Riemann-Finsler Geometry for Diffusion Weighted Magnetic Resonance Imaging

Luc Florack and Andrea Fuster

Abstract We consider Riemann-Finsler geometry as a potentially powerful mathematical framework in the context of diffusion weighted magnetic resonance imaging. We explain its basic features in heuristic terms, but also provide mathematical details that are essential for practical applications, such as tractography and voxel-based classification. We stipulate a connection between the (dual) Finsler function and signal attenuation observed in the MRI scanner, which directly generalizes Stejskal-Tanner's solution of the Bloch-Torrey equations and the diffusion tensor imaging (DTI) model inspired by this. The proposed model can therefore be regarded as an extension of DTI. Technically, reconstruction of the (dual) Finsler function from diffusion weighted measurements is a fairly straightforward generalization of the DTI case. The extension of the Riemann differential geometric paradigm for DTI analysis is, however, nontrivial.

1 Introduction

Diffusion weighted magnetic resonance imaging (dwMRI) has become a standard MRI technique for in vivo imaging of apparent water diffusion processes in fibrous tissue (for an introductory tutorial, cf. Hagmann et al. [1]). Clinical use of dwMRI is hampered by the fact that radically new approaches and abstract representations are required for its analysis. Examples are rank-2 symmetric positive-definite tensor representations in diffusion tensor imaging (DTI), pioneered by Basser, Mattiello and Le Bihan et al. [2–8] and explored by many others [9–20], higher order fully symmetric tensor representations [21–27] and spherical harmonic representations in

L. Florack (✉) • A. Fuster
Eindhoven University of Technology, PO Box 513, NL-5600 MB Eindhoven, The Netherlands
e-mail: L.M.J.Florack@tue.nl; A.Fuster@tue.nl

C.-F. Westin et al. (eds.), *Visualization and Processing of Tensors and Higher Order Descriptors for Multi-Valued Data*, Mathematics and Visualization,
DOI 10.1007/978-3-642-54301-2_8,

high angular resolution diffusion imaging (HARDI) [28–32], and SE(3) Lie group representations [33–35].

In this chapter we concentrate on an extension of the Riemannian paradigm [16, 36], used in the context of DTI, in order to account explicitly for the unconstrained number of local directional degrees of freedom of general dwMRI representations. *Riemann-Finsler geometry* appears to be ideally suited for this purpose, as has already been hinted upon in earlier work [22, 37–41]. However, foregoing work is either driven by heuristics or merely scratches the surface of Riemann-Finsler geometry. For instance, no rigorous connection between the pivotal Finsler function and the physics of dwMRI acquisition has yet been proposed.

More specifically, Melonakos et al. [41] have pioneered Finsler geometry in the context of contours, only briefly touching upon application in dwMRI. Astola et al. [37–40] have applied Finsler geometry, and in particular geodesic tractography, to dwMRI using a fully symmetric fourth order tensor model. Florack et al. [22] have proposed a tensor representation of arbitrary order, discussing operational issues such as spatial and angular regularization. The Cartan geometric approach developed by Duits et al. [33–35] likewise appears intimately related to the theory outlined in this chapter. In all of the above cases the exact connection between Riemann-Finsler geometry and dwMRI is deemphasized, while applications are typically limited to contour detection or tractography.

Our primary goal is to provide a generic model for dwMRI, with potential applications beyond tractography, which manifestly incorporates the Riemannian paradigm for DTI as a limiting case. Secondly, we wish to convey the gist of Riemann-Finsler geometry without dodging mathematical details that are necessary for algorithmic implementation. This does not imply that our treatment of the subject will be self-contained; for a thorough understanding one will find it necessary to consult additional sources. The books by Bao et al. [42] and Rund [43] are especially recommended. Shen and Mo provide additional insight [44, 45]. We hope that our overview will encourage researchers to further contribute to a systematic study and practical application of Riemann-Finsler geometry in the context of dwMRI (and elsewhere).

Riemann-Finsler geometry has its roots in Riemann's "Habilitation" [46]. Riemann focused on a special case, nowadays known as Riemannian geometry. Important (pseudo-Riemannian) application areas, such as Maxwell's theory and Einstein's theory of general relativity, greatly contributed to its popularity. The general case was taken up by Finsler in his PhD thesis [47], and subsequently by Cartan [48] (who was the first to refer to it as "Finsler geometry"), and others.

Although potentially much more powerful, Riemann-Finsler geometry has not yet become nearly as popular as its Riemannian counterpart. To some extent this may be explained by its rather mind-boggling technicalities and heavy computational demands. This should no longer withhold practitioners in our technological era, for both symbolic as well as large-scale numerical manipulations can be readily performed on state-of-the-art computers. Progress in enabling technologies, such as compressed sensing for fast imaging [49], are also likely to contribute to practical feasibility of dwMRI, yet we believe that the major hurdle is still methodological.

2 Theory

2.1 Diffusion Weighted MRI

Recall the Stejskal-Tanner signal attenuation formula in the spin-echo experiment on spin diffusion in an isotropic medium [50] (cf. also [51–55]):

$$\ln \frac{A(2\tau)}{A(0)} = -\gamma^2 D\delta^2(\Delta - \frac{1}{3}\delta)G^2 \,, \tag{1}$$

in which γ is the gyromagnetic ratio of hydrogen, δ the duration of a diffusion-sensitizing gradient pulse (with $\delta < \tau$), Δ the time between a pair of balanced gradient pulses, and G the gradient magnitude. The echo occurs at time 2τ after the onset of the first gradient pulse, and the formula represents the relative signal loss due to diffusion of water molecules over a time interval τ between the 90° pulse and the 180° pulse, a process characterized by the diffusion coefficient D. The positive factor by which D is multiplied on the right hand side of the above expression is known in the trade as the "b-factor" (an allusion to Le Bihan): $b \equiv \gamma^2\delta^2(\Delta - \frac{1}{3}\delta)G^2$.

Brain white matter consists mostly of water (>70 %), but diffusion turns out to be anisotropic as a result of its fibrous architecture, facilitating diffusion along axonal fibers relative to transverse directions. The Stejskal-Tanner experiment inspired Moseley, Basser, Le Bihan, and others, [2–8] to capture this anisotropy in terms of a symmetric positive definite rank-2 diffusion tensor (with components D^{ij}, $i, j = 1, 2, 3$, relative to a coordinate basis) as opposed to the scalar D. This technique is the basis of *diffusion tensor imaging* (DTI).

In order to connect to the mathematical notation in the remainder of this chapter we will denote the signal as a function of position x and the applied normalized diffusion-sensitizing gradient $q = \gamma\delta G$:

$$S(x, q, \tau^*) = S_0(x) \exp\left(-\tau^* H^2(x, q)\right) \,. \tag{2}$$

Here τ^* denotes a time constant related to Δ and δ (in Stejskal-Tanner's scheme we have $\tau^* = \Delta - \delta/3$, cf. Sinnaeve [55] for alternative schemes and associated time constants). Furthermore, the so-called Hamiltonian[1] $H(x, q)$ generalizes the quadratic form encountered in the DTI case, in which it assumes the form[2]

$$H^2_{\mathrm{DTI}}(x, q) = D^{ij}(x) q_i q_j \,. \tag{3}$$

[1] We neglect, but do not a priori exclude, a mild dependence of $H(x, q)$ on τ^*.

