n,m}$, represents the *Legendre spectrum* of T. Practically, it is known only up to some finite degree, $n_{\max}$; for example, the model, EGM2008, has $n_{\max} = 2{,}190$ (Pavlis et al. 2012a,b). The harmonic coefficients of this model refer to a sphere of boundary values whose radius is equated with the semimajor axis of the best-fitting Earth ellipsoid. The uniform convergence of the infinite series, Eq. (69), is guaranteed for $r \geq a$, but effects of divergence are evident in the truncated series, EGM2008, when $r < a$, and due care should be exercised in evaluations on or near the Earth's surface.

The *disturbing potential* may also be defined with respect to higher-degree reference potentials, although in this case one may need to account for significant errors in the coefficients, $C_{n,m}^{\mathrm{ref}}$. In particular, the local interpretation of the field as a stationary random process usually requires removal of a higher-degree reference field.

In the Cartesian formulation, the disturbing potential in free space ($z \geq 0$) is expressed in terms of its Fourier spectrum, $\tau(f_1, f_2)$, on the plane, $z = 0$, as

$$T(x_1, x_2; z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tau(f_1, f_2)\, e^{-2\pi f z} e^{i2\pi(f_1 x_1 + f_2 x_2)} df_1 df_2. \tag{70}$$

where $f = \sqrt{f_1^2 + f_2^2}$.

3.2 The Disturbing Potential as a Stochastic Process

In addition to the well-grounded reasoning already cited, an alternative justification of the stochastic nature of T is argued here based on the fractal (self-similar) characteristics of Earth's topography (see also Turcotte 1987). This will provide also a basis for modeling the covariance function of T and its derivatives. The *fractal* geometry of the Earth's topography (among fractals in general) was investigated and popularized by Mandelbrot in a number of papers and reviewed in his book (Mandelbrot 1983) using fundamentally the concept of Brownian motion, which is the process of a random walk. Thus, without going into the details of fractals, we have at least a connection between topography and randomness. Next, we may appeal to the well-known (in physical geodesy and geophysics) high degree of linear correlation between gravity anomalies and topographic height. This correlation stems from the theory of *isostasy* that explains the existence of topography on the Earth whose state generally tends toward one of hydrostatic equilibrium. Although this correlation is not perfect (or almost nonexistent in regions of tectonic subsidence and rifting), empirical evidence suggests that in many areas the correlation is quite faithful to this theory, even with a number of seemingly crude approximations.

The gravity anomaly, Δg, and its *isostatic reduction* are defined in Hofmann-Wellenhof and Moritz (2005). At a point, P, the isostatically reduced gravity anomaly is given by

$$\Delta g_I(P) = \Delta g(P) - C(P) + A(P), \tag{71}$$

where $C(P)$ is the gravitational effect of all masses above the geoid and $A(P)$ is the effect of their isostatic compensation. Several models for isostatic compensation have been developed by geophysicists (Watts 2001). *Airy's model* treats the compensation locally and assumes that there is no regional flexural rigidity in the lithosphere. With this model, the *topography* presumably floats in the denser mantle, and equilibrium is established according to the buoyancy principle (Fig. 1):

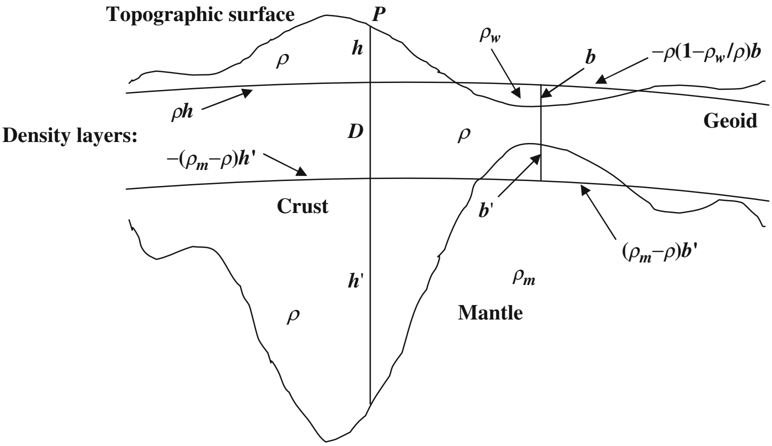

Fig. 1 Isostatic compensation of topography according to the Airy model

$$\rho h = (\rho_m - \rho)\, h' = \Delta\rho h', \tag{72}$$

where h' is the (positive) depth of the "root" with respect to the depth of compensation, D (typically, $D = 30\,\text{km}$), and the crust density, ρ, and the mantle density, ρ_m, are assumed constant. Similarly, in ocean areas, the lower density of water relative to the crust allows the mantle to intrude into the crust, where equilibrium is established if $(\rho - \rho_w)\, b = \Delta\rho b'$, and b is the (positive) bathymetric distance to the ocean floor, b' is the height of the "anti-root" of mantle material, and ρ_w is the density of seawater.

Removing the mass that generates $C\,(P)$ makes the space above the geoid homogeneous (empty). According to Airy's model, the attraction, $A\,(P)$, is due, in effect, to adding that mass to the root so as to make the mantle below D homogeneous. If the isostatic compensation is perfect according to this model, then the isostatic anomaly would vanish because of this created homogeneity; and, indeed, isostatic anomalies tend to be small. Therefore, the free-air gravity anomaly according to Eq. (71) with $\Delta g_I\,(P) \approx 0$ is generated by the attraction due to the topographic masses above the geoid, with density, ρ, and by the attraction due to the lack of mass below the depth of compensation, with density, $-\Delta\rho$:

$$\Delta g\,(P) \approx C\,(P) - A\,(P)\,. \tag{73}$$

Expressions for the terms on the right side can be found using various approximations. One such approximation (Forsberg 1985) "condenses" the topography onto the *geoid* (Helmert condensation, Helmert 1884; Martinec 1998), and the gravitational effect is then due to a two-dimensional mass layer with density, $\kappa_H = \rho h$. Likewise, the gravitational effect of the ocean bottom topography can be modeled by forming a layer on the geoid that represents the ocean's *deficiency* in density relative to the crust. The density of this layer is negative: $\kappa_B = -(\rho - \rho_w)\, b = -\rho\,(1 - \rho_w/\rho)\, b$. The gravitational potential, v, at a point, P, due to a layer condensed from topography (or bathymetry) is given by

$$v\,(P) = G\rho R^2 \iint\limits_{\sigma_Q} \frac{\bar{h}\,(Q)}{\ell} d\sigma_Q, \qquad \bar{h}\,(Q) = \begin{cases} h\,(Q)\,, & Q \in \text{land} \\ -\left(1 - \frac{\rho_w}{\rho}\right) b\,(Q)\,, & Q \in \text{ocean} \end{cases} \tag{74}$$

where ℓ is the distance between P and the integration point.

Similarly, the potential of the mass added below the depth of compensation can be approximated by that of another layer at level D with density, $\kappa'_H = -\Delta\rho h'$, representing a condensation of material that is *deficient* in density with respect to the mantle and extends a depth, h', below D (see Fig. 1). For ocean areas, the anti-root is condensed onto the depth of compensation with density, $\kappa'_B = \Delta\rho b'$.

