Independent and Hitting Sets of Rectangles Intersecting a Diagonal Line

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Abstract. Finding a maximum independent set of a given family of axis-parallel rectangles is a basic problem in computational geometry and combinatorics. This problem has attracted significant attention since the sixties, when Wegner conjectured that the corresponding duality gap, i.e., the maximum possible ratio between the maximum independent set and the minimum hitting set, is bounded by a universal constant. In this paper we improve upon recent results of Chepoi and Felsner and prove that when the given family of rectangles is intersected by a diagonal, this ratio is between 2 and 4. For the upper bound we derive a simple combinatorial argument that first allows us to reprove results of Hixon, and Chepoi and Felsner and then we adapt this idea to obtain the improved bound in the diagonal intersecting case. From a computational complexity perspective, although for general rectangle families the problem is known to be NP-hard, we derive an $O(n^2)$-time algorithm for the maximum weight independent set when, in addition to intersecting a diagonal, the rectangles intersect below it. This improves and extends a classic result of Lubiw. As a consequence, we obtain a 2-approximation algorithm for the maximum weight independent set of rectangles intersecting a diagonal.

1 Introduction

Given a family of axis-parallel rectangles, two natural objects of study are the maximum number of rectangles that do not overlap and the minimum set of points stabbing every rectangle. These problems are known as maximum independent set MIS and minimum hitting set MHS respectively, and in the associated intersection graph they correspond to the maximum independent set and the minimum clique covering. We study these problems for restricted classes of rectangles, and focus on designing algorithms and on evaluating the *duality gap* δ_{GAP}, i.e., the maximum ratio between these quantities. This term arises as MHS is the integral version of the dual of the natural linear programming relaxation of MIS.

From a computational complexity viewpoint, MIS and MHS of rectangles are strongly NP-hard [11,13], so attention has been put into approximation algorithms and polynomial time algorithms for special classes. The current best

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known approximation factor for MIS are $O(\log\log n)$ [3], and $O(\log n/\log\log n)$ for weighted MIS (WMIS) [4]. Very recently, Adamaszek and Wiese [1] designed a pseudo-polynomial time algorithm finding a $(1+\varepsilon)$-approximate solution for WMIS, but it is unknown whether there exist polynomial time constant factor approximation algorithms. A similar situation occurs for MHS: the current best approximation factor is $O(\log\log n)$ [2], while in general, the existence of a constant factor approximation is open. Polynomial time algorithms for these problems have been obtained for special classes. When all rectangles are intervals, the underlying intersection graph is an interval graph and even linear time algorithms are known for MIS, MHS and WMIS [12]. Moving beyond interval graphs, Lubiw [15] devised a cubic-time algorithm for computing a maximum weight independent family of point-intervals, which can be seen as families of rectangles having their upper-right corner along the same diagonal. More recently, Soto and Telha [17] considered the case where the upper-right and lower-left corners of all rectangles are two prescribed point sets of total size m. They designed an algorithm that computes both MIS and MHS in the time required to do m by m matrix multiplication, and showed that WMIS is NP-hard on this class. Finally, there are also known PTAS for special cases, including the results of Chan [4] for squares, and Mustafa and Ray [16] for unit height rectangles.

It is straightforward to observe that given a family of rectangles the size of a maximum independent set is at most that of a minimum hitting set. In particular, for interval graphs this inequality is actually an equality, and this still holds in the case studied by Soto and Telha [17], so that the duality gap is 1 for these classes. A natural question to ask is whether the duality gap for general families of rectangles is bounded. Indeed, already in the sixties Wegner [19] conjectured that the duality gap for arbitrary rectangles families equals 2, whereas Gyárfás and Lehel [9] proposed the weaker conjecture that this gap is bounded by a universal constant. Although these conjectures are still open, Károlyi and Tardos [14] proved that the gap is within $O(\log(\text{mis}))$, where mis is the size of a maximum independent set. For some special classes, the duality gap is indeed a constant. In particular, when all rectangles intersect a given diagonal line, Chepoi and Felsner [5] prove that the gap is between 3/2 and 6, and the upper bound has been further improved for more restricted classes [5,10].