[2] We use Einstein's summation convention throughout, i.e. explicit summation symbols, such as $\sum_{i,j=1}^{3}$ on the r.h.s. of Eq. (3), are suppressed for pairs of identical upper and lower indices.

In general, $H(x,q) \neq H_{\text{DTI}}(x,q)$, but a strong analogy with DTI remains in the form of a homogeneity condition[3] in q-space, viz. we shall require that

$$H^2(x,q) = D^{ij}(x,q)q_i q_j\,, \tag{4}$$

in which the coefficients are zero-homogeneous, i.e.

$$D^{ij}(x,\lambda q) = D^{ij}(x,q) \tag{5}$$

for all $\lambda \neq 0$. This expresses our hypothesis that $\ln(S(x,q,\tau^*)/S_0(x))$ scales quadratically in the *magnitude* of the diffusion-sensitizing gradient, but, unlike DTI, is not necessarily a quadratic form. This assumption is approximately correct for certain ranges of (q,τ^*), and encompasses the validity domain of DTI.

By virtue of homogeneity and mirror symmetry one may, in Eq. (5), think of q as a point on the projective plane, or on the unit sphere with antipodal points identified. Homogeneity also implies that the "highly anisotropic" diffusion tensor $D^{ij}(x,q)$ does *not* in fact—within the domain of validity of our extended model—depend on acquisition details, such as the magnitude of the applied gradients. That is, it is intended to capture tissue intrinsic properties (probed along a certain direction), just like the classical, "mildy anisotropic" diffusion tensor $D^{ij}(x)$. (This does *not* hold for $H(x,q)$ and some related functions that will be introduced below, which do depend on the magnitude of q, and thus on the entire experimental setup.)

Note that the number of degrees of freedom contained in the DTI tensor coefficients $D^{ij}(x)$ at any given point x equals 6 (the number of independent components of a symmetric 2-tensor in 3 dimensions), whereas there are, a priori, ∞ degrees of freedom in $D^{ij}(x,q)$, one for each position in space and each point on the projective plane. Also note that Eq. (2) relies on a mono-exponential signal decay; in this sense it "naturally" extends DTI. It complements alternative DTI refinements, such as multi-compartment models [56]. Our homogeneity condition,

$$H(x,\lambda q) = |\lambda| H(x,q) \qquad \text{for all } \lambda \in \mathbb{R}, \tag{6}$$

also distinguishes our model from diffusional kurtosis imaging (DKI), cf. [57]. (A comparison of our model with these models as well as other, HARDI-like schemes, in relation to their respective validity domains, remains to be made.)

2.2 *The Riemannian Paradigm*

The Riemannian paradigm was introduced by O'Donnell et al. [36] and by Lenglet et al. [16] in the context of DTI. In its original formulation it identifies the diffusion

[3] A function $f(z)$ is called homogeneous of degree r if it satisfies $f(\lambda z) = \lambda^r f(z)$ for all $\lambda > 0$. According to Euler's theorem such a function obeys the first order partial differential equation $z^i \partial f(z)/\partial z^i = r f(z)$.

tensor $D^{ij}(x)$, recall Eqs. (2) and (3), up to a constant proportionality factor, with the dual (or inverse) Riemann metric tensor $g^{ij}(x)$:

$$\tau^* D^{ij}(x) = g^{ij}(x)\,. \tag{7}$$

This defines a Riemannian manifold in which a relatively increased directional diffusion observed along some curve is tantamount to a shortening of Riemannian path length. In this way the problem of tractography can be restated as the problem of finding certain[4] shortest paths (via geodesic equations), or related to level set methods for distance functions induced by geodesic congruences (via Hamilton-Jacobi equations). The motivation for Eq. (7) is heuristic, cf. [58, 59] for conformal adaptations of the metric, arguing for a nontrivial local scaling factor.

Due to its limited angular resolution DTI can only handle mild anisotropies that are believed to be induced by "single fiber coherence", i.e. a local alignment of axonal fibers forcing anisotropy to be more or less axially symmetric, with one dominant diffusion eigenaxis along the fibers (and two minor eigenaxes perpendicular to the fibers). Due to complex fiber architecture in significant parts of the brain, such as fiber crossings, observed diffusivity patterns are highly anisotropic, rendering the DTI hypothesis invalid in such cases. On the positive side, the same limitation (viz. of a priori limited angular resolution) contributes to robustness, especially if a redundant set of diffusion weighted measurements is used for DTI reconstruction.

If we drop the quadratic restriction, Eq. (3), we can invoke the powerful machinery of Riemann-Finsler geometry in a way that mimics the Riemannian paradigm for DTI, viz. recall Eq. (4) and identify the coefficients in this equation with the so-called *dual Riemann-Finsler metric tensor*:

$$\tau^* D^{ij}(x,q) = g^{ij}(x,q)\,. \tag{8}$$

Clearly this is at best an approximation of reality due to mono-exponential decay, Eq. (2), and the homogeneity hypothesis on the physical scaling behavior of the Hamiltonian, Eq. (6). The conditions under which this approximation is realistic are deemphasized here, but will need to be made explicit in future work (cf. Basser [6] for a discussion in the DTI case, to some extent applicable to the general case as well). Suffice it to say that, by construction, the domain of validity certainly reaches beyond that of DTI, which arises in the limiting scenario of mild anisotropy $D^{ij}(x,q) \to D^{ij}(x)$.

In the rest of this chapter we consider the basics of Riemann-Finsler geometry and point out its theoretical relevance for tractography and voxel classification.

[4]Please note that the Riemannian paradigm does *not* stipulate that geodesics are biologically meaningful tracts, cf. Astola et al. [10] for a connectivity criterion that could be used for a deterministic or probabilistic labelling of biologically plausible neural tracts among all possible geodesic tracts. Indeed, in a geodesically complete space *any* two points can be connected by at least one geodesic.

The Hamiltonian framework appears to be most directly related to the physics of dwMRI. In DTI this is reflected by the fact that it is the *inverse* of the diffusion tensor that defines the Riemann metric tensor. The Riemann metric tensor itself is equivalently captured by a (limiting case of a) so-called *Finsler function*, which, in its most general form, constitutes the pivot of Riemann-Finsler geometry. Let us therefore start with the axiomatics of the Finsler function.

2.3 The Finsler Function

Recall that the geometric paradigm for DTI hinges on Riemannian geometry, Eq. (7), stipulating that the diffusion tensor is proportional to the dual Riemann metric tensor $g^{ij}(x)$, with 6 degrees of freedom per point in 3 spatial dimensions. For state-of-the-art dwMRI, in which local signal attenuations are recorded under a multitude of magnetic gradient directions, this limitation on angular resolution is too restrictive. The Riemann-Finsler paradigm removes this limitation altogether.