Equation (74) for a fixed height of the point, P, is a convolution of h and the inverse distance. Further making the planar approximation (for local, or high-frequency applications), this distance is $\ell \approx \sqrt{\left(x_1 - x'_1\right)^2 + \left(x_2 - x'_2\right)^2 + z^2}$, with $\left(x'_1, x'_2\right)$ being the planar coordinates of point Q. Applying the convolution theorem, the Fourier transform of the potential at the level of $z > 0$ is given by

$$V\,(f_1, f_2; z) = \frac{G\rho}{f} \bar{H}\,(f_1, f_2)\, e^{-2\pi f z}. \tag{75}$$

Including the layer at the compensation depth, D, below the geoid with density, $\kappa'_H = -\rho h$ (in view of Eqs. (72); and similarly $\kappa'_B = \rho\,(1 - \rho_w/\rho)\, b$ for ocean areas), the Fourier transform of the total potential due to both the *topography* and its *isostatic compensation* is approximately

$$V\,(f_1, f_2; z) = \frac{G\rho}{f} \bar{H}\,(f_1, f_2) \left(e^{-2\pi f z} - e^{-2\pi f (D+z)}\right). \tag{76}$$

Since the gravity anomaly is approximately the radial derivative of this potential, multiplying by $2\pi f$ yields its Fourier transform:

$$\Delta G\,(f_1, f_2; z) = 2\pi G\rho \bar{H}\,(f_1, f_2) \left(e^{-2\pi f z} - e^{-2\pi f (D+z)}\right). \tag{77}$$

Neglecting the upward continuation term, as well as the isostatic term (which is justified only for very short wavelengths), confirms the empirical linear relationship between the heights and the gravity anomaly.

Figure 2 compares the PSDs of the topography and the gravitational field both globally and locally. The global PSDs were computed from *spherical harmonic expansions* EGM2008 for the gravitational potential and DTM2006 for the topography (Pavlis et al. 2012a) according to Eq. (14) but converted to spatial frequency

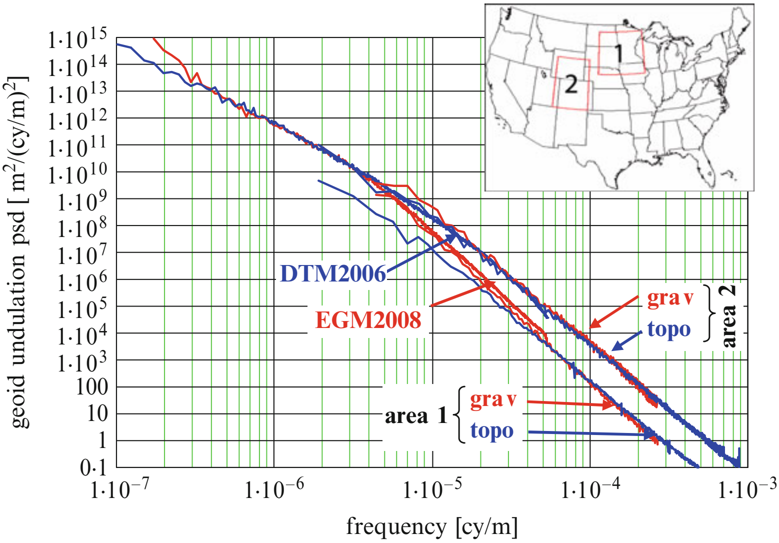

Fig. 2 Comparison of gravitational and topographic PSDs, scaled to the geoid undulation PSD. Global models are EGM2008 and DTM2006, and local PSDs were derived from gravity and topographic data in the indicated areas 1 and 2

using Eq. (36). In addition, both were scaled to the PSD of the *geoid undulation*, which is related to the disturbing potential as $N = T/\gamma$, where $\gamma = 9.8\,\mathrm{m/s^2}$ is an average value of gravity. The topographic height is related to the potential through Eq. (76). Both expansions are complete up to degree, $n_{\max} = 2{,}160$ ($f_{\max} = 5.4\times10^{-5}$ cy/m). DTM2006 is an expansion for both the topographic height above mean sea level and the depth of the ocean and, therefore, does not exactly correspond to $\bar{h}$, as defined in Eq. (74). This contributes to an overestimation of the power at lower frequencies. The obviously lower power of EGM2008 at higher frequencies results from the higher altitude, on average, to which its spectrum refers, that is, the sphere of radius, a.

The other PSDs in Fig. 2 correspond to the indicated regions and were derived according to Eq. (31) from local terrain elevation and gravity anomaly data provided by the US National Geodetic Survey. The data grids in latitude and longitude have resolution of 30 arcsec for the topography and 1 arcmin for the gravity. With a planar approximation for these areas, the Fourier transforms were calculated using their discrete versions. The PSDs were computed by neglecting the limit (and expectation) operators and were averaged in azimuth. Dividing the gravity PSD by $(2\pi\gamma f)^2$ then yields the geoid undulation PSD; and, as before, Eq. (76) relates the topography PSD to the potential PSD that scales to the geoid undulation PSD by $1/\gamma^2$. In these regions, the gravity and topography PSDs match well at the higher frequencies at least, attesting to their high linear correlation. Moreover, these PSDs follow a power law in accord with the presumed *fractal* nature of the topography. These examples then offer a validation of the stochastic interpretation of the gravitational field and also provide a starting point to model its covariance function.

4 Covariance Models

Since the true *covariance function* of a process, such as the Earth's gravity field, rarely is known and local functions can vary from region to region (thus we allow global non-stationarity in local applications), it must usually be modeled from data. We consider here primarily the modeling of the autocovariance function, that is, when $g = h$. Models for the cross covariance function could follow similar procedures, but usually g and h are linearly related and the method of propagation of covariances (see Sect. 2.4) should be followed to derive ϕ_{gh} from ϕ_{gg}.

Modeling the covariance (or correlation) function of a process on the plane or sphere can proceed with different assumptions and motivations. We distinguish in the first place between empirical and analytic methods and in the second between global and local models. Global models describe the correlation of functions on the sphere; whereas, local models usually are restricted to applications where a planar approximation suffices. *Empirical models* are derived directly from data distributed on the presumably spherical or planar surface of the Earth. Rarely, if ever, are global empirical covariance models determined for the sphere according to the principal definition, Eq. (9). Instead, such models are given directly by the degree variances, Eq. (59). For *local modeling* with the planar approximation, the empirical model comes from a discretization of Eq. (27), where we neglect also the limit processes,

$$\hat{\phi}_{gg}(s_1, s_2) = \frac{1}{M} \sum_{x_1'} \sum_{x_2'} g\left(x_1', x_2'\right) g\left(x_1' + s_1, x_2' + s_2\right), \tag{78}$$

and where M is the total number of summed products for each (s_1, s_2). A corresponding approximation for an isotropic model additionally averages the products of g that are separated by a given distance, $s = \sqrt{\left(x_1 - x_1'\right)^2 + \left(x_2 - x_2'\right)^2}$, over all directions. Typically, the maximum s considered is much smaller than (perhaps 10–20 % of) the physical dimension of the data area, since the approximation of Eq. (27) by Eq. (78) worsens as the number of possible summands within a finite area decreases. Also, M may be fixed at the largest possible value (the number of products for $s = 0$) in order to avoid a numerical non*positive definiteness* of the covariance function (Marple 1987, p. 148). However, this creates a biased estimate of the covariance, particularly for the larger distances.

