Chapter 3
Momentum Flux

We have seen in Chapter 1 that a fluid (either a gas or a liquid) is a substance that takes the shape of the vessel containing it. We have also seen that all real fluids have a property called viscosity associated with them.

Let us consider two parallel flat plates with a fluid (say water) in between them. Now let us consider the situation when the bottom plate is carefully moved in the x direction with a reasonably small velocity, v_x. If the velocity is small enough, we can assume that the bottom-most liquid layer adhering to the plate will also move with the same velocity as that of the plate. The shear stress due to the shear force exerted by the bottom-most layer of fluid influences the velocity of the fluid layer above it. The shear stress exerted by the layer above the bottom-most layer influences the velocity of the layer above it, and so on. The resulting steady state velocity profile of the fluid between the two plates is given in Fig. 3-1.

Stress is denoted by τ_{yx}, where yx refers to the fact that the stress (force per unit area) that arises due to a force acting in the x direction on a surface, causes an effect in the y direction. Thus, τ_{yx} is a shear stress – the direction of effect is orthogonal to the direction of motion, and, as we shall better understand later, τ_{xx} is a normal stress. The idea of the flow being in layers as shown in Fig. 3-1, and the shear stress idea subsequently conveyed are simplistic, only for didactic purposes. We will make the ideas more general when needed, later in the chapter.

It can be recalled that the shear or normal stress is force per unit area, and that force is rate of momentum change (from Newton's second law). Or

$$\tau_{yx} = \frac{\text{Force}}{\text{Area}} = \frac{\text{MLT}^{-2}}{\text{L}^2} = \frac{[\text{M}(\text{LT}^{-1})]\text{T}^{-1}}{\text{L}^2}$$

$$= \frac{\text{Rate of momentum change}}{\text{Area}}$$

$$= \text{Momentum flux}$$

G.K. Suraishkumar, *Continuum Analysis of Biological Systems,* Biosystems & Biorobotics 5,
DOI: 10.1007/978-3-642-54468-2_3, © Anamaya Publishers, New Delhi, India 2014

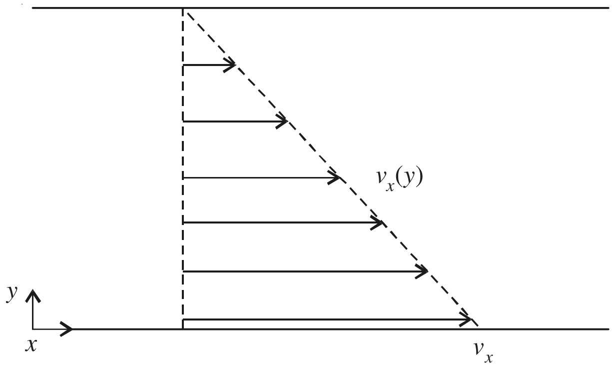


Fig. 3-1 Velocity distribution of a fluid between two parallel, closely spaced plates with the bottom plate being moved at a velocity, v_x

Note that although the direction of action is orthogonal in the example mentioned above, the change happens in the *x momentum* of the subsequent layers.

3.1 Rheology

The relationship between the shear stress, τ_{yx} and a 'shear rate', or velocity gradient $\frac{dv_x}{dy}$ was experimentally observed by Isaac Newton as

$$\tau_{yx} = -\mu \frac{dv_x}{dy} \tag{3.1-1}$$

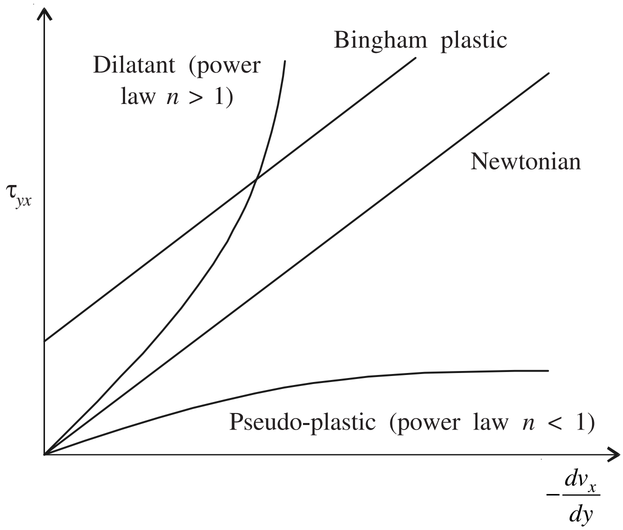
The constant of proportionality, μ, is called viscosity, and is a fundamental material property. Readers interested in acquainting themselves with the methods to estimate viscosity of gases and liquids, and to evaluate the effect of temperature and pressure on viscosity can refer to books like *Transport Phenomena* (Bird et al. 2002). The above equation is a constitutive equation, and is called the 'Newton's law of viscosity'. Recall that Fick's I law was also a constitutive equation. As generalised in Section 2.2.1, it follows the following relationship: Flux is proportional to the gradient of its primary driving force.

Dimensionally, shear stress can be written from the introductory section as

$$\left(\frac{\mathrm{MT}^{-1}}{\mathrm{L}}\right)\left(\frac{(\mathrm{LT}^{-1})}{\mathrm{L}}\right)$$

Thus, the dimensions of viscosity are $\mathrm{ML}^{-1}\mathrm{T}^{-1}$.

Fig. 3.1-1 The shear stress-shear rate behaviour of various fluid types



If we plot τ_{yx} vs $\left(\dfrac{dv_x}{dy}\right)$ for the above fluid (water), we get a straight line passing through the origin as shown in Fig. 3.1-1. Fluids that exhibit such behaviour are known as Newtonian fluids. As can be expected, not all fluids are Newtonian – they may exhibit different stress-shear rate behaviours. Nevertheless, a Newtonian fluid approximation is a good one for many fluids under certain conditions.

Bingham Plastic

A Bingham plastic fluid exhibits a rheology different from a Newtonian one. It does not flow until a certain minimum shear stress, τ_0, is applied i.e. the shear rate is zero until $\tau_{yx} < \tau_0$. τ_0 is called the 'yield stress' for the material.

It can be represented as

$$\tau_{yx} = \tau_0 - \mu \frac{dv_x}{dy} \quad \text{if } |\tau_{yx}| > \tau_0 \tag{3.1-2a}$$

$$\frac{dv_x}{dy} = 0 \quad \text{if } |\tau_{yx}| < \tau_0 \tag{3.1-2b}$$

Power Law Fluids

Newtonian fluids and Bingham plastics have viscosities that are independent of shear rate. Some fluid viscosities, though, are dependent on their shear rates. This means that the fluid will either become easier to flow, or more difficult to flow, with an increase in shear rate. Such fluids are known as power law fluids because the variation of a particular, 'apparent viscosity' with shear rate, is expressed as a power law

$$\tau_{yx} = -m\left|\frac{dv_x}{dy}\right|^{n-1}\frac{dv_x}{dy} \tag{3.1-3}$$

where the apparent viscosity, μ_{app}, is given as

$$\mu_{app} = m\left|\frac{dv_x}{dy}\right|^{n-1} \tag{3.1-4}$$

where m and n are parameters that are dependent on the fluid.

- If $n = 1$, the fluid is Newtonian and $m = \mu$ (Newtonian viscosity)
- If $n < 1$, the fluid is shear-thinning or pseudo-plastic
- If $n > 1$, the fluid is shear-thickening or dilatant

Viscoelastic Fluids

Some fluids show time-dependent behaviour – the shear stress depends on the shear rate (viscous) as well as on the strain (elastic or Hookean). A common constitutive equation to describe viscoelastic fluids is the Maxwell model

$$\tau_{yx} + \frac{\mu}{G}\frac{\partial\tau_{yx}}{\partial t} = \mu\left(-\frac{dv_x}{dy}\right) \tag{3.1-5}$$

where G is the shear elastic modulus (Nm^{-2}).

The synovial fluid that lubricates joints in the human body shows viscoelastic behaviour. It is a complex fluid consisting of proteins out of which hyaluronic acid is the most important. Mucus and vitreous fluid in the eye also show viscoelastic behaviour.

Blood

Blood is a complex biological fluid that consists of plasma, which is a mixture of liquids, proteins, with cells such as erythrocytes, leukocytes, and others suspended in it. It behaves partially as a Bingham plastic, i.e. it exhibits a yield stress, and partially as a viscoelastic fluid. Besides the composition, the complex rheological behaviour of blood also arises from the 'clumping' of erythrocytes (red blood cells) due to the presence of fibrinogen on their surface.

The Casson model can be used to describe blood rheology. It can be stated as

$$\tau^{1/2} = \tau_0^{1/2} + \mu^{1/2} \left| \frac{-dv_x}{dy} \right|^{1/2} \tag{3.1-6}$$

where τ_0 is the yield stress.

The yield stress depends on the volume fraction of erythrocytes in the blood. The volume fraction of erythrocytes in blood is usually referred to as the 'hematocrit' and has a typical value of 0.4.

At lower shear rates, say $< 20\ s^{-1}$, blood shows a complex behaviour that necessitates the use of Eq. 3.1-6, whereas at higher shear rates, say $> 100\ s^{-1}$, blood can be assumed, without loss in accuracy, to behave as a Newtonian fluid. Blood rheology is highly complex and a lot of work has been done on this aspect alone so much so that an entire field of study – hemorheology – is dedicated to it.

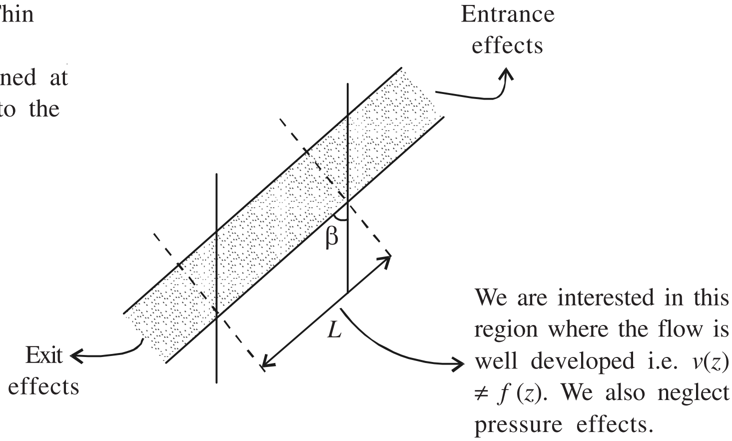
3.2 Types of Flows

Osborne Reynolds studied flows at various flow rates and found that the nature of flow changes with flow rate. Through his now classic, flow visualisation experiment (Reynolds 1883), Reynolds reported that at low flow rates, the flow in a pipe is in layers or laminae, and hence can be called 'laminar flow'. Above a certain flow rate, the flow becomes chaotic, and is called 'turbulent flow'. There is a range of flow rates where one cannot say beforehand whether the flow would be 'laminar' or 'turbulent'. This range/region is called the 'transition region'.

A non-dimensional number, called the Reynolds number, can be used to predict whether the flow will be laminar or turbulent. The Reynolds number is defined as

$$N_{\mathrm{Re}} = \frac{\rho v d}{\mu} \tag{3.2-1}$$

Fig. 3.3-1 Thin flow over a surface inclined at an angle β to the vertical



where ρ is density of the fluid, v is velocity of the fluid, d is pipe diameter and μ is viscosity of the fluid.

In pipe flow (*and only in pipe flow*), the following numbers hold:

$N_{\mathrm{Re}} < 2100$	Laminar flow
$2100 < N_{\mathrm{Re}} < 4000$	Transition
$N_{\mathrm{Re}} > 4000$	Turbulent flow

In the initial part of this chapter, we will deal with laminar flow, and then explicitly address ways to deal with turbulent flows.

3.3 Shell Momentum Balances

Since momentum is a conserved quantity, momentum balance can be used as a principle to obtain useful relationships. In this section, let us do momentum balances over a thin shell of fluid. In other words, the thin shell is the 'system' or 'control volume' over which the momentum balance is written. This technique for solving relevant problems is called the 'shell balance' technique.

To illustrate the technique, let us consider the case of flow in a falling film over an inclined surface (Fig. 3.3-1). Characteristics of such flow are used to evaluate the rheological properties of biological solutions. For example, the 'Bostwick viscometer' is based on the principle of flow over an inclined surface.

We know from basic physics that momentum is a conserved quantity in the absence of external forces. When external forces are present, according

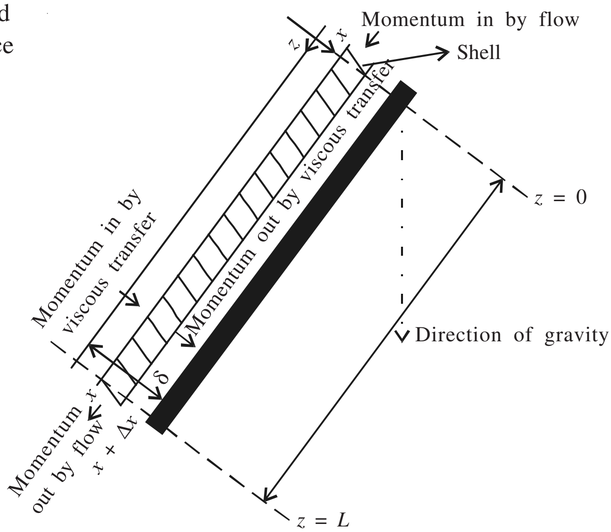


Fig. 3.3-2 Details of the shell considered for the shell balance

to Newton's second law, the rate of change of momentum is equal to the (vector) sum of the forces that act on the system or the control volume, in the direction of motion. In the case of a balance on the total mass, we could write

$$\begin{pmatrix}\text{Rate of total mass}\\ \text{out of the system}\end{pmatrix} - \begin{pmatrix}\text{Rate of total mass}\\ \text{into the system}\end{pmatrix} + \begin{pmatrix}\text{Rate of total mass}\\ \text{accumulation in the system}\end{pmatrix} = 0$$

For a system (or a control volume) that has momentum being brought into it and taken out of it by flowing streams (by convection), a useful form of Newton's second law can be written as

$$\begin{pmatrix}\text{Rate of momentum}\\ \text{out of the system}\end{pmatrix} - \begin{pmatrix}\text{Rate of momentum}\\ \text{into the system}\end{pmatrix} + \begin{pmatrix}\text{Rate of momentum}\\ \text{accumulation in the system}\end{pmatrix} = \begin{pmatrix}\text{Sum of forces acting}\\ \text{on the system}\end{pmatrix} \tag{3.3-1}$$

At steady state, the accumulation rate can be set to zero, and the balance becomes

$$\begin{pmatrix}\text{Rate of momentum}\\ \text{into the system}\end{pmatrix} - \begin{pmatrix}\text{Rate of momentum}\\ \text{out of the system}\end{pmatrix} + \begin{pmatrix}\text{Sum of forces acting}\\ \text{on the system}\end{pmatrix} = 0$$

Momentum can enter the shell (system) by: (i) molecular means (momentum flux) and/or (ii) convection (bulk fluid motion), as illustrated in Fig. 3.3-2.

Further, there could be many forces that act on the system. For illustration, let us consider only the gravity forces that act on the whole volume. The pressure force and the normal force may not be relevant to the direction considered.

We are interested in $v_z(x)$ and $\tau_{xz}(x)$. Let us first acknowledge that the rate of momentum is area × momentum flux.

By Molecular Mechanism

Rate of z momentum in, across the surface at x: $(LW)\ \tau_{xz}|_x$
Rate of z momentum out, across the surface at $x + \Delta x$: $(LW)\ \tau_{xz}|_{x+\Delta x}$

By Convection

Rate of z momentum in, across the surface at $z = 0$: $(W\ \Delta x\ v_z)\ (\rho v_z)|_{z=0}$
Rate of z momentum out, across the surface at $z = L$: $(W\ \Delta x\ v_z)\ (\rho v_z)|_{z=L}$
Gravity force acting on the fluid in the direction of motion: $(L\ W\ \Delta x)\ (\rho\ g \cos\beta)$

Substituting the above in the momentum balance, Eq. 3.3-1, at steady state, we get

$$LW\,\tau_{xz}\big|_x - LW\,\tau_{xz}\big|_{x+\Delta x} + W\,\Delta x\,\rho\,v_z^2\big|_{z=0} - W\,\Delta x\,\rho\,v_z^2\big|_{z=L} + LW\,\Delta x\,\rho\,g\cos\beta = 0 \tag{3.3-2}$$

Since we have chosen conditions such that $v_z \neq f\,(z)$, the third and fourth terms on the LHS cancel each other. Then, if we divide the equation by $LW\Delta x$ and take the limit as $\Delta x \to 0$

$$\lim_{\Delta x\to 0}\left(\frac{\tau_{xz}\big|_{x+\Delta x} - \tau_{xz}\big|_x}{\Delta x}\right) = \rho\,g\cos\beta$$

i.e.

$$\frac{d\tau_{xz}}{dx} = \rho\,g\cos\beta \tag{3.3-3}$$

The solution of the above first order differential equation (DE) is

$$\tau_{xz} = \rho\ g\ x\ \cos\beta + C_1 \tag{3.3-4}$$

To evaluate C_1, we need a boundary condition.

Notice that $x = 0$ is the liquid-gas interface. A standard boundary condition that can be used at *liquid-gas interfaces* is that the momentum flux (hence the velocity gradient) in the liquid phase can be assumed to be zero for most calculations. i.e.

$$\text{at } x = 0,\ \tau_{xz} = 0 \tag{3.3-5}$$

This boundary condition applied on to the solution given in Eq. 3.3-4 yields, $C_1 = 0$. Thus

$$\tau_{xz} = \rho\ g\ x\ \cos\beta \tag{3.3-6}$$

Thus, we have the shear stress distribution, i.e. $\tau_{xz} = f\ (x)$.

To obtain the velocity distribution from the shear stress distribution, we need a link between the two. That link is conveniently provided by the constitutive equation. For example, if the fluid is Newtonian, we know that

$$\tau_{xz} = -\mu \frac{dv_z}{dx}$$

By substituting the constitutive equation in Eq. 3.3-6, we get

$$\frac{dv_z}{dx} = -\left(\frac{\rho\, g \cos\beta}{\mu}\right) x \tag{3.3-7}$$

The solution of the above DE is

$$v_z = -\left(\frac{\rho\, g \cos\beta}{2\mu}\right) x^2 + C_2 \tag{3.3-8}$$

C_2 can be found by another standard boundary condition – at the solid-fluid interface, the fluid velocity equals the velocity with which the surface itself is moving. It is assumed that the fluid will cling to any solid surface with which it is in contact.