The pivot of Riemann-Finsler geometry is a generalised notion of length of a spatial curve C (*Hilbert's invariant integral* [42]):

$$\mathscr{L}(C) = \int_C F(x, dx) \,. \tag{9}$$

The Lagrangian $F(x, dx)$ is called the *Finsler function*. This function cannot be chosen arbitrarily. In order to interpret Eq. (9) properly as an integral over a one-form, one has to require $F(x, dx) = F(x, \dot{x})dt$ for a parametrized curve $x = x(t)$, with $\dot{x} = dx(t)/dt$, so that the functional $\mathscr{L}(C)$ is well-defined and parameter invariant. More specifically, $F(x, \dot{x})$ is required to be smooth for $\dot{x} \neq 0$ and to satisfy the following properties[5]:

$$F(x, \lambda\dot{x}) = |\lambda| F(x, \dot{x}) \qquad \text{for all } \lambda \in \mathbb{R}, \tag{10}$$

$$F(x, \dot{x}) > 0 \qquad \text{if } \dot{x} \neq 0, \tag{11}$$

$$g_{ij}(x, \dot{x})\xi^i \xi^j > 0 \qquad \text{if } \xi \neq 0, \tag{12}$$

in which the *Riemann-Finsler metric tensor* is defined as

$$g_{ij}(x, \dot{x}) = \frac{1}{2} \frac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^i \partial \dot{x}^j} \,. \tag{13}$$

[5]Instead of the norm condition, Eq. (10), one sometimes requires $F(x, \lambda\dot{x}) = \lambda F(x, \dot{x})$. What matters in diffusion processes without convection is *orientation*, not signed direction, so it is natural to require mirror symmetry $\dot{x} \longleftrightarrow -\dot{x}$ a priori.

In these definitions and below, $\dot{x}$ is an a priori *independent* vector argument, not a tangent vector $\dot{x}(t)$ to some underlying parametrized curve $x(t)$, unless explicitly stated otherwise. But it helps intuition to keep in mind the role of this vector argument in an expression like Eq. (9). In particular, when considering a smooth spatial curve $x(t)$, there is a "distinguished" vector $\dot{x} \propto \dot{x}(t)$ associated with any position $x(t)$ along the curve. The extended base manifold with coordinates $(x, \dot{x})$, with $\dot{x} \neq 0$, is referred to as the *slit tangent bundle*. The word "slit" refers to the excluded strip $\dot{x} = 0$. In the context of zero-homogeneous functions, a vector $\dot{x} \neq 0$ represents an equivalence class of points on the line through the origin with direction vector $\dot{x}$. In that case one also refers to the extended base manifold as the *projectivized tangent bundle*, cf. the concept of an *orientation score* by Duits et al. [33–35].

Using Eqs. (10)–(13), it is not difficult to show (with the help of Euler's theorem for homogeneous functions, recall footnote 3) that

$$F^2(x, \dot{x}) = g_{ij}(x, \dot{x})\dot{x}^i \dot{x}^j \,. \tag{14}$$

Riemann's *quadratic restriction* pertains to the "mildly anisotropic" case, $g_{ij}(x, \dot{x}) = g_{ij}(x)$. In general, the Riemann-Finsler metric tensor, Eq. (13), is homogeneous of degree 0: $g_{ij}(x, \lambda\dot{x}) = g_{ij}(x, \dot{x})$ for all $\lambda \in \mathbb{R}$. It may be viewed as being defined on the 5-dimensional projectivized tangent bundle.

Since, in principle, only positions and orientations are of interest, all geometrically relevant quantities will be zero-homogeneous. Although the Finsler function itself does not qualify as such (its domain of definition is the 6-dimensional *slit tangent bundle* of positions and nonzero vectors), it serves as the basic object from which such quantities can be constructed.

The role played by the 3-dimensional (co)tangent spaces erected at each point x of a 3-dimensional Riemannian manifold is replaced by likewise 3-dimensional fibers that collectively constitute a so-called *pulled-back (co)bundle* or *Finsler (co)bundle* in Riemann-Finsler geometry. The major difference is that a pulled-back (co)bundle sits over the 5-dimensional projectivized tangent bundle or 6-dimensional slit tangent bundle, rather than over the 3-dimensional spatial manifold. Given x-coordinates on the spatial manifold the coordinate induced basis sections

$$\left.\frac{\partial}{\partial x^i}\right|_{(x,\dot{x})} \quad \text{respectively} \quad \left. dx^i \right|_{(x,\dot{x})} \tag{15}$$

for its tangent and cotangent bundles can be transplanted to the pulled-back (co)bundle. That is, $\dot{x}$ plays no role in the construction of a fiber at a fiducial point $(x, \dot{x})$.

2.4 Riemann-Finsler Geometry and Its Riemannian Limit

The nontrivial nature of the *Cartan tensor* [42, 43, 48, 60],

$$C_{ijk}(x,\dot{x}) = \frac{1}{4}\frac{\partial^3 F^2(x,\dot{x})}{\partial \dot{x}^i\,\partial \dot{x}^j\,\partial \dot{x}^k}\,, \tag{16}$$

distinguishes Riemann-Finsler geometry from its Riemannian counterpart. One can show that $C_{ijk}(x,\dot{x}) = 0$ if and only if space (the x-manifold) is Riemannian. In fact it suffices to inspect the *Cartan one-form*

$$C_i(x,\dot{x}) = g^{jk}(x,\dot{x})C_{ijk}(x,\dot{x})\,, \tag{17}$$

in which the *dual Riemann-Finsler metric tensor*[6] $g^{ij}(x,\dot{x})$ is the inverse of $g_{ij}(x,\dot{x})$:

$$g^{ik}(x,\dot{x})g_{kj}(x,\dot{x}) = \delta^i_j\,. \tag{18}$$

Indeed, one can show that space is Riemannian if and only if the Cartan one-form, Eq. (17), vanishes identically. In view of the significance of zero-homogeneous functions one often encounters the alternative definitions

$$A_{ijk}(x,\dot{x}) = F(x,\dot{x})C_{ijk}(x,\dot{x}) \quad \text{resp.} \quad A_i(x,\dot{x}) = F(x,\dot{x})C_i(x,\dot{x})\,. \tag{19}$$

The dual Riemann-Finsler metric may be used for index raising and lowering, e.g.

$$C^k_{ij}(x,\dot{x}) = g^{k\ell}(x,\dot{x})C_{ij\ell}(x,\dot{x})\,. \tag{20}$$

(There is no ambiguity here by virtue of symmetry of the covariant Cartan tensor.)

Thus the Cartan tensor measures the degree in which the local structure of the Riemann-Finsler manifold deviates from Riemannian. In view of Eqs. (3), (4) (7), and (8) this boils down to a measure for the degree in which the recorded dwMRI data—matched to the basic paradigm, Eq. (2)—violate the validity conditions for DTI. In other words, it provides a (fuzzy) classification of voxels as "DTI-like" (i.e. mildly anisotropic) versus otherwise (i.e. complex or highly anisotropic).

2.5 Connections in Riemann-Finsler Geometry

There is no "obvious" connection (mechanism for parallel transport) on a Riemann-Finsler manifold. The so-called Berwald, Cartan, Chern-Rund and Hashiguchi

[6] It will be seen later, cf. Eqs. (45)–(47), that it is more natural to think of g^{ij} as a metric in q-space, as opposed to the $\dot{x}$-space metric g_{ij}.

connections may all be considered "natural" extensions of the Levi-Civita connection in Riemannian geometry. For instance, the (torsion-free) Chern-Rund connection is defined by[7]

$$\Gamma^i_{jk}(x,\dot{x}) = \frac{1}{2} g^{i\ell}(x,\dot{x}) \left(\frac{\delta g_{\ell k}(x,\dot{x})}{\delta x^j} + \frac{\delta g_{j\ell}(x,\dot{x})}{\delta x^k} - \frac{\delta g_{jk}(x,\dot{x})}{\delta x^\ell} \right) . \tag{21}$$