1.1 Notation and Classes of Rectangle Families

Throughout this paper, $\mathcal{R}$ denotes a family of n closed, axis-parallel rectangles in $\mathbb{R}^2$. A rectangle $r \in \mathcal{R}$ is defined by its lower-left corner ℓ^r and its upper-right corner u^r. For a point $v \in \mathbb{R}^2$ we let v_x and v_y be its x-coordinate and y-coordinate, respectively. Also, each rectangle $r \in \mathcal{R}$ is associated with a non-negative weight w_r. We also consider a monotone curve, given by a decreasing bijective real function, so that the boundary of each $r \in \mathcal{R}$ intersects the curve in at most 2 points. We use a^r and b^r to denote the higher and lower of these points respectively (which may coincide). We identify the rectangles in $\mathcal{R}$ with the set $[n] = \{1, \dots, n\}$ so that $a^1_x < a^2_x < \cdots < a^n_x$. For any rectangle i, we define $f(i)$ as the rectangle j (if it exists) following i in the order of the b-points, that is, $b_x^i < b_x^j$ and no rectangle k is such that $b_x^i < b_x^k < b_x^j$. For reference, see Figure 1.

A set of rectangles $\mathcal{Q} \subseteq \mathcal{R}$ is called independent if and only if no two rectangles in $\mathcal{Q}$ intersect. On the other hand, a set $H \subseteq \mathbb{R}^2$ of points is a hitting set of $\mathcal{R}$ if every rectangle $r \in \mathcal{R}$ contains at least one point in H. In this paper we consider the problem of finding an independent set of rectangles in $\mathcal{R}$ of maximum cardinality (MIS), and its weighted version (WMIS). We also consider the problem of finding a hitting set of $\mathcal{R}$ of minimum size (MHS). Let us denote by $\text{mis}(\mathcal{R})$, $\text{wmis}(\mathcal{R})$, $\text{mhs}(\mathcal{R})$ the solutions to the above problems, respectively.

Since the solutions of the previous problems depend on properties of the intersection graph $\mathcal{I}(\mathcal{R}) = (\mathcal{R}, \{rr' \colon r \cap r' \neq \emptyset\})$ of the family $\mathcal{R}$, we will assume that no two defining corners in $\{\ell^1, \ell^2, \ldots, \ell^n, u^1, u^2, \ldots, u^n\}$ have the same x-coordinates or y-coordinates (this is done without loss of generality by individually perturbing each rectangle). We will also assume that the curve mentioned in the first paragraph is the diagonal line D given by the equation $y = -x$. This is assumed without loss of generality: by applying suitable piecewise linear transformations on both coordinates we can transform the rectangle family into one with the same intersection graph such that every rectangle intersects D. In what follows, call the closed halfplanes given by $y \geq -x$ and $y \leq -x$, the *halfplanes* of D. Note that both halfplanes intersect in D. The points in the bottom (resp. top) halfplane are said to be below (resp. above) the diagonal.

We study four special classes of rectangle families intersecting D.

Definition 1 (Classes of rectangle families)**.**

1. $\mathcal{R}$ is *diagonal-intersecting* if for all $r \in \mathcal{R}$, $r \cap D \neq \emptyset$.
2. $\mathcal{R}$ is *diagonal-splitting* if there is a side (upper, lower, left, right) such that D intersects all $r \in R$ on that particular side.
3. $\mathcal{R}$ is *diagonal-corner-separated* if there is a halfplane of D containing the same three corners of all $r \in \mathcal{R}$.
4. $\mathcal{R}$ is *diagonal-touching* if there is a corner (upper-right or lower-left) such that D intersects all $r \in R$ exactly on that corner (in particular, either all the upper-right corners, or all the lower-left corners are in D.)

By rotating the plane, we can make the following assumptions: In the second class, we assume that the common side of intersection is the upper one; in the third class, that the upper-right corner is on the top halfplane of D and the other three are in the bottom one; and in the last class, that the corner contained in D is the upper-right one. Under these assumption, each type of rectangle family is more general than the next one. It is worth noting that in terms of their associated intersection graphs, the second and third classes coincide. Indeed, two rectangles of a diagonal-splitting rectangle family $\mathcal{R}$ intersect if and only if they have a point in common in the bottom halfplane of D. Therefore, we can replace each rectangle r with the minimal possible one containing the region of r that is below the diagonal, obtaining a diagonal-corner-separated family with the same intersection graph.

Definition 2 (diagonal-lower-intersecting)**.** A diagonal-intersecting family $\mathcal{R}$ is *diagonal-lower-intersecting* if whenever two rectangles in $\mathcal{R}$ intersect, they have a common point in the bottom halfplane of D.

The next lemma describes the relation between the graph classes associated to the families just defined. Its proof is deferred to the full version of the paper [7].

