Generating Functions for Uniform B-Splines

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Abstract. We derive a closed formula for the generating functions of the uniform B-splines. We begin by constructing a PDE for these generating functions starting from the de Boor recurrence. By solving this PDE, we find that we can express these generating functions explicitly as sums of polynomials times exponentials. Using these generating functions, we derive some known identities, including the Schoenberg identity, the two term formula for the derivatives in terms of B-splines of lower degree, and the partition of unity property. We also derive several new identities for uniform B-splines not previously available from classical methods, including formulas for sums and alternating sums, for moments and reciprocal moments, and for Laplace transforms and convolutions with monomials.

1 Introduction

Generating functions are a powerful tool for investigating the properties of discrete sequences. Explicit formulas and identities for elements of the sequence can often be readily derived once we have an explicit formula for their generating function [1].

The goal of this paper is to compute an explicit formula for the generating function of the uniform B-splines over arbitrary intervals. We shall then use these generating functions to derive several well known identities—including the Schoenberg identity, the two term formula for the derivatives in terms of B-splines of lower degree, and the partition of unity property—for the uniform B-splines. We will also derive several new identities for uniform B-splines not previously available from classical methods such as blossoming or the de Boor recurrence, including formulas for sums and alternating sums, for moments and reciprocal moments, and for Laplace transforms and convolutions with monomials.

This work is inspired by the papers of Y. Simsek [4–6], who computed explicit formulas for a novel collection of generating functions for the classical Bernstein bases

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k} \qquad 0 \le k \le n, \quad 0 \le n < \infty,$$

by summing over the degree n instead of over the index k. He then used these generating functions to derive many known and some new identities for the

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Bernstein basis functions. We shall take a similar approach with the uniform B-splines $N_{k,n}(x)$, again summing over the degree n instead of the index k.

We proceed in the following fashion. In Section 2 we consider the simple special case of B-splines of degree n with knots at the integers $0, \ldots, n+1$ but restricted to the interval $[0,1]$. Over this interval we find that the generating function has an especially simple form as an exponential, so we are encouraged to study the generating function over arbitrary intervals. In Section 3 we apply the de Boor recurrence to derive a PDE for the generating function, and in Section 4 we solve this PDE to find an explicit formula for the generating function over arbitrary intervals. This solution reveals a novel connection between uniform B-splines and exponentials. In Section 5 we show how to use this generating function to derive some classical identities for the uniform B-splines, including Schoenberg's identity, the formula for the derivatives of the B-splines in terms of B-splines of lower degree, and the fact that the B-splines form a the partition of unity. In Section 6 we apply the generating function to derive several new identities for the uniform B-splines not previously accessible from classical methods such as blossoming or the de Boor recurrence. These new identities for uniform B-splines include formulas for sums and alternating sums, for moments and reciprocal moments, and for Laplace transforms and convolutions with monomials. We close in Section 7 with a brief summary of our work along a short discussion of the limitations of our approach to deriving identities for the B-splines using generating functions. We also list a few natural problems involving generating functions and B-splines for future research.

2 A Simple Example: The Generating Function over the Interval $[0, 1]$

We shall begin by investigating the uniform B-splines with knots at the integers when restricted to the interval $[0,1]$.

To fix our notation, let

$N_{k,n}(x)$ = the uniform B-spline of degree n with support $[k, k+n+1]$ and knots at the integers $\{k, k+1, \ldots, k+n+1\}$.

We also introduce the generating functions

$$G_k(x,t) = \sum_{n=0}^{\infty} N_{k,n}(x) t^n.$$

Recall that for uniform B-splines, the functions $N_{k,n}(x)$ are just shifts of the functions $N_{0,n}(x)$—that is,

$$N_{k,n}(x) = N_{0,n}(x-k),$$

so

$$G_k(x,t) = G_0(x-k,t).$$

Thus to investigate the B-splines $N_{k,n}(x)$ and their generating functions $G_k(x,t)$, it is enough to study the B-splines $N_{0,n}(x)$ and their generating functions $G_0(x,t)$.

To investigate the B-splines $N_{0,n}(x)$, consider de Boor recurrence:

$$N_{0,n}(x) = \frac{x}{n} N_{0,n-1}(x) + \frac{n+1-x}{n} N_{1,n-1}(x).$$

For $x \leq 1$, we have $N_{1,n}(x) = 0$. Therefore

$$N_{0,n}(x) = \frac{x}{n} N_{0,n-1}(x) \qquad 0 \leq x \leq 1.$$

Hence in the interval $[0,1]$:

$$N_{0,0}(x) = 1,\ N_{0,1}(x) = x,\ N_{0,2}(x) = \frac{x^2}{2!},\ \ldots,\ N_{0,n}(x) = \frac{x^n}{n!}.$$

Thus over the interval $[0,1]$, we have a remarkably simple explicit formula for the generating function $G_0(x,t)$ of the B-splines $N_{0,n}(x)$:

$$G_0(x,t) = \sum_{n=0}^{\infty} \frac{x^n t^n}{n!} = e^{xt} \qquad 0 \leq x \leq 1.$$

Our goal is to find explicit formulas for the generating function $G_0(x,t)$ over arbitrary intervals.