This expression is obtained from the "classical" Christoffel symbols of Riemannian geometry by formally replacing the Riemann metric $g_{ij}(x)$ by the Riemann-Finsler metric $g_{ij}(x,\dot{x})$, Eq. (13), and spatial derivatives by the *horizontal vectors*

$$\frac{\delta}{\delta x^i} \stackrel{\text{def}}{=} \frac{\partial}{\partial x^i} - N_i^j(x,\dot{x}) \frac{\partial}{\partial \dot{x}^j} . \tag{22}$$

The coefficients $N_i^j(x,\dot{x})$ define the so-called *nonlinear connection* [42]:

$$N_i^j(x,\dot{x}) = \gamma^j_{ik}(x,\dot{x})\dot{x}^k - C^j_{ik}(x,\dot{x})\gamma^k_{\ell m}(x,\dot{x})\dot{x}^\ell \dot{x}^m , \tag{23}$$

in which the *formal Christoffel symbols of the second kind* are introduced as

$$\gamma^i_{jk}(x,\dot{x}) = \frac{1}{2} g^{i\ell}(x,\dot{x}) \left(\frac{\partial g_{\ell k}(x,\dot{x})}{\partial x^j} + \frac{\partial g_{j\ell}(x,\dot{x})}{\partial x^k} - \frac{\partial g_{jk}(x,\dot{x})}{\partial x^\ell} \right) . \tag{24}$$

Note that in the Riemannian limit, both Eq. (21) as well as Eq. (24) simplify to

$$\Gamma^i_{jk}(x,\dot{x}) , \; \gamma^i_{jk}(x,\dot{x}) \longrightarrow \Gamma^i_{jk}(x) = \frac{1}{2} g^{i\ell}(x) \left(\frac{\partial g_{\ell k}(x)}{\partial x^j} + \frac{\partial g_{j\ell}(x)}{\partial x^k} - \frac{\partial g_{jk}(x)}{\partial x^\ell} \right) , \tag{25}$$

the standard *Christoffel symbols of the second kind* defining the torsion-free *Levi-Civita connection* in Riemannian geometry. A computation reveals that[8]

$$\Gamma_{ijk}(x,\dot{x}) = \tag{26}$$
$$\gamma_{ijk}(x,\dot{x}) - C_{hjk}(x,\dot{x}) G^h_{\dot{x}^i}(x,\dot{x}) - C_{hji}(x,\dot{x}) G^h_{\dot{x}^k}(x,\dot{x}) + C_{hik}(x,\dot{x}) G^h_{\dot{x}^j}(x,\dot{x}) ,$$

in which indices have been lowered via the Riemann-Finsler metric tensor:

$$\Gamma_{ijk}(x,\dot{x}) = g_{j\ell}(x,\dot{x})\Gamma^\ell_{ik}(x,\dot{x}) \quad \text{resp.} \quad \gamma_{ijk}(x,\dot{x}) = g_{j\ell}(x,\dot{x})\gamma^\ell_{ik}(x,\dot{x}) , \tag{27}$$

[7]Caveat: In [43] Rund defines these symbols as $\Gamma^{*i}_{jk}(x,\dot{x})$.

[8]Caveat: In [43] Rund defines these symbols as $\Gamma^*_{ijk}(x,\dot{x})$.

and in which the *geodesic coefficients* are defined as[9]

$$G^i_{\dot{x}^j}(x,\dot{x}) = \frac{\partial G^i(x,\dot{x})}{\partial \dot{x}^j} \quad \text{with} \quad G^i(x,\dot{x}) = \frac{1}{2}\gamma^i_{jk}(x,\dot{x})\dot{x}^j\dot{x}^k\,. \tag{28}$$

In fact we have

$$G^i_{\dot{x}^j}(x,\dot{x}) = N^i_j(x,\dot{x}) \tag{29}$$

recall Eq. (23).

2.6 Horizontal-Vertical Splitting

The coupling of position and orientation is formalized in terms of the so-called *horizontal* and *vertical basis vectors*, recall Eq. (22),

$$\frac{\delta}{\delta x^i} \stackrel{\text{def}}{=} \frac{\partial}{\partial x^i} - N^\ell_i(x,\dot{x})\frac{\partial}{\partial \dot{x}^\ell} \quad \text{and} \quad \frac{\partial}{\partial \dot{x}^i}\,. \tag{30}$$

These constitute a basis for the *horizontal* and the *vertical tangent bundle over the slit tangent bundle*:

$$\mathrm{H}_{(x,\dot{x})}\mathrm{TM} = \operatorname{span}\left\{ \left.\frac{\delta}{\delta x^i}\right|_{(x,\dot{x})} \right\} \quad \text{and} \quad \mathrm{V}_{(x,\dot{x})}\mathrm{TM} = \operatorname{span}\left\{ \left.\frac{\partial}{\partial \dot{x}^i}\right|_{(x,\dot{x})} \right\}\,. \tag{31}$$

Their direct sum yields the complete tangent bundle (pointwise):

$$\mathrm{TTM}\backslash\{0\} = \mathrm{HTM} \oplus \mathrm{VTM}\,. \tag{32}$$

By the same token one considers the *horizontal* and *vertical basis covectors*,

$$dx^i \quad \text{and} \quad \delta\dot{x}^i \stackrel{\text{def}}{=} d\dot{x}^i + N^i_\ell(x,\dot{x})dx^\ell\,, \tag{33}$$

yielding the corresponding horizontal and vertical cotangent bundles:

$$\mathrm{H}^*_{(x,\dot{x})}\mathrm{TM} = \operatorname{span}\left\{ dx^i\big|_{(x,\dot{x})} \right\} \quad \text{and} \quad \mathrm{V}^*_{(x,\dot{x})}\mathrm{TM} = \operatorname{span}\left\{ \delta\dot{x}^i\big|_{(x,\dot{x})} \right\}\,, \tag{34}$$

such that, pointwise,

[9] Caveat: In [42] Bao et al. write $G^i(x,\dot{x}) = \gamma^i_{jk}(x,\dot{x})\dot{x}^j\dot{x}^k$.

$$\mathrm{T^*TM}\backslash\{0\} = \mathrm{H^*TM} \oplus \mathrm{V^*TM}\,. \tag{35}$$

The above vectors and covectors satisfy the following duality relations:

$$dx^i\left(\frac{\delta}{\delta x^j}\right) = \delta\dot{x}^i\left(\frac{\partial}{\partial\dot{x}^j}\right) = \delta^i_j \quad \text{and} \quad dx^i\left(\frac{\partial}{\partial\dot{x}^j}\right) = \delta\dot{x}^i\left(\frac{\delta}{\delta x^j}\right) = 0\,. \tag{36}$$

Incorporating a natural scaling so as to ensure zero-homogeneity with respect to $\dot{x}$ (so that it indeed represents orientation rather than "velocity" or a "displacement") we conclude that

$$\mathrm{TTM}\backslash\{0\} = \mathrm{span}\left\{\frac{\delta}{\delta x^i}\,,\; F(x,\dot{x})\frac{\partial}{\partial\dot{x}^i}\right\}\,, \tag{37}$$

and similarly

$$\mathrm{T^*TM}\backslash\{0\} = \mathrm{span}\left\{dx^i\,,\; \frac{\delta\dot{x}^i}{F(x,\dot{x})}\right\}\,. \tag{38}$$

The so-called *Sasaki metric* furnishes the slit tangent bundle with a Riemann metric:

$$g(x,\dot{x}) = g_{ij}(x,\dot{x})\,dx^i \otimes dx^j + g_{ij}(x,\dot{x})\,\frac{\delta\dot{x}^i}{F(x,\dot{x})} \otimes \frac{\delta\dot{x}^j}{F(x,\dot{x})}\,. \tag{39}$$

The horizontal and vertical tangent bundles, Eq. (31), are orthogonal relative to this metric.