Lemma 1. Let $\mathcal{G}_{\text{int}} = \{\mathcal{I}(\mathcal{R}) \colon \mathcal{R} \text{ is diagonal-intersecting}\}$ be the class of intersection graphs arising from diagonal-intersecting families of rectangles. Let also $\mathcal{G}_{\text{low-int}}$, $\mathcal{G}_{\text{split}}$, $\mathcal{G}_{\text{c-sep}}$ and $\mathcal{G}_{\text{touch}}$ be the classes arising from diagonal-lower-intersecting, diagonal-splitting, diagonal-corner-separated, and diagonal-touching families of rectangles, respectively. Then

$$\mathcal{G}_{\text{touch}} \subsetneq \mathcal{G}_{\text{low-int}} = \mathcal{G}_{\text{split}} = \mathcal{G}_{\text{c-sep}} \subsetneq \mathcal{G}_{\text{int}}.$$

We observe that these classes have appeared in the literature under different names. For instance, Hixon [10] call the graphs in $\mathcal{G}_{\text{touch}}$ *hook graphs*, Soto and Thraves [18] call them AND(1) *graphs*, while those in $\mathcal{G}_{\text{int}}$ are called *separable rectangle graphs* by Chepoi and Felsner [5].

1.2 Our Results

Our main results, given in §2, are a quadratic-time algorithm to compute a wmis($\mathcal{R}$) when $\mathcal{R}$ is diagonal-lower-intersecting and a 2-approximation for the same problem when $\mathcal{R}$ is diagonal-intersecting. As far as we know, the former is the first polynomial time algorithm for WMIS on a natural class containing diagonal-touching rectangle families. Our algorithm improves upon previous work in the area. Specifically, for diagonal-touching rectangle families, the best known algorithm to solve WMIS is due to Lubiw [15], who designed a cubic-time algorithm for the problem in the context of *interval systems*. More precisely, a collection of *point-intervals* $Q = \{(p_i, I_i)\}_{i=1}^n$ is a family such that for all i, $p_i \in I_i$ and $I_i = [\text{left}(I_i), \text{right}(I_i)] \subseteq \mathbb{R}$ are a point and an interval, respectively. Q is called *independent* if for $k \neq j$, $p_k \notin I_j$ or $p_j \notin I_k$. Given a finite collection Q of weighted point-intervals, Lubiw designed a dynamic programming based algorithm to find a maximum weighted independent subfamily of Q. It is easy to see[1] that this problem is equivalent to that of finding wmis($\mathcal{R}$) for the diagonal-touching family $\mathcal{R} = \{r_i\}_{i=1}^n$ where r_i is the rectangle with upper right corner $(p_i, -p_i)$ and lower left corner $(\text{left}(I_i), -\text{right}(I_i))$ and having the same weight as that of (p_i, I_i). Lubiw's algorithm was recently rediscovered by Hixon [10].

As in Lubiw's, our algorithm is based on dynamic programming. However, rather than decomposing the instance into small triangles and computing the optimal solution for every possible triangle, our approach involves computing the optimal solutions for what we call a *harpoon*, which is defined for every pair of rectangles. We show that the amortized cost of computing the optimal solution for all harpoons is constant, leading to an overall quadratic time. Interestingly, it

[1] This equivalence has been noticed before [17].

is possible to show that our algorithm is an extension of the linear-time algorithm for maximum weighted independent set of intervals [12].

In §3 we give a short proof that the duality gap δ_{GAP}, i.e., the maximum ratio mhs / mis, is always at most 2 for diagonal-touching families; we also show that $\delta_{\mathrm{GAP}} \leq 3$ for diagonal-lower-intersecting families, and $\delta_{\mathrm{GAP}} \leq 4$ for diagonal-intersecting families. These bounds yields simple 2, 3, and 4-approximation polynomial time algorithms for MHS on each class (they can also be used as approximation algorithms for MIS with the same guarantee, however, as discussed in the previous paragraph, we have an exact algorithm for WMIS on the two first classes, and a 2-approximation for the last one). The 4-approximation for MHS in diagonal-intersecting families is the best approximation known and improves upon the bound of 6 of Chepoi and Felsner [5], who also give a bound of 3 for diagonal-splitting families based on a different method. For diagonal-touching families, Hixon [10] independently showed that $\delta_{\mathrm{GAP}} \leq 2$. To complement the previous results, we show that the duality gap for diagonal-lower-intersecting families is at least 2. We do this by exhibiting an infinite family of instances whose gap is arbitrarily close to 2. Similar instances were obtained, and communicated to us, by Cibulka et al. [6]. Note that this lower bound of 2 improves upon the 5/3 by Fon-Der-Flaass and Kostochka [8] which was the best known lower bound for the duality gap of general rectangle families.