3 A PDE for the Generating Functions Built from the de Boor Recurrence

For a discrete sequence generated by a recurrence one classical way to derive an explicit formula for the generating function is first to use the recurrence to construct a functional equation for the generating function. One can then often solve this functional equation to find an explicit formula for the generating function. This technique works, for example, to derive an explicit formula for the generating function of the fibonacci numbers [1]. Here we shall apply this method to derive a PDE for the generating functions of the uniform B-splines. In the next section we will solve this PDE to find an explicit formula for the generating functions of the uniform B-splines over arbitrary intervals.

Theorem 1.

$$\frac{\partial G_0}{\partial t}(x,t) = xG_0(x,t) + (2-x)G_1(x,t) + t\frac{\partial G_1}{\partial t}(x,t). \tag{1}$$

Proof. We begin with the de Boor recurrence:

$$N_{0,n}(x) = \frac{x}{n} N_{0,n-1}(x) + \frac{n+1-x}{n} N_{1,n-1}(x).$$

Multiplying both sides by nt^{n-1} yields:

$$nN_{0,n}(x)t^{n-1} = xN_{0,n-1}(x)t^{n-1} + (n+1-x)N_{1,n-1}(x)t^{n-1}.$$

Now summing over n, we find that:

$$\begin{aligned}\sum_n nN_{0,n}(x)t^{n-1} =& x\sum_n N_{0,n-1}(x)t^{n-1} + (2-x)\sum_n N_{1,n-1}(x)t^{n-1} \\ &+ t\sum_n (n-1)N_{1,n-1}(x)t^{n-2}.\end{aligned}$$

Therefore

$$\frac{\partial G_0}{\partial t}(x,t) = xG_0(x,t) + (2-x)G_1(x,t) + t\frac{\partial G_1}{\partial t}(x,t).$$

□

4 Solving the PDE for the Generating Functions

We shall now derive an explicit formula for the generating function $G_0(x,t)$ by solving the PDE in Theorem 1. We begin with some special cases.

Over the interval $[0,1]$, we have $N_{1,n}(x) = 0$. so $G_1(x,t) = 0$. Hence the PDE in Equation (1) reduces to

$$\frac{\partial G_0}{\partial t}(x,t) = xG_0(x,t) \qquad 0 \le x \le 1. \tag{2}$$

Therefore, as we observed in Section 2,

$$G_0(x,t) = e^{xt} \qquad 0 \le x \le 1. \tag{3}$$

Over the interval $[1,2]$, we have

$$G_1(x,t) = G_0(x-1,t) = e^{(x-1)t} \qquad 1 \le x \le 2. \tag{4}$$

Therefore the PDE in Equation (1) reduces to

$$\frac{\partial G_0}{\partial t}(x,t) = xG_0(x,t) + (2-x)e^{(x-1)t} + t(x-1)e^{(x-1)t} \quad 1 \le x \le 2. \tag{5}$$

One can now guess the solution must have terms with the exponentials e^{xt} and $e^{(x-1)t}$. By trial and error one soon finds that:

$$G_0(x,t) = e^{xt} - \big((x-1)t+1\big)e^{(x-1)t} \qquad 1 \le x \le 2, \tag{6}$$

which is easily verified by substituting Equation (6) into Equation (5) and seeing that the PDE is indeed satisfied. Proceeding in this manner, we find that we have the following general result.

Theorem 2. *For $x \in [p, p+1]$, the function*

$$G_0(x,t) = \sum_{j=0}^{p} (-1)^j \left(\frac{(x-j)^j t^j}{j!} + \frac{(x-j)^{j-1} t^{j-1}}{(j-1)!} \right) e^{(x-j)t} \tag{7}$$

satisfies the PDE in Equation (1).

Proof. We proceed by induction on p. The cases $p = 0, 1$ have already been discussed. Suppose then that this result is true for $p-1$; then we must verify that this result is also valid for p. To simplify our notation, let

$$G_{0,p-1}(x,t) = \sum_{j=0}^{p-1} (-1)^j \left(\frac{(x-j)^j t^j}{j!} + \frac{(x-j)^{j-1} t^{j-1}}{(j-1)!} \right) e^{(x-j)t},$$

$$g_{0,p}(x,t) = \left(\frac{(x-p)^p t^p}{p!} + \frac{(x-p)^{p-1} t^{p-1}}{(p-1)!} \right) e^{(x-p)t}.$$

Then for $x \in [p, p+1]$

$$G_0(x,t) = G_{0,p-1}(x,t) + (-1)^p g_{0,p}(x,t),$$
$$G_1(x,t) = G_{0,p-1}(x-1,t) + (-1)^{p-1} g_{0,p-1}(x-1,t).$$