Therefore

$$\text{at } x = \delta,\ v_z = 0 \tag{3.3-9}$$

By substituting the boundary condition into the solution, Eq. 3.3-8, we get

$$C_2 = \left(\frac{\rho\, g \cos\beta}{2\mu}\right) \delta^2$$

Therefore

$$v_z = \frac{\rho\, g\, \delta^2 \cos\beta}{2\mu}\left[1 - \left(\frac{x}{\delta}\right)^2\right] \tag{3.3-10}$$

It can be seen that the maximum velocity occurs at $x = 0$. Therefore

$$v_{z,\max} = \frac{\rho g \delta^2 \cos\beta}{2\mu} \tag{3.3-11}$$

Now, the average velocity over a cross-section of a film can be computed using

$$v_{z,\text{avg}} = \frac{\int_0^w \int_0^\delta v_z dxdy}{\int_0^w \int_0^\delta dxdy} = \frac{1}{\delta}\int_0^\delta v_z dx \tag{3.3-12}$$

(since W can be cancelled in the numerator and the denominator). By substituting Eq. 3.3-10 in Eq. 3.3-12, we get

$$\begin{aligned} v_{z,\text{avg}} &= \frac{\rho g \delta^2 \cos\beta}{2\mu} \int_0^1 \left[1 - \left(\frac{x}{\delta}\right)^2\right] d\left(\frac{x}{\delta}\right) \\ &= \frac{\rho g \delta^2 \cos\beta}{2\mu} \left[\left(\frac{x}{\delta}\right) - \frac{1}{3}\left(\frac{x}{\delta}\right)^3\right]_0^1 \\ &= \frac{\rho g \delta^2 \cos\beta}{3\mu} \end{aligned} \tag{3.3-13}$$

The volume flow rate Q is given by

$$Q = \int_0^w \int_0^\delta v_z dxdy = W\delta v_{z,\text{avg}} = W\delta \frac{\rho g \delta^2 \cos\beta}{3\mu} \tag{3.3-14}$$

3.4 Equation of Motion

Let us consider doing momentum balance in three dimensions with the realisation that momentum is a vector. To do that, let us first consider Cartesian coordinates and take the same cuboidal element that we considered for mass balance (Fig. 3.4-1).

As seen earlier, momentum flows into and out of the volume element by two means:

- convection (by virtue of bulk fluid flow)
- molecular aspects (by virtue of velocity gradients)

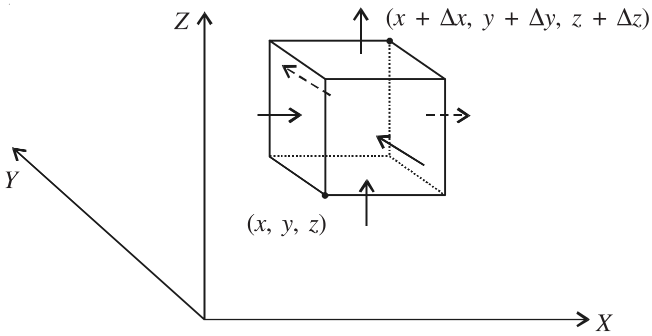


Fig. 3.4-1 Control volume (system) to consider for momentum balances

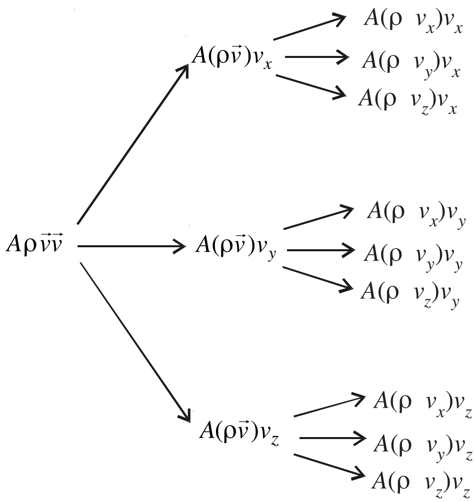


Fig. 3.4-2 The various components of the momentum rate

Momentum Rate by Convection

For momentum transport by convection, note that $(\rho\vec{v})\vec{v}$ is momentum flux (the units can be written down and checked). Thus, the rate of momentum (momentum per time) is $A(\rho\vec{v})\vec{v}$, where A is the area. The units work out as $\cancel{\text{m}}^2\left(\frac{\text{kg}}{\cancel{\text{m}}^3}\frac{\text{m}}{\text{s}}\right)\frac{\cancel{\text{m}}}{\text{s}}$.

There are three components to the rate of momentum: x, y, and z. Each of these components is, in turn, composed of three other components, as shown in Fig. 3.4-2.

Now, let us consider only the *x component* of the momentum rate due to convection:

Entry Rates
x direction (through the face at x) = $(\rho v_x)v_x|_x\ \Delta y\Delta z$
y direction (through the face at y) = $(\rho v_y)v_x|_y\ \Delta x\Delta z$
z direction (through the face at z) = $(\rho v_z)v_x|_z\ \Delta x\Delta y$

Exit Rates
x direction (through the face at $x + \Delta x$) = $(\rho v_x)v_x|_{x+\Delta x}\ \Delta y\Delta z$
y direction (through the face at $y + \Delta y$) = $(\rho v_y)v_x|_{y+\Delta y}\ \Delta x\Delta z$
z direction (through the face at $z + \Delta z$) = $(\rho v_z)v_x|_{z+\Delta z}\ \Delta x\Delta y$

Thus, the net x momentum rate due to convection is

$$\Delta y\Delta z[(\rho v_x)v_x|_x - (\rho v_x)v_x|_{x+\Delta x}] + \Delta x\Delta z[(\rho v_y)v_x|_y - (\rho v_y)v_x|_{y+\Delta y}]$$
$$+ \Delta x\Delta y[(\rho v_z)v_x|_z - (\rho v_z)v_x|_{z+\Delta z}]$$

Momentum Rate by Molecular Aspects

Now, let us look at the momentum rate through molecular aspects. It can be recalled, from earlier in this chapter, that shear stress is momentum flux. Thus, area × shear stress will provide an expression for the momentum rate through molecular aspects.

To begin, let us consider the force that causes the shear stress. Say that the force that acts on the face at x (refer to Fig. 3.4-1) is $\overrightarrow{F^s}_x$, the force that acts on the face at y is $\overrightarrow{F^s}_y$, and the force that acts on the face at z is $\overrightarrow{F^s}_z$. Each of these forces would have three (x, y and z) components, and the components are detailed below:

$$\left.\begin{matrix} F^S{}_{xx} \\ F^S{}_{xy} \\ F^S{}_{xz} \end{matrix}\right\} \text{components of } \overrightarrow{F^S}_x$$

$$\left.\begin{matrix} F^S{}_{yx} \\ F^S{}_{yy} \\ F^S{}_{yz} \end{matrix}\right\} \text{components of } \overrightarrow{F^S}{}_y$$

$$\left.\begin{matrix} F^S{}_{zx} \\ F^S{}_{zy} \\ F^S{}_{zz} \end{matrix}\right\} \text{components of } \overrightarrow{F^S}{}_z$$

Now, let us divide the force components by the appropriate areas to get the components of the stresses.

$$\left.\begin{matrix} \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{matrix}\right\} \text{components of } \vec{\tau}_x$$

$$\left.\begin{matrix} \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \end{matrix}\right\} \text{components of } \vec{\tau}_y$$

$$\left.\begin{matrix} \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \end{matrix}\right\} \text{components of } \vec{\tau}_z$$

τ_{ij} denotes shear stress when $i \neq j$, and normal stress when $i = j$; both shear stress and normal stress arise due to molecular aspects.

Let us first consider only the x *component* of momentum rate due to molecular aspects.

Entry Rates

x direction = $\tau_{xx}|_x \Delta y \Delta z$
y direction = $\tau_{yx}|_y \Delta x \Delta z$
z direction = $\tau_{zx}|_z \Delta x \Delta y$

Exit Rates

x direction = $\tau_{xx}|_{x+\Delta x} \Delta y \Delta z$
y direction = $\tau_{yx}|_{y+\Delta y} \Delta x \Delta z$
z direction = $\tau_{zx}|_{z+\Delta z} \Delta x \Delta y$

Thus, the net x momentum rate due to molecular aspects is

$$\Delta y \Delta z[\tau_{xx}|_x - \tau_{xx}|_{x+\Delta x}] + \Delta x \Delta z[\tau_{yx}|_y - \tau_{yx}|_{y+\Delta y}] + \Delta x \Delta y[\tau_{zx}|_z - \tau_{zx}|_{z+\Delta z}]$$

Forces

Let us consider two important forces that usually act on the volume element, namely fluid pressure force and gravity. If there are other forces acting on the volume element, we need to consider them as additive terms in each direction.

The resultant force in the x direction is

$$\Delta y \Delta z(p|_x - p|_{x+\Delta x}) + \rho g_x \Delta x \Delta y \Delta z$$

where $p = f(\rho, T)$.

Accumulation

The accumulation of the x momentum within the volume element is

$$\Delta x \Delta y \Delta z\left(\frac{\partial \rho v_x}{\partial t}\right)$$

If we substitute the above terms for the x direction in the general momentum balance equation Eq. 3.3-1, divide by $\Delta x \Delta y \Delta z$, and take the limit as Δx, Δy, and $\Delta z \to 0$, we get

$$\frac{\partial(\rho v_x)}{\partial t} = -\left(\frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z}\right) - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) - \frac{\partial p}{\partial x} + \rho g_x \tag{3.4-1}$$

A similar exercise in the y and z directions would give

$$\frac{\partial(\rho v_y)}{\partial t} = -\left(\frac{\partial(\rho v_x v_y)}{\partial x} + \frac{\partial(\rho v_y v_y)}{\partial y} + \frac{\partial(\rho v_z v_y)}{\partial z}\right) - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) - \frac{\partial p}{\partial y} + \rho g_y \tag{3.4-2}$$

$$\frac{\partial(\rho v_z)}{\partial t} = -\left(\frac{\partial(\rho v_x v_z)}{\partial x} + \frac{\partial(\rho v_y v_z)}{\partial y} + \frac{\partial(\rho v_z v_z)}{\partial z}\right) - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) - \frac{\partial p}{\partial z} + \rho g_z \tag{3.4-3}$$

In compact, vectorial notation

$\dfrac{\partial(\rho\vec{v})}{\partial t}$ $=$	$-[\vec{\nabla}.\rho\vec{v}\vec{v}]$	$-[\vec{\nabla}.\tilde{\tau}]$	$-\vec{\nabla}\, p$	$+\rho\,\vec{g}$
Rate of increase in momentum per unit volume	Rate of gain in momentum by convection per unit volume	Rate of gain in momentum by viscous effects per unit volume	Pressure force on the element per unit volume	Gravitational force on the element per unit volume

(3.4-4)

There are a couple of terms in Eq. 3.4-4 that could be new. A review of Eqs. 3.4-1, 3.4-2 and 3.4-3 will reveal that $\tilde{\tau}$ has 9 terms. τ is a second order tensor with 9 components that can be represented by

$$\tilde{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

See Appendix 1 for more on tensor algebra.

Similarly, $\vec{v}\vec{v}$ is a new concept. Note that it is neither a dot product nor a cross product. A review of Eqs. 3.4-1 to 3.4-3 (first terms on the LHS) will reveal that $\vec{v}\vec{v}$ has 9 terms. $\vec{v}\vec{v}$ is known as the 'dyadic product' and is a special form of second order tensor. The dyadic product of two vectors $\vec{v}$ and $\vec{w}$ is

$$\vec{v}\vec{w} = \begin{pmatrix} v_x w_x & v_x w_y & v_x w_z \\ v_y w_x & v_y w_y & v_y w_z \\ v_z w_x & v_z w_y & v_z w_z \end{pmatrix}$$

See Appendix 1 for more on dyad algebra.

Now, let us consider Eq. 3.4-1 written as

$$\frac{\partial(\rho v_x)}{\partial t} + \left(\frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z}\right) = -\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) - \frac{\partial p}{\partial x} + \rho g_x$$

The LHS can be expanded as

$$\rho\frac{\partial v_x}{\partial t} + v_x\frac{\partial \rho}{\partial t} + \left(\rho v_x\frac{\partial v_x}{\partial x} + v_x\frac{\partial \rho v_x}{\partial x} + \rho v_y\frac{\partial v_x}{\partial y} + v_x\frac{\partial \rho v_y}{\partial y} + \rho v_z\frac{\partial v_x}{\partial z} + v_x\frac{\partial \rho v_z}{\partial z}\right)$$

$$= \rho\frac{\partial v_x}{\partial t} + v_x\frac{\partial \rho}{\partial t} + v_x\left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right) + \left(\rho v_x\frac{\partial v_x}{\partial x} + \rho v_y\frac{\partial v_y}{\partial y} + \rho v_z\frac{\partial v_x}{\partial z}\right)$$

$$= \rho \frac{\partial v_x}{\partial t} + v_x \frac{\partial \rho}{\partial t} + v_x \left(\rho \frac{\partial v_x}{\partial x} + v_x \frac{\partial \rho}{\partial x} + \rho \frac{\partial v_y}{\partial y} + v_y \frac{\partial \rho}{\partial y} + \rho \frac{\partial v_z}{\partial z} + v_z \frac{\partial \rho}{\partial z} \right)$$

$$+ \rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right)$$

$$= \left\{ v_x \frac{\partial \rho}{\partial t} + \rho v_x \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) + v_x \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) \right\}$$

$$+ \rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right)$$

$$= \{E\} + \rho \frac{D v_x}{Dt}$$

where

$$E = v_x \left(\frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho v_x \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$

Using the equation of continuity, Eq. 1.4.3-6, the first term on the RHS of the equation above can be written as the negative of the second term on the RHS. Thus

$$E = v_x \left[-\rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \right] + \rho v_x \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Thus, Eq. 3.4-1 can be written as

$$\rho \frac{D v_x}{Dt} = -\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) - \frac{\partial p}{\partial x} + \rho g_x$$

The other two components (y and z) of momentum rate can be similarly expressed and added together, to get a 3-D representation

$\rho \frac{D\vec{v}}{Dt}$	$=$	$-[\vec{\nabla} . \tilde{\tau}]$	$-\vec{\nabla} p$	$+\rho \vec{g}$
$\frac{\text{Mass}}{\text{Volume}} \times \text{Acceleration}$		Viscous forces on the element per unit volume	Pressure force on the element per unit volume	Gravitational force on the element per unit volume

(3.4-5)

Table 3.4-1 The equations of motion in rectangular Cartesian coordinates

x direction

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p}{\partial x}-\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)+\rho g_x \qquad \text{(A1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x \qquad \text{(A2)}$$

y direction

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial p}{\partial y}-\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)+\rho g_y \qquad \text{(B1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\rho g_y \qquad \text{(B2)}$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}-\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right)+\rho g_z \qquad \text{(C1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z \qquad \text{(C2)}$$

The components of Eq. 3.4-5, in different coordinate systems are given in Tables 3.4-1 to 3.4-3; the order of the first two terms on the RHS of Eq. 3.4-5 has been reversed in the tables. To determine velocity distributions and to derive further useful expressions, we need to represent the stresses in terms of velocity gradients and fluid properties. The equations in Tables 3.4-4 to 3.4-6, which give the components of the stress tensor for a Newtonian fluid in the three coordinate systems, can be used toward this objective.

Substituting the expressions from Table 3.4-4 in the momentum balances for the three directions, we get

$$\begin{aligned}\rho\frac{Dv_x}{Dt}&=\frac{\partial}{\partial x}\left[2\mu\frac{\partial v_x}{\partial x}-\frac{2}{3}\mu(\vec{\nabla}.\vec{v})\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)\right]\\&\quad+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right)\right]-\frac{\partial p}{\partial x}+\rho\, g_x\end{aligned} \qquad \text{(3.4-6)}$$

Table 3.4-2 The equations of motion in cylindrical coordinates

r direction

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=$$
$$-\frac{\partial p}{\partial r}-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}-\frac{\tau_{\theta\theta}}{r}+\frac{\partial \tau_{rz}}{\partial z}\right)+\rho g_r \qquad \text{(A1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=$$
$$-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho g_r \qquad \text{(A2)}$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=$$
$$-\frac{1}{r}\frac{\partial p}{\partial \theta}-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta})+\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta}+\frac{\partial \tau_{\theta z}}{\partial z}\right)+\rho g_\theta \qquad \text{(B1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=$$
$$-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\rho g_\theta \qquad \text{(B2)}$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=$$
$$-\frac{\partial p}{\partial z}-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \tau_{zz}}{\partial z}\right)+\rho g_z \qquad \text{(C1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=$$
$$-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z \qquad \text{(C2)}$$

Table 3.4-3 The equations of motion in spherical coordinates*

r direction

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right)=-\frac{\partial p}{\partial r}$$
$$-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr})+\frac{1}{r\sin\theta}\frac{\partial(\tau_{r\theta}\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\tau_{r\theta}}{\partial\phi}-\frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r}\right)+\rho g_r \qquad \text{(A1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right)=-\frac{\partial p}{\partial r}$$
$$+\mu\left(\nabla^2 v_r-\frac{2}{r^2}v_r-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{2}{r^2}v_\theta\cot\theta-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)+\rho g_r \qquad \text{(A2)}$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta})+\frac{1}{r\sin\theta}\frac{\partial(\tau_{\theta\theta}\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi}+\frac{\tau_{r\theta}}{r}-\frac{\cot\theta}{r}\tau_{\phi\phi}\right)+\rho g_\theta \qquad \text{(B1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+\mu\left(\nabla^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2\sin^2\theta}-\frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial v_\phi}{\partial\phi}\right)+\rho g_\theta \qquad \text{(B2)}$$

ϕ direction

$$\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi v_r}{r}+\frac{v_\theta v_\phi}{r}\cot\theta\right)=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}$$
$$-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\phi})+\frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\tau_{\phi\phi}}{\partial\phi}+\frac{\tau_{r\phi}}{r}+\frac{2\cot\theta}{r}\tau_{\theta\phi}\right)+\rho g_\phi \qquad \text{(C1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi v_r}{r}+\frac{v_\theta v_\phi}{r}\cot\theta\right)=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}$$
$$+\mu\left(\nabla^2 v_\phi-\frac{v_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\phi}\right)+\rho g_\phi \qquad \text{(C2)}$$

*Note that $\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)$

Table 3.4-4 Components of the stress tensor for Newtonian fluids in rectangular Cartesian coordinates

$$\tau_{xy} = \tau_{yx} = -\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \quad \text{(A)}$$

$$\tau_{yz} = \tau_{zy} = -\mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \quad \text{(B)}$$

$$\tau_{zx} = \tau_{xz} = -\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) \quad \text{(C)}$$

$$\tau_{xx} = -2\mu\frac{\partial v_x}{\partial x} + \frac{2}{3}\mu\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \quad \text{(D)}$$

$$\tau_{yy} = -2\mu\frac{\partial v_y}{\partial y} + \frac{2}{3}\mu\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \quad \text{(E)}$$

$$\tau_{zz} = -2\mu\frac{\partial v_z}{\partial z} + \frac{2}{3}\mu\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \quad \text{(F)}$$

Table 3.4-5 Components of the stress tensor for Newtonian fluids in cylindrical coordinates

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) \quad \text{(A)}$$

$$\tau_{\theta z} = \tau_{z\theta} = -\mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) \quad \text{(B)}$$

$$\tau_{zr} = \tau_{rz} = -\mu\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) \quad \text{(C)}$$

$$\tau_{rr} = -2\mu\frac{\partial v_r}{\partial r} + \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right) \quad \text{(D)}$$

$$\tau_{\theta\theta} = -2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right) \quad \text{(E)}$$

$$\tau_{zz} = -2\mu\frac{\partial v_z}{\partial z} + \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right) \quad \text{(F)}$$

Table 3.4-6 Components of the stress tensor for Newtonian fluids in spherical coordinates

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) \tag{A}$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = -\mu\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right) \tag{B}$$

$$\tau_{\phi r} = \tau_{r\phi} = -\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right) \tag{C}$$

$$\tau_{rr} = -2\mu\frac{\partial v_r}{\partial r} + \frac{2}{3}\mu\left(\frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(v_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right) \tag{D}$$

$$\tau_{\theta\theta} = -2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right) + \frac{2}{3}\mu\left(\frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(v_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right) \tag{E}$$

$$\begin{aligned}\tau_{\phi\phi} = &-2\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right)\\ &+\frac{2}{3}\mu\left(\frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(v_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)\end{aligned} \tag{F}$$

$$\begin{aligned}\rho\frac{Dv_y}{Dt} = &\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[2\mu\frac{\partial v_y}{\partial y} - \frac{2}{3}\mu(\vec{\nabla}.\vec{v})\right]\\ &+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)\right] - \frac{\partial p}{\partial y} + \rho g_y\end{aligned} \tag{3.4-7}$$

$$\begin{aligned}\rho\frac{Dv_z}{Dt} = &\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)\right]\\ &+\frac{\partial}{\partial z}\left[2\mu\frac{\partial v_z}{\partial z} - \frac{2}{3}\mu(\vec{\nabla}.\vec{v})\right] - \frac{\partial p}{\partial z} + \rho g_z\end{aligned} \tag{3.4-8}$$

The equations of motion (Eqs. 3.4-6 to 3.4-8), equation of state, $p = f(\rho)$, and variation of $\mu = f(\rho)$ completely determine the pressure, density and velocity components in the flowing Newtonian fluid.