Another form of empirical covariance model is its Fourier (or Legendre) transform, derived directly from the data, as was illustrated for the gravity anomaly and topography in Fig. 2. The inverse transform then yields immediately the covariance function. The disadvantage of the empirical covariance model, Eq. (78), is the limited ability (or inability) to derive consistent covariances of functionally related quantities, such as the derivatives of g through the law of propagation of covariances, Eq. (43). This could only be accomplished by working with its transform (see Eqs. (46)), but generally, an analytic model eventually simplifies the computational aspects of determining auto- and cross covariances.

4.1 Analytic Models

Analytic covariance models are constructed from relatively simple mathematical functions that typically are fit to empirical data (either in the spatial or frequency domains) and have the benefit of easy computation and additional properties useful to a particular application (such as straightforward propagation of covariances). An analytic model should satisfy all the basic properties of the covariance function (Sect. 2.4), although depending on the application some of these may be omitted (such as the harmonic extension into space for the Gaussian model, $\phi(s) = \sigma^2 e^{-\beta s^2}$). An analytic model may be developed for the PSD or the covariance function. Ideally (but not always), one leads to a mathematically rigorous formulation of the other.

Perhaps the most famous global analytic model is known as *Kaula's rule*, proposed by W. Kaula (1966, p. 98) in order to develop the idea of a stochastic interpretation of the spherical spectrum of the disturbing potential:

$$(\Phi_{TT})_n = \left(\frac{GM}{R}\right)^2 \frac{10^{-10}}{n^4}\ \mathrm{m}^4/\mathrm{s}^4, \tag{79}$$

where R is the mean Earth radius. It roughly described the attenuation of the harmonic coefficients known at that time from satellite tracking observations, but it is reasonably faithful to the spectral attenuation of the field even at high degrees (see Fig. 4). Note that Kaula's rule is a *power-law model* for the PSD of the geopotential, agreeing with our arguments above for such a characteristic based on the fractal nature of the topography.

The geodetic literature of the latter part of the last century is replete with different types of global and local covariance and PSD models for the Earth's residual gravity field (e.g., Jordan 1972; Tscherning and Rapp 1974; Heller and Jordan 1979; Forsberg 1987; Milbert 1991; among others); but, it is not the purpose here to review them. Rather the present aim is to promote a single elemental prototype model that (1) satisfies all the properties of a covariance model for a stochastic process, (2) has harmonic extension into free space, (3) has both spherical and planar analytic expressions for all derivatives of the potential in both the space and frequency domains, and (4) is sufficiently adaptable to any strength and attenuation of the gravitational field. This is the reciprocal distance model introduced by Moritz (1976, 1980), so called because the covariance function resembles an inverse-distance weighting function. It was also independently studied by Jordan et al. (1981).

4.2 The Reciprocal Distance Model

Consider the disturbing potential, T, as a stochastic process on each of two possibly different horizontal parallel planes or concentric spheres. Given a realization of T on one plane (or sphere), its realization on the other plane (or sphere) is well defined by a solution to Laplace's equation, provided both surfaces are on or outside the Earth's surface (approximated as a plane or sphere). The *reciprocal distance covariance model* between T on one plane and T on the other is given by

$$\phi_{TT}(s; z_1, z_2) = \frac{\sigma^2}{\sqrt{\alpha^2 s^2 + (1 + \alpha(z_1 + z_2))^2}}, \tag{80}$$

where with Eq. (19), $s = \sqrt{s_1^2 + s_2^2}$; z_1, z_2 are heights of the two planes; and σ^2, α are parameters. The Fourier transform, or the PSD, is given by

$$\Phi_{TT}(f; z_1, z_2) = \frac{\sigma^2}{\alpha f} e^{-2\pi f(z_1 + z_2 + 1/\alpha)}, \quad f \neq 0. \tag{81}$$

For spheres with radii, $r_1 \geq R$ and $r_2 \geq R$, the spherical covariance model is

$$\phi_{TT}(\psi; r_1, r_2) = \frac{\sigma^2 (1 - \rho_0)\, \rho/\rho_0}{\sqrt{1 + \rho^2 - 2\rho \cos\psi}}, \tag{82}$$

where ψ is given by Eq. (7), $\rho_0 = (R_0/R)^2$ and σ^2 are parameters, and $\rho = R_0^2 \big/ (r_1 r_2)$. The Legendre transform, or PSD, is given by

$$(\Phi_{TT})_n = \frac{\sigma^2 (1 - \rho_0)}{(2n + 1)\, \rho_0} \rho^{n+1}. \tag{83}$$

In all cases, the heights (or radii) refer to fixed surfaces that define the spatial domain of the corresponding stochastic process. Since we allow $z_1 \neq z_2$ or $r_1 \neq r_2$, the models, Eqs. (80) and (82), technically are cross covariances between two different (but related) processes; and Eqs. (81) and (83) are cross PSDs.

The equivalence of the models, Eqs. (80) and (82), as the spherical surface approaches a plane, is established by identifying $z_1 = r_1 - R$, $z_2 = r_2 - R$ and $s^2 \approx 2R^2 (1 - \cos\psi)$, from which it can be shown (Moritz 1980, p. 183) that

$$\frac{1}{\sqrt{1 + \rho^2 - 2\rho \cos\psi}} \approx \frac{R}{\sqrt{s^2 + (1/\alpha + z_1 + z_2)^2}}, \tag{84}$$

where $1/\alpha = 2(R - R_0)$ and terms of order $(R - R_0)/R$ are neglected. The variance parameter, σ^2, is the same in both versions of the model. It is noted that this model, besides having analytic forms in both the space and frequency domains, is isotropic, depending only on the horizontal distance. Moreover, it correctly incorporates the harmonic extension for the potential at different levels. It is also positive definite since the transform is positive for all frequencies.

The analytic forms permit exact propagation of covariances as elaborated in Sect. 2.4. Since many applications involving the stochastic interpretation of the field nowadays are more local than global, only the (easier) planar propagation is given here (Appendix A) up to second-order derivatives. The covariance propagation of derivatives for similar spherical models was developed by Tscherning (1976). Note that the covariances of the horizontal derivatives are not isotropic.

One further useful feature of the reciprocal distance model is that it possesses analytic forms for hybrid PSD/covariance functions, those that give the PSD in one dimension and the covariance in the other:

$$S_{TT}\left(f_1; s_2; z_1, z_2\right) = \int_{-\infty}^{\infty} \Phi_{TT}\left(f_1, f_2; z_1, z_2\right) e^{i2\pi f_2 s_2} df_2$$
$$= \int_{-\infty}^{\infty} \phi_{TT}\left(s_1, s_2; z_1, z_2\right) e^{-i2\pi f_2 s_2} ds_1 \tag{85}$$

The first integral transforms the PSD to the covariance in the second variable, while the second equivalent integral transforms the covariance function to the frequency domain in the first variable. When a process is given only on a single profile (e.g., along a data track), one may wish to model its *along-track PSD*, which is the hybrid PSD/covariance function with $s_2 = 0$. Appendix B gives the corresponding analytic forms for the (planar) reciprocal distance model.