In the full version of the paper [7], besides proving Lemma 1, we prove that computing a MIS on a diagonal-intersecting family is NP-complete. In light of our polynomial-time algorithm for diagonal-lower-intersecting families, the latter hardness result exhibits what is, in a way, a class at the boundary between polynomial-time solvability and NP-completeness. On the other hand, combining the results of Chalermsook and Chuzhoy [3] and Aronov et al. [2], we show that the duality gap is $O((\log\log \mathrm{mis}(\mathcal{R}))^2)$ for a general family $\mathcal{R}$ of rectangles, improving on the logarithmic bound of Károlyi and Tardos [14].

2 Algorithms for WMIS

The idea behind Lubiw's algorithm [15] for WMIS on diagonal-touching families is to compute the optimal independent set OPT_{ij} included in every possible triangle defined by the points u^i, u^j (which are on D), and (u^i_x, u^j_y) for two rectangles $i < j$. The principle exploited is that in OPT_{ij} there exists one rectangle, say $i < k < j$, such that OPT_{ij} equals the union of OPT_{ik}, the rectangle k, and OPT_{kj}. With this idea the overall complexity of the algorithm turns out to be cubic in n. We now present our algorithm, which works for the more general diagonal-lower-intersecting families, and that is based in a more elaborate idea involving what we call *harpoons*.

2.1 Algorithm for Diagonal-Lower-Intersecting Families

Let us first define some geometric objects that will be used in the algorithm. For any pair of rectangles $i < j$ we define $H_{i,j}$ and $H_{j,i}$, two shapes that we call

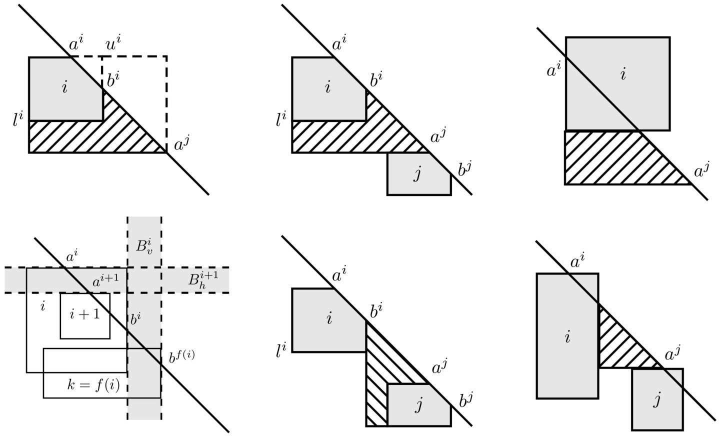

Fig. 1. On the left, the construction of a harpoon and the construction of the strips. On the middle, the harpoons H_{ij} and H_{ji}, with $i < j$. On the right, other particular cases for the harpoon H_{ij} with $i < j$ (the symmetric cases occur for H_{ji}).

harpoons. See Fig. 1. More precisely, the *horizontal harpoon* $H_{i,j}$ consists of the points below the diagonal D obtained by subtracting rectangle i from the closed box defined by the points (ℓ_x^i, a_y^i) and a^j. Similarly, the *vertical harpoon* $H_{j,i}$ are the points below D obtained by subtracting j from the box defined by the points (b_x^j, ℓ_y^j) and b^i. Also, for every rectangle i with $i \geq 1$ (resp. such that $f(i)$ exists) we define B_h^i (resp. B_v^i) as the open horizontal strip that goes through a_{i-1} and a_i (resp. as the open vertical strip that goes through b_i and $b_{f(i)}$).

We say that a rectangle r is contained in the set $H_{i,j}$ (and abusing notation, we write $r \in H_{i,j}$) if the region of r below the diagonal is contained in $H_{i,j}$.

In our algorithm we will compute $S(i,j)$, the weight of the maximum independent set for the subset of rectangles contained in the harpoon $H_{i,j}$. We define two dummy rectangles 0 and $n+1$, at the two ends of the diagonal such that the harpoons defined by these rectangles contain every other rectangle. As previously observed, two rectangles intersect in $\mathcal{R}$ if and only if they intersect below the diagonal. Therefore, $\text{wmis}(\mathcal{R}) = S(0, n+1)$.

