A Survey on Bogoliubov Generating Functionals for Interacting Particle Systems in the Continuum

Dmitri L. Finkelshtein and Maria João Oliveira

1 Bogoliubov Generating Functionals

Let $\Gamma := \Gamma_{\mathbb{R}^d}$ be the configuration space over $\mathbb{R}^d$, $d \in \mathbb{N}$,

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d : |\gamma \cap \Lambda| < \infty \text{ for every compact } \Lambda \subset \mathbb{R}^d \right\},$$

where $|\cdot|$ denotes the cardinality of a set. As usual we identify each $\gamma \in \Gamma$ with the non-negative Radon measure $\sum_{x\in\gamma} \delta_x$ on the Borel σ-algebra $\mathscr{B}(\mathbb{R}^d)$, where δ_x is the Dirac measure with mass at x, $\sum_{x\in\emptyset} \delta_x := 0$. This allows to endow Γ with the vague topology, that is, the weakest topology on Γ with respect to which all mappings

$$\Gamma \ni \gamma \longmapsto \langle f, \gamma \rangle := \int_{\mathbb{R}^d} d\gamma(x)\, f(x) = \sum_{x\in\gamma} f(x)$$

are continuous for all continuous functions f on $\mathbb{R}^d$ with compact support. In the sequel we denote the corresponding Borel σ-algebra on Γ by $\mathscr{B}(\Gamma)$.

Definition 1. Let μ be a probability measure on $(\Gamma, \mathscr{B}(\Gamma))$. The Bogoliubov generating functional (shortly GF) B_μ corresponding to μ is a functional defined at each $\mathscr{B}(\mathbb{R}^d)$-measurable function θ by

D.L. Finkelshtein
Department of Mathematics, Swansea University, Singleton Park, Swansea SA2 8PP, UK
e-mail: d.l.finkelshtein@swansea.ac.uk

M.J. Oliveira (✉)
Universidade Aberta and CMAF, University of Lisbon, P 1649-003 Lisbon, Portugal
e-mail: oliveira@cii.fc.ul.pt

C. Bernardin and P. Gonçalves (eds.), *From Particle Systems to Partial Differential Equations*, Springer Proceedings in Mathematics & Statistics 75,
DOI 10.1007/978-3-642-54271-8_6,

$$B_\mu(\theta) := \int_\Gamma d\mu(\gamma) \prod_{x\in\gamma}(1+\theta(x)), \tag{1}$$

provided the right-hand side exists for $|\theta|$, i.e., $B_\mu(|\theta|) < \infty$.

Observe that for each $\theta > -1$ such that the right-hand side of (1) exists, one may equivalently rewrite (1) as

$$B_\mu(\theta) := \int_\Gamma d\mu(\gamma)\, e^{\langle \ln(1+\theta),\gamma\rangle},$$

showing that B_μ is a modified Laplace transform.

From Definition 1, it is clear that the existence of $B_\mu(\theta)$ for $\theta \neq 0$[1] depends on the underlying probability measure μ. However, it follows also from Definition 1 that if the GF B_μ corresponding to a probability measure μ exists, then the domain of B_μ depends on μ. Conversely, the domain of B_μ reflects special properties over the measure μ [21]. For instance, if μ has finite local exponential moments, i.e., for all $\alpha > 0$ and all bounded Borel sets $\Lambda \subseteq \mathbb{R}^d$,

$$\int_\Gamma d\mu(\gamma)\, e^{\alpha|\gamma\cap\Lambda|} < \infty,$$

then B_μ is well-defined, for instance, on all bounded functions θ with compact support. The converse is also true and it follows from the fact that, for each $\alpha > 0$ and for each Λ described as before, the latter integral is equal to $B_\mu((e^\alpha - 1)\mathbb{1}_\Lambda)$, where $\mathbb{1}_\Lambda$ is the indicator function of Λ. In this situation, to a such measure μ one may associate the so-called correlation measure ρ_μ.

In order to introduce the notion of correlation measure, for any $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ let

$$\Gamma^{(n)} := \{\gamma \in \Gamma : |\gamma| = n\},\ n \in \mathbb{N}, \quad \Gamma^{(0)} := \{\emptyset\}.$$

Clearly, each $\Gamma^{(n)}$, $n \in \mathbb{N}$, can be identified with the symmetrization of the set $\{(x_1,\ldots,x_n) \in (\mathbb{R}^d)^n : x_i \neq x_j \text{ if } i \neq j\}$ under the permutation group over $\{1,\ldots,n\}$, which induces a natural (metrizable) topology on $\Gamma^{(n)}$ and the corresponding Borel σ-algebra $\mathscr{B}(\Gamma^{(n)})$. Moreover, for the Lebesgue product measure $(dx)^{\otimes n}$ fixed on $(\mathbb{R}^d)^n$, this identification yields a measure $m^{(n)}$ on $(\Gamma^{(n)}, \mathscr{B}(\Gamma^{(n)}))$. This leads to the space of finite configurations

$$\Gamma_0 := \bigsqcup_{n=0}^{\infty} \Gamma^{(n)}$$

[1] Of course, for any probability measure μ on $(\Gamma, \mathscr{B}(\Gamma))$ one has $B_\mu(0) = 1$.

endowed with the topology of disjoint union of topological spaces and the corresponding Borel σ-algebra $\mathscr{B}(\Gamma_0)$, and to the so-called Lebesgue-Poisson measure on $(\Gamma_0, \mathscr{B}(\Gamma_0))$,

$$\lambda := \lambda_{dx} := \sum_{n=0}^{\infty} \frac{1}{n!} m^{(n)}, \quad m^{(0)}(\{\emptyset\}) := 1. \tag{2}$$

Given a probability measure μ on $(\Gamma, \mathscr{B}(\Gamma))$ with finite local exponential moments, the correlation measure ρ_μ corresponding to μ is a measure on $(\Gamma_0, \mathscr{B}(\Gamma_0))$ defined for all complex-valued exponentially bounded $\mathscr{B}(\Gamma_0)$-measurable functions G with local support[2] by

$$\int_{\Gamma_0} d\rho_\mu(\eta)\, G(\eta) = \int_\Gamma d\mu(\gamma) \sum_{\substack{\eta \subset \gamma \\ |\eta| < \infty}} G(\eta). \tag{3}$$

As a consequence, for every bounded $\mathscr{B}(\mathbb{R}^d)$-measurable function θ with compact support and $G = e_\lambda(\theta)$,

$$e_\lambda(\theta, \eta) := \prod_{x \in \eta} \theta(x), \; \eta \in \Gamma_0 \setminus \{\emptyset\}, \quad e_\lambda(\theta, \emptyset) := 1,$$

definition (3) leads to

$$B_\mu(\theta) = \int_\Gamma d\mu(\gamma) \prod_{x \in \gamma} (1 + \theta(x)) = \int_\Gamma d\mu(\gamma) \sum_{\substack{\eta \subset \gamma \\ |\eta| < \infty}} e_\lambda(\theta, \eta) = \int_{\Gamma_0} d\rho_\mu(\eta)\, e_\lambda(\theta, \eta),$$

yielding a description of the GF B_μ in terms of either the correlation measure ρ_μ or the so-called correlation function $k_\mu := \frac{d\rho_\mu}{d\lambda}$ corresponding to μ, if ρ_μ is absolutely continuous with respect to the Lebesgue-Poisson measure λ:

$$B_\mu(\theta) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta, \eta) k_\mu(\eta). \tag{4}$$