4.3 Parameter Determination

The reciprocal distance PSD model, Eq. (81), clearly does not have the form of a *power law*, but it nevertheless serves in modeling the PSD of the gravitational field when a number of these models are combined linearly:

$$\Phi_{TT}\left(f; z_1, z_2\right) = \sum_{j=1}^{J} \frac{\sigma_j^2}{\alpha_j f} e^{-2\pi f\left(z_1+z_2+1/\alpha_j\right)}. \tag{86}$$

The parameters, α_j, σ_j^2, are chosen appropriately to yield a power-law attenuation of the PSD. This selection is based on the *empirical PSD* of data that in the case of the gravitational field are usually *gravity anomalies*, $\Delta g \approx -\partial T/\partial r$, on the Earth's surface ($z_1 = z_2 = 0$).

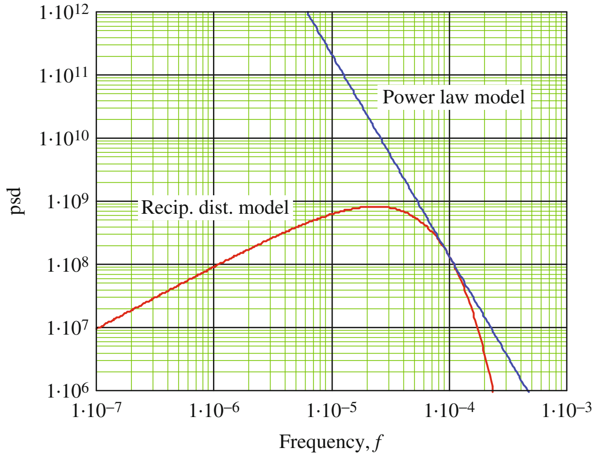

Fig. 3 Fitting a reciprocal distance model component to a power-law PSD

Multiplying the PSD for the disturbing potential by $(2\pi f)^2$, we consider reciprocal distance components of the PSD of the gravity anomaly (from Eq. (86)) in the form

$$\Phi(f) = Afe^{-Bf}, \tag{87}$$

where $A = (2\pi)^2 \sigma^2 / \alpha$ and $B = 2\pi/\alpha$ are constants to be determined such that the model is tangent to the empirical PSD. Here we assume that the latter is a power-law model (see Fig. 3),

$$p(f) = Cf^{-\beta}, \tag{88}$$

where the constants, C and β, are given. In terms of natural logarithms, the reciprocal distance PSD component is $\ln(\Phi(f)) = \ln(A) + \omega - Be^{\omega}$, where $\omega = \ln f$; and its slope is $d(\ln(\Phi(f)))/d\omega = 1 - Be^{\omega}$. The slope should be $-\beta$, which yields $Be^{\omega} = 1 + \beta$. Also, the reciprocal distance and power-law models should intersect, say, at $f = \bar{f}$, which requires $\ln(C) - \beta\omega = \ln(A) + \omega - Be^{\omega}$. Solving for A and B, we find:

$$A = C\left(\frac{e}{\bar{f}}\right)^{1+\beta}, \quad B = \frac{1+\beta}{\bar{f}}. \tag{89}$$

With a judicious selection of discrete frequencies, $\bar{f}_j$, a number of PSD components may be combined to approximate the power law over a specified domain. Due to the overlap of the component summands in Eq. (86), an appropriate scale factor may still be required for a proper fit.

This modeling technique was applied to the two regional PSDs shown in Fig. 2. Additional low-frequency components were added to model the field at frequencies,

Table 1 Reciprocal distance PSD parameters

Area 1						Area 2					
j	σ^2 (m^4/s^4)	α (1/m)	j	σ^2 (m^4/s^4)	α (1/m)	j	σ^2 (m^4/s^4)	α (1/m)	j	σ^2 (m^4/s^4)	α (1/m)
1	10^5	3×10^{-7}	9	1.59×10^{-4}	5.03×10^{-4}	1	10^5	3×10^{-7}	9	3.98×10^{-3}	5.47×10^{-4}
2	3,300	9.69×10^{-7}	10	9.97×10^{-6}	1.13×10^{-3}	2	3,300	9.69×10^{-7}	10	3.34×10^{-4}	1.23×10^{-3}
3	650	4.76×10^{-6}	11	6.26×10^{-7}	2.52×10^{-3}	3	640	7.56×10^{-6}	11	2.81×10^{-5}	2.74×10^{-3}
4	162	8.94×10^{-6}	12	3.93×10^{-8}	5.64×10^{-3}	4	951	9.73×10^{-6}	12	2.36×10^{-6}	6.14×10^{-3}
5	10.2	2.00×10^{-5}	13	2.47×10^{-9}	1.26×10^{-2}	5	79.9	2.18×10^{-5}	13	1.98×10^{-7}	1.37×10^{-2}
6	0.641	4.48×10^{-5}	14	1.55×10^{-10}	2.83×10^{-2}	6	6.71	4.88×10^{-5}	14	1.66×10^{-8}	3.08×10^{-2}
7	4.02×10^{-2}	1.00×10^{-4}	15	9.74×10^{-12}	6.33×10^{-2}	7	0.564	1.09×10^{-4}	15	1.40×10^{-9}	6.89×10^{-2}
8	2.53×10^{-3}	2.25×10^{-4}				8	4.74×10^{-2}	2.44×10^{-4}			

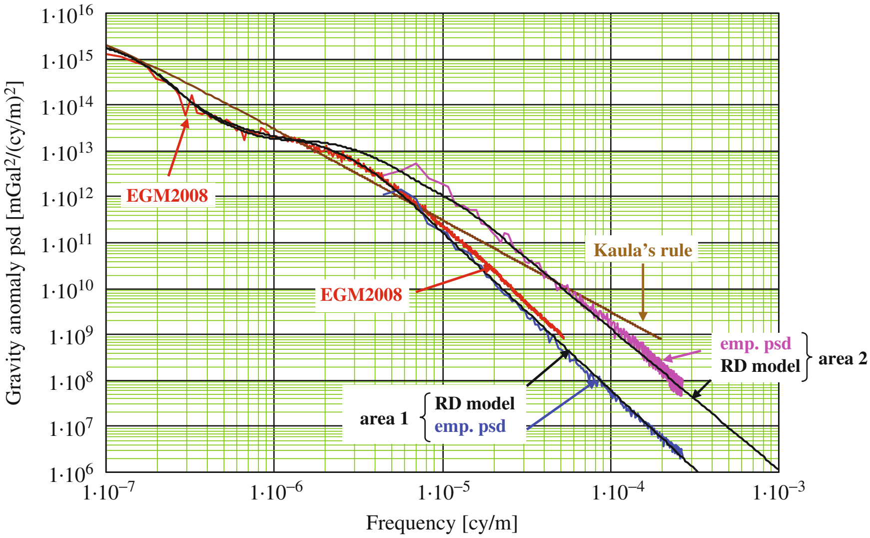

Fig. 4 Comparison of empirical and reciprocal distance (RD) model PSDs for the gravity anomaly in the two areas shown in Fig. 2

$f < 10^{-5}$ cy/m. Table 1 lists the reciprocal distance parameters for each of the regions in Fig. 2; and Fig. 4 shows various true and corresponding modeled PSDs for the *gravity anomaly*. The parameters may be used to define consistently the cross PSDs and cross covariances of any of the derivatives of the disturbing potential in the respective regions.