When ρ and μ are constant, since $\vec{\nabla}.\vec{v} = 0$ according to the continuity equation, the equation of motion can be written as

$$\rho\frac{D\vec{v}}{Dt} = \mu\vec{\nabla}^2\vec{v} - \vec{\nabla}p + \rho\vec{g} \tag{3.4-9}$$

Equation 3.4-9 is called the Navier-Stokes equation.

If viscous effects are not important, $\vec{\nabla} \cdot \tilde{\tau} = 0$. Then, Eq. 3.4-5 becomes

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} p + \rho \vec{g} \tag{3.4-10}$$

Equation 3.4-10 is called the Euler equation.

3.4.1 Applications of the Equations of Motion: Steady State Falling Film

The equations of motion given in Tables 3.4-1 to 3.4-3 can be used to solve problems more easily compared to using shell balances. Note that simpler equations are available in certain cases (e.g. for Eq. A2 in Table 3.4-3), which are given in other texts.

To illustrate this, let us solve the steady state falling film problem that we did in Section 3.3, using the equation of motion.

For convenience in this system geometry, let us use rectangular coordinates. Let us use Eq. C1 of Table 3.4-1 to get the shear stress profile. Note that $v_x = 0$, $v_y = 0$. Therefore, only Eq. C1 with v_z is relevant.

$$\rho\left(\overset{=0,\ \text{SS}}{\cancel{\frac{\partial v_z}{\partial t}}} + \overset{\substack{=0,\\ v_x=0}}{\cancel{v_x}}\frac{\partial v_z}{\partial x} + \overset{\substack{=0,\\ v_y=0}}{\cancel{v_y}}\frac{\partial v_z}{\partial y} + v_z \overset{\substack{=0,\ v_z \text{ is}\\ \text{not a } f(z)}}{\cancel{\frac{\partial v_z}{\partial z}}}\right) = -\overset{\substack{=0,\ \text{chosen}\\ \text{condition}}}{\cancel{\frac{\partial p}{\partial z}}} - \left(\frac{\partial \tau_{xz}}{\partial x} + \overset{\substack{=0,\ \tau_{yz} \text{ is}\\ \text{not a } f(y)}}{\cancel{\frac{\partial \tau_{yz}}{\partial y}}} + \overset{\substack{=0,\ \tau_{zz} \text{ is}\\ \text{not a } f(z)}}{\cancel{\frac{\partial \tau_{zz}}{\partial z}}}\right) + \rho g_z \tag{3.4.1-1}$$

Since

$$g_z = g \cos\beta$$

we get

$$0 = -\frac{\partial \tau_{xz}}{\partial x} + \rho g \cos\beta$$

which is the same equation as Eq. 3.3-3.

To get the velocity profile of a Newtonian fluid, we can directly begin from Eq. C2 of Table 3.4-1.

$$\rho\left(\overset{=0,\ \text{SS}}{\cancel{\frac{\partial v_z}{\partial t}}} + \overset{\substack{=0,\\ v_x=0}}{\cancel{v_x}}\frac{\partial v_z}{\partial x} + \overset{\substack{=0,\\ v_y=0}}{\cancel{v_y}}\frac{\partial v_z}{\partial y} + v_z \overset{\substack{=0,\ v_z \text{ is}\\ \text{not a } f(z)}}{\cancel{\frac{\partial v_z}{\partial z}}}\right) = -\overset{\substack{=0,\ \text{chosen}\\ \text{condition}}}{\cancel{\frac{\partial p}{\partial z}}} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \overset{\substack{=0,\ v_z \text{ is}\\ \text{not a } f(y)}}{\cancel{\frac{\partial^2 v_z}{\partial y^2}}} + \overset{\substack{=0,\ v_z \text{ is}\\ \text{not a } f(z)}}{\cancel{\frac{\partial^2 v_z}{\partial z^2}}}\right) + \rho g_z$$

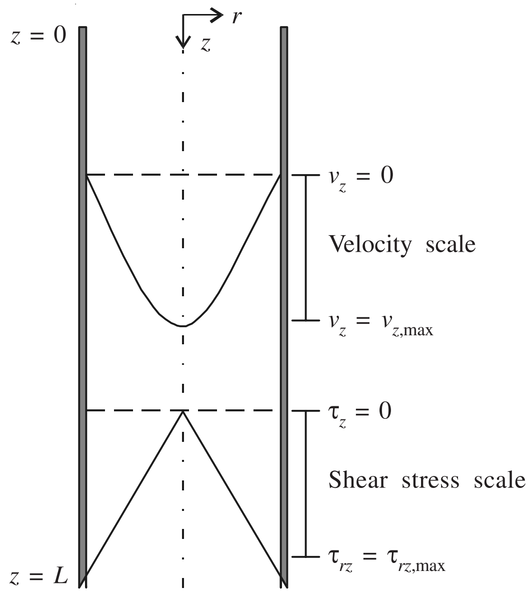


Fig. 3.4.2-1 Velocity profile and shear stress profile for a laminar flow in a cylindrical pipe

$$0 = \mu \frac{\partial^2 v_z}{\partial x^2} + \rho g \cos \beta \tag{3.4.1-2}$$

i.e.

$$\mu \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial x}\right) = -\rho g \cos \beta$$

or

$$\frac{\partial v_z}{\partial x} = -\left(\frac{\rho g \cos \beta}{\mu}\right) x$$

which is the same equation as Eq. 3.3-7.

3.4.2 Flow in a Cylindrical Pipe

Let us consider the laminar flow through a pipe of cylindrical cross-section. The results have significance in a variety of situations ranging from flow in micro-devices, flow of body fluids in the human body, and at least as a first approximation, to the flow of liquids and gases in the bio-process industry.

Figure 3.4.2-1 shows the laminar flow of a Newtonian fluid down a cylindrical pipe placed vertically. Let the flow be well-developed, i.e. the axial velocity at any particular radial position in the pipe is not dependent on the length, $v_z \neq f(z)$.

Let us derive the profiles of shear rates and velocities across the tube diameter.

Since the system of interest is cylindrical, it is best to use cylindrical coordinates here. Thus, the Table 3.4-2 is relevant: since there is no radial flow, the r component is irrelevant. Similarly, since there is no flow around the axis of the cylinder, the θ component is irrelevant. The only relevant component is, thus, the z component.

Let us first use Eq. A2 from Table 3.4-2

$$\rho\left(\overset{(\text{SS})}{\cancelto{0}{\frac{\partial v_r}{\partial t}}} + v_r \overset{(v_r=0)}{\cancelto{0}{\frac{\partial v_r}{\partial r}}} + \overset{(v_\theta=0)}{\cancelto{0}{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}} - \overset{(v_\theta=0)}{\cancelto{0}{\frac{v_\theta^2}{r}}} + v_z \overset{\substack{(v_r \neq f(z))\\(v_r=0)}}{\cancelto{0}{\frac{\partial v_r}{\partial z}}}\right)$$

$$= -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\overset{(v_r=0)}{\cancelto{0}{(r v_r)}}\right) + \frac{1}{r^2}\overset{\substack{v_r \neq f(\theta)\\(v_r=0)}}{\cancelto{0}{\frac{\partial^2 v_r}{\partial \theta^2}}} + \frac{2}{r^2}\overset{(v_\theta=0)}{\cancelto{0}{\frac{\partial v_\theta}{\partial \theta}}} + \overset{\substack{(v_r \neq f(z))\\(v_r=0)}}{\cancelto{0}{\frac{\partial^2 v_r}{\partial z^2}}}\right] + \overset{(g_r=0)}{\cancelto{0}{\rho g_r}} \tag{3.4.2-1}$$

The equation reduces to

$$\frac{\partial p}{\partial r} = 0 \text{ or } p \neq f(r) \tag{3.4.2-2}$$

This is an important insight, i.e. the pressure across the cross-section of a pipe at a particular length in laminar flow through a pipe does not depend on the radial position.

Let us next consider Eq. B2 from Table 3.4-2

$$\rho\left(\overset{(\text{SS})}{\cancelto{0}{\frac{\partial v_\theta}{\partial t}}} + v_r \overset{(v_r=0)}{\cancelto{0}{\frac{\partial v_\theta}{\partial r}}} + \overset{(v_\theta=0)}{\cancelto{0}{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}} + \overset{(v_r,\, v_\theta=0)}{\cancelto{0}{\frac{v_r v_\theta}{r}}} + v_z \overset{(v_\theta \neq f(z))}{\cancelto{0}{\frac{\partial v_\theta}{\partial z}}}\right)$$

$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\overset{(v_\theta=0)}{\cancelto{0}{(r v_\theta)}}\right) + \frac{1}{r^2}\overset{\substack{v_\theta \neq f(\theta)\\(v_\theta=0)}}{\cancelto{0}{\frac{\partial^2 v_\theta}{\partial \theta^2}}} + \frac{2}{r^2}\overset{\substack{v_r \neq f(\theta)\\(v_r=0)}}{\cancelto{0}{\frac{\partial v_r}{\partial \theta}}} + \overset{\substack{v_\theta \neq f(z)\\(v_\theta=0)}}{\cancelto{0}{\frac{\partial^2 v_\theta}{\partial z^2}}}\right] + \overset{(g_\theta=0)}{\cancelto{0}{\rho g_\theta}} \tag{3.4.2-3}$$

The equation reduces to

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} = 0$$

Thus

$$\frac{\partial p}{\partial \theta} = 0$$

or

$$p \neq f(\theta) \tag{3.4.2-4}$$

Thus, the pressure does not depend on the angular position in the pipe.

Now, let us consider Eq. C2 from Table 3.4-2

$$\rho\left(\underbrace{\cancel{\frac{\partial v_z}{\partial t}}}_{\substack{0\\(\text{SS})}} + \underbrace{\cancel{v_r\frac{\partial v_z}{\partial r}}}_{\substack{0\\(v_r = 0)}} + \underbrace{\cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}}_{\substack{0\\(v_\theta = 0)}} + \underbrace{\cancel{v_z\frac{\partial v_z}{\partial z}}}_{\substack{0\\(v_z \neq f(z))}}\right)$$

$$= -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \underbrace{\frac{1}{r^2}\cancel{\frac{\partial^2 v_z}{\partial \theta^2}}}_{\substack{0\\v_z \neq f(\theta)}} + \underbrace{\cancel{\frac{\partial^2 v_z}{\partial z^2}}}_{\substack{0\\(v_z \neq f(z))}}\right] + \rho g_z \tag{3.4.2-5}$$

While considering the terms in the above equation, $v_z \neq f(\theta)$ because the flow, in this case, occurs in cylindrical layers. In other words, the axial velocities at all points at a particular radius, and length do not vary with θ.

$$-\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = -\frac{\partial p}{\partial z} + \rho g_z \tag{3.4.2-6}$$

Let us define

$$P = p - \rho g_z z$$

Since $g_z = g$, we can write

$$\frac{\partial p}{\partial z} - \rho g = \frac{\partial (P - \rho g z)}{\partial z} = \frac{\partial P}{\partial z}$$

Therefore

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{\partial P}{\partial z} \tag{3.4.2-7}$$

We know from Eqs. 3.4.2-2 and 3.4.2-4 that $p \neq f(r)$ and $p \neq f(\theta)$.

Thus, $P = p + \rho g z \neq f(r)$ and $\neq f(\theta)$.

Since $p = f(z)$ alone, the partial derivative on the RHS can be replaced by an ordinary derivative.

Similarly, v_z and r are only $f(r)$; they are not $f(\theta)$ or $f(z)$. Thus the partial derivative on the LHS can also be replaced by ordinary derivative, and the equation becomes

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)=\frac{dP}{dz} \tag{3.4.2-8}$$

Besides, the LHS is a function of r and the RHS is a function of z, i.e.

$$\frac{\mu}{r}\frac{df(r)}{dr}=\frac{df(z)}{dz} \tag{3.4.2-9}$$

This is possible only if each derivative equals a constant, say C_1.

Let us take the RHS of Eq. 3.4.2-8 first

$$\frac{dP}{dz}=C_1 \tag{3.4.2-10}$$

Then

$$P = C_1 z + C_2 \tag{3.4.2-11}$$

The relevant boundary conditions are

$$\text{At } z = 0, \quad P = P_0$$
$$\text{At } z = L, \quad P = P_L$$

Thus

$$C_2 = P_0$$

$$C_1 = \frac{P_L - P_0}{L}$$

Therefore

$$P=\left(\frac{P_L-P_0}{L}\right)z+P_0 \tag{3.4.2-12}$$

Now let consider the LHS and equate it to the same C_1

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)=C_1=\frac{\Delta P}{L}$$

where $\Delta P = P_L - P_0$.

Thus

$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)=\frac{\Delta P}{L}\times\frac{r}{\mu}$$

Upon integration, we get

$$r\frac{dv_z}{dr} = \frac{\Delta P}{L}\frac{r^2}{2\mu} + C_2$$

At $r = 0$, C_2 must be equal to 0.

Therefore

$$\frac{dv_z}{dr} = \frac{\Delta P\, r}{2\mu L} \qquad (3.4.2\text{-}13)$$

Integrating this, we get

$$v_z = \frac{\Delta P\, r^2}{4\mu\, L} + C_3 \qquad (3.4.2\text{-}14)$$

Now, using the BC that at $r = R$, $v_z = 0$ ('no slip boundary condition')

$$C_3 = -\frac{\Delta P\, R^2}{4\mu\, L}$$

Thus

$$v_z = \frac{\Delta P}{4\mu L}(r^2 - R^2) = \frac{(-\Delta P)R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right] \qquad (3.4.2\text{-}15)$$

Therefore, the velocity profile is parabolic across the diameter, as shown in Fig. 3.4.2-1.

Note that $\Delta P = P_L - P_0$; typically, for the flow to occur, $P_L < P_0$, and thus $(-\Delta P)$ is positive.

The maximum velocity occurs at $r = 0$ (from Eq. 3.4.2-15), i.e., at the centre line (axis) of the tube.

$$v_{z,\max} = \frac{(-\Delta P)R^2}{4\mu L} \qquad (3.4.2\text{-}16)$$

The average velocity across the cross-section

$$v_{z,\text{avg}} = \frac{\int_0^{2\pi}\int_0^R v_z\, r\,dr\,d\theta}{\int_0^{2\pi}\int_0^R r\,dr\,d\theta}$$

$$= \frac{\displaystyle\int_0^{2\pi}\int_0^R \frac{(-\Delta P)R^2}{4\mu L}\left\{1 - \left(\frac{r}{R}\right)^2\right\} r\,dr\,d\theta}{\dfrac{R^2}{2}\times 2\pi}$$

$$= \frac{(-\Delta P)R^2}{\pi R^2 \times 4\mu L}\left[\int_0^{2\pi}\int_0^R r dr d\theta - \int_0^{2\pi}\int_0^R \frac{r^2}{R^2} r dr d\theta\right]$$

$$= \frac{(-\Delta P)}{4\pi\mu L}\left\{\left[\frac{R^2}{2}\times 2\pi - \frac{r^4}{4\,R^2}\right]_0^R 2\pi\right\}$$

$$= \frac{(-\Delta P)}{4\pi\mu L}\left(\frac{R^2}{2} - \frac{R^2}{4}\right)2\pi$$

$$v_{z,\text{avg}} = \frac{(-\Delta P)\times R^2}{2\mu L \times 4} = \frac{(-\Delta P)R^2}{8\mu L} = \frac{1}{2}(v_{z,\max}) \qquad (3.4.2\text{-}17)$$

The volumetric flow rate, Q = Area × $v_{z,\text{avg}}$. Thus

$$Q = \frac{\pi R^2 \times (-\Delta P)R^2}{8\mu L} = \frac{\pi}{8\mu L}R^4(-\Delta P) \qquad (3.4.2\text{-}18)$$

Thus

$$\begin{aligned} Q \quad &\alpha\ (-\ \Delta P) \\ &\alpha\ R^4 \end{aligned}$$

If the radius is doubled at the same ($-\ \Delta P$), the volumetric flow rate increases 16-fold.

Equation 3.4.2-18 is known as the Hagen-Poiseuille (pronounced as Pwah-zoo-yuh; here 'oo' is pronounced as in 'book') equation.

Let us now use Eq. C1 of Table 3.4-2 to derive an expression for the shear stress profile. To visualise $\tau_{\theta z}$, note that the first subscript, θ, refers to the direction of the velocity gradient, and the second subscript, z, refers to the direction of the stress or the force. If v_z is different at different θ, then $\tau_{\theta z}$ could arise. But, that is not the case here, in a laminar flow. A similar visualisation would provide $\tau_{zz} \neq f(z)$, since v_z does not vary with z for this well developed flow.

$$\rho\left(\overset{\substack{(\text{SS})\\0}}{\cancel{\frac{\partial v_z}{\partial t}}} + v_r \overset{\substack{(v_r = 0)\\0}}{\cancel{\frac{\partial v_z}{\partial r}}} + \overset{\substack{(v_\theta = 0)\\0}}{\cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}} + v_z \overset{\substack{(v_z \neq f(z))\\0}}{\cancel{\frac{\partial v_z}{\partial z}}}\right)$$

$$= -\frac{\partial p}{\partial z} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{rz}) + \overset{\substack{\tau_{\theta z} \neq f(\theta)\\0}}{\cancel{\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}}} + \overset{\substack{\tau_{zz} \neq\\(f(z))\\0}}{\cancel{\frac{\partial \tau_{zz}}{\partial z}}}\right] + \rho g_z \qquad (3.4.2\text{-}19)$$

The terms that remain yield

$$\frac{1}{r}\left(\frac{\partial}{\partial r}(r\tau_{rz})\right) = -\frac{\partial p}{\partial z} + \rho g_z \tag{3.4.2-20}$$

If we define $P = p - \rho g_z z$, with the recognition that $g_z = g$, we can write the above equation as

$$\frac{1}{r}\left(\frac{\partial}{\partial r}(r\tau_{rz})\right) = -\frac{\partial P}{\partial z} \tag{3.4.2-21}$$

Using the same argument that we used for solving Eq. 3.4.2-7, the solution becomes

$$\tau_{rz} = -\frac{\Delta P\, r}{2L} + C' \tag{3.4.2-22}$$

B.C.: $\tau_{rz} = 0$ at $r = 0$.

Thus

$$\tau_{rz} = \left(-\frac{\Delta P}{2L}\right) r \tag{3.4.2-23}$$

The linear profile for τ_{rz} is shown in Fig. 3.4.2-1.