Throughout this work we will consider GF defined on the whole $L^1 := L^1(\mathbb{R}^d, dx)$ space of complex-valued functions. Furthermore, we will assume that the GF are entire. For a comprehensive presentation of the general theory of holomorphic functionals on Banach spaces see e.g. [1,5]. We recall that a functional $A : L^1 \to \mathbb{C}$ is entire on L^1 whenever A is locally bounded and for all $\theta_0, \theta \in L^1$

---

[2] That is, $G \upharpoonright_{\Gamma_0 \setminus \Gamma_\Lambda} \equiv 0$, $\Gamma_\Lambda := \{\eta \in \Gamma : \eta \subset \Lambda\}$, for some bounded Borel set $\Lambda \subseteq \mathbb{R}^d$ and there are $C_1, C_2 > 0$ such that $|G(\eta)| \leq C_1 e^{C_2 |\eta|}$ for all $\eta \in \Gamma_0$.

the mapping $\mathbb{C} \ni z \mapsto A(\theta_0 + z\theta) \in \mathbb{C}$ is entire. Thus, at each $\theta_0 \in L^1$, every entire functional A on L^1 has a representation in terms of its Taylor expansion,

$$A(\theta_0 + z\theta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} d^n A(\theta_0; \theta, \ldots, \theta), \quad z \in \mathbb{C}, \theta \in L^1.$$

Theorem 1. *Let A be an entire functional on L^1. Then each differential $d^n A(\theta_0; \cdot), n \in \mathbb{N}, \theta_0 \in L^1$ is defined by a symmetric kernel*

$$\delta^n A(\theta_0; \cdot) \in L^\infty(\mathbb{R}^{dn}) := L^\infty\left((\mathbb{R}^d)^n, (dx)^{\otimes n}\right)$$

called the variational derivative of n-th order of A at the point θ_0. More precisely,

$$\begin{aligned} d^n A(\theta_0; \theta_1, \ldots, \theta_n) &:= \frac{\partial^n}{\partial z_1 \ldots \partial z_n} A\left(\theta_0 + \sum_{i=1}^n z_i \theta_i\right)\Bigg|_{z_1 = \ldots = z_n = 0} \\ &=: \int_{(\mathbb{R}^d)^n} dx_1 \ldots dx_n \, \delta^n A(\theta_0; x_1, \ldots, x_n) \prod_{i=1}^n \theta_i(x_i) \end{aligned}$$

for all $\theta_1, \ldots, \theta_n \in L^1$. Moreover, the operator norm of the bounded n-linear functional $d^n A(\theta_0; \cdot)$ is equal to $\|\delta^n A(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^{dn})}$ and for all $r > 0$ one has

$$\|\delta A(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{r} \sup_{\|\theta'\|_{L^1} \leq r} |A(\theta_0 + \theta')| \tag{5}$$

and, for $n \geq 2$,

$$\|\delta^n A(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^{dn})} \leq n! \left(\frac{e}{r}\right)^n \sup_{\|\theta'\|_{L^1} \leq r} |A(\theta_0 + \theta')|. \tag{6}$$

Remark 1. 1. According to Theorem 1, the Taylor expansion of an entire functional A at a point $\theta_0 \in L^1$ may be written in the form

$$\begin{aligned} A(\theta_0 + \theta) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} dx_1 \ldots dx_n \, \delta^n A(\theta_0; x_1, \ldots, x_n) \prod_{i=1}^n \theta(x_i) \\ &= \int_{\Gamma_0} d\lambda(\eta) \, \delta^n A(\theta_0; \eta) e_\lambda(\theta, \eta), \end{aligned}$$

where λ is the Lebesgue-Poisson measure defined in (2).
2. Concerning Theorem 1, we observe that the analogous result does not hold neither for other L^p-spaces, nor Banach spaces of continuous functions, or

Sobolev spaces. For a detailed explanation see the proof of Theorem 1 and Remark 7 in [21].

The first part of Theorem 1 stated for GF and their variational derivatives at $\theta_0 = 0$ yields the next result. In particular, it shows that the assumption of entireness on L^1 is a natural environment, namely, to recover the notion of correlation function.

Proposition 1. *Let B_μ be an entire GF on L^1. Then the measure ρ_μ is absolutely continuous with respect to the Lebesgue-Poisson measure λ and the Radon-Nykodim derivative $k_\mu = \frac{d\rho_\mu}{d\lambda}$ is given by*

$$k_\mu(\eta) = \delta^{|\eta|} B_\mu(0;\eta) \quad \text{for } \lambda\text{-a.a. } \eta \in \Gamma_0.$$

Remark 2. Proposition 1 shows that the correlation functions $k_\mu^{(n)} := k_\mu \upharpoonright_{\Gamma^{(n)}}$ are the Taylor coefficients of the GF B_μ. In other words, B_μ is the generating functional for the correlation functions $k_\mu^{(n)}$. This was also the reason why N. N. Bogoliubov [4] introduced these functionals. Furthermore, GF are also related to the general infinite dimensional analysis on configuration spaces, cf., e.g. [19]. Namely, through the unitary isomorphism S_λ defined in [19] between the space $L^2(\Gamma_0, \lambda)$ of complex-valued functions and the Bargmann-Segal space one finds $B_\mu = S_\lambda(k_\mu)$.

Concerning the second part of Theorem 1, namely, estimates (5) and (6), we note that A being entire does not ensure that for every $r > 0$ the supremum appearing on the right-hand side of (5), (6) is always finite. This will hold if, in addition, the entire functional A is of bounded type, that is,

$$\forall\, r > 0, \quad \sup_{\|\theta\|_{L^1} \le r} |A(\theta_0 + \theta)| < \infty, \ \forall\, \theta_0 \in L^1.$$

Hence, as a consequence of Proposition 1, it follows from (5) and (6) that the correlation function k_μ of an entire GF of bounded type on L^1 fulfills the so-called generalized Ruelle bound, that is, for any $0 \le \varepsilon \le 1$ and any $r > 0$ there is some constant $C \ge 0$ depending on r such that

$$k_\mu(\eta) \le C\, (|\eta|!)^{1-\varepsilon} \left(\frac{e}{r}\right)^{|\eta|}, \quad \lambda\text{–a.a. } \eta \in \Gamma_0. \tag{7}$$

In our case, $\varepsilon = 0$. We observe that if (7) holds for $\varepsilon = 1$ and for at least one $r > 0$, then condition (7) is the classical Ruelle bound. In terms of GF, the latter means that $|B_\mu(\theta)| \le C \exp\left(\frac{e}{r}\|\theta\|_{L^1}\right)$, as it can be easily checked using representation (4) and the following equality [20],

$$\int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(f,\eta) = \exp\left(\int_{\mathbb{R}^d} dx\, f(x)\right), \quad f \in L^1.$$

This special case motivates the definition of the family of Banach spaces $\mathscr{E}_\alpha$, $\alpha > 0$, of all entire functionals B on L^1 such that

$$\|B\|_\alpha := \sup_{\theta \in L^1} \left(|B(\theta)| \, e^{-\frac{1}{\alpha}\|\theta\|_{L^1}} \right) < \infty, \tag{8}$$

cf. [21, Proposition 23], which plays an essential role in the study of stochastic dynamics of infinite particle systems (Sect. 2).