Moreover, by the inductive hypothesis

$$\frac{\partial G_{0,p-1}}{\partial t}(x,t) = xG_{0,p-1}(x,t) + (2-x)G_{1,p-1}(x,t) + t\frac{\partial G_{1,p-1}}{\partial t}(x,t).$$

Therefore it is enough to verify that

$$\frac{\partial g_{0,p}}{\partial t}(x,t) = xg_{0,p}(x,t) - (2-x)g_{0,p-1}(x-1,t) - t\frac{\partial g_{0,p-1}}{\partial t}(x-1,t).$$

But by direct computation:

$$\begin{aligned}\frac{\partial g_{0,p}}{\partial t}(x,t) =& (x-p)\left(\frac{(x-p)^p t^p}{p!} + \frac{(x-p)^{p-1} t^{p-1}}{(p-1)!} \right) e^{(x-p)t} \\ &+ \left(\frac{(x-p)^p t^{p-1}}{(p-1)!} + \frac{(x-p)^{p-1} t^{p-2}}{(p-2)!} \right) e^{(x-p)t}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial g_{0,p}}{\partial t}(x,t) =& xg_{0,p}(x,p) - \left(\frac{(x-p)^p t^p}{(p-1)!} + \frac{p(x-p)^{p-1} t^{p-1}}{(p-1)!} \right) e^{(x-p)t} \\ &+ (x-p)\left(\frac{(x-p)^{p-1} t^{p-1}}{(p-1)!} + \frac{(x-p)^{p-2} t^{p-2}}{(p-2)!} \right) e^{(x-p)t},\end{aligned}$$

or equivalently

$$\begin{aligned}\frac{\partial g_{0,p}}{\partial t}(x,t) =& xg_{0,p}(x,p) - \left(\frac{(x-p)^p t^p}{(p-1)!} + \frac{p(x-p)^{p-1} t^{p-1}}{(p-1)!} \right) e^{(x-p)t} \\ &+ \big((x-2) + (2-p)\big)\left(\frac{(x-p)^{p-1} t^{p-1}}{(p-1)!} + \frac{(x-p)^{p-2} t^{p-2}}{(p-2)!} \right) e^{(x-p)t}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial g_{0,p}}{\partial t}(x,t) = xg_{0,p}(x,t) &- (2-x)g_{0,p-1}(x-1,t)\\ &- \left(\frac{(x-p)^p t^p}{(p-1)!} + \frac{p(x-p)^{p-1}t^{p-1}}{(p-1)!}\right)e^{(x-p)t}\\ &+ (2-p)\left(\frac{(x-p)^{p-1}t^{p-1}}{(p-1)!} + \frac{(x-p)^{p-2}t^{p-2}}{(p-2)!}\right)e^{(x-p)t},\end{aligned}$$

so

$$\begin{aligned}\frac{\partial g_{0,p}}{\partial t} &= xg_{0,p}(x,t) - (2-x)g_{0,p-1}(x-1,t) - \frac{(x-p)^p t^p}{(p-1)!}e^{(x-p)t}\\ &+ (2-2p)\left(\frac{(x-p)^{p-1}t^{p-1}}{(p-1)!}\right)e^{(x-p)t} - \frac{(x-p)^{p-2}t^{p-2}}{(p-3)!}e^{(x-p)t}.\end{aligned}$$

Therefore it is enough to verify that

$$t\frac{\partial g_{0,p-1}(x-1,t)}{\partial t} = \left(\frac{(x-p)^p t^p}{(p-1)!} + 2\frac{(x-p)^{p-1}t^{p-1}}{(p-2)!} + \frac{(x-p)^{p-2}t^{p-2}}{(p-3)!}\right)e^{(x-p)t}.$$

But by definition

$$g_{0,p-1}(x-1,t) = \left(\frac{(x-p)^{p-1}t^{p-1}}{(p-1)!} + \frac{(x-p)^{p-2}t^{p-2}}{(p-2)!}\right)e^{(x-p)t}.$$

Hence

$$\begin{aligned}t\frac{\partial g_{0,p-1}(x-1,t)}{\partial t} =& \left(\frac{(x-p)^{p-1}t^{p-1}}{(p-2)!} + \frac{(x-p)^{p-2}t^{p-2}}{(p-3)!}\right)e^{(x-p)t}\\ &+ (x-p)t\left(\frac{(x-p)^{p-1}t^{p-1}}{(p-1)!} + \frac{(x-p)^{p-2}t^{p-2}}{(p-2)!}\right)e^{(x-p)t},\end{aligned}$$

so indeed

$$t\frac{\partial g_{0,p-1}(x-1,t)}{\partial t} = \left(\frac{(x-p)^p t^p}{(p-1)!} + 2\frac{(x-p)^{p-1}t^{p-1}}{(p-2)!} + \frac{(x-p)^{p-2}t^{p-2}}{(p-3)!}\right)e^{(x-p)t}.$$

□

5 Deriving Identities for the Uniform B-Splines from their Generating Functions

With explicit formulas for the generating functions now in hand, we are finally ready to derive some identities for the uniform B-splines.