3.4.2.1 Capillary Flow

Flow through capillaries, i.e. tubes of very small linear dimension (radius, in the case of capillaries with circular cross-section) of the order of microns, is usually laminar. Capillary flows have great significance in microfluidics, flow through vasculature, flow through porous media, and many other situations of biological interest. Since the flow is laminar, the equations developed in the previous section are also applicable for flow through capillaries of circular cross-section.

Capillary flow arises because the force of attraction (adhesion force) between the liquid molecules and the molecules of the walls of the capillary duct are stronger than the attractive forces between the liquid molecules (cohesive forces). This causes the edges of the fluid near the capillary wall to rise, and due to cohesion, the liquid follows (or is dragged along by the stronger adhesion) as a whole, which results in the flow. We know from high/higher secondary school physics that cohesion results in a force that is usually represented as a force per unit length, or surface tension, γ. The capillary pressure due to surface tension at that point, or the meniscus, in a capillary of radius, r, is given by the appropriate simplification of the

Young-Laplace equation ($p_{st} = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$ where R_1 and R_2 are the radii of curvature (see Berg 2009) for a derivation of the Young-Laplace equation) as

$$p_{st} = \frac{2\gamma}{r}\cos\theta \tag{3.4.2.1-1}$$

where θ is the contact angle (wetting angle) between the liquid and the capillary wall.

Note that this pressure is inversely proportional to the radius of the duct. It becomes predominant in capillaries, and provides the ‘driving force’ for the bulk flow of the liquid through the capillary, even if other ‘driving forces’ are absent. When other ‘driving forces’ such as those provided by a liquid column, or an external pump are present, the pressures can be added to get the total pressure difference for the flow ($-\Delta P$). For example, to obtain the flow rate in capillary flow when no other ‘driving forces’ for the flow are present, the use of the Hagen-Poiseuille relationship, Eq. 3.4.2-18, gives

$$Q = \frac{\pi}{8\mu L} r^4 \left(\frac{2\gamma}{r}\right)\cos\theta = \frac{\pi\gamma}{4\mu L} r^3 \cos\theta \tag{3.4.2.1-2}$$

Since the flow rate is a product of the cross-sectional area and the penetration velocity, the penetration velocity (v_p) can be obtained by dividing the above equation by the cross-sectional area πr^2

$$v_p = \frac{dL}{dt} = \frac{\gamma}{4\mu L} r\cos\theta \tag{3.4.2.1-3}$$

In microfluidic situations, the above equation can be integrated to get the position of the liquid front along the capillary as a function of time.

Capillary flow in porous media: Porous media is a term that refers to any medium that has a solid matrix with interconnected interstitial spaces, through which there is movement of some species of interest. For example, soil is a porous medium through which water, pollutants, fines, etc., can travel. Sometimes, interstitial spaces can be considered as a set of capillary tubes, and thus capillary flow through porous media is an area with wide applications.

Interestingly, many substances of biological interest can be considered as porous media. For example, any tissue, including whole organs such as liver, kidney, heart, brain, etc., can be treated as porous media because they contain cells that are dispersed, and connected voids through which nutrients, drugs and other substances travel to reach the cells. Tissue regeneration, which is used to grow artificial organs, typically happens on a scaffold, and this system can be considered a porous medium. Similarly, biological pollution treatment systems such as the trickling filter, or the matrix in which cells are immobilised in a type of bioreactor, can be treated as porous media.

To obtain the kinetics of liquid movement by capillary flow into a porous medium, the medium is typically treated as consisting of cylindrical capillary tubes. Then, the distance penetrated by the liquid into the porous medium L can be obtained by the Washburn (Washburn 1921) or Rideal (Rideal 1922) equation, or by the integration of Eq. 3.4.2.1-3 as

$$L = \left(\frac{\gamma}{2\mu} r \cos\theta \right)^{0.5} t^{0.5} \qquad (3.4.2.1\text{-}4)$$

There have been many improvements to this equation that take into account tortuosity (crooked or non-straight nature of the capillary channels in the porous medium), wettability of the liquid, and other relevant parameters. As a starting point for further reading, the interested reader is referred to the paper by Yang et al. (1988).

3.4.3 Tangential Annular Flow

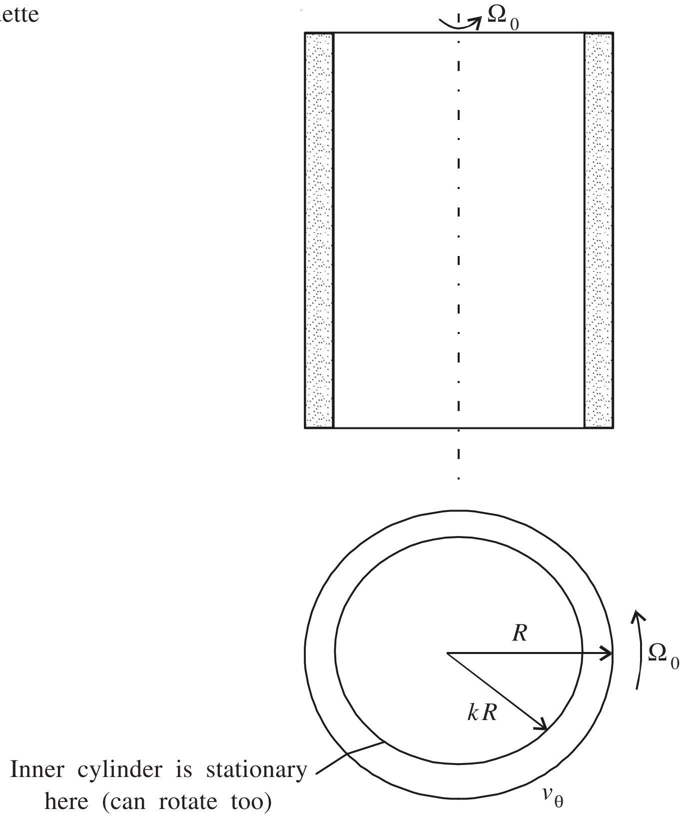
Tangential annular flow between two concentric cylinders is used in couette-flow rheometers to measure viscosity of a variety of biological fluids or bio-products such as xanthan gum. It is also used to study the effects of a 'defined' shear on cells (Sahoo et al. 2003). For our study, let us first consider the tangential annular flow of a Newtonian fluid (Fig. 3.4.3-1). We are interested in the tangential velocity profile between the cylinders, the relevant shear stress distribution, and the torque which is required to turn the outer shaft at steady state.

This is a cylindrical system, and hence it is most convenient to use cylindrical coordinates for analysis. From Eq. A2 of Table 3.4-2, we get the equation of motion in the r direction as

$$\rho\left(\overset{=0,\ \text{SS}}{\cancel{\frac{\partial v_r}{\partial t}}} + \overset{=0,\ v_r=0}{\cancel{v_r}} \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \overset{=0,\ v_r \neq f(\theta)}{\cancel{\frac{\partial v_r}{\partial \theta}}} - \frac{v_\theta^2}{r} + \overset{=0,\ v_z=0}{\cancel{v_z}} \frac{\partial v_r}{\partial z} \right)$$

$$= -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r} \overset{=0,\ v_r=0}{\cancel{(r v_r)}} \right) + \frac{1}{r^2} \overset{=0,\ v_r=0}{\cancel{\frac{\partial^2 v_r}{\partial \theta^2}}} - \frac{2}{r^2} \overset{=0,\ v_\theta \neq f(\theta)}{\cancel{\frac{\partial v_\theta}{\partial \theta}}} + \overset{=0,\ v_r=0}{\cancel{\frac{\partial^2 v_r}{\partial z^2}}} \right) + \overset{=0,\ g_r=0}{\cancel{\rho g_r}}$$

Fig. 3.4.3-1 Couette viscometer



Thus

$$-\rho\frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r} \tag{3.4.3-1}$$

From Eq. B2 of Table 3.4-2 (θ component)

$$\rho\left(\overset{=0\ (\mathrm{SS})}{\cancel{\frac{\partial v_\theta}{\partial t}}} + v_r\overset{(v_r=0)}{\cancelto{0}{\frac{\partial v_\theta}{\partial r}}} + \frac{v_\theta}{r}\overset{v_\theta \neq f(\theta)}{\cancelto{0}{\frac{\partial v_\theta}{\partial \theta}}} + \overset{(v_r=0)}{\cancelto{0}{\frac{v_r v_\theta}{r}}} + v_z\overset{(v_z=0)}{\cancelto{0}{\frac{\partial v_\theta}{\partial z}}}\right) = -\frac{1}{r}\overset{=0\ (p\ \text{not a}\ f(\theta))}{\cancel{\frac{\partial p}{\partial \theta}}}$$

$$+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\overset{v_\theta \neq f(\theta)}{\cancelto{0}{\frac{\partial^2 v_\theta}{\partial \theta^2}}} + \frac{2}{r^2}\overset{v_r \neq f(\theta)}{\cancelto{0}{\frac{\partial v_r}{\partial \theta}}} + \overset{v_\theta \neq f(\theta)}{\cancelto{0}{\frac{\partial^2 v_\theta}{\partial z^2}}}\right] + \overset{(g_\theta=0)}{\cancelto{0}{\rho g_\theta}}$$

In the above equation, $p \neq f(\theta)$ because the points at different angles at the same radial position cannot have different pressures.

Thus

$$0 = \mu\left[\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right)\right] \tag{3.4.3-2}$$

Since r is the only variable, the partial derivatives have been converted into ordinary derivatives.

For the z component

$$\rho\left(\overset{=0\ \text{(SS)}}{\cancel{\frac{\partial v_z}{\partial t}}} + v_r\overset{(v_r=0)}{\cancelto{0}{\frac{\partial v_z}{\partial r}}} + \frac{v_\theta}{r}\overset{v_z \neq f(\theta)}{\cancelto{0}{\frac{\partial v_z}{\partial \theta}}} + v_z\overset{(v_z=0)}{\cancelto{0}{\frac{\partial v_z}{\partial z}}}\right)$$

$$= -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\overset{(v_z=0)}{\cancelto{0}{\frac{\partial v_z}{\partial r}}}\right) + \frac{1}{r^2}\overset{(v_z=0)}{\cancelto{0}{\frac{\partial^2 v_z}{\partial \theta^2}}} + \overset{(v_z=0)}{\cancelto{0}{\frac{\partial^2 v_z}{\partial z^2}}}\right] + \rho g_z$$

Thus

$$0 = -\frac{\partial p}{\partial z} + \rho g_z \tag{3.4.3-3}$$

Integrating Eq. 3.4.3-3 with the boundary conditions (BCs) given below

At $r = kR$, $v_\theta = 0$ (inner cylinder is stationary)

At $r = R$, $v_\theta = \Omega_0 R$

$$\left[\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right)\right] = 0$$

$$\frac{1}{r}\frac{d}{dr}(rv_\theta) = C_1$$

$$\frac{d}{dr}(rv_\theta) = C_1 r$$

Let $(rv_\theta) = m$

$$\frac{d}{dr}m = C_1 r$$

$$m = \frac{C_1 r^2}{2} + C_2$$

$$rv_\theta = \frac{C_1 r^2}{2} + C_2$$

$$v_\theta = \frac{C_1 r}{2} + \frac{C_2}{r}$$

Using the first BC, we get

$$0 = \frac{C_1}{2}(kR) + \frac{C_2}{kR}$$

$$C_1 = -\left(\frac{C_2}{kR}\right) \times \frac{2}{kR} = -\frac{2C_2}{k^2 R^2}$$

Using the second BC, we get

$$\Omega_0 R = -\frac{\cancel{2}C_2}{k^2 R^{\cancel{2}}} \times \frac{\cancel{R}}{\cancel{2}} + \frac{C_2}{R}$$

$$\Omega_0 R = C_2\left(\frac{1}{R} - \frac{1}{Rk^2}\right) = \frac{C_2}{R}\left(1 - \frac{1}{k^2}\right) = \frac{C_2}{R}\left(\frac{k^2 - 1}{k^2}\right)$$

$$C_2 = \frac{\Omega_0 R^2 k^2}{k^2 - 1}$$

$$C_1 = -\frac{2}{k^2 R^2}\left(\frac{\Omega_0 R^2 k^2}{k^2 - 1}\right)$$

$$= -\left(\frac{2\Omega_0}{k^2 - 1}\right)$$

Therefore

$$v_\theta = -\frac{2\Omega_0}{k^2 - 1} \cdot \frac{r}{2} + \frac{\Omega_0 k^2 R^2}{(k^2 - 1)r}$$

$$= \frac{\Omega_0 R^2}{(1 - k^2)}\left(\frac{r}{R^2} - \frac{k^2}{r}\right)$$

$$= \frac{\Omega_0 k R^2}{(1 - k^2)}\left(\frac{r}{kR^2} - \frac{k}{r}\right)$$

$$= \frac{\Omega_0 R^2}{\frac{1}{k} - k}\frac{1}{R}\left(\frac{r}{kR} - \frac{kR}{r}\right)$$

$$v_\theta = \frac{\Omega_0 R\left(\dfrac{kR}{r} - \dfrac{r}{kR}\right)}{\left(k - \dfrac{1}{k}\right)} \tag{3.4.3-4}$$

The relevant shear stress distribution can also be obtained by using the expression for the shear stress components in cylindrical coordinates as given in Table 3.4-5. From Eq. A

$$\tau_{r\theta} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

In this case

$$\tau_{r\theta} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\cancelto{0(v_r \neq f(\theta))}{\frac{\partial v_r}{\partial \theta}}\right]$$

Using Eq. 3.4.3-4 (since r is the only independent variable, the partial derivatives can be replaced with total derivatives), we get

$$\tau_{r\theta} = -\mu\left[r\frac{d}{dr}\left\{\frac{\Omega_0 R\left(\dfrac{kR}{r} - \dfrac{r}{kR}\right)}{r\ \left(k - \dfrac{1}{k}\right)}\right\}\right]$$

$$= -\frac{\mu\Omega_0 R}{\left(k - \dfrac{1}{k}\right)}\left[r\frac{d}{dr}\left(\frac{1}{r}\left\{\frac{kR}{r} - \frac{r}{kR}\right\}\right)\right]$$

$$= -\frac{\mu\Omega_0 R}{\left(k - \dfrac{1}{k}\right)}\left[r\frac{d}{dr}\left(\frac{kR}{r^2} - \frac{1}{kR}\right)\right]$$

$$= -\frac{\mu\Omega_0 R}{\left(k - \dfrac{1}{k}\right)}\left[r\left(-\frac{2kR}{r^3}\right)\right]$$

$$= -\frac{2\mu\Omega_0 R^2 k}{\left(k - \dfrac{1}{k}\right)}\left(\frac{1}{r^2}\right)$$

$$\tau_{r\theta} = -\,2\mu\Omega_0 R^2\left(\frac{k^2}{k^2 - 1}\right)\left(\frac{1}{r^2}\right) \tag{3.4.3-5}$$

The torque that is needed to turn the outer cylinder

$$= \text{Force} \times \text{Lever arm distance}$$

$$= -\,\tau_{r\theta}|_{r=R} \times \text{Area} \times \text{Lever arm distance}$$

(the negative sign before $\tau_{r\theta}$ is to overcome the shear stress by the fluid on the wall)

$$= -\,\tau_{r\theta}|_{r=R} \times (2\pi RL) \times R$$

$$= 2\mu\Omega_0 R^2 \left(\frac{k^2}{1-k^2}\right)\left(\frac{1}{R^2}\right) \times 2\pi RL \times R$$

$$= 4\pi\mu\Omega_0 LR^2 \left(\frac{k^2}{1-k^2}\right) \qquad (3.4.3\text{-}6)$$

3.4.4 Dimensionless Numbers and Non-dimensional Analysis

As briefly mentioned in Chapter 2, when certain physical quantities are combined suitably, the resulting quantity or 'number' does not possess any dimensions. For example, as we have seen in Eq. 3.2-1, the quantity, $\dfrac{\rho v d}{\mu}$ is dimensionless, and is called the Reynolds number.

There are many advantages in using non-dimensional numbers, or in expressing relations in terms of non-dimensional numbers. Such relations may be applied more generally rather than be restricted to a particular, say, tube diameter, as we have already seen in Section 3.2 for the occurrence of, say, laminar flow in tubes. To explain further, it does not matter what the particular values of fluid density, fluid velocity, fluid viscosity and pipe diameter are; as long as their appropriate combination, the Reynolds number, is less than 2100, it will result in laminar flow.

Other advantages in using dimensionless numbers will become evident in the following sections. Since we have brought up the aspect of non-dimensional numbers, let us explore one more powerful possibility of obtaining useful relationships for design and operation, with them. The basis for this powerful possibility is Buckingham's pi theorem.

Buckingham's Pi Theorem

If there are n variables in a problem, and these variables contain m primary dimensions (e.g. M, L, T, and so three for this combination of primary dimensions), the equation relating all the variables will have $(n - m)$ dimensionless groups. Buckingham called these dimensionless groups as π groups.

Mathematically, it can be expressed as

$$f(\pi_1, \pi_2, \ldots, \pi_{n-m}) = 0 \tag{3.4.4-1}$$

The π groups must be independent of each other. In other words, it must not be possible to express any π group as some combination of the other π groups.

Let us look at dimensions first:

Fundamental	
Mass	[M]
Length	[L]
Time	[T]
Flow Geometry	
Area	$[L^2]$
Volume	$[L^3]$
Kinematic	
Velocity	$[LT^{-1}]$
Acceleration	$[LT^{-2}]$
Kinematic viscosity	$[L^2T^{-1}]$
Dynamic	
Force	$[MLT^{-2}]$
Pressure	$[ML^{-1}T^{-2}]$
Work	$[ML^2T^{-2}]$
Energy	$[ML^2T^{-2}]$
Power	$[ML^2T^{-3}]$
Momentum	$[MLT^{-1}]$
Density	$[ML^{-3}]$
Viscosity	$[ML^{-1}T^{-1}]$
Surface tension	$[MT^{-2}]$

Two conditions need to be satisfied to successfully apply the method to get useful relationships:

1. Each of the fundamental dimensions must appear in at least one of the n variables.

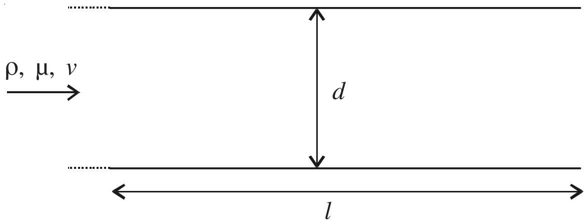


Fig. 3.4.4-1 Flow in a cylindrical pipe

2. It must not be possible to form a dimensionless group by using some of the variables themselves or the variables raised to some powers, within a recurring set. A recurring set is a group of variables that form the dimensionless groups.

Let us apply this method to a situation that we have already seen – to get an expression for pressure drop Δp in a straight pipe (Fig. 3.4.4-1).

Δp is expected to depend on d, l, ρ, μ, v. In other words

$$f(\Delta p,\ d,\ l,\ \rho,\ \mu,\ v) = 0$$

The set of variables within the brackets is the recurring set.

Number of variables (n): 6
Number of fundamental dimensions (m): 3 (M, L, T)

Therefore, from the Buckingham pi theorem, the number of dimensionless groups: $n - m = 6 - 3 = 3$.

Also, from experience, it is known that the recurring set must contain 3 (the same number as the number of dimensionless groups) variables that cannot themselves be formed into a dimensionless group. Thus, l and d cannot be chosen together since (l/d) is dimensionless. Δp, ρ and v cannot be chosen together since ($\Delta p/\rho v^2$) is dimensionless. Therefore, let us choose d, v and ρ.