5 Summary and Future Directions

The preceding sections have developed the theory for correlation functions on the sphere and plane for deterministic functions and stochastic processes using standard spherical harmonic (Legendre) and Fourier basis functions. Assuming an ergodic (hence stationary) stochastic process, its covariance function (with zero means) is essentially the correlation function defined for a particular realization of the process. These concepts were applied to the disturbing gravitational potential. Based on the fractal nature of Earth's topography and its relationship to the gravitational field, the power spectral density (PSD) of the disturbing potential was shown to behave like a power law at higher spatial frequencies. This provides the basis for the definition and determination of an analytic model for the covariance function that offers mutually consistent cross covariances (and PSDs) among its various derivatives, including vertical derivatives.

Once established for a particular region, such models have numerous applications from least-squares collocation (and the related kriging) to more mundane procedures such as interpolation and filtering. Furthermore, they are ideally suited to generating a synthetic field for use in simulation studies of potential theory, as well as Monte Carlo statistical analyses in estimation theory. The details of such applications are beyond the present scope but are readily formulated.

The developed reciprocal distance model is quite versatile when combined linearly using appropriate parameters and is able to represent the PSD of the disturbing potential (and any of its derivatives) with different spectral amplitudes depending on the region in question. Two examples are provided in which a combination of 15 such reciprocal distance components is fitted accurately to the empirical gravity anomaly PSD in either smooth or rough regional fields. Although limited to some extent by being isotropic (for the vertical derivatives, only), the resulting models are completely analytic in both spatial and frequency domains; and thus, the computed cross covariances and cross PSDs of all derivatives of the disturbing potential are mutually consistent, which is particularly important in estimation and error analysis studies.

The global representation of the gravitational field in terms of spherical harmonics has many applications that are, in fact, becoming more and more local as the computational capability increases and models are expanded to higher maximum degree, $n_{\max}$. The most recent global model, EGM2008, includes coefficients complete up to $n_{\max} = 2{,}190$, and the historical trend has been to develop models with increasingly high global resolution as more and more globally distributed data become available. However, such high-degree models also face the potential problem of divergence near the Earth's surface (below the sphere of convergence) and must always submit to the justifiable criticism that they are inefficient local representations of the field. In fact, the two PSDs presented here are based on the classical local approximation, the planar approximation, with traditional Fourier (sinusoidal) basis functions.

Besides being limited by the planar approximation, the Fourier basis functions, in the strictest sense, still have global support for nonperiodic functions. However, there exists a vast recent development of local-support representations of the gravitational field using splines on the sphere, including tensor-product splines (e.g., Schumaker and Traas 1991), radial basis functions (Schreiner 1997; Freeden et al. 1998), and splines on sphere-like surfaces (Alfeld et al. 1996); see also Jekeli (2005). Representations of the gravitational field using these splines, particularly the radial basis functions and the Bernstein-Bézier polynomials used by Alfeld et al. depend strictly on local data, and the models can easily be modified by the addition or modification of individual data. Thus, they also do not depend on regularly distributed data, as do the spherical harmonic and Fourier series representations.

On the other hand, these global support models based on regular data distributions lead to particularly straightforward and mutually consistent transformations among the PSDs of all derivatives of the gravitational potential, which greatly facilitates the modeling of their correlations. For irregularly scattered data, the splines lend themselves to a multiresolution representation of the field on the sphere, analogous to wavelets in Cartesian space. This has been developed for the tensor-product splines by Lyche and Schumaker (2000) and for the radial basis splines by Freeden et al. (1998); see also Fengler et al. (2004). For the Bernstein-Bézier polynomial splines, a multiresolution model is also possible. How these newer constructive approximations can be adapted to correlation modeling with mutually consistent transformations (propagation of covariances and analogous PSDs) among all derivatives of the gravitational potential represents a topic for future development and analysis.

Appendix A

The planar reciprocal distance model, Eq. (80), for the covariance function of the disturbing potential is repeated here for convenience with certain abbreviations

$$\phi_{T,T}\left(s_1, s_2; z_1, z_2\right) = \frac{\sigma^2}{M^{1/2}} \tag{90}$$

where

$$M = \beta^2 + \alpha^2 s^2,\ \ \beta = 1 + \alpha\left(z_1 + z_2\right),\ \ s^2 = s_1^2 + s_1^2,\ \ s_1 = x_1 - x_1',\ \ s_2 = x_2 - x_2'. \tag{91}$$

The primed coordinates refer to the first subscripted function in the covariance, and the unprimed coordinates refer to the second function. The altitude levels for these functions are z_1 and z_2, respectively. Derivatives of the disturbing potential with respect to the coordinates are denoted $\partial T/\partial x_1 = T_{x_1}$, $\partial^2 T\big/(\partial x_1 \partial z) = T_{x_1 z}$, etc. The following expressions for the cross covariances are derived by repeatedly using Eqs. (42) and (54). The arguments for the resulting function are omitted but are the same as in Eq. (90):

$$\phi_{T_{x_1},T} = \frac{\sigma^2 \alpha^2 s_1}{M^{3/2}} = -\phi_{T,T_{x_1}} \tag{92}$$

$$\phi_{T_{x_2},T} = \frac{\sigma^2 \alpha^2 s_2}{M^{3/2}} = -\phi_{T,T_{x_2}} \tag{93}$$

$$\phi_{T_z,T} = -\frac{\sigma^2\alpha\beta}{M^{3/2}} = \phi_{T,T_z} \tag{94}$$

$$\phi_{T_{x_1},T_{x_1}} = \frac{\sigma^2\alpha^2}{M^{5/2}}\left(M - 3\alpha^2 s_1^2\right) \tag{95}$$

$$\phi_{T_{x_1},T_{x_2}} = -3\frac{\sigma^2\alpha^4}{M^{5/2}} s_1 s_2 = \phi_{T_{x_2},T_{x_1}} \tag{96}$$

$$\phi_{T_{x_1},T_z} = -3\frac{\sigma^2\alpha^3\beta}{M^{5/2}} s_1 = -\phi_{T_z,T_{x_1}} \tag{97}$$

$$\phi_{T_{x_2},T_{x_2}} = \frac{\sigma^2\alpha^2}{M^{5/2}}\left(M - 3\alpha^2 s_2^2\right) \tag{98}$$

$$\phi_{T_{x_2},T_z} = -3\frac{\sigma^2\alpha^3\beta}{M^{5/2}} s_2 = -\phi_{T_z,T_{x_2}} \tag{99}$$