Description of the algorithm:

1. *Initialization.* In the execution of the algorithm we will need to know what rectangles have their lower-left corner in which strips. To compute this we do a preprocessing step. Define $\hat{B}_v^i$ and $\hat{B}_h^i$ as initially empty. For each rectangle $r \in \mathcal{R}$, check if ℓ^r is in B_h^i. If so, we add r to the set $\hat{B}_h^i$. Similarly, if ℓ^r is in B_v^i, we add r to the set $\hat{B}_v^i$.
2. *Main loop.* We compute the values $S(i,j)$ corresponding to the maximum-weight independent set of rectangles in $\mathcal{R}$ strictly contained in $H_{i,j}$. We do this by dynamic programming starting with the values $S(i,i) = 0$. Assume

that we have computed all $S(i,j)$ for all i, j such that $|i-j|<\ell$. We now show how to compute these values when $|i-j|=\ell$.

2.1 Set $S(i,j)=S(i,j-1)$ if $i<j$ and $S(i,j)=S(i,f(j))$ if $i>j$.

2.2 Define $\hat{B}_{i,j}$ as $\hat{B}_h^j$ if $i<j$, or $\hat{B}_v^j$ if $i>j$.

2.3 For each rectangle $k\in\hat{B}_{i,j}$ and strictly contained in harpoon $H_{i,j}$ do:

2.3.1. Compute $m=w_k+\max\{S(i,k),S(k,i)\}+S(k,j)$.

2.3.2. If $m>S(i,j)$, then $S(i,j):=m$.

3. *Output.* $S(0,n+1)$.

It is trivial to modify the algorithm to return not only wmis($\mathcal{R}$) but also the independent set of rectangles attaining that weight. We now establish the running time of our algorithm.

Theorem 1. *The previous algorithm runs in* $O(n^2)$.

Proof. The pre-processing stage needs linear time if the rectangles are already sorted, otherwise we require $O(n\log n)$ time. The time to compute $S(i,j)$ is $O(1+|\hat{B}_{i,j}|)$ since checking if a rectangle is in a harpoon takes constant time. As the index of a rectangle is at most once in some $\hat{B}_h$ and at most once in some $\hat{B}_v$, the time to fill all the table $S(\cdot,\cdot)$ is:

$$\sum_{(i,j)\in[n]^2} O(1+|\hat{B}_{i,j}|)=O(n^2).$$

The algorithm is then quadratic in the number of rectangles. □

In order to analyze the correctness of our algorithm we define a partial order over the rectangles in $\mathcal{R}$.

Definition 3. The (strict) *onion ordering* $\prec$ in $\mathcal{R}$ is defined as

$$i\prec j \iff \text{rectangles } i \text{ and } j \text{ are disjoint}, \ell_x^i<\ell_x^j, \text{ and } \ell_y^i<\ell_y^j.$$

It is immediate to see that $\prec$ is a strict partial ordering in $\mathcal{R}$. We say that i is dominated by j if $i\prec j$.

For any rectangle k in a harpoon $H_{i,j}$, let $S_k(i,j)$ be the value of the maximum-weight independent set containing k and rectangles in $H_{i,j}$ which are not dominated by k in the onion ordering, and $\mathcal{S}_k(i,j)$ be the corresponding set of rectangles.

Lemma 2. For any rectangle k in $H_{i,j}$, the following relation holds:

$$S_k(i,j)=w_k+\max\{S(i,k),S(k,i)\}+S(k,j).$$

Proof. Since $k\in H_{i,j}$, we have that i, k and j are mutually non-intersecting, and as indices, $\min(i,j)<k<\max(j,i)$. Assume that the harpoon is horizontal, i.e., $i<j$ (the proof for $i>j$ is analogous). In particular, we know that a^i,b^i,a^k,b^k,a^j,b^j appear in that order on the diagonal. There are three cases for the positioning of the two rectangles i and k. See Fig. 2.

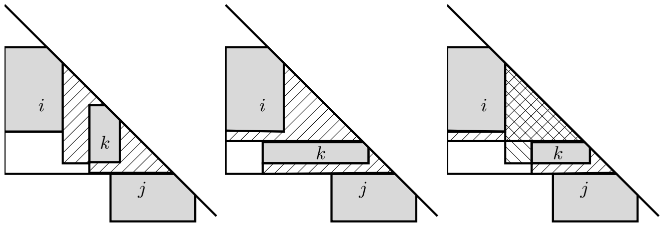

Fig. 2. The three cases for a rectangle in a horizontal harpoon

First case: i and k are separated by a vertical line, but not separated by a horizontal one. Noting that $H_{i,k} \subseteq H_{k,i}$, we conclude that all the rectangles of $\mathcal{S}_k(i,j) \setminus \{k\}$ are in $H_{k,i}$ or in $H_{k,j}$. Since $H_{k,i}$ and $H_{k,j}$ are disjoint, as shown on the first picture, we conclude the correctness of the formula.