The dimensions are

$$d = [\mathrm{L}]$$
$$v = [\mathrm{LT}^{-1}]$$
$$\rho = [\mathrm{ML}^{-3}]$$

Let us rewrite the dimensions in terms of the chosen variables.

$$[\mathrm{L}] = d$$
$$[\mathrm{M}] = \rho d^3$$
$$[\mathrm{T}] = dv^{-1}$$

Now let us take the remaining variables, Δp, l and μ, in turn. First

$$\Delta p = [ML^{-1}T^{-2}]$$

Therefore

$$\Delta p[M^{-1}LT^{2}] \text{ is dimensionless}$$

Thus

$$\pi_1 = \Delta p \ (\rho d^3)^{-1}(d) \ (d^{-1}v)^2$$
$$= \Delta p/\rho v^2$$

Now, let us consider the length, l

$$l = [L]$$

Therefore

$$l[L]^{-1} \text{ is dimensionless}$$

Thus

$$\pi_2 = l/d$$

Now, finally, let us consider μ

$$\mu = [ML^{-1}T^{-1}]$$

Therefore

$$\mu \ [M^{-1}LT] \text{ is dimensionless}$$

Thus

$$\pi_3 = \mu(\rho d^3)^{-1}(d) \ (dv^{-1})$$

$$= \frac{\mu}{\rho v d} \text{ or } 1/N_{Re}$$

From the Buckingham's pi theorem

$$\frac{\Delta p}{\rho v^2} = f\left(\frac{1}{d}, \frac{1}{N_{Re}}\right)$$

or

$$\frac{\Delta p}{\rho v^2} = k\left(\frac{1}{d}\right)^a \left(\frac{1}{N_{Re}}\right)^b$$

Thus, from a mere dimensional analysis, we know the form of the relationship between the relevant variables. Let us see the validity of what we have got by comparing the above relation to what we already know. We had seen earlier, in pipe flow, the volumetric flow rate

$$Q = \frac{\pi \Delta p}{8\mu l} r^4$$

or

$$Av = \frac{\pi \Delta p}{8\mu l}\left(\frac{d}{2}\right)^4$$

$$(\cancel{\pi} d^2 / 4)v = \frac{\cancel{\pi}\Delta p}{2\cancel{8}\mu l}\left(\frac{d}{2}\right)^4$$

$$\Delta p = (v\cancel{d}^{\cancel{2}}\mu l \times 32)/(d^{\cancel{4}\,2}) = 32\left(\frac{\mu v}{d}\right)\left(\frac{l}{d}\right)$$

$$\frac{\Delta p}{\rho v^2} = 32\left(\frac{\mu}{\rho v d}\right)\left(\frac{l}{d}\right)$$

$$= 32\left(\frac{1}{N_{\mathrm{Re}}}\right)\left(\frac{l}{d}\right)$$

$$\frac{\Delta p}{\rho v^2} = f\left(\frac{1}{N_{\mathrm{Re}}}, \frac{l}{d}\right) \text{ and } a = b = 1;\ k = 32$$

3.5 Unsteady State Flow

Let us consider a fluid that is initially at rest in a circular tube. At $t = 0$, the fluid is set in motion by an axial pressure gradient, say $\frac{\Delta p}{L}$ where Δp is the difference in pressure (pressure drop) across the tube of length L. From the time the pressure gradient is applied to the time the steady state is achieved, the velocity profile across the cross-section of the tube at, say a certain location on the length of the tube varies. At that location, let us study the time-dependant (unsteady state) variation of velocity profiles. Also, note that we have implicitly assumed that at any time in the tube the flow will be in cylindrical layers (laminar).

Let us first take Eq. C2 of Table 3.4-2 (the z component of the equation of motion in cylindrical coordinates), and simplify it by cancelling the irrelevant terms.

$$\rho\left(\frac{\partial v_z}{\partial t}+\cancelto{v_r=0}{v_r\frac{\partial v_z}{\partial r}}+\cancelto{v_\theta=0}{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}+v_z\frac{\partial v_z}{\partial z}\right)$$

$$=-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\cancelto{v_z\neq f(\theta)}{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z$$

$$\rho\frac{\partial v_z}{\partial t}+v_z\frac{\partial v_z}{\partial z}=-\frac{\partial(p-\rho g z)}{\partial z}+\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\mu\frac{\partial^2 v_z}{\partial z^2}$$

The above equation is very complex to solve. To be able to get some insights, let us make it amenable to an analytical solution, however tedious – this can be done by making an approximation, $v_z \neq f(z)$. In other words, it is assumed that at a particular time, the axial velocity at a particular radial position does not vary with the length of the tube – this may not be a bad assumption. Making suitable assumptions and approximations are essential in engineering practice, and is mostly an art. Thus, the equation to solve becomes

$$\rho\frac{\partial v_z}{\partial t}=-\frac{\partial(p-\rho g z)}{\partial z}+\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \tag{3.5-1}$$

with

IC: At $t = 0$, $v_z = 0$

BC 1: At $r = 0$, v_z = finite or $\dfrac{\partial v_z}{\partial z} = 0$

BC 2: At $r = R$, $v_z = 0$

Since p does not vary with time once the flow begins or with r (as seen earlier, and which is also valid here), $p - \rho g z = P = f(z)$ alone. Thus, the partial derivative $\dfrac{\partial P}{\partial z}$ can be replaced with the total derivative, $\dfrac{dP}{dz}$.

Therefore

$$\frac{dP}{dz}=\text{Constant}=\frac{\Delta P}{L}$$

where

$$\Delta P = P_L - P_0$$

Thus, Eq. 3.5-1 can be written as

$$\rho \frac{\partial v_z}{\partial t} = -\frac{\Delta P}{L} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \tag{3.5-2}$$

Use of dimensionless variables usually simplifies analysis. Let us define the following dimensionless variables.

$$\phi = \frac{v_z}{(-\Delta P)R^2/4\mu L} = \frac{v_z}{v_{z,\max}} \tag{3.5-3}$$

$$\xi = r/R \tag{3.5-4}$$

$$\tau = \frac{v t}{R^2} \tag{3.5-5}$$

where $v = \frac{\mu}{\rho}$, the kinematic viscosity.

From the above definitions

$$v_z = \frac{(-\Delta P)R^2}{4\mu L}\phi$$

$$r = \xi R$$

$$t = \frac{R^2\tau}{v}$$

Thus

$$\frac{\partial v_z}{\partial t} = \frac{(-\Delta P)\cancel{R^2}}{4\mu L} \times \frac{v}{\cancel{R^2}}\frac{\partial \phi}{\partial \tau}$$

$$\rho\frac{\partial v_z}{\partial t} = \frac{\cancel{\rho}(-\Delta P)\times\cancel{\mu}}{4\cancel{\mu}L\cancel{\rho}}\frac{\partial \phi}{\partial \tau} = \frac{(-\Delta P)}{4L}\frac{\partial \phi}{\partial \tau}$$

Further

$$r\frac{\partial v_z}{\partial r} = \xi R\frac{\partial\left(\frac{(-\Delta P)R^2}{4\mu L}\phi\right)}{\partial(\xi R)} = \frac{\left(\xi R\frac{(-\Delta P)R^2}{4\mu L}\partial\phi\right)}{R\quad\partial(\xi)} = \frac{\xi(-\Delta P)R^2}{4\mu L}\frac{\partial\phi}{\partial\xi}$$

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{\mu}{R\xi}\frac{\partial}{\partial(R\xi)}\left(\frac{\xi(-\Delta P)R^2}{4\mu L}\frac{\partial\phi}{\partial\xi}\right)$$

$$= \frac{\mu}{R^2}\frac{1}{\xi}\frac{(-\Delta P)R^2}{4\mu L}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\phi}{\partial\xi}\right)$$

$$= \frac{(-\Delta P)}{4L}\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\phi}{\partial\xi}\right)$$

Through substitution of the above expressions in Eq. 3.5-2 we get

$$\frac{(-\Delta P)}{4L}\frac{\partial\phi}{\partial\tau} = \frac{(-\Delta P)}{L} + \frac{(-\Delta P)}{4L}\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\phi}{\partial\xi}\right)$$

$$\frac{\partial\phi}{\partial\tau} = 4 + \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\phi}{\partial\xi}\right) \tag{3.5-6}$$

IC: At $\tau = 0$, $\phi = 0$

BC 1: At $\xi = 0$, ϕ = finite, or $\dfrac{\partial\phi}{\partial\xi} = 0$

BC 2: At $\xi = 1$, $\phi = 0$

For a steady state flow, we can use Eq. 3.4.2-15 to get

$$v_z = \frac{(-\Delta P)R^2}{4\mu L}\left\{1 - \left(\frac{r}{R}\right)^2\right\}$$

We can write the relationship in terms of dimensionless variables as

$$\phi_\infty = 1 - \xi^2 \tag{3.5-7}$$

where $\phi_\infty = \phi(\tau = \infty)$ i.e. when steady state is reached.

ϕ can be written in terms of a 'steady state' value and a 'deviation' value i.e.

$$\phi(\xi, \tau) = \phi_\infty(\xi) - \phi_t(\xi, \tau) \tag{3.5-8}$$

where $\phi_t(\xi, \tau)$ is the deviation value that represents the 'deviation from steady state'. Thus

$$\frac{\partial\phi}{\partial\tau} = \frac{\partial(\phi_\infty - \phi_t)}{\partial\tau}$$

$$= -\frac{\partial\phi_t}{\partial\tau} \quad \because \phi_\infty \neq f(\tau)$$

Also

$$\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\phi}{\partial\xi}\right) = \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial(\phi_\infty - \phi_t)}{\partial\xi}\right)$$

$$= \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial(1 - \xi^2 - \phi_t)}{\partial\xi}\right)$$

$$= \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\left\{-2\xi - \frac{\partial\phi_t}{\partial\xi}\right\}\right)$$

$$= \frac{1}{\xi}\frac{\partial}{\partial \xi}\left((-2\xi^2) - \xi\frac{\partial \phi_t}{\partial \xi}\right)$$

$$= \frac{1}{\xi}\left[-4\xi - \frac{\partial}{\partial \xi}\left(\xi\frac{\partial \phi_t}{\partial \xi}\right)\right]$$

$$= -4 - \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial \phi_t}{\partial \xi}\right)$$

Substituting the above in Eq. 3.5-6, we get

$$-\frac{\partial \phi_t}{\partial \tau} = 4 - 4 - \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial \phi_t}{\partial \xi}\right)$$

$$\frac{\partial \phi_t}{\partial \tau} = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial \phi_t}{\partial \xi}\right) \tag{3.5-9}$$

Now, the initial and boundary conditions are

IC: At $\tau = 0$, $\phi_t = \phi_\infty$ (by substituting $\phi = 0$ in Eq. 3.5-8)
BC 1: At $\xi = 0$, ϕ = finite i.e. (since $\phi_\infty = 0$ from Eq. 3.5-7 and $\phi = 0$ when $\xi = 1$)
BC 2: At $\xi = 1$, $\phi_t = 0$

If we assume that $\phi_t(\xi, \tau)$ is separable as

$$\phi_t(\xi, \tau) = f(\xi) . g(\tau)$$

then

$$\frac{\partial \phi_t}{\partial \tau} = f\frac{dg}{d\tau} \quad \text{and} \quad \frac{\partial \phi_t}{\partial \xi} = g \times \frac{df}{d\xi}$$

Therefore

$$f\frac{dg}{d\tau} = \frac{1}{\xi}\frac{d}{d\xi}\left(\xi g\frac{df}{d\xi}\right)$$

$$f\frac{dg}{d\tau} = g\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{df}{d\xi}\right)$$

$$\frac{1}{g}\frac{dg}{d\tau} = \frac{1}{f}\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{df}{d\xi}\right) \tag{3.5-10}$$

Since the LHS is a function of τ alone and the RHS is a function of ξ alone, for Eq. 3.5-10 to hold at all times, each side must be equal to a constant, say $-k^2$ (negative); the reason for a negative value will be apparent shortly.

$$\frac{1}{g}\frac{dg}{d\tau} = -k^2 \tag{3.5-11}$$

This implies

$$g = C_1 \exp(-k^2\tau) \tag{3.5-12}$$

If $(-k^2)$ is not negative, then g, and consequently ϕ_t cannot diminish to zero at steady state ($\tau = \infty$); thus the constant $(-k^2)$ needs to be negative.

Equation 3.5-10 can be written as

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{df}{d\xi}\right) + k^2 f = 0 \tag{3.5-13}$$

The boundary conditions are given below

BC 1: At $\xi = 0$, f = finite, i.e. $\dfrac{df}{d\xi} = 0$

BC 2: At $\xi = 1$, $f = 0$ (since $\phi_t = 0$ for all g, note that $g = g(\tau)$)

The solution for Eq. 3.5-13 requires knowledge of Bessel functions and their relationships. The student is directed to other appropriate books (e.g. Lih 1974) for a better understanding of the same. Here, we merely present the solution.

The solution is of the form

$$f = c_2 J_0(k\xi) + c_3 Y_0(k\xi) \tag{3.5-14}$$

where J_0 is a Bessel function of the I kind

$$J_0(k\xi) = \sum_{k=0}^{\alpha} \frac{(-1)^r \left(\frac{k\xi}{2}\right)^{2r}}{(r!)^2}$$

Y_0 is a Weber's Bessel function of the II kind

$$Y_0(k\xi) = \frac{2}{\pi}\left[\bar{Y}_0(k\xi) - (\ln 2 - \Gamma) J_0(k\xi)\right]$$

where $\Gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right) = 0.57721\ldots$ (Euler's constant),

and $\bar{Y}_0$ is a Neumann's Bessel function of the II kind.

$$\bar{Y}_0(k\xi) = J_0(k\xi)\int \frac{d\xi}{\xi[J_0(k\xi)]^2}$$

$C_3 = 0$ (from BC 1; otherwise the term would not be finite since $Y_0(0) = -\infty$)

$C_2\, J_0\,(k) = 0$ (from BC 2).

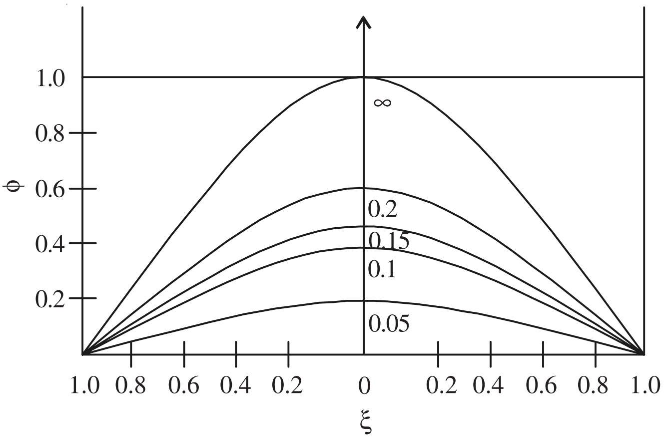


Fig. 3.5-1 Profiles of ϕ at a particular length position on the tube, versus ξ for various values of τ

Now, C_2 cannot be zero since that would result in a trivial solution, ($f = 0$). Therefore

$$J_0\ (k) = 0$$

This happens multiple times when $k = 2.4048...$ ($= k_1$), 5.52009 ($= k_2$), 8.6537... ($= k_3$), and so on.

Thus, there are infinite solutions

$$f_n = C_{2n}\ J_0\ (k_n\xi)\ n = 1,\ 2,\ 3,\ldots\ \infty$$

This implies that

$$\phi_{tn} = C'_n J_0(k_n\xi)\ \exp\ (-\ k_n^2\ \tau),\ n = 1,\ 2,\ 3,\ldots\ \infty$$

where $C'_n = C_1 C_{2n}$.

Using the principles of superimposition, orthogonality relationships, and other relevant aspects of Bessel functions, the final solution is

$$\phi(\xi,\tau) = (1-\xi^2) - 8\sum_{n=1}^{\infty} \frac{J_0(k_n\xi)}{k_n^3 J_1(k_n)} \exp(-\,k_n^2\tau) \tag{3.5-15}$$

A representative plot of ϕ versus ξ for various values of τ is given in Fig. 3.5-1.

3.6 Pulsatile Flow

In the earlier cases, we had considered a linear, time invariant pressure gradient. However, flows in the body, e.g. blood flow through the vasculature are pulsatile in nature because of the pumping of the heart. In other words, the pressure gradient varies with time.

Let us consider here a sinusoidal pressure gradient. Although not strictly valid for blood flow, a time-varying sinusoidal pressure gradient does provide valuable insights into the nature of pulsatile biological flows. Also, let us assume that the axial velocity at a particular radial position does not change with the length of the tube at any given time.

From Eq. C2 of Table 3.4-2 (the z component of the equation of motion)

$$\rho\left(\frac{\partial v_z}{\partial t} + \overset{\substack{(v_r = 0)\\ 0}}{\cancel{v_r \frac{\partial v_z}{\partial r}}} + \overset{\substack{(v_\theta = 0)\\ 0}}{\cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}} + \overset{\substack{(v_z \neq f(z))\\ 0}}{\cancel{v_z \frac{\partial v_z}{\partial z}}}\right)$$

$$= -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \overset{\substack{v_z \neq f(\theta)\\ 0}}{\cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}} + \overset{\substack{(v_z \neq f(z))\\ 0}}{\cancel{\frac{\partial^2 v_z}{\partial z^2}}}\right] + \rho g_z$$

Note that we have taken $v_z \neq f(z)$ at a particular time. Thus, the remaining terms yield

$$\rho\frac{\partial v_z}{\partial t} = -\frac{\partial(p - \rho g z)}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \tag{3.6-1}$$

By the same arguments as in Section 3.5 that led to Eq. 3.5-2, we can write

$$\frac{\partial P}{\partial z} = \frac{-\Delta P}{L} + A\sin\omega t \tag{3.6-2}$$

where $\dfrac{-\Delta P}{L}$ is the average pressure gradient; A and ω are the frequency and amplitude, respectively, of the periodic pressure function.

Since $\dfrac{\mu}{\rho} = \nu$, the equation of motion can be written as

$$\frac{1}{\nu}\frac{\partial v_z}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{(-\Delta P)}{\mu L} + \frac{A}{\mu}\sin\omega t \tag{3.6-3}$$

We can guess that the solution for v_z consists of a steady state part (average value) and a periodic part (fluctuating value) corresponding to the average and fluctuating pressure gradients, i.e.

$$v_z\,(r,\,t) = \bar{v}_z(r) + v_z'\,(r,\,t) \tag{3.6-4}$$

Substituting Eq. 3.6-4 in Eq. 3.6-3, and using separation of variables as in the Section 3.5 with the recognition that $\bar{v}_z(r)$ is not a function of t, gives two equations

$$0 = \frac{1}{r}\left(\frac{d}{dr} r \frac{d\bar{v}_z}{dr}\right) + \frac{(-\Delta P)}{\mu L} \tag{3.6-5}$$

$$\frac{1}{v}\frac{\partial v_z'}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial v_z'}{\partial r} + \frac{A}{\mu}\sin\omega t \tag{3.6-6}$$

The solution of Eq. 3.6-5, as seen in an earlier section is

$$\overline{v_z} = \frac{(-\Delta P)R^2}{4\mu L}\left\{1 - \left(\frac{r}{R}\right)^2\right\} \tag{3.6-7}$$

Equation 3.6-6 can be solved by Laplace transforms through a lengthy procedure, with the boundary conditions as

$$\text{BC 1: At } r = 0,\ \frac{\partial \overline{v_z'}}{\partial r} = 0$$

$$\text{BC 2: At } r = R,\ \overline{v_z'} = 0$$

to yield the following as the other part of the solution – the first part being the parabolic profile given by Eq. 3.6-7. Here we merely state the combined solution as follows

$$v_z(r,t) = \frac{(-\Delta P)R^2}{4\mu L}\left\{1 - \left(\frac{r}{R}\right)^2\right\} + \frac{2A}{\rho}\sum_{k=1}^{\infty}\frac{J_0\left(\frac{\alpha_k r}{R}\right)}{\alpha_k J_1(\alpha_k)}\left\{\frac{\omega\exp(-\alpha_k^2 vt/R^2)}{\left(\frac{\alpha_k^4 v^2}{R^4}\right)+\omega^2} + \frac{\sin(\omega t - \phi)}{\sqrt{\left(\frac{\alpha_k^4 v^2}{R^4}\right)+\omega^2}}\right\} \tag{3.6-8}$$

Thus, the velocity profile at a cross-section varies with time from the basal parabolic profile. The variation is cyclic, as can be expected from a cyclic pressure gradient.