For more details and proofs and for further results concerning GF see [21] and the references therein.

2 Stochastic Dynamic Equations

The stochastic evolution of an infinite particle system might be described by a Markov process on Γ, which is determined heuristically by a Markov generator L defined on a suitable space of functions on Γ. If such a Markov process exists, then it provides a solution to the (backward) Kolmogorov equation

$$\frac{d}{dt} F_t = LF_t, \qquad F_t|_{t=0} = F_0. \tag{9}$$

However, the construction of the Markov process seems to be often a difficult question and at the moment it has been successfully accomplished only for very restrictive classes of generators, see [16] and [24].

Besides this technical difficulty, in applications it turns out that one needs a knowledge on certain characteristics of the stochastic evolution in terms of mean values rather than pointwise, which do not follow neither from the construction of the Markov process nor from the study of (9). These characteristics concern e.g. observables, that is, functions defined on Γ, for which expected values are given by

$$\langle F, \mu \rangle = \int_\Gamma d\mu(\gamma) \, F(\gamma),$$

where μ is a probability measure on Γ, that is, a state of the system. This leads to the time evolution problem on states,

$$\frac{d}{dt} \langle F, \mu_t \rangle = \langle LF, \mu_t \rangle, \quad \mu_t|_{t=0} = \mu_0. \tag{10}$$

Technically, to proceed further, first we shall exploit definition (3), namely, the sum appearing therein, which concerns the so-called K-transform introduced by A. Lenard [26]. That is a mapping which maps functions defined on Γ_0 into functions defined on the space Γ. More precisely, given a complex-valued bounded

$\mathscr{B}(\Gamma_0)$-measurable function G with bounded support[3] (shortly $G \in B_{\mathrm{bs}}(\Gamma_0)$), the K-transform of G is a mapping $KG : \Gamma \to \mathbb{C}$ defined at each $\gamma \in \Gamma$ by

$$(KG)(\gamma) := \sum_{\substack{\eta \subset \gamma \\ |\eta| < \infty}} G(\eta). \tag{11}$$

It has been shown in [18] that the K-transform is a linear and invertible mapping. Thus, definition (3) shows, in particular, that for any probability measure μ on $(\Gamma, \mathscr{B}(\Gamma))$ with finite local exponential moments, one has $B_{\mathrm{bs}}(\Gamma_0) \subseteq L^1(\Gamma_0, \rho_\mu)$. Moreover, on the dense set $B_{\mathrm{bs}}(\Gamma_0)$ in $L^1(\Gamma_0, \rho_\mu)$ the inequality $\|KG\|_{L^1(\mu)} \leq \|G\|_{L^1(\rho_\mu)}$ holds, which allows an extension of the K-transform to a bounded operator $K : L^1(\Gamma_0, \rho_\mu) \to L^1(\Gamma, \mu)$ in such a way that equality (3) still holds for any $G \in L^1(\Gamma_0, \rho_\mu)$. For the extended operator the explicit form (11) still holds, now μ-a.e. This means, in particular,

$$(Ke_\lambda(f))(\gamma) = \prod_{x \in \gamma} (1 + f(x)), \quad \mu\text{–a.a. } \gamma \in \Gamma,$$

for all $\mathscr{B}(\mathbb{R}^d)$-measurable functions f such that $e_\lambda(f) \in L^1(\Gamma_0, \rho_\mu)$, cf. e.g. [18].

In terms of the time evolution description (10) on the states μ_t of an infinite particle system, these considerations imply that for F being of the type $F = KG$, $G \in B_{\mathrm{bs}}(\Gamma_0)$, (10) may be rewritten in terms of the correlation functions $k_t := k_{\mu_t}$ corresponding to the states μ_t, provided these functions exist (or, more generally, in terms of correlation measures $\rho_t := \rho_{\mu_t}$), yielding

$$\frac{d}{dt} \langle\langle G, k_t \rangle\rangle = \langle\langle \hat{L}G, k_t \rangle\rangle, \quad k_t|_{t=0} = k_{\mu_0}, \tag{12}$$

where $\hat{L} := K^{-1}LK$ and $\langle\langle \cdot, \cdot \rangle\rangle$ is the usual pairing

$$\langle\langle G, k \rangle\rangle := \int_{\Gamma_0} d\lambda(\eta)\, G(\eta) k(\eta). \tag{13}$$

Of course, a stronger version of (12) is

$$\frac{d}{dt} k_t = \hat{L}^* k_t, \quad k_t|_{t=0} = k_{\mu_0}, \tag{14}$$

for $\hat{L}^*$ being the dual operator of $\hat{L}$ in the sense defined in (13).

[3] That is, $G \upharpoonright_{\Gamma_0 \setminus \left(\bigsqcup_{n=0}^{N} \Gamma_\Lambda^{(n)}\right)} \equiv 0$, $\Gamma_\Lambda^{(n)} := \{\eta \in \Gamma : \eta \subset \Lambda\} \cap \Gamma^{(n)}$, for some $N \in \mathbb{N}_0$ and for some bounded Borel set $\Lambda \subseteq \mathbb{R}^d$.

Representation (4) combined with (12), (13) gives us a way to widen the dynamical description towards the GF $B_t := B_{\mu_t}$ corresponding to μ_t [13, 21], provided these functionals exist. Informally,

$$\frac{\partial}{\partial t}B_t(\theta) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta)\frac{\partial}{\partial t}k_t(\eta) = \int_{\Gamma_0} d\lambda(\eta)\,(\hat{L}e_\lambda(\theta))(\eta)k_t(\eta). \quad (15)$$

In other words, given the operator $\tilde{L}$ defined at $B(\theta) := \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta)k(\eta)$, $k : \Gamma_0 \to [0,+\infty)$, by

$$(\tilde{L}B)(\theta) := \int_{\Gamma_0} d\lambda(\eta)\,(\hat{L}e_\lambda(\theta))(\eta)k(\eta), \quad (16)$$

heuristically (15) means that $B_t, t \geq 0$, is a solution to the Cauchy problem

$$\frac{\partial}{\partial t}B_t = \tilde{L}B_t, \qquad B_t|_{t=0} = B_{\mu_0}. \quad (17)$$

According to the considerations above, there is a close connection between the Markov evolution (10) and the Cauchy problems (12), (14), and (17). More precisely, given a solution $\mu_t, t \geq 0$, to (10), if additionally the correlation function k_{μ_t} corresponding to each state μ_t exists, then $k_t := k_{\mu_t}$ is a solution to (12). Similarly, the informal sequence of equalities (15) shows that if the GF B_{μ_t} exists for each time $t \geq 0$, then $B_t := B_{\mu_t}$ solves (17). Conversely, given a solution k_t to (12), or to (14), or a solution $B_t, t \geq 0$, to (17), for k_{μ_0} and B_{μ_0} being, respectively, the correlation function and the GF corresponding to the initial state μ_0 of the system, an additional analysis is needed in order to check that each k_t (resp., B_t) is indeed a correlation function (resp., a GF) corresponding to some measure μ_t. If so, then, by construction, $\mu_t, t \geq 0$, is a solution to (10) and $k_t = k_{\mu_t}$ (resp., $B_t = B_{\mu_t}$). For more details concerning the aforementioned analysis see e.g. [10] for the case of correlation functions, and [21, 25] for the GF case.