Cf. the Appendix for the formal motivation of horizontal and vertical basis vectors and covectors. The heuristics behind them is that they permit a coordinate independent, geometrically meaningful splitting into "horizontal" (pertaining to spatial position) and "vertical" (complementary) dimensions. As a counterexample, Eq. (66) in the Appendix shows what happens if we would use the standard coordinate bases in $(x,\dot{x})$-space. A change of spatial coordinates, $x = x(\xi)$, causes the new spatial coordinate vectors to be a linear superposition of *all* coordinate basis vectors that we started out from, whence they do not induce a coordinate independent splitting.

2.7 Horizontal Curves and Finsler Geodesics

Spatial trajectories $x(t)$ have a "natural" (sparse) manifestation in the "vertical" (orientation) dimension, viz. through identification of the trajectory's tangent vector $\dot{x}(t)$ with the vector $\dot{x}$. In other words, interpreted as a curve along the Finsler manifold a spatial curve, $x = \xi(t)$, say, has a natural parametrization $(x,\dot{x}) = (\xi(t),\dot{\xi}(t))$. A tangent vector of this curve is given by (with $\dot{\xi}(t) \equiv d\xi(t)/dt$ and $\ddot{\xi}(t) \equiv d^2\xi(t)/dt^2$)

$$\boldsymbol{T}(t) = \dot{\xi}^i(t)\frac{\partial}{\partial x^i} + \ddot{\xi}^i(t)\frac{\partial}{\partial \dot{x}^i}\,. \tag{40}$$

Note that the individual terms in this equation do not have an intrinsic meaning, to the extent that a splitting of the six dimensional tangent space into span $\{\partial/\partial x^i\}$ and span $\{\partial/\partial \dot{x}^i\}$ is not preserved after a spatial coordinate transformation, cf. Eq. (66) in the Appendix. The aforementioned, geometrically meaningful splitting suggests that we rather decompose the tangent vector as follows, recall Eq. (30):

$$\boldsymbol{T}(t) = \dot{\xi}^i(t)\frac{\delta}{\delta x^i} + \left(\ddot{\xi}^i(t) + N^i_j(\xi(t),\dot{\xi}(t))\,\dot{\xi}^j(t)\right)\frac{\partial}{\partial \dot{x}^i}\,. \tag{41}$$

The requirement of *horizontality* then entails that the vertical component vanishes:

$$\delta\dot{x}^i\,(\boldsymbol{T}(t)) = 0\,. \tag{42}$$

Using the basic duality relations, Eqs. (36), this means that

$$\ddot{\xi}^i(t) + N^i_j(\xi(t),\dot{\xi}(t))\,\dot{\xi}^j(t) = 0\,. \tag{43}$$

By virtue of Eqs. (23) and (24), using the fact that $C^j_{ik}(\xi,\dot{\xi})\dot{\xi}^k = 0$ (a consequence of the homogeneity property $g_{ij}(x,\lambda\dot{x}) = g_{ij}(x,\dot{x})$ and Euler's theorem for homogeneous functions, recall footnote 3), this simplifies to

$$\ddot{\xi}^i(t) + \gamma^i_{jk}(\xi(t),\dot{\xi}(t))\,\dot{\xi}^j(t)\dot{\xi}^k(t) = 0\,. \tag{44}$$

This *geodesic equation* has the same form as in the Riemannian case, except for the fact that Christoffel symbols have been replaced by their formal counterparts, Eq. (24) (or, equivalently, Eq. (21)).

We conclude this section by noting that Eq. (44) provides us with the Finslerian analogue of the geodesic tractography method previously proposed in the Riemannian setting for DTI. We stress that it will likewise need to be complemented with a way to sift geodesics into fibers and non-fibers, either deterministically or probabilistically. The Finslerian analogues of the connectivity measures proposed by Astola et al. [10] are quite straightforward.

2.8 Lagrangian Versus Hamiltonian Frameworks

The non-singular Riemann-Finsler metric enables the same kind of index gymnastics in Riemann-Finsler geometry as the Riemann metric does in the Riemannian case. In particular we have the "velocity"–"momentum" (or $\dot{x}$–q) duality expressed by the equations

$$q_i = g_{ij}(x, \dot{x})\dot{x}^j \quad \text{and} \quad \dot{x}^i = g^{ij}(x, q)q_j \,, \tag{45}$$

in which the *dual Riemann-Finsler metric* has now been prototyped such that

$$g^{ik}(x, q)g_{kj}(x, \dot{x}) = \delta^i_j \,, \tag{46}$$

assuming the aforementioned relationship between $\dot{x}$ and q. Note that, unlike in Eq. (18), the dual metric tensor has been expressed as a function of momentum, not velocity.

The foregoing formulation of the theory, with geometric quantities expressed as functions of position x and velocity $\dot{x}$, is known as the *Lagrangian framework*. The alternative formulation, in which the velocity variable is replaced by momentum q, is known as the *Hamiltonian framework*. The connection between the Lagrangian and corresponding Hamiltonian frameworks is particularly elegant in Riemann-Finsler geometry, in which the *Hamiltonian function* (or *dual Finsler function*) is given by

$$H(x, q) = F(x, \dot{x}) \,, \tag{47}$$

again assuming Eq. (45) to hold. As a consequence, the dual Riemann-Finsler metric tensor plays a similar role in the Hamiltonian framework as the Riemann-Finsler metric tensor does in the Lagrangian framework.[10]

The physical interpretations of the dual formalisms differ and depend on context. The Lagrangian formalism highlights the role of *geodesic congruences*, families of geodesics viewed as "particle trajectories", for which the vector variable $\dot{x}$ expresses "particle velocity". In the Hamiltonian formalism one considers "wave phenomena" induced by such geodesic congruences, in which case the covector variable q enters as "wave momentum", which, by definition, is the normal along which wave fronts propagate. Recall that in anisotropic media wave fronts induced by the interference of the disturbances caused by individual particles do not travel in the same direction as the particles themselves (cf. Huygens' principle, [61]). This is expressed by Eq. (45), as the (dual) metric is not necessarily diagonal.

2.9 Indicatrix and Figuratrix

The *indicatrix* at a fixed point x is the level set, or "glyph", of the Riemann-Finsler unit sphere, $F(x, \dot{x}) = 1$, or, by virtue of Eq. (14),

$$g_{ij}(x, \dot{x})\dot{x}^i\dot{x}^j = 1 \,. \tag{48}$$

[10]One sometimes reserves the terms Lagrangian and Hamiltonian for the *squared* Finsler and dual Finsler function. The associated "energy" functionals are *not* parametrization invariant.

The *figuratrix* at a fixed point x is the Hamiltonian counterpart, i.e. the level set given by $H(x, q) = 1$, recall Eqs. (4) and (8):

$$g^{ij}(x, q) q_i q_j = 1 \,. \tag{49}$$

One can show that, as a result of zero-homogeneity of the Riemann-Finsler (dual) metric tensor, both indicatrix as well as figuratrix represent *convex glyphs*.