3.7 Solutions to Equations

As was evident in the Sections 3.5 and 3.6, the mathematical effort to solve the formulated equations could be significant. An analytical solution is reasonably complete, and capable of rendering itself to confident interpretations due to the continuous nature of this approach. However, an analytical solution may not be available for all situations. Thus, in many research problems, it is common to take a numerical approach, such as the finite element method, to solve the formulated equations. Some level of expertise is needed for the appropriate use and interpretation of numerical solutions. Even if one does not possess such expertise, one can team up with a suitable expert for the solution.

In this section, let us see some formulations for simplifying the solutions of the differential equations. Two of the common approaches are merely mentioned in this section. The reader is referred to other texts (Bird et al. 2002) for the details on these approaches.

3.7.1 Stream Function Approach

Velocity is expressed as the gradient of a 'stream function' say, ψ. For example

$$v_x = -\frac{\partial \psi}{\partial y}$$

$$v_y = +\frac{\partial \psi}{\partial x}$$

ψ = constant (mathematical representation) indicates streamlines (physical significance) i.e. the path traced by the particles of fluid under steady flow.

3.7.2 Boundary Layer Theory

The flow is split into two parts:

- Potential flow (away from the wall) (ρ = constant; μ = 0; flow is irrotational ($\vec{\nabla} \times \vec{v} = 0$))
- Boundary layer flow (close to the wall)

3.8 Turbulent Flow

As seen in Section 3.2, above a certain Reynolds number that is dependent on the system (4000 for pipe flow), the flow turns chaotic or turbulent.

Many flows in the bio-industry occur in the turbulent regime. Turbulent flow can also occur near artificial valves of the heart, which may result in wasteful expenditure of pumping energy.

By careful measurements, it has been experimentally shown that for turbulent flow *in a pipe*, the time-smoothed components (average quantities at a point), $\bar{v}_z$ and $\bar{v}_{z,\text{avg}}$ (these terms will become clearer soon) are related as

Turbulent flow		Laminar flow	
$\dfrac{\bar{v}_z}{\bar{v}_{z,\max}} \cong \left(1-\dfrac{r}{R}\right)^{\frac{1}{7}}$	$=$	$\left[1-\left(\dfrac{r}{R}\right)^2\right]$	(3.8-1)
$\dfrac{\bar{v}_{z,\text{avg}}}{\bar{v}_{z,\max}} \cong \dfrac{4}{5}$	$=$	$\dfrac{1}{2}$	(3.8-2)
$\Delta P \propto Q^{\frac{7}{4}}$	$\propto$	Q	(3.8-3)

Turbulent flow can be visualised as the random motion of packets of fluid (eddies). For turbulent flow in a tube, the flow is entirely random at the centre of the tube, i.e. far away from the wall. Near the wall, the fluctuations of velocity in the axial direction is greater than the fluctuations in the radial direction. At the wall, the fluctuations are zero.

Let us take a closer look at these fluctuations. We will focus our attention on the fluid behaviour at one point in the tube (pipe) where turbulent flow exists. As we are watching it, let us say that the mean velocity decreases (probably due to a change in the pressure drop causing the flow, by turning down the pump speed).

The variation of the axial component of the velocity, v_z, at the point of observation, would look like that given in Fig. 3.8-1.

$\bar{v}_z$ is called the time-smoothed velocity, i.e. the average of v_z over a time interval large enough with respect to the time of turbulent oscillation, but small enough with respect to the time changes in the pressure drop causing the flow.

$$\bar{v}_z = \frac{1}{t_a}\int_t^{t+t_a} v_z dt \tag{3.8-4}$$

Thus

$$v_z = \bar{v}_z + v_z' \text{ (average + fluctuation)} \tag{3.8-5}$$

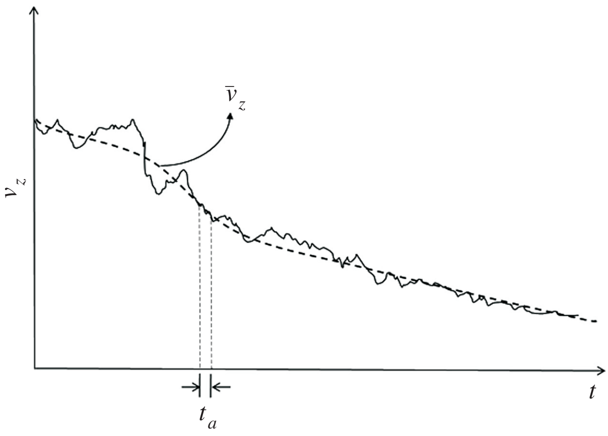


Fig. 3.8-1 Velocity (v_z) and time-smoothed velocity ($\bar{v}_z$) at a point in turbulent flow

The pressure at a point will also vary in a similar fashion

$$p = \bar{p} + p' \tag{3.8-6}$$

If we take the average of the fluctuations, $\bar{v}_z$, since the positive values will balance the negative values

$$\bar{v}'_z = 0 \tag{3.8-7}$$

Thus, we cannot use $\bar{v}'_z$ as a measure of turbulence. However, the average of the squares of the fluctuation values, $\overline{v_z'^2}$, will not be zero and can be used as a measure of turbulence. In fact

$$\text{Intensity of turbulence} \equiv \frac{\sqrt{\overline{v_z'^2}}}{\bar{v}_{z,\text{avg}}} \tag{3.8-8}$$

The intensity of turbulence is typically between 0.01 and 0.1.

At the wall, since the fluctuations in the radial component will be different from those in the axial direction, we need to differentiate between the two. Researchers have found that near the wall

$$(\text{Axial})\ \frac{\sqrt{\overline{v_z'^2}}}{\bar{v}_{z,\text{avg}}} > \frac{\sqrt{\overline{v_r'^2}}}{\bar{v}_{z,\text{avg}}}\ (\text{Radial})$$

At the centre of the tube the above values are comparable (isotropic condition).

As long as the eddy size is greater than the mean free path of the molecules (continuum holds), the

- equation of continuity (mass balance)
- equation of motion (momentum balance)

are applicable for turbulent flow. Let us consider the conservation equations for turbulent flow, at a point and take the case for which we have an intuitive feel, i.e. the equation in rectangular Cartesian coordinates. For illustration, let us consider an incompressible flow.

Equation of Continuity

$$\frac{\partial}{\partial x}(\bar{v}_x + v_x') + \frac{\partial}{\partial y}(\bar{v}_y + v_y') + \frac{\partial}{\partial z}(\bar{v}_z + v_z') = 0 \tag{3.8-9}$$

Equation of Motion

x direction

$$\frac{\partial}{\partial t}\rho(\bar{v}_x + v_x') = -\frac{\partial}{\partial x}(\bar{p} + p') - \left[\frac{\partial}{\partial x}\rho(\bar{v}_x + v_x')(\bar{v}_x + v_x') + \frac{\partial}{\partial y}\rho(\bar{v}_y + v_y')(\bar{v}_x + v_x') \right.$$
$$\left. + \frac{\partial}{\partial z}\rho(\bar{v}_z + v_z')(\bar{v}_x + v_x')\right] + \mu\nabla^2(\bar{v}_x + v_x') + \rho g_x \tag{3.8-10}$$

Taking the time average of the velocity components, i.e. $\bar{v} = \frac{1}{t_a}\int_0^{t_a} v\, dt$ over t_as that are large with respect to turbulent oscillations but small with respect to macro variations, the time-smoothed equation of continuity can be written as follows (note that the time averaged fluctuations will tend to zero)

$$\frac{\partial \bar{v}_x}{\partial x} + \frac{\partial \bar{v}_y}{\partial y} + \frac{\partial \bar{v}_z}{\partial z} = 0 \tag{3.8-11}$$

Similarly, the time-smoothed equation of motion can be written as

$$\frac{\partial}{\partial t}\rho\bar{v}_x = -\frac{\partial \bar{p}}{\partial x} - \left[\frac{\partial}{\partial x}\rho\bar{v}_x\bar{v}_x + \frac{\partial}{\partial y}\rho\bar{v}_y\bar{v}_x + \frac{\partial}{\partial z}\rho\bar{v}_z\bar{v}_x\right]$$
$$-\left[\frac{\partial}{\partial x}\rho\overline{v_x'v_x'} + \frac{\partial}{\partial y}\rho\overline{v_y'v_x'} + \frac{\partial}{\partial z}\rho\overline{v_z'v_x'}\right] + \mu\nabla^2\bar{v}_x + \rho g_x \tag{3.8-12}$$

The third term in brackets on the RHS of Eq. 3.8-12 is the only extra term when compared to the equation of continuity for laminar flow.

Now, since $\rho\vec{v}\,\vec{v}$ = momentum flux or stress, let us say that

$$\bar{\tau}_{xx}^{(t)} = \rho\overline{v'_x v'_x}$$

$$\bar{\tau}_{xy}^{(t)} = \rho\overline{v'_x v'_y}$$

and so on.

These are the components of the turbulent momentum flux tensor $\bar{\tilde{\tau}}^{(t)}$. The stresses are also known as Reynolds stresses.

In vector notation, the time-smoothed equation of continuity is

$$\vec{\nabla}.\bar{\vec{v}} = 0 \tag{3.8-13}$$

and the time-smoothed equation of motion is

$$\rho\frac{D\bar{\vec{v}}}{Dt} = -\vec{\nabla}\,\bar{p} - \left[\vec{\nabla}.\bar{\tilde{\tau}}^{(l)}\right] - \left[\vec{\nabla}.\bar{\tilde{\tau}}^{(t)}\right] + \rho\vec{g} \tag{3.8-14}$$

The above Eqs. (3.8-9) to (3.8-14) are valid for an incompressible flow. Similarly, it can be shown that the earlier equations and the tables for laminar flow are valid if we replace

$$v_i \text{ by } \bar{v}_i$$

$$p \text{ by } \bar{p}$$

and

$$\tau_{ij} \text{ by } \bar{\tau}_{ij}^{(l)} + \bar{\tau}_{ij}^{(t)}$$

However, to get the velocity profile, we need a relationship between τ and the velocity gradient.

For laminar flow, we had a theoretical base in terms of constitutive equations. For turbulent flow, we do not have that luxury. Nevertheless, many expressions based on experimental observations have been proposed. Two are given below.

The first is on the same lines as for the laminar case.

$$\bar{\tau}_{yx}^{(t)} = -\mu^{(t)}\frac{d\bar{v}_x}{dy} \tag{3.8-15}$$

where $\mu^{(t)}$ is termed as 'eddy viscosity' and its value could be hundreds of times the molecular viscosity.

Another popular expression was formulated by Prandtl. For this expression, it is assumed that the eddies in the fluid move around in a fashion similar to that of the molecules in a gas. A 'mixing length', l, which is a function of position represents an idea similar to the 'mean free path' in the kinetic theory of gases. The relationship is given as

$$\overline{\tau}_{yx}^{(t)} = -\rho l^2 \left|\frac{d\overline{v}_x}{dy}\right| \frac{d\overline{v}_x}{dy} \tag{3.8-16}$$

For flow in pipes/tubes, the relationship between velocity and distance (velocity profile) in turbulent flow through Deissler's empirical formulation is as follows:

If we define

$$v^+ = \frac{\overline{v}_z}{\sqrt{\frac{\tau_0}{\rho}}}$$

and

$$s^+ = s\left(\sqrt{\frac{\tau_0}{\rho}}\right)\frac{\rho}{\mu}$$

where $s = R - r$ i.e. the radial distance from the wall and τ_0 is wall shear stress at $s = 0$.

For $s^+ > 26$

$$v^+ = \frac{1}{0.36}\ln s^+ + 3.8 \tag{3.8-17}$$

For $0 \leq s^+ \leq 5$

$$v^+ = s^+ \tag{3.8-18}$$

And for $0 \leq s^+ \leq 26$

$$v^+ = \int_0^{s^+} \frac{ds^+}{1 + n^2 v^+ s^+ (1 - \exp\{-n^2 v^+ s^+\})} \tag{3.8-19}$$

where n is Deissler's constant for tube flow, near the wall. It was found empirically to be equal to 0.124.

3.9 Macroscopic Aspects: The Engineering Bernoulli Equation

Although the understanding of fluid flow thus far was in good depth, the mathematical effort was significant. If we can reduce the effort, but still get acceptable answers, it may be good for engineering design and operation. The 'engineering Bernoulli equation' is useful for this purpose.

To arrive at the engineering Bernoulli equation, one can begin with the equation of motion, Eq. 3.4-4. The details of the lengthy and mathematically involved derivation are indicated in different sections of Bird et al. (2002). Some details are highlighted here.

First, the dot product of the velocity vector, $\vec{v}$, is taken with the equation of motion, i.e. Eq. 3.4-4. Then, skilful rearrangement of terms, the application of the equation of continuity, and representation of the acceleration due to gravity term as the negative gradient of a scalar potential per unit mass (we will do operations similar to some of the above, later in Chapter 4), followed by further rearrangement of terms, leads to an equation of mechanical energy (the kinetic energy alone, the potential energy alone, and the sum of kinetic and potential energies are examples of 'mechanical energy'), which is given below as Eq. 3.9-1. Please note that the mechanical energy is not conserved.

$$\frac{\partial\left(\frac{1}{2}\rho v^2+\rho\phi\right)}{\partial t}=-\left(\vec{\nabla}\cdot\left(\frac{1}{2}\rho v^2+\rho\phi\right)\vec{v}\right)-(\vec{\nabla}\cdot p\vec{v})-p(-\vec{\nabla}\cdot\vec{v})-(\vec{\nabla}\cdot(\tilde{\tau}\cdot\vec{v}))-(-\tilde{\tau}:\vec{\nabla}\vec{v}) \tag{3.9-1}$$

Equation 3.9-1 is a differential equation, which can be integrated over the volume of the macroscopic system of interest. The integration procedure needs the knowledge of the three-dimensional Leibniz formula, Gauss divergence theorem, etc. After further rearrangement of terms, the integrated equation can be written as

$$\frac{d\left(\int_V\left(\frac{1}{2}\rho v^2+\rho\phi\right)\right)}{dt}=-\Delta\left[\dot{m}\left(\frac{p}{\rho}+\frac{1}{2}v^2+gz\right)\right]+\int_V p(\vec{\nabla}\cdot\vec{v})dV+\int_V(\tilde{\tau}:\vec{\nabla}\vec{v})dV-(-W_s) \tag{3.9-2}$$

where the Δ represents the difference in the relevant variables between the two positions, say the ends of the volume of interest e.g. entry and exit points of a pipe through which a fluid is flowing. W_s refers to the work done on the fluid, say by a pump, and is the negative of the work done by the system/control volume on the fluid. Also note that the LHS is the time derivative of the sum of kinetic and potential energies that is obtained by integrating Eq. 3.9-1 over the relevant volume.

The term $\int_V p(\vec{\nabla}\cdot\vec{v})dV$ denotes compression or expansion experienced by the fluid in the relevant volume of interest. It is zero for incompressible fluids.

The term $\int_V(\tilde{\tau}:\vec{\nabla}\vec{v})dV$ represents what can be simplistically said to be energy loss due to viscous effects, or viscous dissipation. For Newtonian fluids, this term is negative, and thus, represents a loss. However, the same cannot be generalised to all fluids.

Now, at steady state, the LHS of Eq. 3.9-2 is zero. Further, under the assumption of a 'representative streamline' through the system, and for a constant mass flow rate $(\dot{m})$ between the two positions of interest (say 1 and 2), the following combination of terms can be approximately made:

$$\dot{m}\Delta\left(\frac{p}{\rho}\right) - \int_V p(\vec{\nabla}\cdot\vec{v})\,dV \approx \dot{m}\int_1^2 \frac{1}{\rho}\,dp \tag{3.9-3}$$

Further, with the assumption that

$$\frac{(v^3)_{\text{avg}}}{v_{\text{avg}}} \approx v_{\text{avg}}^2 \text{ or say, } v^2 \tag{3.9-4}$$

and by division throughout by $\dot{m}$, we can write Eq. 3.9-2, under all the above assumptions, including that of an incompressible fluid as

$$\frac{\Delta p}{\rho} + \frac{\Delta v^2}{2} + g\Delta x + \widehat{FL} + \widehat{W_s} = 0 \tag{3.9-5}$$

where

$$\widehat{FL} = -\frac{1}{\dot{m}}\int(\tilde{\tau}:\vec{\nabla}\vec{v})\,dV$$

$$\widehat{W_s} = \frac{1}{\dot{m}}W_s$$

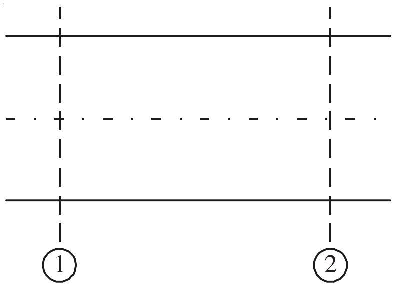
Equation 3.9-5 is a useful form of the engineering Bernoulli equation.

For design and operation, what is called the *friction factor approach* would be the easiest, with an acceptable balance between rigour and ease of usability. Let us use the friction factor approach for a few practical situations. As can be seen in the following sections, we invoke the just developed engineering Bernoulli equation quite extensively.

3.9.1 Friction Factor for Flow through a Straight Horizontal Pipe

Let us consider a well-developed flow through a straight horizontal pipe (Fig. 3.9.1-1).

Fig. 3.9.1-1 A section of the straight horizontal pipe taken for analysis



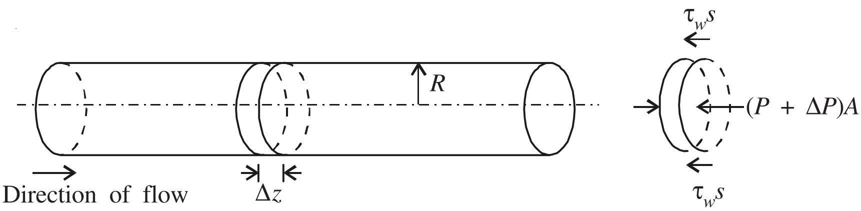


Fig. 3.9.1-2 Flow through a straight pipe, and a differential, disc-shaped fluid element taken for analysis

Let us apply the engineering Bernoulli equation between cross-sections 1 and 2

$$\frac{\Delta p}{\rho}+\overset{0(v_1 = v_2)}{\cancel{\frac{\Delta v^2}{2}}}+\overset{0(z_1 = z_2)}{\cancel{g\Delta z}}+\hat{F}_L+\overset{\text{no shaft work}}{\cancel{\hat{W}_s}}=0$$

Thus

$$\widehat{FL}=-\frac{\Delta p}{\rho} \tag{3.9.1-1}$$

Note that we have made no assumption regarding the nature of flow (i.e. whether it is laminar or turbulent). Thus, the above is applicable to both laminar and turbulent flows.