5.1 Schoenberg's Identity

Theorem 3. *(Schoenbergs Identity [3])*

$$N_{0,n}(x) = \frac{1}{n!}\sum_{j=0}^{p}(-1)^j \binom{n+1}{j}(x-j)^n \qquad p \le x \le p+1. \tag{8}$$

Proof. Schoenberg's identity for the B-splines follows immediately from the explicit formula for the generating functions. We simply compare coefficients of t^n on both sides of the generating function:

$$G_0(x,t) = \sum_{j=0}^{p}(-1)^j\left(\frac{(x-j)^j t^j}{j!} + \frac{(x-j)^{j-1}t^{j-1}}{(j-1)!}\right)e^{(x-j)t} \qquad p \le x \le p+1.$$

Expanding the exponential function on the right hand side, we find that

$$G_0(x,t) = \sum_{j=0}^{p}(-1)^j\left(\frac{(x-j)^j t^j}{j!} + \frac{(x-j)^{j-1}t^{j-1}}{(j-1)!}\right)\left(\sum_{k=0}^{\infty}\frac{(x-j)^k t^k}{k!}\right) \quad p \le x \le p+1.$$

Now equating the terms with t^n on both side of this equation yields

$$\begin{aligned} N_{0,n}(x)t^n =& \sum_{j=0}^{p}\Bigg((-1)^j\frac{(x-j)^j t^j}{j!}\frac{(x-j)^{n-j}t^{n-j}}{(n-j)!} + \\ & (-1)^j\frac{(x-j)^{j-1}t^{j-1}}{(j-1)!}\frac{(x-j)^{n-j+1}t^{n-j+1}}{(n-j+1)!}\Bigg), \end{aligned}$$

so

$$N_{0,n}(x) = \frac{1}{n!}\sum_{j=0}^{p}(-1)^j\left(\binom{n}{j} + \binom{n}{j-1}\right)(x-j)^n = \frac{1}{n!}\sum_{j=0}^{p}(-1)^j\binom{n+1}{j}(x-j)^n.$$

□

5.2 The Derivative Formula

To derive a formula for the derivative of the uniform B-splines, we begin by deriving a functional equation for the derivative of their generating function.

Lemma 1.

$$\frac{\partial G_0}{\partial x}(x,t) = tG_0(x,t) - tG_1(x,t). \tag{9}$$

Proof. By Theorem 2:

$$G_0(x,t) = \sum_{j=0}^{p}(-1)^j\left(\frac{(x-j)^j t^j}{j!} + \frac{(x-j)^{j-1}t^{j-1}}{(j-1)!}\right)e^{(x-j)t}.$$

Therefore

$$\frac{\partial G_0}{\partial x}(x,t) = \sum_{j=0}^{p} (-1)^j \left(\frac{(x-j)^{j-1} t^j}{(j-1)!} + \frac{(x-j)^{j-2} t^{j-1}}{(j-2)!} \right) e^{(x-j)t}$$
$$+ t \sum_{j=0}^{p} (-1)^j \left(\frac{(x-j)^j t^j}{j!} + \frac{(x-j)^{j-1} t^{j-1}}{(j-1)!} \right) e^{(x-j)t},$$

or equivalently

$$\frac{\partial G_0}{\partial x}(x,t) = -\, t \sum_{j=0}^{p} (-1)^{j-1} \left(\frac{((x-1)-(j-1))^{j-1} t^{j-1}}{(j-1)!} \right.$$
$$\left. + \frac{((x-1)-(j-1))^{j-2} t^{j-2}}{(j-2)!} \right) e^{((x-1)-(j-1))t}$$
$$+ t \sum_{j=0}^{p} (-1)^j \frac{(x-j)^j t^j}{j!} + \frac{(x-j)^{j-1} t^{j-1}}{(j-1)!} e^{(x-j)t}.$$

Hence

$$\frac{\partial G_0}{\partial x}(x,t) = tG_0(x,t) - tG_0(x-1,t) = tG_0(x,t) - tG_1(x,t).$$

Theorem 4. *(Derivative Formula)*

$$\frac{\partial N_{0,n}(x)}{\partial x} = N_{0,n-1}(x) - N_{1,n-1}(x). \tag{10}$$

Proof. From Lemma 1, we have the functional equation:

$$\frac{\partial G_0}{\partial x}(x,t) = tG_0(x,t) - tG_1(x,t).$$

Comparing the coefficients of t^n on both sides, we find that:

$$\frac{\partial N_{0,n}(x)}{\partial x} t^n = tN_{0,n-1}(x) t^{n-1} - tN_{1,n-1}(x) t^{n-1},$$
$$\frac{\partial N_{0,n}(x)}{\partial x} = N_{0,n-1}(x) - N_{1,n-1}(x).$$

□

5.3 The de Boor Recurrence

Starting from the de Boor recurrence, we derived a PDE for the partial derivative of the generating function with respect to t. We can also go the other way: starting from this functional equation for the partial derivative of the generating function with respect to t, we can derive the de Boor recurrence. Thus this PDE is actually equivalent to the de Boor recurrence.