Second case: i and k are separated by a horizontal line, but not by a vertical one. The proof follows almost exactly as in the first case.

Third case: i and k are separated by both a horizontal line and a vertical line. By geometric and minimality arguments, all the rectangles in $\mathcal{S}_k(i,j) \setminus \{k\}$ are in the union of the three harpoons $H_{i,k}$, $H_{k,i}$ and $H_{k,j}$ depicted. Finally, if there are two rectangles in $H_{i,k} \cup H_{k,i}$ then they must be in the same harpoon, so the formula holds. □

Theorem 2. *Our algorithm returns a maximum weight independent set of $\mathcal{R}$.*

Proof. By induction. For the trivial harpoons $H_{i,i}$, the maximum independent set has weight 0, because this set is empty. The correctness of the theorem follows directly from the previous lemma and the next implications: For $i \neq j$,

$$i < j \Longrightarrow S(i,j) = \max\left\{S(i,j-1), \max_{k \in \hat{B}_h^j \cap H_{i,j}} S_k(i,j)\right\}.$$

$$j < i \Longrightarrow S(i,j) = \max\left\{S(i,f(j)), \max_{k \in \hat{B}_v^j \cap H_{i,j}} S_k(i,j)\right\}.$$

Indeed, assume that $i < j$ (the case $i > j$ is analogous). Let $\mathcal{S}$ be the MIS corresponding to $S(i,j)$, and let $m \in \mathcal{S}$ be minimal with respect to the onion ordering. If m is in $H_{i,j-1}$ then $S(i,j) = S(i,j-1)$. Otherwise, m is in $\hat{B}_h^j$ and since $\mathcal{S} \setminus \{m\}$ does not contain rectangles dominated by m, $S(i,j) = S_m(i,j)$. □

2.2 An Approximation for Diagonal-Intersecting Families

We use the previous algorithm to get a 2-approximation for diagonal-intersecting rectangle families. This improves upon the 6-approximation (which is only for the unweighted case) of Chepoi and Felsner [5].

Theorem 3. *There exists a 2-approximation polynomial algorithm for* WMIS *on diagonal-intersecting rectangle families.*

Proof. Divide $\mathcal{R}$ into two subsets: the rectangle that intersect the diagonal on their upper side, and the ones that don't. It is easy to see that every rectangle in the second subset intersect the diagonal on its left side. Using symmetry, the left side case is equivalent to the upper side case. Therefore we can compute in polynomial time a WMIS in each subset. We output the heaviest one. Its weight is at least half of $\text{wmis}(\mathcal{R})$. This algorithm gives a 2-approximation □

3 Duality Gap and Other Approximation Algorithms

In this section we explore the duality gap, that is, the largest possible ratio between mhs and mis, on some of the rectangle classes defined before.

Theorem 4. *The duality gap for diagonal-touching rectangle families is between 3/2 and 2. For diagonal-lower-intersecting families it is between 2 and 3, and for diagonal-intersecting families it is between 2 and 4.*

We will prove the upper bounds and the lower bounds separately.

Proof of the upper bounds in Theorem 4. Let $\mathcal{R}$ be a rectangle family in the plane, that can be in one of the three classes described on the theorem. In the case which $\mathcal{R}$ is diagonal-lower-intersecting we first replace each rectangle $r \in \mathcal{R}$ by the minimal one containing the region of r that is below the diagonal. The modified family has the same intersection graph as before, but it is diagonal-corner-separated. In particular, the region of each rectangle that is above the diagonal is a triangle or a single point.

We use $\mathcal{R}_x$ and $\mathcal{R}_y$ to denote the projections of the rectangles in $\mathcal{R}$ on the x-axis and y-axis respectively. Both $\mathcal{R}_x$ and $\mathcal{R}_y$ can be regarded as intervals, and so we can compute in polynomial time the minimum hitting sets, P_x and P_y, and the maximum independent sets, $\mathcal{I}_x$ and $\mathcal{I}_y$, of $\mathcal{R}_x$ and $\mathcal{R}_y$ respectively. Since interval graphs are perfect, $|P_x| = |\mathcal{I}_x|$ and $|P_y| = |\mathcal{I}_y|$.