Let us consider a differential fluid volume which is disc-shaped of radius R and thickness, Δz, as shown in Fig. 3.9.1-2. τ_w will be the wall shear stress both in laminar and turbulent flows – this is because even in turbulent flow, the flow closest to the wall is laminar.

A force balance on the differential fluid element yields

$$p(\pi R^2) - (p + \Delta p)\ (\pi R^2) - \tau_\omega\ (2\pi R \Delta z) = 0 \tag{3.9.1-2}$$

$$-\tau_\omega = \frac{(p+\Delta p)(\pi R^2) - p\pi R^2}{(\Delta z)(2\pi R)}$$

$$\tau_\omega = -\left(\frac{\Delta p}{\Delta z}\right)\frac{R}{2}$$

In the limit $\Delta z \to 0$

$$\tau_\omega = -\left(\frac{dp}{dz}\right)\frac{R}{2}$$

$$\frac{dp}{dz} + \frac{2\tau_\omega}{R} = 0 \qquad (3.9.1\text{-}3)$$

For a pipe of length L between points 1 and 2, Eq. 3.9.1-3 can be integrated to yield

$$\frac{p_2 - p_1}{L} + \frac{2\tau_\omega}{R} = 0$$

or

$$\tau_\omega = \frac{-(p_2 - p_1)}{L} \times \frac{R}{2} = \frac{-(\Delta p)}{L} \times \frac{D}{4}$$

or

$$-\Delta p = \frac{4L\tau_\omega}{D}$$

Substituting this into Eq. 3.9.1-1, we get

$$\widehat{FL} = \frac{4\tau_\omega L}{\rho D} \qquad (3.9.1\text{-}4)$$

Let us define a dimensionless parameter, f, as

$$f = \frac{(F_k)}{A} \times \frac{1}{KE'} \qquad (3.9.1\text{-}5)$$

where f is the friction factor, F_k is the force exerted by a fluid due to its motion on the body of interest, A is the appropriate area and KE' is the kinetic energy per unit volume.

(A fluid exerts a force on a body in contact with it and of interest. That force can be thought to consist of two parts, F_s and F_k. F_s is the force that is exerted even when the fluid is stationary. F_k is the force exerted when the fluid is in relative motion compared to the body of interest.)

In our case of tube flow, f can be conveniently defined as

$$f \equiv \frac{\tau_\omega}{\left(\frac{1}{2}\right)\rho v_{\text{avg}}^2} = \frac{-\frac{\Delta p}{L} \times \frac{D}{4}}{\frac{1}{2}\rho v_{\text{avg}}^2} = \frac{(-\Delta p)D}{2L\rho v_{\text{avg}}^2} \tag{3.9.1-6}$$

Thus

$$\tau_\omega = \frac{1}{2}\rho v_{\text{avg}}^2 f \tag{3.9.1-7}$$

Substituting Eq. 3.9.1-7 in Eq. 3.9.1-4, we get

$$\widehat{FL} = \frac{4\left(\frac{1}{2}\rho v_{\text{avg}}^2 f\right)L}{\rho D} = 4f\left(\frac{L}{D}\right)\left(\frac{v_{\text{avg}}^2}{2}\right) \tag{3.9.1-8}$$

This $\widehat{FL}$ accounts for frictional losses at the pipe wall (skin friction). Equation 3.9.1-8 can be written as

$$\widehat{FL} = f\left(\frac{L}{\frac{D}{4}}\right)\left(\frac{v_{\text{avg}}^2}{2}\right)$$

If we define a 'hydraulic radius', r_H as

$$r_H = \frac{\text{Cross-sectional area}}{\text{Wetted perimeter}} \tag{3.9.1-9}$$

for our pipe

$$r_H = \frac{\pi\left(\frac{D^2}{4}\right)}{\pi D} = \frac{D}{4}$$

Thus

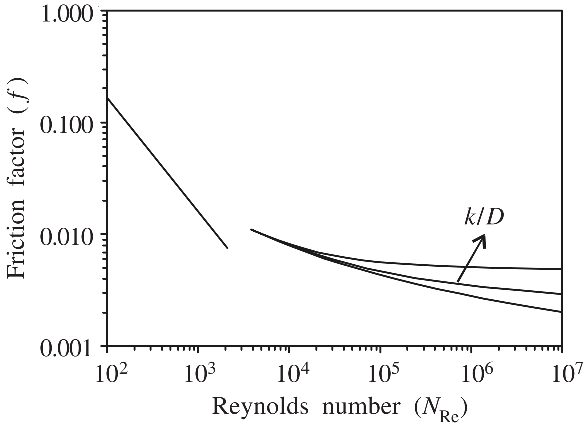
$$\widehat{FL} = f\left(\frac{L}{r_H}\right)\left(\frac{v_{\text{avg}}^2}{2}\right) \tag{3.9.1-10}$$

This equation, in practice, can be extended to all cross-sectional geometries.

To find the above friction factor for pipe flow, a friction factor chart (Fig. 3.9.1-3) can be used. The term 'friction factor' refers to the Fanning friction factor, and not the Moody's friction factor that is normally used in other (e.g. civil engineering) applications.

- For the laminar regions, we can use $f = \dfrac{16}{N_{\text{Re}}}$

Fig. 3.9.1-3 Friction factor chart



- For the turbulent regime, we need to use the chart.
- For the intermediate regime ($2100 < N_{\mathrm{Re}} < 4000$), we usually avoid design.

In the turbulent regime, the friction factor, f is a function of the roughness factor, k/D (represented as different curves on the friction factor chart), where k is the roughness length (effective thickness), and D is the diameter of the pipe.

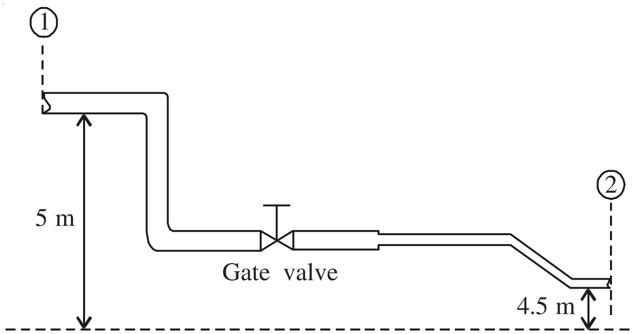
The laminar region is represented by $f = \dfrac{16}{N_{\mathrm{Re}}}$. The turbulent region has different curves that correspond to different k/D values of the pipes. The lowest curve corresponds to a smooth pipe. The curve above that corresponds to a k/D of 10^{-4} and the topmost curve corresponds to a k/D value of 10^{-3}. For values in between, interpolations can be done to obtain estimates of f.

Example 3.9.1-1

A cleaning liquid used in many Bioprocess industries needs to be piped through the pipeline system above the ground as shown in Fig. 3.9.1-4.

The pipeline system consists of 50 m of 12" nominal diameter pipe and 20 m of 8" nominal diameter pipe. All elbows are standard and flanged, and the material used for the piping is schedule 80 wrought iron pipe. Determine the pressure drop needed between points 1 and 2 to maintain a flow rate of 0.05 $m^3 s^{-1}$. What is the pumping power that is needed to maintain the flow rate? The density of the liquid is 870 kg m^{-3} and its viscosity is 1.375×10^{-3} Pa s.

Fig. 3.9.1-4 Details of the pipeline for Example 3.9.1-1



Nominal diameter and schedule numbers are standard terminology used in process industries. They have evolved for historical reasons of communications between the different people working in the industry. The details of the terminology are given in various handbooks (e.g. *Perry's Chemical Engineers' Handbook* 2007) and other books too (McCabe et al. 2004).

In brief, the schedule number refers to the working stress and equals $1000\,\frac{p_{\max}}{S}$, where $p_{\max}$ is allowable working pressure and S is allowable tensile stress.

The correspondence between nominal diameter, internal diameter, and the wall thickness is available in many sources, e.g. the references given above.

For a 12" nominal diameter

$$\text{id} = 0.2889\ \text{m}$$
$$\therefore\ \text{cs area} = 0.066\ \text{m}^2$$

For a 8" nominal diameter

$$\text{id} = 0.1937\ \text{m}$$
$$\therefore\ \text{cs area} = 0.0297\ \text{m}^2$$

Also, for wrought iron, roughness factor (k) = 4.6×10^{-5} m.

Let us apply the engineering Bernoulli equation between points 1 and 2 in the piping network shown in Fig. 3.9.1-4.

$$\frac{\Delta P}{\rho}+\frac{\Delta v^2}{2}+g\Delta z+\hat{F}_L+\overset{0\ \text{(no shaft work)}}{\cancel{\hat{W}_s}}=0$$

$$\frac{p_2-p_1}{\rho}+\frac{(v_2^2-v_1^2)}{2}+g(z_2-z_1)+\widehat{FL}=0$$

We need to find $p_2 - p_1$.

We know that $\rho = 870$ kg m^{-3}

$$v_2 = \frac{\dot{V}_2}{A_2} = \frac{0.05}{0.0297} = 1.7 \text{ m s}^{-1}$$

$$v_1 = \frac{\dot{V}_1}{A_1} = \frac{0.05}{0.066} = 0.763 \text{ m s}^{-1}$$

$$z_2 = 4.5 \text{ m; } z_1 = 5 \text{ m}$$

$$\widehat{FL} = ?$$

For a pipe, and different pipe fittings (valves, etc., which are piping network components), $\widehat{FL}$ can be calculated as $\widehat{FL} = K_f \dfrac{v_{\text{avg}}^2}{2}$ for each fitting, and added together to get the total $\widehat{FL}$. K_f values for some common fittings are given in brackets next to the fitting: straight pipe $\left(4f\dfrac{L}{D}\right)$; 180° bend (2.2); 90° elbow (0.9); 45° elbow (0.4); tee (1.8); wide open globe valve (15); wide open gate valve (0.2). In addition, the K_f values for a sudden contraction and a sudden expansion can be evaluated as follows

$$\text{Sudden contraction: } 0.4\left(1 - \frac{A_b}{A_a}\right)$$

$$\text{Sudden expansion: } \left(1 - \frac{A_b}{A_a}\right)^2$$

where b is smaller diameter and a is larger diameter; v_{avg} is taken at b.

Thus, depending on the fittings in the piping network

$$\widehat{FL} = \left(4f\frac{L}{D} + \sum K_f\right)\frac{v_{\text{avg}}^2}{2}$$

In this case

$$\widehat{FL} = \sum K_f \frac{v_{\text{avg}}^2}{2}\bigg|_{12''} + 4f\frac{L}{D}\frac{v_{\text{avg}}^2}{2}\bigg|_{12''\text{ pipe}} + 4f\frac{L}{D}\frac{v_{\text{avg}}^2}{2}\bigg|_{8''\text{ pipe}} + \sum K_f \frac{v_{\text{avg}}^2}{2}\bigg|_{8''}$$

To find f, let us use the friction factor chart for which we need the Reynolds numbers.

$$N_{\text{Re},12''\text{ pipe}} = \frac{\rho v_{av} D}{\mu} = \frac{870 \times 0.763 \times 0.289}{1.375 \times 10^{-3}} = 1.39 \times 10^5$$

$$N_{\mathrm{Re,8"\,pipe}} = \frac{870 \times 1.7 \times 0.194}{1.375 \times 10^{-3}} = 1.47 \times 10^5$$

Both are turbulent flows.

Now, for reading the appropriate curve on the friction factor chart in the case of a turbulent flow, we need $\frac{k}{d}$.

$$\frac{k}{d_{12"}} = \frac{0.000046}{0.2889} = 1.6 \times 10^{-4}$$

$$\frac{k}{d_{8"}} = \frac{0.000046}{0.194} = 2.37 \times 10^{-4}$$

From the friction factor chart, $f_{12"} = 0.0045$; $f_{8"} = 0.00445$.

Pipe fittings: 2(12", 90°) + 2(8", 45°) elbows, 1(12") gate valve, 1(8") sudden contraction

$$\sum K_f \Big|_{12"} = 2 \times 0.9 + 1 \times 0.2 = 2.0$$

$$\sum K_f \Big|_{8"} = 2 \times 0.4 + 0.4\left(1 - \frac{0.0297}{0.066}\right) = 1.02$$

Substituting the above in the engineering Bernoulli equation, we get

$$\frac{\Delta p}{870} + \frac{1.7^2 - 0.763^2}{2} + 9.8(45-5) + \left(4 \times 0.0045 \times \frac{50}{0.289} \times \frac{0.763^2}{2}\right)$$

$$+ \left(4 \times 0.0045 \times \frac{20}{0.194} \times \frac{1.7^2}{2}\right) + \left(2 \times \frac{0.763^2}{2}\right) + 1.02 \times \frac{1.7^2}{2} = 0$$

$$\frac{\Delta p}{870} = 4.9 - 1.154 - 0.91 - 2.65 - 0.582 - 1.474$$

$$\Delta p = -\ 1626.9 \text{ Pa or } -\ 1.63 \text{ kPa}$$

Pumping power required

$$= (-\ \Delta p) \times \dot{V}$$

$$= 1626.9 \times 0.05 = 81.3 \text{ W} = 0.081 \text{ kW}$$

Example 3.9.1-2

Stenosis or narrowing of the arteries can cause health difficulties especially cardiac related ones. If the stenosis happens to be at the place of expansion in

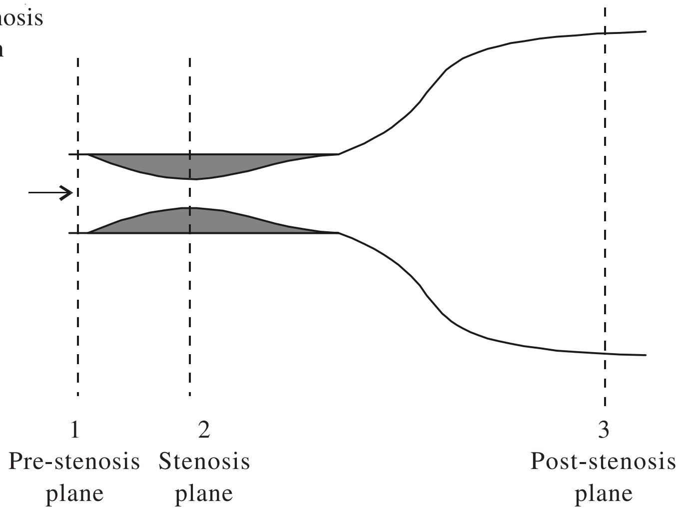


Fig. 3.9.1-5 Stenosis (narrowing) in an artery

the arterial cross-section, other difficulties could arise. One of the difficulties is due to “cavitation” or gas bubble formation followed by rupture. Rupture releases an enormous amount of energy that can even destroy metallic surfaces. Develop a criterion in terms of the flow velocities, pressure, and areas for the concept of cavitation at the stenosis.

Let us apply the engineering Bernoulli equation between planes 1 and 2 in Fig. 3.9.1-5. Upon cancelling the terms that are not relevant, we get

$$\frac{P_2 - P_1}{\rho} + \frac{(v_{2,\text{avg}}^2 - v_{1,\text{avg}}^2)}{2} + \overset{\substack{\text{(no change in} \\ \text{elevation)} \\ 0}}{g\Delta z} + \hat{F}_L + \overset{0\ \text{(no shaft work)}}{\hat{W}_s} = 0 \qquad (3.9.1.2\text{-}1)$$

$\widehat{FL}$ here corresponds to the loss due to contraction. Approximating this contraction to a sudden contraction, from Example 3.9.1-1 we get

$$K_f = 0.4\left(1 - \frac{A_2}{A_1}\right)$$

where A is CS area.

Thus

$$\widehat{FL} = 0.4\left(1 - \frac{A_2}{A_1}\right)\frac{v_{2,\text{avg}}^2}{2}$$

Note the velocity we use here to calculate $\widehat{FL}$.

Thus

$$\frac{p_2 - p_1}{\rho} + \frac{(v_{2,\text{avg}}^2 - v_{1,\text{avg}}^2)}{2} + 0.4\left(1 - \frac{A_2}{A_1}\right)\frac{v_{2,\text{avg}}^2}{2} = 0 \tag{3.9.1.2-2}$$

For cavitation to occur, bubbles of gas need to form or nucleation of gas bubbles need to take place. To understand the conditions for gas bubble formation, consider the case of boiling water. In boiling, bubbles begin to appear when the pressure increases due to temperature increase, and finally equals the saturated vapour pressure (note that this is not an equilibrium situation, and thus we cannot apply the phase diagram to find the relevant temperature-pressure relationship for the vapour and liquid phases). In the case of cavitation, the approach is from the other direction; the pressure decreases with increase in velocity of the fluid, and when the pressure equals or becomes lower than the saturated vapour pressure, bubbles form and cavitation occurs. Let us define the difference between the actual pressure and the saturated vapour pressure as p_g.

Thus

$$p_2 - p_1 = p_{2g} - p_{1g} \tag{3.9.1.2-3}$$

Also for continuity

$$A_1 v_1 = A_2 v_2 \quad \text{or} \quad v_2 = \frac{A_1}{A_2} v_1 \tag{3.9.1.2-4}$$

Since the pressure and velocity are inversely related, and since $v_2 > v_1$, $p_{g2} = 0$ is the condition for the onset of cavitation.

Substituting the above in Eq. 3.9.1.2-2, we get

$$-\frac{p_{g1}}{\rho} + \frac{1}{2} v_{1,\text{avg}}^2 \left(\frac{A_1^2}{A_2^2} - 1\right) + \frac{0.4}{2}\left(1 - \frac{A_2}{A_1}\right)\frac{A_1^2}{A_2^2} v_{1,\text{avg}}^2 = 0$$

$$-\frac{p_{g1}}{\rho} + \frac{v_{1,\text{avg}}^2}{2}\left(\frac{A_1^2}{A_2^2} - 1 + 0.4\frac{A_1^2}{A_2^2} - 0.4\frac{A_1}{A_2}\right) = 0$$

$$1.4\left(\frac{A_1}{A_2}\right)^2 - 0.4\left(\frac{A_1}{A_2}\right) - 1 = \frac{p_{g1}}{\rho} \times \frac{2}{v_{1,\text{avg}}^2}$$

$$1.4\left(\frac{A_1}{A_2}\right)^2 - 0.4\left(\frac{A_1}{A_2}\right) - \left(1 + \frac{p_{g1}}{\rho}\frac{2}{v_{1,\text{avg}}^2}\right) = 0$$

The solution of this quadratic equation provides the condition in terms of $\dfrac{A_1}{A_2}$ for cavitation to occur, i.e.

$$\frac{A_1}{A_2} = \frac{\left(0.4 \pm \sqrt{0.4^2 - 4 \times 1.4 \times \left(-1 + \frac{p_{g1}}{\rho} \times \frac{2}{v_{1,\text{avg}}^2}\right)}\right)}{2 \times 1.4}$$

$$= \frac{\left(0.4 \pm \sqrt{0.16 + 5.6\left(\frac{2p_{g1}}{\rho v_{1,\text{avg}}^2} - 1\right)}\right)}{2.8}$$

If $\frac{A_1}{A_2} \geq$ the above RHS, $p_{g2} \leq 0$, and cavitation will occur.