Remark 3. Although correlation functions appear in this work as a side remark, we note that the study of the properties of correlation functions of a dynamics is a classical problem in mathematical physics. In order to analyze the existence of solutions to (12), (14), and the properties of such solutions, some approaches have been proposed. One of them is based on semigroup techniques, which for birth-and-death dynamics has been accomplished in e.g. [7, 10, 12, 22, 23] and summarized in a recent article [11]. Another approach is based on the so-called Ovsyannikov technique and it has been successfully applied in the analysis of birth-and-death as well as hopping particle systems (on a finite time interval), see e.g. [2, 3, 6].

In most concrete applications, to find a solution to (17) on a Banach space seems to be often a difficult question. However, this problem may be simplified within the framework of scales of Banach spaces. We recall that a scale of Banach spaces is a one-parameter family of Banach spaces $\{\mathbb{B}_s : 0 < s \leq s_0\}$ such that $\mathbb{B}_{s''} \subseteq \mathbb{B}_{s'}$,

$\|\cdot\|_{s'} \leq \|\cdot\|_{s''}$ for any pair s', s'' such that $0 < s' < s'' \leq s_0$, where $\|\cdot\|_s$ denotes the norm in $\mathbb{B}_s$. As an example, it is clear from definition (8) that for each $\alpha_0 > 0$ the family $\{\mathscr{E}_\alpha : 0 < \alpha \leq \alpha_0\}$ is a scale of Banach spaces.

Within this framework, one has the following existence and uniqueness result (see e.g. [27]). For concreteness, in subsections below we will analyze two examples of applications.

Theorem 2. *On a scale of Banach spaces $\{\mathbb{B}_s : 0 < s \leq s_0\}$ consider the initial value problem*

$$\frac{du(t)}{dt} = Au(t), \quad u(0) = u_0 \in \mathbb{B}_{s_0} \tag{18}$$

where, for each $s \in (0, s_0)$ fixed and for each pair s', s'' such that $s \leq s' < s'' \leq s_0$, $A : \mathbb{B}_{s''} \to \mathbb{B}_{s'}$ is a linear mapping so that there is an $M > 0$ such that for all $u \in \mathbb{B}_{s''}$

$$\|Au\|_{s'} \leq \frac{M}{s'' - s'}\|u\|_{s''}.$$

Here M is independent of s', s'' and u, however it might depend continuously on s, s_0.

Then, for each $s \in (0, s_0)$, there is a constant $\delta > 0$ (which depends on M) such that there is a unique function $u : [0, \delta(s_0 - s)) \to \mathbb{B}_s$ which is continuously differentiable on $(0, \delta(s_0 - s))$ in $\mathbb{B}_s$, $Au \in \mathbb{B}_s$, and solves (18) in the time-interval $0 \leq t < \delta(s_0 - s)$.

2.1 The Glauber Dynamics

The Glauber dynamics is an example of a birth-and-death model where, in this special case, particles appear and disappear according to a death rate identically equal to 1 and to a birth rate depending on the interaction between particles. More precisely, let $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a pair potential, that is, a $\mathscr{B}(\mathbb{R}^d)$-measurable function such that $\phi(-x) = \phi(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$, which we will assume to be non-negative and integrable. Given a configuration $\gamma \in \Gamma$, the birth rate of a new particle at a site $x \in \mathbb{R}^d \setminus \gamma$ is given by $\exp(-E(x, \gamma))$, where $E(x, \gamma)$ is a relative energy of interaction between a particle located at x and the configuration γ defined by

$$E(x, \gamma) := \sum_{y \in \gamma} \phi(x - y) \in [0, +\infty]. \tag{19}$$

Informally, in terms of Markov generators, this means that the behavior of such an infinite particle system is described by

$$(L_G F)(\gamma) := \sum_{x\in\gamma} (F(\gamma \setminus \{x\}) - F(\gamma)) + z \int_{\mathbb{R}^d} dx\, e^{-E(x,\gamma)} (F(\gamma \cup \{x\}) - F(\gamma)), \tag{20}$$

where $z > 0$ is an activity parameter (for more details see e.g. [13, 23]). Thus, according to Sect. 2, the operator $\tilde{L}_G$ defined in (16) is given cf. [13] by

$$(\tilde{L}_G B)(\theta) = -\int_{\mathbb{R}^d} dx\, \theta(x) \Big(\delta B(\theta; x) - zB\Big(\theta e^{-\phi(x-\cdot)} + e^{-\phi(x-\cdot)} - 1\Big)\Big). \tag{21}$$

The Glauber dynamics is an example where semigroups theory can be apply to study the time evolution in terms of correlation functions, see e.g. [10, 12, 23]. However, within the context of GF, semigroup techniques seem do not work (see e.g. [17]). This is partially due to the fact that given the natural class of Banach spaces $\mathscr{E}_\alpha$, the operator $\tilde{L}_G$ maps elements of a Banach space $\mathscr{E}_\alpha$, $\alpha > 0$, on elements of larger Banach spaces $\mathscr{E}_{\alpha'}$, $0 < \alpha' < \alpha$ [14]:

$$\|\tilde{L}_G B\|_{\alpha'} \leq \frac{\alpha'}{\alpha - \alpha'} \left(1 + z\alpha e^{\frac{\|\phi\|_{L^1}}{\alpha}-1} \right) \|B\|_\alpha, \quad B \in \mathscr{E}_\alpha.$$

However, this estimate of norms and an application of Theorem 2 lead to the following existence and uniqueness result.