$$\phi_{T_z,T_z} = \frac{\sigma^2\alpha^2}{M^{5/2}}\left(2M - 3\alpha^2 s^2\right) = \phi_{T_{x_1},T_{x_1}} + \phi_{T_{x_2},T_{x_2}} \tag{100}$$

$$\phi_{T,T_{x_1x_1}} = -\phi_{T_{x_1},T_{x_1}} = \phi_{T_{x_1x_1},T} \tag{101}$$

$$\phi_{T,T_{x_1x_2}} = -\phi_{T_{x_1},T_{x_2}} = \phi_{T_{x_1x_2},T} \tag{102}$$

$$\phi_{T,T_{x_1z}} = -\phi_{T_{x_1},T_z} = -\phi_{T_{x_1z},T} \tag{103}$$

$$\phi_{T,T_{x_2x_2}} = -\phi_{T_{x_2},T_{x_2}} = \phi_{T_{x_2x_2},T} \tag{104}$$

$$\phi_{T,T_{x_2z}} = -\phi_{T_{x_2},T_z} = -\phi_{T_{x_2z},T} \tag{105}$$

$$\phi_{T,T_{zz}} = \phi_{T_z,T_z} = \phi_{T_{zz},T} \tag{106}$$

$$\phi_{T_{x_1},T_{x_1x_1}} = \frac{3\sigma^2\alpha^4 s_1}{M^{7/2}}\left(-3M + 5\alpha^2 s_1^2\right) = -\phi_{T_{x_1x_1},T_{x_1}} \tag{107}$$

$$\begin{aligned}\phi_{T_{x_1},T_{x_1x_2}} &= \frac{3\sigma^2\alpha^4 s_2}{M^{7/2}}\left(-M + 5\alpha^2 s_1^2\right) = -\phi_{T_{x_1x_2},T_{x_1}} = \phi_{T_{x_2},T_{x_1x_1}} \\ &= -\phi_{T_{x_1x_1},T_{x_2}}\end{aligned} \tag{108}$$

$$\begin{aligned}\phi_{T_{x_1},T_{x_1z}} &= \frac{3\sigma^2\alpha^3\beta}{M^{7/2}}\left(-M + 5\alpha^2 s_1^2\right) = \phi_{T_{x_1z},T_{x_1}} = -\phi_{T_z,T_{x_1x_1}} \\ &= -\phi_{T_{x_1x_1},T_z}\end{aligned} \tag{109}$$

$$\begin{aligned}\phi_{T_{x_1},T_{x_2x_2}} &= \frac{3\sigma^2\alpha^4 s_1}{M^{7/2}}\left(-M + 5\alpha^2 s_2^2\right) = -\phi_{T_{x_2x_2},T_{x_1}} = \phi_{T_{x_2},T_{x_1x_2}} \\ &= -\phi_{T_{x_1x_2},T_{x_2}}\end{aligned} \tag{110}$$

$$\begin{aligned}\phi_{T_{x_1},T_{x_2z}} &= \frac{15\sigma^2\alpha^5\beta}{M^{7/2}} s_1 s_2 = \phi_{T_{x_2z},T_{x_1}} = -\phi_{T_z,T_{x_1x_2}} = \phi_{T_{x_2},T_{x_1z}} = -\phi_{T_{x_1x_2},T_z} \\ &= \phi_{T_{x_1z},T_{x_2}}\end{aligned} \tag{111}$$

$$\phi_{T_{x_1},T_{zz}} = \frac{3\sigma^2\alpha^4 s_1}{M^{7/2}}\left(4\beta^2 - \alpha^2 s^2\right) = -\phi_{T_{zz},T_{x_1}} = -\phi_{T_z,T_{x_1 z}} = \phi_{T_{x_1 z},T_z} \tag{112}$$

$$\phi_{T_{x_2},T_{x_2 x_2}} = \frac{3\sigma^2\alpha^4 s_2}{M^{7/2}}\left(-3M + 5\alpha^2 s_2^2\right) = -\phi_{T_{x_2 x_2},T_{x_2}} \tag{113}$$

$$\phi_{T_{x_2},T_{x_2 z}} = \frac{3\sigma^2\alpha^3\beta}{M^{7/2}}\left(-M + 5\alpha^2 s_2^2\right) = \phi_{T_{x_2 z},T_{x_2}} = -\phi_{T_z,T_{x_2 x_2}} = -\phi_{T_{x_2 x_2},T_z} \tag{114}$$

$$\phi_{T_{x_2},T_{zz}} = \frac{3\sigma^2\alpha^4 s_2}{M^{7/2}}\left(4\beta^2 - \alpha^2 s^2\right) = -\phi_{T_{zz},T_{x_2}} = -\phi_{T_z,T_{x_2 z}} = \phi_{T_{x_2 z},T_z} \tag{115}$$

$$\phi_{T_z,T_{zz}} = \frac{3\sigma^2\alpha^3\beta}{M^{7/2}}\left(-2M + 5\alpha^2 s^2\right) = \phi_{T_{zz},T_z} \tag{116}$$

$$\phi_{T_{x_1 x_1},T_{x_1 x_1}} = \frac{3\sigma^2\alpha^4}{M^{9/2}}\left(3M^2 - 30M\alpha^2 s_1^2 + 35\alpha^4 s_1^4\right) \tag{117}$$

$$\phi_{T_{x_1 x_1},T_{x_1 x_2}} = \frac{15\sigma^2\alpha^6 s_1 s_2}{M^{9/2}}\left(-3M + 7\alpha^2 s_1^2\right) = \phi_{T_{x_1 x_2},T_{x_1 x_1}} \tag{118}$$

$$\phi_{T_{x_1 x_1},T_{x_1 z}} = \frac{15\sigma^2\alpha^5\beta s_1}{M^{9/2}}\left(-3M + 7\alpha^2 s_1^2\right) = -\phi_{T_{x_1 z},T_{x_1 x_1}} \tag{119}$$

$$\phi_{T_{x_1 x_1},T_{x_2 x_2}} = \frac{3\sigma^2\alpha^4}{M^{9/2}}\left(M^2 - 5M\alpha^2 s^2 + 35 s_1^2 s_2^2\right) = \phi_{T_{x_2 x_2},T_{x_1 x_1}} = \phi_{T_{x_1 x_2},T_{x_1 x_2}} \tag{120}$$

$$\begin{aligned}\phi_{T_{x_1 x_1},T_{x_2 z}} &= \frac{15\sigma^2\alpha^5\beta s_2}{M^{9/2}}\left(-M + 7\alpha^2 s_1^2\right) = -\phi_{T_{x_2 z},T_{x_1 x_1}} = -\phi_{T_{x_1 z},T_{x_1 x_2}}\\ &= \phi_{T_{x_1 x_2},T_{x_1 z}}\end{aligned} \tag{121}$$

$$\phi_{T_{x_1 x_1},T_{zz}} = \frac{3\sigma^2\alpha^4}{M^{9/2}}\left(-4M^2 + 5M\alpha^2 s_2^2 + 35\beta^2\alpha^2 s_1^2\right) = \phi_{T_{zz},T_{x_1 x_1}} = -\phi_{T_{x_1 z},T_{x_1 z}} \tag{122}$$