3.9.2 Friction Factor for Solids Moving Relative to a Fluid

For solids with a projected area, A_p (area projected on a plane that is normal to the relative motion direction)

$$F_k = (A_p)\left(\frac{1}{2}\rho v_\infty^2\right)f \tag{3.9.2-1}$$

where v_∞ is the free stream velocity or the approach velocity at a large distance from the object.

F_k is often referred to as the drag force, while f is usually represented as C_D, the drag coefficient. A plot of variation of C_D with Reynolds number, N_{Re}, is available in handbooks, e.g. the one referred to in the earlier section.

When a sphere ($A_p = \pi R^2 = \pi D^2/4$) of density ρ_p falls through a fluid of density ρ, at a terminal velocity of v_t ($= v_\infty$), a simple force balance provides

$$F_k = \left(\frac{4}{3}\pi R^3\right)\rho_p g - \left(\frac{4}{3}\pi R^3\right)\rho g \tag{3.9.2-2}$$

Using Eq. 3.9.2-1

$$F_k = (\pi R^2)\left(\frac{1}{2}\rho v_t^2\right)f$$

Equating the above two expressions, we can get an expression for the friction factor for this case, as

$$f = \frac{4}{3}\frac{gD}{v_t^2}\left(\frac{\rho_p - \rho}{\rho}\right) \tag{3.9.2-3}$$

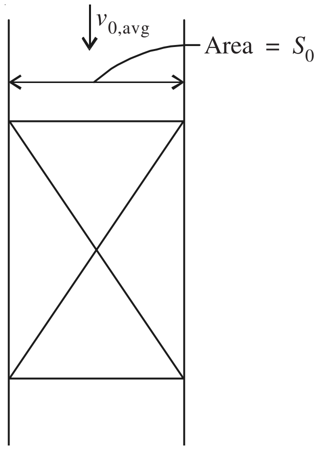


Fig. 3.9.3-1 A packed bed. $v_{0,avg}$ is average superficial velocity; the average velocity before the flow reaches the bed when there are no additional restrictions to flow

3.9.3 Friction Factor in Packed Beds

Packed beds are used in biological processes, especially in water and waste water processing. Certain stages of such processes involve removing undesirables by microorganisms or other agents in a packed bed.

A rigorous analysis of a packed bed is difficult, because even if an effort leads to a representative set of mathematical equations, they may not be easily solvable.

So, let us attempt a simpler analysis using the following assumptions:

- Replace the tortuous flow path inside the bed (Fig. 3.9.3-1) through the voids by a set of identical parallel conduits of the same length as that of the bed. Let the radius of each conduit be R, and the total cross-sectional area of the conduits (number of conduits times the CS area of each conduit) be S.
- Use a representative hydraulic radius to make the results somewhat extendable to many cross-sectional geometries.
- Let the particles be uniform with point contacts between them.
- Assume laminar flow in the conduits.

Let us consider that the total drag force (F_D) per unit total cross-sectional area in the parallel conduits is the sum of viscous drag forces (F_V), and inertial drag forces (F_I) per unit total CS area (S).

Now, let us focus our attention on each conduit with radius R, for a while. From Eq. 3.4.2-17 the average velocity in the conduit is

$$v_{\text{avg}} = \frac{(-\Delta P)R^2}{8\mu L} \qquad (3.4.2\text{-}17)$$

We also know from the equation just before Eq. 3.9.1-4 that

$$(-\Delta P) = \frac{4L\tau_{w,V}}{D} \quad \text{or} \quad \frac{2L\tau_{w,V}}{R}$$

The subscript V refers to the viscous component. Substituting the above expression for (– ΔP) into Eq. 3.4.2-17, we get

$$v_{\text{avg}} = \left(\frac{2L\tau_{w,V}}{R}\right)\frac{R^2}{8\mu L}$$

Transposing, we get

$$\tau_{w,V} = \frac{4\mu\, v_{\text{avg}}}{R} \text{ and we know that } \tau_{w,V} = \frac{F_V}{S}$$

In terms of the hydraulic radius, r_H (to generalise it to channels of any cross-sectional shape)

$$\frac{F_V}{S} = \frac{k\,\mu\, v_{\text{avg}}}{r_H} \tag{3.9.3-1}$$

Now, let us look at the inertial component. The inertial force per unit cross-sectional area of the conduit

$$\frac{F_I}{S} = \tau_{w,I}$$

From Eq. 3.9.1-7, we get

$$\tau_{w,I} = \frac{1}{2}\rho v_{\text{avg}}^2 f = k_2 \rho v_{\text{avg}}^2 \tag{3.9.3-2}$$

Thus, the total drag force per unit conduit area according to the summative consideration of the viscous and inertial components, is

$$\frac{F_D}{S} = \frac{k_1 \mu v_{\text{avg}}}{r_H} + k_2 \rho v_{\text{avg}}^2 \tag{3.9.3-3}$$

Now, let us focus on the entire bed. Let us define

$$\frac{\text{Volume of voids in the bed}}{\text{Total bed volume}} = \in \tag{3.9.3-4}$$

In other words

$$\frac{\text{CS area of imaginary conduits in bed} \times L_{\text{imaginary conduits}}}{\text{CS area of bed} \times L_{\text{bed}}} = \in$$

By one of our earlier assumptions

$$L_{\text{imaginary conduits}} = L_{\text{bed}}$$

Thus

$$\frac{\text{CS area of imaginary conduits in bed}}{\text{CS area of bed}} = \in \qquad (3.9.3\text{-}5)$$

By mass conservation, since the mass flow rates through the conduits are additive, and S = total number of conduits × cross-sectional area of each conduit.

$$\rho v_{0,\text{avg}} S_0 = \rho v_{\text{avg}} S$$

Since the density is a constant

$$\frac{v_{0,\text{avg}}}{v_{\text{avg}}} = \frac{S}{S_0} = \in$$

or

$$v_{\text{avg}} = \frac{v_{0,\text{avg}}}{\in} \qquad (3.9.3\text{-}6)$$

$v_{0,\text{avg}}$ i.e. 'superficial' or 'empty tower' velocity is much easier to measure compared to v_{avg}.

Now, let us relate the pressure drop across the bed to measurable parameters. To do that let us focus on the particles in the bed for a while. The aim is to express the relevant equations in terms of the measurable/ calculable particle parameters.

The total surface area of the particles is A_S

$$A_s = N_p \, s_p \qquad (3.9.3\text{-}7)$$

where N_p is total number of particles in the bed and s_p is area of one particle.

Assuming uniform particles

$$N_p \text{ is also} = \frac{\text{Volume of solids in bed}}{\text{Volume of one particle}} = \frac{S_0 L(1-\in)}{v_p} \qquad (3.9.3\text{-}8)$$

where S_0 is cross-section of empty tower and L is bed length.

Substituting Eq. 3.9.3-7 in Eq. 3.9.3-8, we get

$$\frac{A_s}{s_p} = \frac{S_0 L(1-\in)}{v_p}$$

$$A_s = \frac{S_0 L(1-\in) s_p}{v_p} \qquad (3.9.3\text{-}9)$$

Now

$$r_H = \frac{\text{CS area}}{\text{Wetted perimeter}} = \frac{\text{CS area} \times L}{\text{Wetted perimeter} \times L} = \frac{S\,L}{A_s} = \frac{(\in S_0)L}{A_s} \tag{3.9.3-10}$$

In the equation above, since we have assumed point contacts between particles, and hence there is no loss in surface area due to contact, the total surface area of the particles will equal the total surface area of the conduits.

Substituting A_s from Eq. 3.9.3-9 into Eq. 3.9.3-10, we get

$$r_H = \frac{\in S_0 L}{S_0 L(1-\in)\,s_p/v_p} = \frac{\in v_p}{(1-\in)s_p} \tag{3.9.3-11}$$

Substituting the above equation in Eq. 3.9.3-3, we get

$$F_D = \frac{S_0 \rho L(1-\in)s_p}{\in^2 v_p}\left[\frac{k_1 \mu v_{0,\text{avg}}(1-\in)s_p}{\rho v_p} + k_2 v_{0,\text{avg}}^2\right] \tag{3.9.3-12}$$

We can also express the drag force as the product of (pressure drop) and (effective area), i.e.

$$F_D = (-\,\Delta p)(S_0 \in)$$

Equating the two expressions for the drag force, we get

$$(-\,\Delta p)\,S_0 \in = S_0 \rho L\left(\frac{1-\in}{\in^2}\right)\left(\frac{s_p}{v_p}\right)\left[\frac{k_1 \mu v_{0,\text{avg}}(1-\in)s_p}{\rho v_p} + k_2 v_{0,\text{avg}}^2\right]$$

$$\frac{(-\,\Delta p)}{\rho L} = \left(\frac{1-\in}{\in^3}\right)\left(\frac{s_p}{v_p}\right)\left[\frac{k_1 \mu v_{0,\text{avg}}(1-\in)}{\rho}\,\frac{s_p}{v_p} + k_2 v_{0,\text{avg}}^2\right] \tag{3.9.3-13}$$

For a sphere

$$\frac{s_p}{v_p} = \frac{\pi D_p^2}{\frac{\pi}{6} D_p^3} = \frac{6}{D_p} \tag{3.9.3-14}$$

For any particle, let us define an equivalent diameter D_p as the diameter of the sphere having the same volume as that of the particle.

Let us also define sphericity, ϕ_s as

$$\phi_s = \frac{\text{Surface area of the equivalent sphere}}{\text{Actual surface area}} \tag{3.9.3-15}$$

$$\phi_s = \frac{\pi D_p^2}{s_p}$$

Thus

$$s_p = \frac{\pi D_p^2}{\phi_s}$$

and therefore

$$\therefore \frac{\Delta p}{v_p} = \frac{\pi D_p^2}{\phi_s \frac{\pi}{6} D_p^3} = \frac{6}{\phi_s D_p} \tag{3.9.3-16}$$

Values of ϕ_s for various commonly used particles are available in handbooks.

Ergun correlated experimental data and found that

$$k_1 = \frac{150}{36} \quad \text{and} \quad k_2 = \frac{1.75}{6}$$

Thus, the pressure drop equation can be written as

$$\frac{(-\Delta p)}{\rho v_{0,\text{avg}}^2} \cdot \frac{\phi_s D_p}{L} \cdot \frac{\in^3}{(1-\in)} = \frac{150}{\phi_s D_p} \frac{1-\in}{\rho v_{0,\text{avg}}^2 / \mu} + 1.75 \tag{3.9.3-17}$$

Equation 3.9.3-17 is called the Ergun equation. The above equation works well for most packings – except for packings of extreme shape such as needles, rings or saddles.

By comparison with the friction factor defined earlier, Eq. 3.9.1-6, we can define the friction factor for a packed bed as

$$f_{pb} \equiv \frac{(-\Delta p)\phi_s D_p \in^3}{\rho v_{0,\text{avg}}^2 L(1-\in)} \tag{3.9.3-18}$$

Substituting this back into Eq. 3.9.3-17, we get

$$f_{pb} = \frac{150(1-\in)}{\phi_s N_{\text{Re},p}} + 1.75 \tag{3.9.3-19}$$

At low $N_{\text{Re},p}$, 1.75 is negligible in comparison with the other term. Thus, at low $N_{\text{Re},p}$

$$f_{pb} = \frac{150(1-\in)}{\phi_s N_{\text{Re},p}} \tag{3.9.3-20}$$

This implies that (through substitution of the expression for f_{pb} back into the above equation)

$$\frac{(-\Delta p)\phi_s D_p^2 \in^3}{L v_{0,\text{avg}} \mu (1-\in)^2} = 150$$

$$(-\Delta p)\frac{S_0\phi_s D_p^2 \in^3}{S_0 v_{0,\text{avg}} L\mu(1-\in)^2} = 150$$

where $S_0 v_{0,\text{avg}}$ is volumetric flow rate, Q. The above equation is called the Kozeny-Carman equation.

If ϕ_s, D_p, and $\in$ are constants

$$Q \propto (-\Delta p) \text{ and } \propto \frac{1}{\mu L} \tag{3.9.3-21}$$

This is known as Darcy's law and has many applications.

At large N_{Re}, the first term in the RHS in Eq. 3.9.3-19 becomes negligible. Under such condition, we get the Blake-Plummer equation, i.e.

$$\frac{(-\Delta p)}{\rho v_{0,\text{avg}}^2}.\frac{\phi_s D_p}{L}.\frac{\in^3}{(1-\in)} = 1.75 \tag{3.9.3-22}$$

The above equations can be used to predict pressure drop across beds. Hence, the pumping requirements across packed beds can be estimated.

Exercises

1. Succinctly differentiate between

   (a) Laminar and turbulent flows
   (b) Pseudoplastic and dilatant fluids
   (c) Pseudoplastic and viscoelastic fluids
   (d) Bingham plastic and power law fluids
   (e) Viscosity and kinematic viscosity

2. Which model is an applicable constitutive equation for blood?

3. How do the equation of continuity in Chapter 2 and the equations of motion given in the tables in this chapter for laminar flow have to be modified so that they become applicable for turbulent flow?

4. There exists a concept called 'dynamic similarity' that makes it possible to use non-dimensional analysis for scale-up. Read up about the concept of dynamic similarity.

5. In a micro-processing unit for biological analytes that is based on microfluidics, multiple channels feed into a heating device that consists of a thin box-like structure with top and bottom faces that can be heated to increase the temperature of the fluid flowing through it. Even in the micro-dimensions,

the thickness of the box, d, or the space between the top and the bottom faces can be considered very small compared to the length and breadth of the heating box. An incompressible fluid of viscosity, μ, is flowing through it, and the heating is not turned on. If the pressure drop between the inlet and outlet of the heating box is a constant, K, derive an expression for the velocity profile of the fluid between the top and bottom faces of the boxes, when the flow is well developed; ignore entrance and exit effects. In a circular capillary section of the same set-up, what would be the flow rate, if the pressure due to surface tension is the only driving force for bulk flow in that section?

6. Two glass plates are placed horizontally at a distance of $3d$ from each other with a fluid in between them. A third thin rigid sheet (of negligible mass) is fixed between the plates, at a distance of d from the top plate – the sheet cannot move. The top plate is moved at a velocity of v ms^{-1}, and the bottom plate at 2 v ms^{-1}. Determine the velocity profiles at steady state when the plates are moved (a) in the same direction, and (b) in opposite directions.

7. It is well known that skiing in the snow-filled mountains in the northern regions of our country is possible because there is water formation from the snow/ice under the ski, due to pressure. This thin water layer provides the lubrication needed and makes skiing possible. Interestingly, it can be looked at as a water layer with the top part being bounded by the ski, and the bottom by the stationary solid ground (covered with snow/ice). Consider a person weighing 60 kg skiing on a 15° slope at a speed of 30 km h^{-1}. The ski bottom surface can be approximated to a rectangle of dimensions 14×80 cm^2. If the viscosity of water in the water layer is 1.8×10^{-3} N s m^{-2}, estimate the thickness of the water layer.

8. Set up the differential equations to obtain the velocity profile of a Newtonian fluid flowing in laminar flow, in a duct of square cross-section.

9. In one of our earlier studies in our lab that investigated the effects of a physical stress on the metabolic and genetic responses of cells, we needed to grow cells over extended periods at defined shear stress. We had used a co-axial cylinder set-up, and grew cells in the thin annular space between the cylinders. The outer cylinder was rotated at different rpms, thereby providing different shear rates, and hence different shear stresses on the cells. Develop expressions for the shear stress and the shear rate on the cells in terms of operational parameters, and discuss the expressions developed.

10. A lab deals with a particularly shear sensitive cell line which needs to be transported from one point to another in a pipe of length L, diameter, D. It was decided to use laminar flow for transport. Let the critical shear stress that the cells can tolerate be τ_{crit}, and the corresponding shear rate, assuming that the fluid is Newtonian be $\dot{\gamma}_{crit}$. Let the critical shear stress/rate be in the

range of shear that occurs in the laminar flow regime. Derive an expression for the maximum flow rate that can be used to transfer a uniform solution of the cells, of viscosity, μ, to ensure 80 % survival at the exit.

11. Semi-circular canals in the ear help the brain sense orientations of the head. They are three half-circular, inter-connected tubes inside the ear which are orthogonal (perpendicular) to each other. Each semi-circular canal is approximately a torus with a radius of curvature, say R, and inner radius of cross-section, a; $a << R$. Each canal is filled with a fluid called the endolymph, and contains a gelatinous membrane called cupula, a motion sensor with hair cells (cilia). The cilia move when the endolymph rushes past it and send a signal to the brain. The movement in the endolymph is induced as a result of twisting of the head. Hence a direct relation between angle of twist and cupula movement helps the brain sense rotations. The endolymph and cupula densities are constant and equal, under normal conditions, to avoid the effect of gravity on the cupula. Assume the endolymph to be Newtonian, with a density, r, and viscosity, n. Formulate a relationship between angular acceleration of the head with the rotation axis vertically located through the centre of the head, and cupula deflection in the corresponding semi-circular canal.

12. Atherosclerosis is a disease caused by the rise in the level of cholesterol in the body. The proteoglycans carried along the arteries are able to bind to the lining in the arteries. This leads to plaque build-up in the arteries, which results in the decrease of artery radius, and abnormal blood pressures. In a patient with atherosclerosis, the pressure drop in the artery, by some non-invasive method, was found to be $\Delta P'$, instead of ΔP under normal conditions. What is the thickness of the plaque built-up on the inner wall of the artery, if it can be assumed that the plaque covers the entire inner wall at the relevant region of the artery. Further assume that blood can be approximated to a Newtonian fluid for this purpose.

13. A student working on the SDS-page experiment prepared the required reagents for a stacking gel and kept it for his/her partner to process further. When the partner got to it after about 10 min, (s)he loaded the stacking gel into the pipette. Due to the time lapsed, the fluid behaves as a Bingham plastic now. Neglecting the tip of the pipette, find the steady state velocity profile in the pipette and the mass flow rate of the gel for a given constant pressure drop exerted by the mechanism in the pipette.

14. The lachrymal sac stores tears, and the tears flow through the lachrymal duct to the eye. By assuming that the lachrymal duct is a straight pipe of 12 mm length and 1 mm diameter, and by neglecting gravity, estimate the pressure needed to force the tears into the eyes through the larchrymal duct at a flow rate of 1.2 mL min^{-1} in laminar flow. Take the viscosity of the tears to be 4.4 Pa. s, and its density to be 1000 kg m^{-3}.

15. Pulsatile drug delivery systems have been used to deliver a desired amount of drug at the desired time and location. Develop an expression for the time dependent velocity of the drug solution for a given pressure drop, and other needed parameters.

Some of the exercise problems given above were suggested/formulated by G. Shashank, G. Vivek Sathvik, D. Divya Vani, I. Pradeep Kumar (6, 8), Akhil Sai Valluri (7), S. Kousik, Sagar Laygude, Utsav Saxena (10–12), Uma Maheswari, Namrata Kamat, Kiran, Kemun Khimun, Rashmi Kumari (13, 15), P. Raghavendran, P. Vivek, K. Ramasamy and M. Ashok (14).

Fully Open-ended Exercise

Estimate the power needed to overcome the frictional drag in the trachea. Use this to determine the efficiency of the trachea in the respiratory circuit, and propose an easily measurable physical parameter that can be used to decide whether a person with respiratory difficulties needs ventilator support. This problem was formulated by Akhil Sai Valluri, for his CFA exercise (CFA stands for choose-focus-analyse exercise). See end of Chapter 1 for some details.

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