Theorem 5. *(de Boor Recurrence)*

$$N_{0,n}(x) = \frac{x}{n} N_{0,n-1}(x) + \frac{n+1-x}{n} N_{1,n-1}(x).$$

Proof. By Theorem 2, the generating function $G_0(x,t)$ satisfies the functional equation:

$$\frac{\partial G_0}{\partial t}(x,t) = xG_0(x,t) + (2-x)G_1(x,t) + t\frac{\partial G_1}{\partial t}(x,t).$$

Therefore

$$\begin{aligned}\sum_n nN_{0,n}(x)t^{n-1} =& x\sum_n N_{0,n-1}(x)t^{n-1} + (2-x)\sum_n N_{1,n-1}(x)t^{n-1} \\ &+ t\sum_n (n-1)N_{1,n-1}(x)t^{n-2}.\end{aligned}$$

Comparing the coefficients of t^{n-1} on both sides yields:

$$nN_{0,n}(x) = xN_{0,n-1}(x) + (n+1-x)N_{1,n-1}(x).$$

Now dividing both sides by n, we conclude that:

$$N_{0,n}(x) = \frac{x}{n} N_{0,n-1}(x) + \frac{n+1-x}{n} N_{1,n-1}(x).$$

□

5.4 Partition of Unity

Here we shall use the generating functions to show that the uniform B-splines form a partition of unity. We begin with some technical results.

Lemma 2.

$$\sum_{k=-d}^{0} G_k(x,t) = \sum_{k=-d}^{0} (-1)^{k+d} \left(\frac{(x-k)^{k+d} t^{k+d}}{(k+d)!} \right) e^{(x-k)t} \qquad 0 \le x \le 1. \tag{11}$$

Proof. We proceed by induction on d. For $d = 0$, this formula reduces to

$$G_0(x,t) = e^{xt},$$

which is just Equation (3). Now by the inductive hypothesis:

$$\sum_{k=-d}^{0} G_k(x,t) = \sum_{k=-d}^{0} (-1)^{k+d} \left(\frac{(x-k)^{k+d} t^{k+d}}{(k+d)!} \right) e^{(x-k)t}. \tag{12}$$

Moreover,

$$
\begin{aligned}
G_{-(d+1)}(x,t) &= \sum_n N_{-(d+1),n}(x)t^n \\
&= \sum_{j=0}^{d+1}(-1)^j\Bigg(\frac{(x+d+1-j)^j t^j}{j!} \\
&\qquad + \frac{(x+d+1-j)^{j-1}t^{j-1}}{(j-1)!}\Bigg)e^{(x+d+1-j)t}.
\end{aligned}
$$

Reindexing by setting $i = j-1$, we get

$$
\begin{aligned}
G_{-(d+1)}(x,t) =& e^{(x+d+1)t} + \sum_{i=0}^{d}(-1)^{i+1}\Bigg(\frac{(x+d-i)^{i+1}t^{i+1}}{(i+1)!} \\
&+ \frac{(x+d-i)^i t^i}{i!}\Bigg)e^{(x+d-i)t}.
\end{aligned}
$$

Now setting $k = i-d$, we arrive at

$$
\begin{aligned}
G_{-(d+1)}(x,t) =& e^{(x+d+1)t} + \sum_{k=-d}^{0}(-1)^{k+d+1}\Bigg(\frac{(x-k)^{k+d+1}t^{k+d+1}}{(k+d+1)!} \\
&+ \frac{(x-k)^{k+d}t^{k+d}}{(k+d)!}\Bigg)e^{(x-k)t}.
\end{aligned}
$$

Adding this last equation to (12) yields our result. □

Lemma 3.

$$
\sum_{-n\le k\le 0} N_{k,n}(x) = \sum_{k=0}^{n}(-1)^{n-k}\left(\frac{(x+k)^n}{k!(n-k)!}\right) \qquad 0 \le x \le 1. \tag{13}
$$

Proof. This result follows directly from Lemma 2 by setting $d = n$ and comparing the coefficients of t^n of both sides of Equation (11). □

Lemma 4.

$$
\sum_{k=0}^{n}(-1)^{n-k}\left(\frac{(x+k)^n}{k!(n-k)!}\right) = 1.
$$

Proof. To establish this result, we shall use a divided difference argument. The following divided difference formula follows easily by induction on n:

$$
f[0,1,\ldots,n] = \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(k).
$$

Therefore

$$\sum_{k=0}^{n}(-1)^{n-k}\left(\frac{(x+k)^n}{k!(n-k)!}\right)=\frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(x+k)^n=(x+\cdot)^n[0,1,\ldots,n].$$

But

$f[0,1,\ldots,n]=$ highest order coefficient of the polynomial interpolant,

so

$$(x+\cdot)^n[0,1,\ldots,n]=1.$$

□

Proposition 1. *(Partition of Unity)*

$$\sum_{-n\le k\le 0} N_{k,n}(x)=1.$$

Proof. By translation invariance, it is enough to prove this result for $0 \le x \le 1$. But for $0 \le x \le 1$, this result follows immediately from Lemmas 3 and 4. □

6 New Identities for Uniform B-Splines

So far we have used our generating function to derive some well known identities for the uniform B-splines. In this section we shall derive some new identities for uniform B-splines using their generating functions.