Furthermore, since rectangles with disjoint projections over the x-axis (resp. over the y-axis) are disjoint, we also have

$$\text{mis}(\mathcal{R}) \geq \max\{|\mathcal{I}_x|, |\mathcal{I}_y|\} = \max\{|P_x|, |P_y|\}.$$

Observe that the collection $\mathcal{P} = P_x \times P_y \subset \mathbb{R}^2$ hits every rectangle of $\mathcal{R}$. From here we get the (trivial) bound $\text{mhs}(\mathcal{R}) \leq |\mathcal{P}| \leq \text{mis}(\mathcal{R})^2$ which holds for every rectangle family. When $\mathcal{R}$ is in one of the classes studied in this paper, we can improve the bound.

Let $\mathcal{P}^-$ and $\mathcal{P}^+$ be the sets of points in $\mathcal{P}$ that are below or above the diagonal, respectively. Consider the following subsets of $\mathcal{P}$:

$$\mathcal{F}^- = \{p \in \mathcal{P}^- \colon \nexists q \in \mathcal{P}^- \setminus \{p\}, p_x < q_x \text{ and } p_y < q_y\}.$$
$$\mathcal{F}^+ = \{p \in \mathcal{P}^+ \colon \nexists q \in \mathcal{P}^+ \setminus \{p\}, q_x < p_x \text{ and } q_y < p_y\}.$$
$$\mathcal{F}^* = \{p \in \mathcal{P}^+ \colon \nexists q \in \mathcal{P}^+ \setminus \{p\}, q_x \leq p_x \text{ and } q_y \leq p_y\}.$$

The set $\mathcal{F}^-$ (resp. $\mathcal{F}^+$) forms the closest "staircase" to the diagonal that is below (resp. above) it. The set $\mathcal{F}^*$ corresponds to the lower-left bending points of the staircase defined by $\mathcal{F}^+$. From here, it is easy to see that

$$\max\{|\mathcal{F}^-|, |\mathcal{F}^+|\} \leq |P_x| + |P_y| - 1 \leq 2\,\mathrm{mis}(\mathcal{R}) - 1.$$
$$|\mathcal{F}^*| \leq \max\{|P_x|, |P_y|\} \leq \mathrm{mis}(\mathcal{R}).$$

If $r \in \mathcal{R}$ is hit by a point of $\mathcal{P}^-$, let $p_1(r)$ be the point of $\mathcal{P}^- \cap r$ closest to the diagonal (in ℓ_1-distance). Since r intersects the diagonal, and the points of $\mathcal{P}$ form a grid, we conclude that $p_1(r) \in \mathcal{F}^-$. Similarly, if $r \in \mathcal{R}$ is hit by a point of $\mathcal{P}^+$, let $p_2(r)$ be the point of $\mathcal{P}^+ \cap r$ closest to the diagonal. Since r intersects the diagonal, we conclude that $p_2(r) \in \mathcal{F}^+$. Furthermore, if the region of r that is above the diagonal is a triangle, then $p_2(r) \in \mathcal{F}^*$.

If $\mathcal{R}$ is diagonal-touching, then every rectangle is hit by a point of $\mathcal{F}^-$, and so $\mathrm{mhs}(\mathcal{R}) \leq |\mathcal{F}^-| \leq 2\,\mathrm{mis}(\mathcal{R}) - 1$. If $\mathcal{R}$ is diagonal-lower-intersecting (and, after the modification discussed at the beginning of this proof, diagonal-corner-separated), then every rectangle is hit by a point of $\mathcal{F}^- \cup \mathcal{F}^*$, and so $\mathrm{mhs}(\mathcal{R}) \leq |\mathcal{F}^-| + |\mathcal{F}^*| \leq 3\,\mathrm{mis}(\mathcal{R}) - 1$. Finally, if $\mathcal{R}$ is diagonal-intersecting, then every rectangle is hit by a point of $\mathcal{F}^- \cup \mathcal{F}^+$, and so $\mathrm{mhs}(\mathcal{R}) \leq |\mathcal{F}^-| + |\mathcal{F}^+| \leq 4\,\mathrm{mis}(\mathcal{R}) - 2$. □

Proof of the lower bounds of Theorem 4. The lower bound of 3/2 is achieved by any family $\mathcal{R}$ whose intersection graph G is a 5-cycle. It is easy to see that $\mathcal{R}$ can be realized as a diagonal-touching family, that $\mathrm{mis}(\mathcal{R}) = 2$ and $\mathrm{mhs}(\mathcal{R}) = 3$, and so the claim holds.

The lower bound of 2 for diagonal-lower-intersecting and diagonal-intersecting families is asymptotically attained by a sequence of rectangle families $\{\mathcal{R}_k\}_{k \in \mathbb{Z}^+}$. We will describe the sequence in terms of infinite rectangles which intersect the diagonal, but it is easy to transform each $\mathcal{R}_k$ into a family of finite ones by considering a big bounding box.