$$\phi_{T_{x_1 x_2},T_{x_2 x_2}} = \frac{15\sigma^2\alpha^6 s_1 s_2}{M^{9/2}}\left(-3M + 7\alpha^2 s_2^2\right) = \phi_{T_{x_2 x_2},T_{x_1 x_2}} \tag{123}$$

$$\begin{aligned}\phi_{T_{x_1 x_2},T_{x_2 z}} &= \frac{15\sigma^2\alpha^5\beta s_1}{M^{9/2}}\left(-M + 7\alpha^2 s_2^2\right) = -\phi_{T_{x_2 z},T_{x_1 x_2}} = -\phi_{T_{x_1 z},T_{x_2 x_2}}\\ &= \phi_{T_{x_2 x_2},T_{x_1 z}}\end{aligned} \tag{124}$$

$$\phi_{T_{x_1 z},T_{zz}} = \frac{15\sigma^2\alpha^5\beta s_1}{M^{9/2}}\left(3M - 7\beta^2\right) = -\phi_{T_{zz},T_{x_1 z}} \tag{125}$$

$$\phi_{T_{x_2 x_2},T_{x_2 x_2}} = \frac{3\sigma^2\alpha^4}{M^{9/2}}\left(3M^2 - 30M\alpha^2 s_2^2 + 35\alpha^4 s_2^4\right) \tag{126}$$

$$\phi_{T_{x_2 x_2},T_{x_2 z}} = \frac{15\sigma^2\alpha^5\beta s_2}{M^{9/2}}\left(-3M + 7\alpha^2 s_2^2\right) = -\phi_{T_{x_2 z},T_{x_2 x_2}} \tag{127}$$

$$\phi_{T_{x_2 x_2},T_{zz}} = \frac{3\sigma^2\alpha^4}{M^{9/2}}\left(-4M^2 + 5M\alpha^2 s_1^2 + 35\beta^2\alpha^2 s_2^2\right) = \phi_{T_{zz},T_{x_2 x_2}} = -\,\phi_{T_{x_2 z},T_{x_2 z}} \tag{128}$$

$$\phi_{T_{x_2 z},T_{zz}} = \frac{15\sigma^2\alpha^5\beta s_2}{M^{9/2}}\left(3M - 7\beta^2\right) = -\phi_{T_{zz},T_{x_2 z}} \tag{129}$$

$$\phi_{T_{zz},T_{zz}} = \frac{3\sigma^2\alpha^4}{M^{9/2}}\left(8\beta^4 - 24\beta^2\alpha^2 s^2 + 3\alpha^4 s^4\right) = \phi_{T_{x_1 z},T_{x_1 z}} + \phi_{T_{x_2 z},T_{x_2 z}} \tag{130}$$

Appendix B

The hybrid PSD/covariance function of the disturbing potential, given by Eq. (85), can be shown to be

$$S_{T,T}\left(f_1; s_2; z_1, z_2\right) = \frac{2\sigma^2}{\alpha} K_0\left(2\pi f_1 d\right), \tag{131}$$

where K_0 is the modified Bessel function of the second kind and zero order, and

$$d = \sqrt{\frac{\beta^2}{\alpha^2} + s_2^2}. \tag{132}$$

It is the along-track PSD if $s_2 = 0$. In the following hybrid PSD/covariances of the derivatives of T, also the modified Bessel function of the second kind and first order, K_1, appears. Both Bessel function always have the argument, $2\pi f_1 d$; and, the arguments of the hybrid PSD/covariances are the same as in Eq. (131).

$$S_{T,T_{x_1}} = i2\pi f_1 S_{TT} = -S_{T_{x_1},T} \tag{133}$$

$$S_{T,T_{x_2}} = \frac{2\sigma^2\left(2\pi f_1\right)s_2}{\alpha d} K_1 = -S_{T_{x_2},T} \tag{134}$$

$$S_{T,T_z} = -\frac{2\sigma^2\left(2\pi f_1\right)\beta}{\alpha^2 d} K_1 = S_{T_z,T} \tag{135}$$

$$S_{T_{x_1},T_{x_1}} = (2\pi f_1)^2 \, S_{TT} \tag{136}$$

$$S_{T_{x_1},T_{x_2}} = i\,2\pi f_1 S_{T_{x_2},T} = S_{T_{x_2},T_{x_1}} \tag{137}$$

$$S_{T_{x_1},T_z} = -i\,2\pi f_1 S_{T,T_z} = -S_{T_z,T_{x_1}} \tag{138}$$

$$S_{T_{x_2},T_{x_2}} = \frac{2\sigma^2 (2\pi f_1)}{\alpha d}\left(\left(1-\frac{2s_2^2}{d^2}\right)K_1 - 2\pi f_1 \frac{s_2^2}{d} K_0\right) \tag{139}$$

$$S_{T_{x_2},T_z} = -\frac{2\sigma^2 (2\pi f_1)\,\beta s_2}{\alpha^2 d^3}\,(2K_1 + 2\pi f_1 d\ K_0) = -S_{T_z,T_{x_2}} \tag{140}$$

$$S_{T_z,T_z} = S_{T_{x_1},T_{x_1}} + S_{T_{x_2},T_{x_2}} \tag{141}$$

$$S_{T,T_{x_1x_1}} = -S_{T_{x_1},T_{x_1}} = S_{T_{x_1x_1},T} \tag{142}$$

$$S_{T,T_{x_1x_2}} = -S_{T_{x_1},T_{x_2}} = S_{T_{x_1x_2},T} \tag{143}$$

$$S_{T,T_{x_1z}} = -S_{T_{x_1},T_z} = -S_{T_{x_1z},T} \tag{144}$$

$$S_{T,T_{x_2x_2}} = -S_{T_{x_2},T_{x_2}} = S_{T_{x_2x_2},T} \tag{145}$$

$$S_{T,T_{x_2z}} = -S_{T_{x_2},T_z} = -S_{T_{x_2z},T} \tag{146}$$

$$S_{T,T_{zz}} = S_{T_z,T_z} = S_{T_{zz},T} \tag{147}$$

$$S_{T_{x_1},T_{x_1x_1}} = i\,(2\pi f_1)^3 \, S_{T,T} = -S_{T_{x_1x_1},T_{x_1}} \tag{148}$$

$$S_{T_{x_1},T_{x_1x_2}} = (2\pi f_1)^2 \, S_{T,T_{x_2}} = -S_{T_{x_1x_2},T_{x_1}} = S_{T_{x_2},T_{x_1x_1}} = -S_{T_{x_1x_1},T_{x_2}} \tag{149}$$

$$S_{T_{x_1},T_{x_1z}} = (2\pi f_1)^2 \, S_{T,T_z} = S_{T_{x_1z},T_{x_1}} = -S_{T_z,T_{x_1x_1}} = -S_{T_{x_1x_1},T_z} \tag{150}$$