Proposition 2 ([14, Theorem 3.1]). *Given an* $\alpha_0 > 0$*, let* $B_0 \in \mathscr{E}_{\alpha_0}$*. For each* $\alpha \in (0, \alpha_0)$ *there is a* $T > 0$ *(which depends on* α, α_0*) such that there is a unique solution* B_t*,* $t \in [0, T)$*, to the initial value problem* $\frac{\partial B_t}{\partial t} = \tilde{L}_G B_t$*, (21),* $B_{t|t=0} = B_0$ *in the space* $\mathscr{E}_\alpha$*.*

Remark 4. 1. Concerning the initial conditions considered in Proposition 2, observe that, in particular, B_0 can be an entire GF B_{μ_0} on L^1 such that, for some constants $\alpha_0, C > 0$, $|B_{\mu_0}(\theta)| \leq C \exp(\frac{\|\theta\|_{L^1}}{\alpha_0})$ for all $\theta \in L^1$. As we have mentioned before, in such a situation an additional analysis is required in order to guarantee that for each time $t \in [0, T)$ the solution B_t given by Proposition 2 is a GF. If so, then clearly each B_t is the GF corresponding to the state of the particle system at the time t. For more details see [14, Remark 3.6].

2. If the initial condition B_0 is an entire GF on L^1 such that the corresponding correlation function k_0 (given by Proposition 1) fulfills the Ruelle bound $k_0(\eta) \leq z^{|\eta|}$, $\eta \in \Gamma_0$, where z is the activity parameter appearing in definition (20), then the local solution given by Proposition 2 might be extend to a global one, that is, to a solution defined on the whole time interval $[0, +\infty)$. For more details and the proof see [14, Corollary 3.7].

2.2 The Kawasaki Dynamics

The Kawasaki dynamics is an example of a hopping particle model where, in this case, particles randomly hop over the space $\mathbb{R}^d$ according to a rate depending on the interaction between particles. More precisely, let $a : \mathbb{R}^d \to [0, +\infty)$ be an even and integrable function and let $\phi : \mathbb{R}^d \to [0, +\infty]$ be a pair potential, which we will assume to be integrable. A particle located at a site x in a given configuration $\gamma \in \Gamma$ hops to a site y according to a rate given by $a(x-y)\exp(-E(y,\gamma))$, where $E(y,\gamma)$ is a relative energy of interaction between the site y and the configuration γ defined similarly to (19). Informally, the behavior of such an infinite particle system is described by

$$(L_K F)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, a(x-y) e^{-E(y,\gamma)} \left(F(\gamma \setminus \{x\} \cup \{y\}) - F(\gamma) \right), \qquad (22)$$

meaning in terms of the operator $\tilde{L}_K$ defined in (16) that

$$(\tilde{L}_K B)(\theta) = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y) e^{-\phi(x-y)} (\theta(y) - \theta(x)) \delta B(\theta e^{-\phi(y-\cdot)} + e^{-\phi(y-\cdot)} - 1; x), \qquad (23)$$

cf. [13]. In this case the following estimate of norms holds

$$\|\tilde{L}_K B\|_{\alpha'} \le 2e^{\frac{\|\phi\|_{L^1}}{\alpha}} \frac{\alpha'}{\alpha-\alpha'} \|a\|_{L^1} \|B\|_\alpha, \quad B \in \mathscr{E}_\alpha, \alpha' < \alpha,$$

which, by an application of Theorem 2, yields the following statement.

Proposition 3 ([15, Theorem 3.1]). *Given an* $\alpha_0 > 0$, *let* $B_0 \in \mathscr{E}_{\alpha_0}$. *For each* $\alpha \in (0, \alpha_0)$ *there is a* $T > 0$ *(which depends on* α, α_0*) such that there is a unique solution* B_t, $t \in [0, T)$, *to the initial value problem* $\frac{\partial}{\partial t} B_t = \tilde{L}_K B_t$, (23), $B_{t|t=0} = B_0$ *in the space* $\mathscr{E}_\alpha$.

3 Vlasov Scaling

We proceed to investigate the Vlasov-type scaling proposed in [8] for generic continuous particle systems and accomplished in [9] and [2] for the Glauber and the Kawasaki dynamics, respectively, now in terms of GF. As explained in these references, we start with a rescaling of an initial correlation function k_0, denoted, respectively, by $k_{G,0}^{(\varepsilon)}$, $k_{K,0}^{(\varepsilon)}$, $\varepsilon > 0$, which has a singularity with respect to ε of the type $k_{G,0}^{(\varepsilon)}(\eta), k_{K,0}^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|} r_0(\eta)$, $\eta \in \Gamma_0$, being r_0 a function independent of ε. The aim is to construct a scaling for the operator L_G (resp., L_K) defined in (20) (resp., (22)), $L_{G,\varepsilon}$ (resp., $L_{K,\varepsilon}$), $\varepsilon > 0$, in such a way that the following two conditions are fulfilled. The first one is that under the scaling $L \mapsto L_{\#,\varepsilon}$, $\# = G, K$, the solution $k_{\#,t}^{(\varepsilon)}$, $t \geq 0$, to

$$\frac{\partial}{\partial t} k_{\#,t}^{(\varepsilon)} = \hat{L}_{\#,\varepsilon}^* k_{\#,t}^{(\varepsilon)}, \quad k_{\#,t}^{(\varepsilon)}{}_{|t=0} = k_{\#,0}^{(\varepsilon)}$$

preserves the order of the singularity with respect to ε, that is, $k_{\#,t}^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|} r_{\#,t}(\eta)$, $\eta \in \Gamma_0$. The second condition is that the dynamics $r_0 \mapsto r_{\#,t}$ preserves the Lebesgue-Poisson exponents, that is, if r_0 is of the form $r_0 = e_\lambda(\rho_0)$, then each $r_{\#,t}$, $t > 0$, is of the same type, i.e., $r_{\#,t} = e_\lambda(\rho_{\#,t})$, where $\rho_{\#,t}$ is a solution to a non-linear equation (called a Vlasov-type equation). As shown in [8, Example 8], [9], in the case of the Glauber dynamics this equation is given by

$$\frac{\partial}{\partial t} \rho_{G,t}(x) = -\rho_{G,t}(x) + z e^{-(\rho_{G,t} * \phi)(x)}, \quad x \in \mathbb{R}^d, \tag{24}$$

where $*$ denotes the usual convolution of functions. Existence of classical solutions $0 \leq \rho_{G,t} \in L^\infty$ to (24) has been discussed in [6,9]. For the Kawasaki dynamics, the corresponding Vlasov-type equation is given by

$$\frac{\partial}{\partial t} \rho_{K,t}(x) = (\rho_{K,t} * a)(x) e^{-(\rho_{K,t} * \phi)(x)} - \rho_{K,t}(x)(a * e^{-(\rho_{K,t} * \phi)})(x), \quad x \in \mathbb{R}^d, \tag{25}$$

cf. [8, Example 12], [2]. In this case, existence of classical solutions $0 \leq \rho_{K,t} \in L^\infty$ to (25) has been discussed in [2].

Therefore, it is natural to consider the same scalings, but in terms of GF.