For $i \in \mathbb{Z}^+$, define the i-th layer of the instance as $\mathcal{L}_i = \{U(i), D(i), L(i), R(i)\}$, and the k-th instance $\mathcal{R}_k = \bigcup_{i=1}^{k} \mathcal{L}_i$, where:

$$U(i) = [2i, 2i+1] \times [-(2i+\tfrac{1}{3}), +\infty), \quad D(i) = [2i+\tfrac{2}{3}, 2i+\tfrac{5}{3}] \times (-\infty, -2i],$$
$$L(i) = (-\infty, 2i+\tfrac{1}{3}] \times [-2i-1, -2i], \quad R(i) = [2i, \infty) \times [-(2i+\tfrac{5}{3}), -(2i+\tfrac{2}{3})].$$

Consider the instance $\mathcal{R}_k$ depicted in Figure 3 with k layers of rectangles. $\mathcal{R}_k$ can be easily transformed into a diagonal-lower-intersecting family by "straightening" the staircase curve shown in the figure without changing its intersection graph. Let I be a maximum independent set of rectangles in that instance. It is immediately clear that a minimum hitting set has size $2k$ since no point in the plane can hit more that two rectangles.

Let us prove that the size of a maximum independent set is at most $k+2$, amounting to conclude that the ratio is arbitrarily close to 2. To this end, we let $i_D = \min\{i : D(i) \in I\}$ and $i_R = \min\{i : R(i) \in I\}$, and if no $D(i) \in I$ or no $R(i) \in I$, we let $i_D = k+1$ or $i_R = k+1$, respectively. When $i_D = i_R = k+1$, it is immediate that $|I| \leq k$. Assume then, without loss of generality, that $i_D < i_R$.

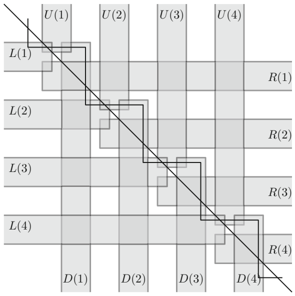

Fig. 3. The family $\mathcal{R}_4$. The diagonal line shows this family is diagonal-intersecting. The staircase line shows that it is actually lower-diagonal-intersecting.

Since for $i = 1, \ldots, i_D - 1$ the set I neither contains rectangle $D(i)$ nor $R(i)$, we have that I contains at most one rectangle on each of these layers. It follows that $\left|I \cap \bigcup_{i=1}^{i_D-1} \mathcal{L}_i\right| \leq i_D - 1$. Similarly, for $i = i_D + 1, \ldots, i_R - 1$ the set I neither contains rectangle $L(i)$ nor $R(i)$, thus $\left|I \cap \bigcup_{i=i_D+1}^{i_R-1} \mathcal{L}_i\right| \leq i_R - i_D - 1$. Finally, we have that for $i = i_R + 1, \ldots, k$ the set I neither contains rectangle $L(i)$ nor $U(i)$, and on layer i_R, I contains at most 2 rectangles; thus $\left|I \cap \bigcup_{i=i_R}^{k} \mathcal{L}_i\right| \leq k - i_R + 2$. To conclude, note that I may contain at most 2 rectangles of layer i_D, then

$$|I| = \sum_{i=1}^{k} |I \cap \mathcal{L}_i| \leq i_D - 1 + i_R - i_D - 1 + k - i_R + 2 + 2 = k + 2. \qquad \square$$

Corollary 1. There is a simple 2-approximation polynomial time algorithm for MHS on diagonal-touching families, a 3-approximation for MHS on diagonal-lower-intersecting families, and a 4-approximation polynomial time algorithm for MHS on diagonal-intersecting families.

Proof. The algorithm consists in computing and returning $\mathcal{F}^-$ for the first case, $\mathcal{F}^- \cup \mathcal{F}^*$ for the second one, and $\mathcal{F}^- \cup \mathcal{F}^+$ for the third one. □

4 Discussion

To conclude the paper we mention open problems that are worth further investigation. First, note that the computational complexity of MHS is open for all classes of rectangle families considered in this paper. The complexity of recognizing the intersection graphs of different rectangles families is also open. It is known that the most general version of this problem, that is recognizing if a graph is the intersection graph of a family of rectangles, is NP-complete [20]. However, little is known for restricted classes. Finally, it would be interesting to determine the duality gap for the classes of rectangle families studied here.

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