6.1 New Identities from Specializing the Generating Functions

Here we derive new identities for the sums and alternating sums as well as for the moments and reciprocal moments of the uniform B-splines by considering special values of t in the generating functions $G_0(x,t)$. The reader may easily construct other identities for the B-splines by considering other specializations of their generating functions.

Theorem 6. *(Sums and Alternating Sums)*

$$\sum_{n=0}^{\infty} N_{0,n}(x)=e^{x}+\sum_{j=1}^{p}(-1)^{j}\frac{x(x-j)^{j-1}}{j!}e^{(x-j)} \qquad p\le x\le p+1. \tag{14}$$

$$\sum_{n=0}^{\infty}(-1)^{n} N_{0,n}(x)=e^{-x}+\sum_{j=1}^{p}\frac{(x-2j)(x-j)^{j-1}}{j!}e^{-(x-j)} \quad p\le x\le p+1. \tag{15}$$

Proof. These results follow immediately by substituting $t=\pm 1$ on both sides of the generating function in Equation (7). □

Theorem 7. *(Moments and Reciprocal Moments)*

$$\sum_{n=0}^{\infty} x^n N_{0,n}(x) = \sum_{j=0}^{p}(-1)^j\left(\frac{(x-j)^j x^j}{j!} + \frac{(x-j)^{j-1}x^{j-1}}{(j-1)!}\right)e^{(x-j)x}, \quad p \le x \le p+1. \tag{16}$$

$$\sum_{n=0}^{\infty} x^{-n} N_{0,n}(x) = \sum_{j=0}^{p}(-1)^j\left(\frac{(1-j/x)^j}{j!}+\frac{(1-j/x)^{j-1}}{(j-1)!}\right)e^{(1-j/x)},\ p \le x \le p+1. \tag{17}$$

Proof. These results follow immediately by substituting $t = x^{\pm 1}$ on both sides of the generating function in Equation (7). □

6.2 New Identities from Manipulating the Generating Functions

Here we derive new identities for the Laplace transform of the B-splines along with new convolution formulas for the B-splines with the monomials by manipulating the generating functions. Limited only by their imagination and ingenuity, readers may seek other identities for the B-splines by manipulating their generating functions.

Our explicit formula for the generating functions over the interval $[p, p+1]$ in Equation (7) is

$$\sum_{n} N_{0,n}(x)t^n = \sum_{j=0}^{p}(-1)^j\left(\frac{(x-j)^j t^j}{j!} + \frac{(x-j)^{j-1}t^{j-1}}{(j-1)!}\right)e^{(x-j)t}.$$

In this section we explore what happens when we move e^{xt} to the left hand side.

Theorem 8. *(Convolution Formulas)*

$$\sum_{j=0}^{n}\frac{(-1)^j}{j!}x^j N_{0,n-j}(x) = \sum_{j=0}^{p}(-1)^n\left(\frac{j^{n-j}(x-j)^j}{j!(n-j)!} - \frac{j^{n-j+1}(x-j)^{j-1}}{(j-1)!(n-j+1)!}\right) \tag{18}$$

for all n and $p \le x \le p+1$.

$$\begin{aligned}\sum_{k=0}^{n}\frac{(-1)^k}{k!}(x+\alpha)^k N_{0,n-k}(x) =&(-1)^n\sum_{j=0}^{\min(p,n)}\left(\frac{(j+\alpha)^{n-j}}{j!(n-j+1)!}(x-j)^{j-1}\right.\\ &\left.\big((x-j)(n-j+1)-(j+\alpha)j\big)\right)\end{aligned} \tag{19}$$

for all n, α and $p \le x \le p+1$.

Proof. To prove the first identity, start with the generating function in Equation (7):

$$\sum_n N_{0,n}(x)t^n = \sum_{j=0}^{p}(-1)^j\left(\frac{(x-j)^j t^j}{j!} + \frac{(x-j)^{j-1}t^{j-1}}{(j-1)!}\right)e^{(x-j)t}.$$

Now multiply both sides by e^{-xt}:

$$\sum_n N_{0,n}(x)t^n e^{-xt} = \sum_{j=0}^{p}(-1)^j\left(\frac{(x-j)^j t^j}{j!} + \frac{(x-j)^{j-1}t^{j-1}}{(j-1)!}\right)e^{-jt}.$$