3.1 The Glauber Dynamics

The previous scheme was accomplished in [9] through the scale transformations $z \mapsto \varepsilon^{-1} z$ and $\phi \mapsto \varepsilon\phi$ of the operator L_G, that is,

$$(L_{G,\varepsilon} F)(\gamma) := \sum_{x \in \gamma} (F(\gamma \setminus \{x\}) - F(\gamma)) + \frac{z}{\varepsilon} \int_{\mathbb{R}^d} dx \, e^{-\varepsilon E(x,\gamma)} (F(\gamma \cup \{x\}) - F(\gamma)).$$

To proceed towards GF, let us consider $k_{G,t}^{(\varepsilon)}$ defined as before and $k_{G,t,\mathrm{ren}}^{(\varepsilon)}(\eta) := \varepsilon^{|\eta|} k_{G,t}^{(\varepsilon)}(\eta)$. In terms of GF, these yield

$$B^{(\varepsilon)}_{G,t}(\theta) := \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta) k^{(\varepsilon)}_{G,t}(\eta),$$

and

$$B^{(\varepsilon)}_{G,t,\mathrm{ren}}(\theta) := \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta) k^{(\varepsilon)}_{G,t,\mathrm{ren}}(\eta) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\varepsilon\theta,\eta) k^{(\varepsilon)}_{G,t}(\eta) = B^{(\varepsilon)}_{G,t}(\varepsilon\theta),$$

leading, as in (16) and (17), to the initial value problem

$$\frac{\partial}{\partial t} B^{(\varepsilon)}_{G,t,\mathrm{ren}} = \tilde{L}_{G,\varepsilon,\mathrm{ren}} B^{(\varepsilon)}_{G,t,\mathrm{ren}}, \quad B^{(\varepsilon)}_{G,t,\mathrm{ren}|t=0} = B^{(\varepsilon)}_{G,0,\mathrm{ren}}, \tag{26}$$

where, for all $\theta \in L^1$,

$$(\tilde{L}_{G,\varepsilon,\mathrm{ren}} B)(\theta) = -\int_{\mathbb{R}^d} dx\, \theta(x) \left(\delta B(\theta,x) - zB\left(\theta e^{-\varepsilon\phi(x-\cdot)} + \frac{e^{-\varepsilon\phi(x-\cdot)} - 1}{\varepsilon} \right) \right),$$

cf. [14]. Concerning this operator, it has been also shown in [14, Proposition 4.2] that if $B \in \mathscr{E}_\alpha$ for some $\alpha > 0$, then, for all $\theta \in L^1$, $(\tilde{L}_{G,\varepsilon,\mathrm{ren}} B)(\theta)$ converges as ε tends zero to

$$(\tilde{L}_{G,V} B)(\theta) := -\int_{\mathbb{R}^d} dx\, \theta(x) \left(\delta B(\theta;x) - zB\left(\theta - \phi(x-\cdot)\right)\right).$$

Furthermore, fixed $0 < \alpha < \alpha_0$, if $B \in \mathscr{E}_{\alpha''}$ for some $\alpha'' \in (\alpha, \alpha_0]$, then $\left\{\tilde{L}_{G,\varepsilon,\mathrm{ren}} B, \tilde{L}_{G,V} B\right\} \subset \mathscr{E}_{\alpha'}$ for all $\alpha \le \alpha' < \alpha''$, and one has

$$\|\tilde{L}_\# B\|_{\alpha'} \le \frac{\alpha_0}{\alpha'' - \alpha'} \left(1 + z\alpha_0 e^{\frac{\|\phi\|_{L^1}}{\alpha} - 1} \right) \|B\|_{\alpha''},$$

where $\tilde{L}_\# = \tilde{L}_{G,\varepsilon,\mathrm{ren}}$ or $\tilde{L}_\# = \tilde{L}_{G,V}$. That is, the estimate of norms for $\tilde{L}_{G,\varepsilon,\mathrm{ren}}$, $\varepsilon > 0$, and the limiting mapping $\tilde{L}_{G,V}$ are similar. Therefore, given any $B_{G,0,V}, B^{(\varepsilon)}_{G,0,\mathrm{ren}} \in \mathscr{E}_{\alpha_0}$, $\varepsilon > 0$, it follows from Theorem 2 that for each $\alpha \in (0, \alpha_0)$, there is a constant $\delta > 0$ such that there is a unique solution $B^{(\varepsilon)}_{G,t,\mathrm{ren}} : [0, \delta(\alpha_0 - \alpha)) \to \mathscr{E}_\alpha$, $\varepsilon > 0$, to each initial value problem (26) and a unique solution $B_{G,t,V} : [0, \delta(\alpha_0 - \alpha)) \to \mathscr{E}_\alpha$ to the initial problem

$$\frac{\partial}{\partial t} B_{G,t,V} = \tilde{L}_{G,V} B_{G,t,V}, \quad B_{G,t,V|t=0} = B_{G,0,V}. \tag{27}$$

In other words, independent of the initial value problem under consideration, the solutions obtained are defined on the same time-interval and with values in the same Banach space. Therefore, it is natural to analyze under which conditions the solutions to (26) converge to the solution to (27). This follows from the following general result [14]:

Theorem 3. *On a scale of Banach spaces* $\{\mathbb{B}_s : 0 < s \le s_0\}$ *consider a family of initial value problems*

$$\frac{du_\varepsilon(t)}{dt} = A_\varepsilon u_\varepsilon(t), \quad u_\varepsilon(0) = u_\varepsilon \in \mathbb{B}_{s_0}, \quad \varepsilon \ge 0, \tag{28}$$

where, for each $s \in (0, s_0)$ *fixed and for each pair* s', s'' *such that* $s \le s' < s'' \le s_0$, $A_\varepsilon : \mathbb{B}_{s''} \to \mathbb{B}_{s'}$ *is a linear mapping so that there is an* $M > 0$ *such that for all* $u \in \mathbb{B}_{s''}$

$$\|A_\varepsilon u\|_{s'} \le \frac{M}{s'' - s'} \|u\|_{s''}.$$

Here M *is independent of* ε, s', s'' *and* u*, however it might depend continuously on* s, s_0*. Assume that there is a* $p \in \mathbb{N}$ *and for each* $\varepsilon > 0$ *there is an* $N_\varepsilon > 0$ *such that for each pair* s', s'', $s \le s' < s'' \le s_0$*, and all* $u \in \mathbb{B}_{s''}$

$$\|A_\varepsilon u - A_0 u\|_{s'} \le \sum_{k=1}^{p} \frac{N_\varepsilon}{(s'' - s')^k} \|u\|_{s''}.$$

In addition, assume that $\lim_{\varepsilon\to 0} N_\varepsilon = 0$ *and* $\lim_{\varepsilon\to 0} \|u_\varepsilon(0) - u_0(0)\|_{s_0} = 0$.