A convenient interpretation of these structures is obtained by "freezing" the (co)vector argument of the Riemann-Finsler (dual) metric tensor in Eqs. (48) and (49), so that one ends up with (parametrized) quadratic forms. These are known as the *osculating indicatrix* and *osculating figuratrix*, respectively:

$$g_{ij}(x, \dot{x}_0) \dot{x}^i \dot{x}^j = 1 \,, \tag{50}$$

$$g^{ij}(x, q_0) q_i q_j = 1 \,. \tag{51}$$

One could think of these as gauge figures of a parametrized family of inner products on the tangent, respectively cotangent space of the spatial domain, each direction (specified by $\dot{x}_0$ or q_0) having its own unique instance. The Cartan tensor, Eq. (16), plays the pivotal role in relating the individual members of such a family.

In the DTI/Riemannian case the coefficients in Eqs. (50) and (51) are independent of the orientation parameters, so that each point in space has an unambiguously defined ellipsoidal shape representing the entire family. Indicatrices have been widely adopted in DTI visualization [62]. They might also be useful for our general case, although by their convex nature they are not likely to reflect the rich amount of information contained in a general Finsler function very clearly. A slick selection of osculating indicatrices might in that case prove more insightful.

2.10 Covariant Derivatives

The horizontal and vertical one-forms given by Eq. (38) can be used as a basis for decomposing the *covariant differential* of an arbitrary tensor field on the slit tangent bundle. For simplicity consider

$$\boldsymbol{T}(x, \dot{x}) = T^i_j(x, \dot{x}) \frac{\partial}{\partial x^i} \otimes dx^j \,, \tag{52}$$

and

$$\nabla \boldsymbol{T}(x, \dot{x}) = (\nabla \boldsymbol{T})^i_j \, (x, \dot{x}) \frac{\partial}{\partial x^i} \otimes dx^j \,. \tag{53}$$

Then each component on the r.h.s. is a one-form, and can thus be written as a sum of horizontal and vertical one-forms. By definition,

$$(\nabla \boldsymbol{T})^i_j\,(x,\dot{x}) = T^i_{j|k}(x,\dot{x})\,dx^k + T^i_{j;k}(x,\dot{x})\,\frac{\delta \dot{x}^k}{F(x,\dot{x})}\,. \tag{54}$$

By evaluation on the corresponding dual basis, Eq. (37), one obtains

$$T^i_{j|k}(x,\dot{x}) = \frac{\delta T^i_j(x,\dot{x})}{\delta x^k} + T^\ell_j(x,\dot{x})\Gamma^i_{\ell k}(x,\dot{x}) - T^i_\ell(x,\dot{x})\Gamma^\ell_{jk}(x,\dot{x})\,, \tag{55}$$

$$T^i_{j;k}(x,\dot{x}) = F(x,\dot{x})\,\frac{\partial T^i_j(x,\dot{x})}{\partial \dot{x}^k}\,. \tag{56}$$

Equations (55) and (56) are the components of the *horizontal covariant derivative* and the *vertical covariant derivative* of the tensor field, respectively (relative to the Chern-Rund connection, recall Eq. (21)). Higher order tensors are treated similarly. Their horizontal covariant derivatives will contain as many "correction terms" involving the Riemann-Finsler Γ-symbols of Eq. (21) as indicated by their order. Note the elegant similarity with the Riemannian case.

Some cases are particularly important, e.g. those involving the Riemann-Finsler metric tensor or its dual. We have

$$g_{ij|k}(x,\dot{x}) = 0\,, \tag{57}$$

$$g_{ij;k}(x,\dot{x}) = 2F(x,\dot{x})\,C_{ijk}(x,\dot{x})\,, \tag{58}$$

$$g^{ij}{}_{|k}(x,\dot{x}) = 0\,, \tag{59}$$

$$g^{ij}{}_{;k}(x,\dot{x}) = -2F(x,\dot{x})\,C^{ij}_k(x,\dot{x})\,. \tag{60}$$

The Kronecker tensor is covariantly constant both horizontally as well as vertically:

$$\delta^i_{j|k} = 0\,, \tag{61}$$

$$\delta^i_{j;k} = 0\,. \tag{62}$$

Thus, unlike in the Riemannian case, the Riemann-Finsler metric tensor is covariantly constant only along horizontal directions, whereas its behavior in vertical directions is governed by the Cartan tensor (the covariant derivative is said to be "almost metric compatible").

3 Conclusion and Discussion

Riemann-Finsler geometry naturally extends the Riemannian rationale used in the context of DTI to general dwMRI representations. It can be equivalently approached from a Lagrangian or Hamiltonian perspective, although the latter appears to be most closely related to the physics of dwMRI acquisition and its underlying model in terms of a generalized mono-exponential Stejskal-Tanner equation.

We have pointed out its potential application to voxel classification based on the Cartan tensor and related quantities, and to dwMRI tractography by deriving the corresponding Finsler geodesic equations, without the quadratic restriction inherent to the DTI model, yet retaining quadratic scaling in the *magnitude* of the gradient magnetic field. Although this does not cover the general (multi-exponential and/or non-homogeneous) case, the conditions for and limitations of this conjecture, and in particular the added value relative to DTI, diffusional kurtosis imaging (DKI), and (other) HARDI schemes, are worthwhile investigating. Future work will concentrate on this, on the reconstruction of the (dual) Finsler function and related quantities, and on experimental validation of Finsler tractography and voxel classification as advocated in this chapter.

Appendix: Horizontal and Vertical Splitting

We may consider the partial derivatives with respect to x^i and $\dot{x}^i$ as coordinate vector fields on the tangent bundle TM, and consider the effect induced by a change of coordinates of the base manifold M, $x = x(\xi)$ say. Since $\dot{x}$ is a vector, this induces the following *vector transformation law* for its components $\dot{x}^i$ expressed in terms of its new components, $\dot{\xi}^p$, say:

$$\dot{x}^i = \frac{\partial x^i}{\partial \xi^p} \dot{\xi}^p \,, \tag{63}$$

or, equivalently,

$$\frac{\partial}{\partial \dot{\xi}^p} = \frac{\partial x^i}{\partial \xi^p} \frac{\partial}{\partial \dot{x}^i} \,, \tag{64}$$

so that, by construction,

$$\dot{x}^i \frac{\partial}{\partial x^i} = \dot{\xi}^p \frac{\partial}{\partial \xi^p} \,. \tag{65}$$

As a result,

$$\frac{\partial}{\partial \xi^p} = \frac{\partial x^i}{\partial \xi^p} \frac{\partial}{\partial x^i} + \frac{\partial^2 x^i}{\partial \xi^p \partial \xi^q} \dot{\xi}^q \frac{\partial}{\partial \dot{x}^i} \,. \tag{66}$$

Given the definition of the horizontal vectors, Eq. (22), and of the nonlinear connection, Eq. (23), it is then a tedious but straightforward exercise to deduce that

$$\frac{\delta}{\delta \xi^p} = \frac{\partial x^i}{\partial \xi^p} \frac{\delta}{\delta x^i} \,, \tag{67}$$

similar to the vector transformation law for the vertical components, recall Eq. (64).

Likewise one has the *covector transformation law* for the components of the horizontal and vertical one-forms, recall Eq. (33):

$$dx^i = \frac{\partial x^i}{\partial \xi^p} d\xi^p \,, \tag{68}$$

respectively

$$\delta \dot{x}^i = \frac{\partial x^i}{\partial \xi^p} \delta \dot{\xi}^i \,. \tag{69}$$

The "natural" transformation behavior expressed by Eqs. (64) and (67)–(69) motivates the definitions of horizontal and vertical vectors and covectors in Sect. 2.6.