Then expand the exponentials on both sides of this equation as power series and compare the coefficients of t^n. Thus

$$\sum_n\sum_{j=0}^{n}\frac{(-1)^j}{j!}x^j N_{0,n-j}(x)t^n = \sum_n\sum_{j=0}^{p}(-1)^n\left(\frac{j^{n-j}(x-j)^j}{j!(n-j)!} - \frac{j^{n-j+1}(x-j)^{j-1}}{(j-1)!(n-j+1)!}\right)t^n,$$

so comparing the coefficients of t^n it follows that for all n and $p \le x \le p+1$

$$\sum_{j=0}^{n}\frac{(-1)^j}{j!}x^j N_{0,n-j}(x) = \sum_{j=0}^{p}(-1)^n\left(\frac{j^{n-j}(x-j)^j}{j!(n-j)!} - \frac{j^{n-j+1}(x-j)^{j-1}}{(j-1)!(n-j+1)!}\right).$$

The second identity can be proved in a similar fashion by initially multiplying both sides of the generating function by $e^{-(x+\alpha)t}$ and proceeding as in the proof of the first identity. □

Next we shall investigate identities generated by taking the Laplace transform of the explicit formula for the generating functions. We begin by recalling a well-known result.

Lemma 5.

$$\int_0^\infty t^k e^{-at}dt = \frac{k!}{a^{(k+1)}}, \ \textit{when } a > 0. \tag{20}$$

Proof. Integrate by parts and apply induction on k. □

Theorem 9. *(Laplace Transforms)*

$$\sum_n \frac{n!N_{0,n}(x)}{(x+1)^{n+1}} = 1 + (x+1)\sum_{j=1}^{p}\frac{(-1)^j}{(j+1)^{j+1}}(x-j)^{j-1}. \tag{21}$$

$$\sum_n \frac{n!N_{0,n}(x)}{(x+\alpha)^{n+1}} = \frac{1}{\alpha} + (x+\alpha)\sum_{j=1}^{p}\frac{(-1)^j}{(j+\alpha)^{j+1}}(x-j)^{j-1} \tag{22}$$

for all $\alpha > 0$ and $p \le x \le p+1$.

Proof. To prove the first result, again we begin with the explicit formula for the generating function given in Equation (7):

$$\sum_n N_{0,n}(x)t^n = \sum_{j=0}^{p}(-1)^j \left(\frac{(x-j)^j t^j}{j!} + \frac{(x-j)^{j-1}t^{j-1}}{(j-1)!}\right)e^{(x-j)t} \quad p \le x \le p+1.$$

Now multiply both sides by $e^{-(x+1)t}$:

$$\sum_n N_{0,n}(x)t^n e^{-(x+1)t} = \sum_{j=0}^{p}(-1)^j \left(\frac{(x-j)^j t^j}{j!} + \frac{(x-j)^{j-1}t^{j-1}}{(j-1)!}\right)e^{-(j+1)t}$$

and integrate with respect to t:

$$\int_0^\infty \sum_n N_{0,n}(x)t^n e^{-(x+1)t}dt = \int_0^\infty \sum_{j=0}^{p}(-1)^j \left(\frac{(x-j)^j t^j}{j!} + \frac{(x-j)^{j-1}t^{j-1}}{(j-1)!}\right)e^{-(j+1)t}dt.$$

Then

$$\sum_n N_{0,n}(x)\int_0^\infty t^n e^{-(x+1)t}dt = \sum_{j=0}^{p}(-1)^j \left(\frac{(x-j)^j}{j!}\int_0^\infty t^j e^{-(j+1)t}dt + \frac{(x-j)^{j-1}}{(j-1)!}\int_0^\infty t^{j-1}e^{-(j+1)t}dt\right),$$

so by Lemma 5:

$$\sum_n \frac{n!N_{0,n}(x)}{(x+1)^{n+1}} = 1 + (x+1)\sum_{j=1}^{p}\frac{(-1)^j}{(j+1)^{j+1}}(x-j)^{j-1}.$$

The second identity can be proved in a similar fashion by initially multiplying both sides of the generating function by $e^{-(x+\alpha)t}$ and proceeding as in the proof of the first identity. □

7 Summary, Conclusions, and Future Research

We derived a closed formula for the generating functions of the uniform B-splines, revealing a novel connection between uniform B-splines and exponential functions. Using this generating function, we established several classical identities for the uniform B-splines. These identities along with the corresponding functional equations for the generating functions are listed in Table 1. We also derived some new identities for uniform B-splines that cannot be derived by standard methods. These new identities are listed in Table 2.