Then, for each $s \in (0, s_0)$*, there is a constant* $\delta > 0$ *(which depends on* M*) such that there is a unique solution* $u_\varepsilon : [0, \delta(s_0 - s)) \to \mathbb{B}_s$, $\varepsilon \ge 0$*, to each initial value problem* (28) *and for all* $t \in [0, \delta(s_0 - s))$ *we have*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon(t) - u_0(t)\|_s = 0.$$

We observe that if $0 \le \phi \in L^1 \cap L^\infty$, then, given $\alpha_0 > \alpha > 0$, for all $B \in \mathscr{E}_{\alpha''}$, $\alpha'' \in (\alpha, \alpha_0]$, one finds [14, Proposition 4.4]

$$\|\tilde{L}_{G,\varepsilon,\mathrm{ren}} B - \tilde{L}_{G,V} B\|_{\alpha'} \le \varepsilon z \|\phi\|_{L^\infty} \|B\|_{\alpha''} e^{\frac{\|\phi\|_{L^1}}{\alpha}} \left(\frac{\|\phi\|_{L^1}\alpha_0}{\alpha'' - \alpha'} + \frac{4\alpha_0^3}{(\alpha'' - \alpha')^2 e} \right)$$

for all α' such that $\alpha \le \alpha' < \alpha''$ and all $\varepsilon > 0$. Thus, given the local solutions $B^{(\varepsilon)}_{G,t,\mathrm{ren}}, B_{G,t,V}$, $t \in [0, \delta(\alpha_0 - \alpha))$, in $\mathscr{E}_\alpha$ to the initial value problems (26) and (27), respectively, with $B^{(\varepsilon)}_{G,0,\mathrm{ren}}, B_{G,0,V} \in \mathscr{E}_{\alpha_0}$, if $\lim_{\varepsilon\to 0} \|B^{(\varepsilon)}_{G,0,\mathrm{ren}} - B_{G,0,V}\|_{\alpha_0} = 0$, then, by an application of Theorem 3, $\lim_{\varepsilon\to 0} \|B^{(\varepsilon)}_{G,t,\mathrm{ren}} - B_{G,t,V}\|_\alpha = 0$, for each $t \in [0, \delta(\alpha_0 - \alpha))$. Moreover [14, Theorem 4.5], if $B_{G,0,V}(\theta) = \exp\left(\int_{\mathbb{R}^d} dx\, \rho_0(x)\theta(x)\right)$, $\theta \in L^1$, for some function $0 \le \rho_0 \in L^\infty$ such that $\|\rho_0\|_{L^\infty} \le \frac{1}{\alpha_0}$, and if $\max\{\frac{1}{\alpha_0}, z\} < \frac{1}{\alpha}$ then, for each $t \in [0, \delta(\alpha_0 - \alpha))$,

$$B_{G,t,V}(\theta) = \exp\left(\int_{\mathbb{R}^d} dx\, \rho_t(x)\theta(x)\right), \quad \theta \in L^1,$$

where $0 \leq \rho_t \in L^\infty$ is a classical solution to Eq. (24) such that, for each $t \in [0, \delta(\alpha_0 - \alpha))$, $\|\rho_t\|_{L^\infty} \leq \frac{1}{\alpha}$. For more results and proofs see [14].

3.2 The Kawasaki Dynamics

In this example one shall consider the scale transformation $\phi \mapsto \varepsilon\phi$ of the operator L_K cf. [2], that is,

$$(L_{K,\varepsilon}F)(\gamma) := \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, a(x-y)e^{-\varepsilon E(y,\gamma)} \left(F(\gamma \setminus \{x\} \cup \{y\}) - F(\gamma)\right).$$

To proceed towards GF we consider $k_{K,t}^{(\varepsilon)}$, $k_{K,t,\mathrm{ren}}^{(\varepsilon)}$ and $B_{K,t}^{(\varepsilon)}$ defined as before, which lead to the Cauchy problem

$$\frac{\partial}{\partial t} B_{K,t,\mathrm{ren}}^{(\varepsilon)} = \tilde{L}_{K,\varepsilon,\mathrm{ren}} B_{K,t,\mathrm{ren}}^{(\varepsilon)}, \quad B_{K,t,\mathrm{ren}}^{(\varepsilon)}{}_{|t=0} = B_{K,0,\mathrm{ren}}^{(\varepsilon)}, \tag{29}$$

with

$$\begin{aligned}(\tilde{L}_{K,\varepsilon,\mathrm{ren}}B)(\theta) = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y)e^{-\varepsilon\phi(x-y)}(\theta(y) - \theta(x)) \\ \times \delta B\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; x\right),\end{aligned}$$

for all $\varepsilon > 0$ and all $\theta \in L^1$. Similar arguments show [15] that given a $B \in \mathcal{E}_\alpha$ for some $\alpha > 0$, then, for all $\theta \in L^1$, $(\tilde{L}_{K,\varepsilon,\mathrm{ren}}B)(\theta)$ converges as ε tends to zero to

$$(\tilde{L}_{K,V}B)(\theta) := \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y)(\theta(y) - \theta(x))\delta B(\theta - \phi(y - \cdot); x).$$

In addition, given $0 < \alpha < \alpha_0$, if $B \in \mathcal{E}_{\alpha''}$ for some $\alpha'' \in (\alpha, \alpha_0]$, then $\left\{\tilde{L}_{K,\varepsilon,\mathrm{ren}}B, \tilde{L}_{K,V}B\right\} \subset \mathcal{E}_{\alpha'}$ for all $\alpha \leq \alpha' < \alpha''$, and the following inequality of norms holds

$$\|\tilde{L}_\# B\|_{\alpha'} \leq 2\|a\|_{L^1} \frac{\alpha_0}{(\alpha'' - \alpha')} e^{\frac{\|\phi\|_{L^1}}{\alpha}} \|B\|_{\alpha''},$$

where $\tilde{L}_\# = \tilde{L}_{K,\varepsilon,\mathrm{ren}}$ or $\tilde{L}_\# = \tilde{L}_{K,V}$. Now, let us assume that $0 \leq \phi \in L^1 \cap L^\infty$ and let $\alpha_0 > \alpha > 0$ be given. Then, for all $B \in \mathcal{E}_{\alpha''}$, $\alpha'' \in (\alpha, \alpha_0]$, the following estimate holds [15, Proposition 4.3]

$$\begin{aligned}&\|\tilde{L}_{K,\varepsilon,\mathrm{ren}}B-\tilde{L}_{K,V}B\|_{\alpha'}\\&\leq 2\varepsilon\|a\|_{L^1}\|\phi\|_{L^\infty}\frac{e\alpha_0}{\alpha}\|B\|_{\alpha''}e^{\frac{\|\phi\|_{L^1}}{\alpha}}\left(\left(2e\|\phi\|_{L^1}+\frac{\alpha_0}{e}\right)\frac{1}{\alpha''-\alpha'}+\frac{8\alpha_0^2}{(\alpha''-\alpha')^2}\right)\end{aligned}$$

for all α' such that $\alpha \leq \alpha' < \alpha''$ and all $\varepsilon > 0$, meaning that one may apply Theorem 3.