References

1. Hagmann, P., Jonasson, L., Maeder, P., Thiran, J.P., Wedeen, V.J., Meuli, R.: Understanding diffusion MR imaging techniques: from scalar diffusion-weighted imaging to diffusion tensor imaging and beyond. RadioGraphics **26**, S205 (2006)
2. Basser, P.J., Mattiello, J., Le Bihan, D.: Estimation of the effective self-diffusion tensor from the NMR spin echo. J. Magn. Reson. **103**, 247 (1994)
3. Basser, P.J., Mattiello, J., Le Bihan, D.: MR diffusion tensor spectroscopy and imaging. Biophys. J. **66**(1), 259 (1994)
4. Basser, P.J., Mattiello, J., Le Bihan, D.: Diffusion tensor MR imaging of the human brain. Radiology **201**(3), 637 (1996)
5. Basser, P.J., Pierpaoli, C.: Microstructural and physiological features of tissues elucidated by quantitative-diffusion-tensor MRI. J. Magn. Reson. Ser. B **111**(3), 209 (1996)
6. Basser, P.J.: Relationships between diffusion tensor and q-space MRI. Magn. Reson. Med. **47**(2), 392 (2002)
7. Le Bihan, D., Mangin, J.F., Poupon, C., Clark, C.A., Pappata, S., Molko, N., Chabriat, H.: Diffusion tensor imaging: concepts and applications. J. Magn. Reson. Imaging **13**, 534 (2001)
8. Moseley, M., Cohen, Y., Kucharczyk, J., Mintorovitch, J., Asgari, H.S., Wendland, M.F., Tsuruda, J., Norman, D.: Diffusion-weighted MR imaging of anisotropic water diffusion in cat central nervous system. Radiology **176**(2), 439 (1990)
9. Arsigny, V., Fillard, P., Pennec, X., Ayache, N.: Log-Euclidean metrics for fast and simple calculus on diffusion tensors. Magn. Reson. Med. **56**(2), 411 (2006)
10. Astola, L., Florack, L., ter Haar Romeny, B.: Measures for pathway analysis in brain white matter using diffusion tensor images. In: Karssemeijer, N., Lelieveldt, B. (eds.) Proceedings of the Twentieth International Conference on Information Processing in Medical Imaging–IPMI 2007, Kerkrade. Lecture Notes in Computer Science, vol. 4584, pp. 642–649. Springer, Berlin (2007)
11. Astola, L., Florack, L.: Sticky vector fields and other geometric measures on diffusion tensor images. In: Proceedings of the 9th IEEE Computer Society Workshop on Mathematical Methods in Biomedical Image Analysis, held in conjuction with the IEEE Computer Society Conference on Computer Vision and Pattern Recognition, Anchorage, 23–28 June 2008. IEEE Computer Society
12. Astola, L., Fuster, A., Florack, L.: A Riemannian scalar measure for diffusion tensor images. Pattern Recognit. **44**(9), 1885 (2011). doi:10.1016/j.patcog.2010.09.009

13. Deriche, R., Calder, J., Descoteaux, M.: Optimal real-time Q-ball imaging using regularized Kalman filtering with incremental orientation sets. Med. Image Anal. **13**(4), 564 (2009)
14. Fillard, P., Pennec, X., Arsigny, V., Ayache, N.: Clinical DT-MRI estimation, smoothing, and fiber tracking with log-Euclidean metrics. IEEE Trans. Med. Imaging **26**(11) (2007)
15. Florack, L.M.J., Astola, L.J.: A multi-resolution framework for diffusion tensor images. In: Aja Fernández, S., de Luis Garcia, R. (eds.) CVPR Workshop on Tensors in Image Processing and Computer Vision, Anchorage, 24–26 June 2008. IEEE (2008). Digital proceedings
16. Lenglet, C., Deriche, R., Faugeras, O.: Inferring white matter geometry from diffusion tensor MRI: application to connectivity mapping. In: Pajdla, T., Matas, J. (eds.) Proceedings of the Eighth European Conference on Computer Vision, Prague, May 2004. Lecture Notes in Computer Science, vol. 3021–3024, pp. 127–140. Springer, Berlin (2004)
17. Lenglet, C., Rousson, M., Deriche, R., Faugeras, O.: Statistics on the manifold of multivariate normal distributions: theory and application to diffusion tensor MRI processing. J. Math. Imaging Vis. **25**(3), 423 (2006)
18. Lenglet, C., Prados, E., Pons, J.P.: Brain connectivity mapping using Riemannian geometry, control theory and PDEs. SIAM J. Imaging Sci. **2**(2), 285 (2009)
19. Pennec, X., Fillard, P., Ayache, N.: A Riemannian framework for tensor computing. Int. J. Comput. Vis. **66**(1), 41 (2006)
20. Prados, E., Soatto, S., Lenglet, C., Pons, J.P., Wotawa, N., Deriche, R., Faugeras, O.: Control theory and fast marching techniques for brain connectivity mapping. In: Proceedings of the IEEE Computer Society Conference on Computer Vision and Pattern Recognition, New York, June 2006, vol. 1, pp. 1076–1083. IEEE Computer Society (2006)
21. Florack, L., Balmashnova, E.: Two canonical representations for regularized high angular resolution diffusion imaging. In: Alexander, D., Gee, J., Whitaker, R. (eds.) MICCAI Workshop on Computational Diffusion MRI, New York, 10 Sept 2008, pp. 85–96 (2008)
22. Florack, L., Balmashnova, E., Astola, L., Brunenberg, E.: A new tensorial framework for single-shell high angular resolution diffusion imaging. J. Math. Imaging Vis. **3**(38), 171 (2010). Published online: doi:10.1007/s10851-010-0217-3
23. Jensen, J.H., Helpern, J.A., Ramani, A., Lu, H., Kaczynski, K.: Diffusional kurtosis imaging: the quantification of non-Gaussian water diffusion by means of magnetic resonance imaging. Magn. Reson. Med. **53**(6), 1432 (2005)
24. Jian, B., Vemuri, B.C., Özarslan, E., Carney, P.R., Mareci, T.H.: A novel tensor distribution model for the diffusion-weighted MR signal. NeuroImage **37**, 164 (2007)
25. Liu, C., Bammer, R., Acar, B., Moseley, M.E.: Characterizing non-Gaussian diffusion by using generalized diffusion tensors. Magn. Reson. Med. **51**(5), 924 (2004)
26. Özarslan, E., Mareci, T.H.: Generalized diffusion tensor imaging and analytical relationships between diffusion tensor imaging and high angular resolution imaging. Magn. Reson. Med. **50**(5), 955 (2003)
27. Özarslan, E., Shepherd, T.M., Vemuri, B.C., Blackband, S.J., Mareci, T.H.: Resolution of complex tissue microarchitecture using the diffusion orientation transform (DOT). NeuroImage **31**, 1086 (2006)
28. Descoteaux, M., Angelino, E., Fitzgibbons, S., Deriche, R.: Apparent diffusion coefficients from high angular resolution diffusion imaging: estimation and applications. Magn. Reson. Med. **56**(2), 395 (2006)
29. Descoteaux, M., Angelino, E., Fitzgibbons, S., Deriche, R.: Regularized, fast, and robust analytical Q-ball imaging. Magn. Reson. Med. **58**(3), 497 (2007)
30. Florack, L.M.J.: Codomain scale space and regularization for high angular resolution diffusion imaging. In: Aja Fernández, S., de Luis Garcia, R. (eds.) CVPR Workshop on Tensors in Image Processing and Computer Vision, Anchorage, 24–26 June 2008. IEEE (2008). Digital proceedings
31. Hess, C.P., Mukherjee, P., Tan, E.T., Xu, D., Vigneron, D.B.: Q-ball reconstruction of multimodal fiber orientations using the spherical harmonic basis. Magn. Reson. Med. **56**, 104 (2006)
32. Tuch, D.S.: Q-ball imaging. Magn. Reson. Med. **52**, 1358 (2004)