$$S_{T_{x_1},T_{x_2x_2}} = i\,2\pi f_1 \, S_{T_{x_2},T_{x_2}} = -S_{T_{x_2x_2},T_{x_1}} = S_{T_{x_2},T_{x_1x_2}} = -S_{T_{x_1x_2},T_{x_2}} \tag{151}$$

$$\begin{aligned} S_{T_{x_1},T_{x_2z}} &= i\,2\pi f_1 \, S_{T_{x_2},T_z} = S_{T_{x_2z},T_{x_1}} = -S_{T_z,T_{x_1x_2}} = S_{T_{x_2},T_{x_1z}} = -S_{T_{x_1x_2},T_z} \\ &= S_{T_{x_1z},T_{x_2}} \end{aligned} \tag{152}$$

$$S_{T_{x_1},T_{zz}} = -i\,2\pi f_1 \, S_{T_z,T_z} = -S_{T_{zz},T_{x_1}} = -S_{T_z,T_{x_1z}} = S_{T_{x_1z},T_z} \tag{153}$$

$$\begin{aligned} S_{T_{x_2},T_{x_2x_2}} &= -\frac{2\sigma^2 (2\pi f_1)\, s_2}{\alpha d^3}\left(\left(6-\frac{8s_2^2}{d^2} - (2\pi f_1 \, s_2)^2\right)K_1 + 2\pi f_1 \, d \left(3-\frac{4s_2^2}{d^2}\right)K_0 \right. \\ &= -S_{T_{x_2x_2},T_{x_2}} \end{aligned} \tag{154}$$

$$S_{T_{x_2},T_{x_2z}} = -\frac{2\sigma^2\,(2\pi f_1)\,\beta}{\alpha^2 d^3}\left(\left(2-\frac{8s_2^2}{d^2}-(2\pi f_1\,s_2)^2\right)K_1+2\pi f_1\,d\left(1-\frac{4s_2^2}{d^2}\right)K_0\right)$$
$$= S_{T_{x_2z},T_{x_2}} = -S_{T_z,T_{x_2x_2}} = -S_{T_{x_2x_2},T_z} \quad (155)$$

$$S_{T_{x_2},T_{zz}} = -S_{T_{x_2},T_{x_1x_1}} - S_{T_{x_2},T_{x_2x_2}} = -S_{T_{zz},T_{x_2}} = -S_{T_z,T_{x_2z}} = S_{T_{x_2z},T_z} \quad (156)$$

$$S_{T_z,T_{zz}} = S_{T_{x_1},T_{x_1z}} + S_{T_{x_2},T_{x_2z}} = S_{T_{zz},T_z} \quad (157)$$

$$S_{T_{x_1x_1},T_{x_1x_1}} = (2\pi f_1)^4\,S_{T,T} \quad (158)$$

$$S_{T_{x_1x_1},T_{x_1x_2}} = i\,(2\pi f_1)^3\,S_{T_{x_2},T} = S_{T_{x_1x_2},T_{x_1x_1}} \quad (159)$$

$$S_{T_{x_1x_1},T_{x_1z}} = -i\,(2\pi f_1)^3\,S_{T,T_z} = -S_{T_{x_1z},T_{x_1x_1}} \quad (160)$$

$$S_{T_{x_1x_1},T_{x_2x_2}} = (2\pi f_1)^2\,S_{T_{x_2},T_{x_2}} = S_{T_{x_2x_2},T_{x_1x_1}} = S_{T_{x_1x_2},T_{x_1x_2}} \quad (161)$$

$$S_{T_{x_1x_1},T_{x_2z}} = (2\pi f_1)^2\,S_{T_{x_2},T_z} = -S_{T_{x_2z},T_{x_1x_1}} = -S_{T_{x_1z},T_{x_1x_2}} = S_{T_{x_1x_2},T_{x_1z}} \quad (162)$$

$$S_{T_{x_1x_1},T_{zz}} = -(2\pi f_1)^2\,S_{T_z,T_z} = S_{T_{zz},T_{x_1x_1}} = -S_{T_{x_1z},T_{x_1z}} \quad (163)$$

$$S_{T_{x_1x_2},T_{x_2x_2}} = i\,2\pi f_1\,S_{T_{x_2x_2},T_{x_2}} = S_{T_{x_2x_2},T_{x_1x_2}} \quad (164)$$

$$S_{T_{x_1x_2},T_{x_2z}} = -i\,2\pi f_1\,S_{T_{x_2},T_{x_2z}} = -S_{T_{x_2z},T_{x_1x_2}} = -S_{T_{x_1z},T_{x_2x_2}} = S_{T_{x_2x_2},T_{x_1z}} \quad (165)$$

$$S_{T_{x_1x_2},T_{zz}} = -i\,2\pi f_1\,S_{T_{x_2z},T_z} = S_{T_{zz},T_{x_1x_2}} = -S_{T_{x_1z},T_{x_2z}} = -S_{T_{x_2z},T_{x_1z}} \quad (166)$$

$$S_{T_{x_1z},T_{zz}} = S_{T_{x_1x_1},T_{x_1z}} + S_{T_{x_2x_2},T_{x_1z}} = -S_{T_{zz},T_{x_1z}} \quad (167)$$

$$S_{T_{x_2x_2},T_{x_2x_2}} = \frac{2\sigma^2\,(2\pi f_1)}{\alpha d^3}\left(2\pi f_1\,d\left(3-\frac{24s_2^2}{d^2}+\frac{24s_2^4}{d^4}+(2\pi f_1\,s_2)^2\,\frac{s_2^2}{d^2}\right)K_0+\right.$$
$$\left.+2\left(3-\frac{24s_2^2}{d^2}-3\,(2\pi f_1 s_2)^2+\frac{24s_2^4}{d^4}+4\,(2\pi f_1\,s_2)^2\,\frac{s_2^2}{d^2}\right)K_1\right) \quad (168)$$

$$S_{T_{x_2x_2},T_{x_2z}} = -\frac{2\sigma^2\,(2\pi f_1)\,\beta s_2}{\alpha^2 d^5}\left(2\pi f_1\,d\left(12-\frac{24s_2^2}{d^2}-(2\pi f_1\,s_2)^2\right)K_0\right.$$
$$\left.+\left(24-\frac{48s_2^2}{d^2}+3\,(2\pi f_1 d)^2-8\,(2\pi f_1\,s_2)^2\right)K_1\right)$$
$$= -S_{T_{x_2z},T_{x_2x_2}} \quad (169)$$

$$S_{T_{x_2x_2},T_{zz}} = -S_{T_{x_1x_1},T_{x_2x_2}} - S_{T_{x_2x_2},T_{x_2x_2}} = S_{T_{zz},T_{x_2x_2}} = -S_{T_{x_2z},T_{x_2z}} \tag{170}$$

$$S_{T_{x_2z},T_{zz}} = S_{T_{x_1x_1},T_{x_2z}} + S_{T_{x_2x_2},T_{x_2z}} = -S_{T_{zz},T_{x_2z}} \tag{171}$$

$$S_{T_{zz},T_{zz}} = S_{T_{x_1z},T_{x_1z}} + S_{T_{x_2z},T_{x_2z}} \tag{172}$$

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