Proposition 4 ([15, Theorem 4.4]). *Given an $0 < \alpha < \alpha_0$, let $B^{(\varepsilon)}_{K,t,\mathrm{ren}}, B_{K,t,V}$, $t \in [0,T)$, be the local solutions in $\mathscr{E}_\alpha$ to the initial value problems (29),*

$$\frac{\partial}{\partial t}B_{K,t,V}=\tilde{L}_{K,V}B_{K,t,V},\quad B_{K,t,V\,|t=0}=B_{K,0,V},$$

with $B^{(\varepsilon)}_{K,0,\mathrm{ren}}, B_{K,0,V} \in \mathscr{E}_{\alpha_0}$. If $0 \leq \phi \in L^1 \cap L^\infty$ and $\lim_{\varepsilon\to 0}\|B^{(\varepsilon)}_{K,0,\mathrm{ren}} - B_{K,0,V}\|_{\alpha_0} = 0$, then, for each $t \in [0,T)$, $\lim_{\varepsilon\to 0}\|B^{(\varepsilon)}_{K,t,\mathrm{ren}} - B_{K,t,V}\|_\alpha = 0$. Moreover, if $B_{K,0,V}(\theta) = \exp\left(\int_{\mathbb{R}^d} dx\, \rho_0(x)\theta(x)\right)$, $\theta \in L^1$, for some function $0 \leq \rho_0 \in L^\infty$ such that $\|\rho_0\|_{L^\infty} \leq \frac{1}{\alpha_0}$, then for each $t \in [0,T)$, $B_{K,t,V}(\theta) = \exp\left(\int_{\mathbb{R}^d} dx\, \rho_t(x)\theta(x)\right)$, $\theta \in L^1$, where $0 \leq \rho_t \in L^\infty$ is a classical solution to Eq. (25).

References

1. Barroso, J.A.: Introduction to Holomorphy. Volume 106 of Mathematics Studies. North-Holland, Amsterdam (1985)
2. Berns, C., Kondratiev, Yu.G., Kozitsky, Yu., Kutoviy, O.: Kawasaki dynamics in continuum: micro- and mesoscopic descriptions. J. Dyn. Diff. Equat. **25**, 1027–1056 (2013)
3. Berns, C., Kondratiev, Yu.G., Kutoviy, O.: Construction of a state evolution for Kawasaki dynamics in continuum. Anal. Math. Phys. **3**(2), 97–117 (2013)
4. Bogoliubov, N.N.: Problems of a Dynamical Theory in Statistical Physics. Gostekhisdat, Moscow (1946) (in Russian). English translation in de Boer, J., Uhlenbeck, G.E. (eds.) Studies in Statistical Mechanics, vol. 1, pp. 1–118. North-Holland, Amsterdam (1962)
5. Dineen, S.: Complex Analysis in Locally Convex Spaces. Volume 57 of Mathematics Studies. North-Holland, Amsterdam (1981)
6. Finkelshtein, D.L., Kondratiev, Yu.G., Kozitsky, Yu.: Glauber dynamics in continuum: a constructive approach to evolution of states. Discret. Contin. Dyn. Syst. **33**(4), 1431–1450 (2013)
7. Finkelshtein, D.L., Kondratiev, Yu.G., Kutoviy, O.: Individual based model with competition in spatial ecology. SIAM J. Math. Anal. **41**, 297–317 (2009)
8. Finkelshtein, D.L., Kondratiev, Yu.G., Kutoviy, O.: Vlasov scaling for stochastic dynamics of continuous systems. J. Stat. Phys. **141**, 158–178 (2010)
9. Finkelshtein, D.L., Kondratiev, Yu.G., Kutoviy, O.: Vlasov scaling for the Glauber dynamics in continuum. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **14**(4), 537–569 (2011)
10. Finkelshtein, D.L., Kondratiev, Yu.G., Kutoviy, O.: Correlation functions evolution for the Glauber dynamics in continuum. Semigroup Forum **85**, 289–306 (2012)
11. Finkelshtein, D.L., Kondratiev, Yu.G., Kutoviy, O.: Semigroup approach to non-equilibrium birth-and-death stochastic dynamics in continuum. J. Funct. Anal. **262**, 1274–1308 (2012)

12. Finkelshtein, D., Kondratiev, Yu., Kutoviy, O., Zhizhina, E.: An approximative approach to construction of the Glauber dynamics in continuum. Math. Nachr. **285**, 223–235 (2012)
13. Finkelshtein, D.L., Kondratiev, Yu.G., Oliveira, M.J.: Markov evolutions and hierarchical equations in the continuum. I: one-component systems. J. Evol. Equ. **9**(2), 197–233 (2009)
14. Finkelshtein, D.L., Kondratiev, Yu.G., Oliveira, M.J.: Glauber dynamics in the continuum via generating functionals evolution. Complex Anal. Oper. Theory **6**(4), 923–945 (2012)
15. Finkelshtein, D.L., Kondratiev, Yu.G., Oliveira, M.J.: Kawasaki dynamics in the continuum via generating functionals evolution. Methods Funct. Anal. Topol. **18**(1), 55–67 (2012)
16. Garcia, N.L., Kurtz, T.G.: Spatial birth and death processes as solutions of stochastic equations. ALEA Lat. Am. J. Probab. Math. Stat. **1**, 281–303 (2006)
17. Kondratiev, Yu., Kozitsky, Yu., Shoikhet, D.: Dynamical systems on sets of holomorphic functions. In: Complex Analysis and Dynamical Systems IV, Part 1. Contemporary Mathematics, vol. 553. American Mathematical Society, Providence (2011)
18. Kondratiev, Yu.G., Kuna, T.: Harmonic analysis on configuration space I. General theory. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **5**(2), 201–233 (2002)
19. Kondratiev, Yu.G., Kuna, T., Oliveira, M.J.: Analytic aspects of Poissonian white noise analysis. Methods Funct. Anal. Topol. **8**(4), 15–48 (2002)
20. Kondratiev, Yu.G., Kuna, T., Oliveira, M.J.: On the relations between Poissonian white noise analysis and harmonic analysis on configuration spaces. J. Funct. Anal. **213**(1), 1–30 (2004)
21. Kondratiev, Yu.G., Kuna, T., Oliveira, M.J.: Holomorphic Bogoliubov functionals for interacting particle systems in continuum. J. Funct. Anal. **238**(2), 375–404 (2006)
22. Kondratiev, Yu., Kutoviy, O., Minlos, R.: On non-equilibrium stochastic dynamics for interacting particle systems in continuum. J. Funct. Anal. **255**, 200–227 (2008)
23. Kondratiev, Yu., Kutoviy, O., Zhizhina, E.: Nonequilibrium Glauber-type dynamics in continuum. J. Math. Phys. **47**(11), 113501 (2006)
24. Kondratiev, Y., Skorokhod, A.: On contact processes in continuum. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **9**(2), 187–198 (2006)
25. Kuna, T.: Studies in configuration space analysis and applications. PhD thesis, Bonner Mathematische Schriften Nr. 324, University of Bonn (1999)
26. Lenard, A.: States of classical statistical mechanical systems of infinitely many particles II. Arch. Ration. Mech. Anal. **59**, 241–256 (1975)
27. Treves, F.: Ovcyannikov Theorem and Hyperdifferential Operators. Volume 46 of Notas de Matemática. IMPA, Rio de Janeiro (1968)