33. Duits, R., Franken, E.M.: Left invariant parabolic evolution equations on SE(2) and contour enhancement via invertible orientation scores, part I: linear left-invariant diffusion equations on SE(2). Q. Appl. Math. **68**(2), 255 (2010)
34. Duits, R., Franken, E.M.: Left invariant parabolic evolution equations on SE(2) and contour enhancement via invertible orientation scores, part II: nonlinear left-invariant diffusion equations on invertible orientation scores. Q. Appl. Math. **68**(2), 293 (2010)
35. Duits, R., Franken, E.: Left-invariant diffusions on the space of positions and orientations and their application to crossing-preserving smoothing of HARDI images. Int. J. Comput. Vis. **12**(3), 231 (2011). Published online: doi:10.1007/s11263-010-0332-z
36. O'Donnell, L., Haker, S., Westin, C.F.: New approaches to estimation of white matter connectivity in diffusion tensor MRI: elliptic PDEs and geodesics in a tensor-warped space. In: Dohi, T., Kikinis, R. (eds.) Proceedings of the 5th International Conference on Medical Image Computing and Computer-Assisted Intervention—MICCAI 2002, Tokyo, 25–28 Sept 2002. Lecture Notes in Computer Science, vol. 2488–2489, pp. 459–466. Springer, Berlin (2002)
37. Astola, L.J., Florack, L.M.J.: Finsler geometry on higher order tensor fields and applications to high angular resolution diffusion imaging. In: Tai, X.C., Mørken, K., Lysaker, M., Lie, K.A. (eds.) Scale Space and Variational Methods in Computer Vision: Proceedings of the Second International Conference, SSVM 2009, Voss. Lecture Notes in Computer Science, vol. 5567, pp. 224–234. Springer, Berlin (2009)
38. Astola, L.J.: Multi-scale Riemann-Finsler geometry: applications to diffusion tensor imaging and high resolution diffusion imaging. Ph.D. thesis, Eindhoven University of Technology, Department of Mathematics and Computer Science, Eindhoven (2010)
39. Astola, L.J., Florack, L.M.J.: Finsler geometry on higher order tensor fields and applications to high angular resolution diffusion imaging. Int. J. Comput. Vis. **92**(3), 325 (2011). doi:10.1007/s11263-010-0377-z
40. Astola, L.J., Jalba, A.C., Balmashnova, E.G., Florack, L.M.J.: Finsler streamline tracking with single tensor orientation distribution function for high angular resolution diffusion imaging. J. Math. Imaging Vis. **41**(3), 170 (2011)
41. Melonakos, J., Pichon, E., Angenent, S., Tannenbaum, A.: Finsler active contours. IEEE Trans. Pattern Anal. Mach. Intell. **30**(3), 412 (2008)
42. Bao, D., Chern, S.S., Shen, Z.: An Introduction to Riemann-Finsler Geometry. Graduate Texts in Mathematics, vol. 2000. Springer, New York (2000)
43. Rund, H.: The Differential Geometry of Finsler Spaces. Springer, Berlin (1959)
44. Shen, Z.: Lectures on Finsler Geometry. World Scientific, Singapore (2001)
45. Mo, X.: An Introduction to Finsler Geometry. Peking University Series in Mathematics, vol. 1. World Scientific, Singapore (2006)
46. Riemann, B.: Über die Hypothesen, welche der Geometrie zu Grunde liegen. In: Weber, H. (ed.) Gesammelte Mathematische Werke, pp. 272–287. Teubner, Leibzig (1892)
47. Finsler, P.: Ueber kurven und Flächen in allgemeinen Räumen. Ph.D. thesis, University of Göttingen,, Göttingen (1918)
48. Cartan, E.: Les Espaces de Finsler. Hermann, Paris (1934)
49. Michailovich, O., Rathi, Y., Dolui, S.: Spatially regularized compressed sensing for high angular resolution diffusion imaging. IEEE Trans. Med. Imaging **30**(5), 1100 (2011). doi:10.1109/TMI.2011.2142189
50. Stejskal, E.O., Tanner, J.E.: Spin diffusion measurements: Spin echoes in the presence of a time-dependent field gradient. J. Comput. Phys. **42**(1), 288 (1965)
51. Bloch, F.: Nuclear induction. Phys. Rev. **70**, 460 (1946)
52. Haacke, E.M., Brown, R.W., Thompson, M.R., Venkatesan, R.: Magnetic Resonance Imaging: Physical Principles and Sequence Design. Wiley, New York (1999)
53. Stejskal, E.O.: Use of spin echoes in a pulsed magnetic-field gradient to study anisotropic, restricted diffusion and flow. J. Comput. Phys. **43**(10), 3597 (1965)
54. Torrey, H.C.: Bloch equations with diffusion terms. Phys. Rev. D **104**, 563 (1956)

55. Sinnaeve, D.: The Stejskal-Tanner equation generalized for any gradient shape—an overview of most pulse sequences measuring free diffusion. Concepts Magn. Reson. Part A **40A**(2), 39 (2012). doi:10.1002/cmr.a.21223
56. Panagiotaki, E., Schneider, T., Siow, B., Hall, M.G., Lythgoe, M.F., Alexander, D.C.: Compartment models of the diffusion MR signal in brain white matter: a taxonomy and comparison. NeuroImage **59**(3), 2241 (2012). doi:10.1016/j.neuroimage.2011.09.081
57. Fieremans, E., Jensen, J.H., Helpern, J.A.: White matter characterization with diffusional kurtosis imaging. NeuroImage **58**(1), 177 (2011). doi:10.1016/j.neuroimage.2011.06.006
58. Hao, X., Whitaker, R.T., Fletcher, P.T.: Adaptive Riemannian metrics for improved geodesic tracking of white matter. In: Székely, G., Hahn, H.K. (eds.) Proceedings of the Twenty-Second International Conference on Information Processing in Medical Imaging–IPMI 2011, Kloster Irsee. Lecture Notes in Computer Science, vol. 6801, pp. 13–24. Springer, Berlin (2011)
59. Fuster, A., Dela Haije, T.C.J., Florack, L.M.J.: On the Riemannian rationale for diffusion tensor imaging. In: Y. Wiaux, E.D. McEwen (eds.) Proceedings of the International BASP Frontiers Workshop 2013, BASP 2013, Villars-Sur-Ollon, 27 Jan–1 Feb 2013, p. 62 (2013) http://baspfrontiers.epfl.ch/proceedings.pdf
60. Rund, H.: The Hamilton-Jacobi Theory in the Calculus of Variations. Robert E. Krieger Publishing Company, Huntington (1973)
61. Huygens, C.: Traité de la Lumière. Pierre van der Aa, Leiden (1690)
62. Kingsley, P.B.: Introduction to diffusion tensor imaging mathematics: part I. Tensors, rotations, and eigenvectors. Concepts Magn. Reson. Part A **28A**(2), 101 (2006). doi:10.1002/cmr.a.20048