Table 1. Some classical B-spline identities and the corresponding functional equations for their generating functions

B-Splines Identities	Generating Functions Functional Equations
$N_{0,n}(x) = \frac{x}{n} N_{0,n-1}(x)$ $+\frac{n+1-x}{n} N_{1,n-1}(x)$	$\frac{\partial G_0}{\partial t}(x,t) = xG_0(x,t)$ $+(2-x)G_1(x,t) + t\frac{\partial G_1}{\partial t}(x,t)$
$\frac{\partial N_{0,n}(x)}{\partial x} = N_{0,n-1}(x) - N_{1,n-1}(x)$	$\frac{\partial G_0}{\partial x}(x,t) = tG_0(x,t) - tG_1(x,t)$
$\sum_k N_{k,n}(x) \equiv 1$	$\sum_{k=-n}^{0} G_k(x,t) =$ $\sum_{k=-n}^{0} (-1)^{k+n} \left(\frac{(x-k)^{k+n} t^{k+n}}{(k+n)!}\right) e^{(x-k)t}$
$N_{0,n}(x) =$ $\frac{1}{n!} \sum_{j=0}^{p} (-1)^j \binom{n+1}{j} (x-j)^n$ $p \le x \le p+1$	$G_0(x,t) =$ $\sum_{j=0}^{p} (-1)^j \left(\frac{(x-j)^j t^j}{j!} + \frac{(x-j)^{j-1} t^{j-1}}{(j-1)!}\right) e^{(x-j)t}$ $p \le x \le p+1$

Sums and Alternating Sums

$$\sum_{n=0}^{\infty} N_{0,n}(x) = e^x + \sum_{j=1}^{p} (-1)^j \frac{x(x-j)^{j-1}}{j!} e^{(x-j)} \qquad p \le x \le p+1$$

$$\sum_{n=0}^{\infty} (-1)^n N_{0,n}(x) = e^{-x} + \sum_{j=1}^{p} \frac{(x-2j)(x-j)^{j-1}}{j!} e^{-(x-j)} \qquad p \le x \le p+1$$

Moments and Reciprocal Moments

$$\sum_{n=0}^{\infty} x^n N_{0,n}(x) = \sum_{j=0}^{p} (-1)^j \left(\frac{(x-j)^j x^j}{j!} + \frac{(x-j)^{j-1} x^{j-1}}{(j-1)!}\right) e^{(x-j)x} \qquad p \le x \le p+1$$

$$\sum_{n=0}^{\infty} x^{-n} N_{0,n}(x) = \sum_{j=0}^{p} (-1)^j \left(\frac{(1-j/x)^j}{j!} + \frac{(1-j/x)^{j-1}}{(j-1)!}\right) e^{(1-j/x)} \qquad p \le x \le p+1$$

Convolution Formulas

$$\sum_{j=0}^{n} \frac{(-1)^j}{j!} x^j N_{0,n-j}(x) = \sum_{j=0}^{p} (-1)^n \left(\frac{j^{n-j}(x-j)^j}{j!(n-j)!} - \frac{j^{n-j+1}(x-j)^{j-1}}{(j-1)!(n-j+1)!} \right)$$

for all n and $p \le x \le p+1$

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} (x+\alpha)^k N_{0,n-k}(x) = (-1)^n \sum_{j=0}^{\min(p,n)} \left(\frac{(j+\alpha)^{n-j}}{j!(n-j+1)!} (x-j)^{j-1} \left((x-j)(n-j+1) - (j+\alpha)j\right) \right)$$

for all n, α and $p \le x \le p+1$

Laplace Transforms

$$\sum_{n} \frac{n! N_{0,n}(x)}{(x+1)^{n+1}} = 1 + (x+1) \sum_{j=1}^{p} \frac{(-1)^j}{(j+1)^{j+1}} (x-j)^{j-1}$$

$$\sum_{n} \frac{n! N_{0,n}(x)}{(x+\alpha)^{n+1}} = \frac{1}{\alpha} + (x+\alpha) \sum_{j=1}^{p} \frac{(-1)^j}{(j+\alpha)^{j+1}} (x-j)^{j-1}$$

for all $\alpha > 0$ and $p \le x \le p+1$

Table 2. Some new identities for the B-splines derived from their generating functions

Yet despite these successes, generating functions are not a panacea for deriving identities for uniform B-splines. The following two well known identities—the Marsden identity and the refinement equation—are not readily established using generating functions:

$$(x-t)^n = \sum_{k} (k+1-t) \cdots (k+n-t) N_{k,n}(x) \qquad (\textit{Marsden Identity})$$

$$N_{0,n}(x) = \sum_{k} \frac{\binom{n+1}{k}}{2^n} N_{0,n}(2x-k) \qquad (\textit{Refinement Equation})$$

We can, however, derive these identities directly or indirectly from the de Boor recurrence, which we have seen is equivalent to the PDE for the generating functions (see Table 1).

Currently our generating functions are restricted to B-splines with uniformly spaced knots— that is, knots in arithmetic progression

$$t_{k+1} = t_k + h \quad \text{(arithmetic progression).}$$

In the future we hope to extend our generating functions to B-splines with knots in geometric or affine progression—that is, to B-splines with knot sequences where

$$\begin{aligned} t_{k+1} &= qt_k && \text{(geometric progression)} \\ t_{k+1} &= qt_k + h && \text{(affine progression).} \end{aligned}$$

B-splines with knots in affine progression would also include B-splines with knots at the q-integers [2].

Finally we would also like to extend our generating functions to multivariate splines such as box splines, where simple recurrences for the basis functions are also available.

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