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# From Particle Systems to Partial Differential Equations

## Particle Systems and PDEs, Braga, Portugal, December 2012

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# Preface

This volume presents the proceedings of the international conference "Particle Systems and Partial Differential Equations I," which took place at the Centre of Mathematics of the University of Minho, Braga, Portugal, from December 5–7, 2012.

The purpose of the conference was to bring together researchers from different areas of mathematics, namely, probability, partial differential equations, and kinetics theory. All of the participants had the opportunity to present their recently obtained results. The goal of the meeting was twofold:

1. To present to a varied public the subject of interacting particle systems, its motivation from the viewpoint of physics, and its relation with partial differential equations or kinetics theory
2. To stimulate discussions and possibly new collaborations among researchers with different backgrounds

The book contains some lecture notes written by François Golse on the derivation of hydrodynamic equations (compressible and incompressible Euler and Navier-Stokes) from the Boltzmann equation and several short papers written by some of the participants in the conference. Among the topics covered in these papers are hydrodynamic limits, fluctuations, phase transitions, motions of shocks and antishocks in exclusion processes, large number asymptotics for systems with self-consistent coupling, quasivariational inequalities, unique continuation properties for PDEs.

This volume will be valuable to probabilists, analysts, and also to mathematicians in general who are interested in statistical physics, stochastic processes, partial differential equations, and kinetics theory. We hope it would also prove useful to physicists.

The editors would like to take this opportunity to express their thanks to all the authors of this volume for their contributions. We would also like to thank FCT for support through the project PTDC/MAT/109844/2009, to CMAT for support by "FEDER" through the "Programa Operacional Factores de Competitividade COMPETE" and by FCT through the project PEst-C/MAT/UI0013/2011. We acknowledge the support of the French Ministry of Education through the grant ANR-10-BLAN 0108 (SHEPI). We are grateful to Égide and FCT for the research project FCT/1560/25/1/2012/S. We also thank all the referees for their painstaking work, which has contributed much to the excellence of this volume.

We really hope you enjoy reading the book!

Nice, France — Cédric Bernardin
Braga, Portugal — Patricia Gonçalves

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# Part I
# Mini-course

# Fluid Dynamic Limits of the Kinetic Theory of Gases

**François Golse**

## 1 Introduction

The purpose of these lecture notes is to introduce the reader to a series of recent mathematical results on the fluid dynamic limits of the Boltzmann equation.

The idea of looking for rigorous derivations of the partial differential equations of fluid mechanics from the kinetic theory of gases goes back to D. Hilbert. In his 6th problem presented in his plenary address at the 1900 International Congress of Mathematicians in Paris [60], he gave this as an example of "axiomatization" of physics. In Hilbert's own words

> [. . .] Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes [. . .] which lead from the atomistic view to the laws of motion of continua.

Hilbert himself studied this problem; his contributions include an important theorem (Theorem 16 below) on the linearization at uniform equilibrium states of the Boltzmann collision integral, together with a systematic asymptotic expansion method still widely used more than 100 years after his article [59] appeared (see Sect. 2.2.1).

Of course, after Hilbert's 1900 address [60], physics evolved in such a way that, while the existence of atoms was no longer questioned as in the days of L. Boltzmann and J.C. Maxwell, the classical kinetic theory of gases could no longer be considered as a good example of an "axiom of physics".

In fact, the Boltzmann equation of the kinetic theory of gases can be rigorously derived as an asymptotic limit of Newton's second law of motion written for each molecule in a gas [65]. Certainly Newton's laws of motion can be regarded as an

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axiom of classical mechanics. However, the idea that the Boltzmann equation could be viewed as a consequence of Newton's laws of motion appeared for the first time in a remarkable paper by H. Grad [54], almost half a century after Hilbert formulated his problems.

But while Hilbert's original question lost some of its interest from the point of view of theoretical physics, it has gained a lot of importance with the various applications of kinetic modeling in modern technology (such as rarefied gas dynamics in the context of space flight, plasma physics, neutron transport in fissile material, semiconductor physics …). Readers interested in applications of rarefied gas dynamics will find a lot of information in [94].

These lectures are focused on fluid dynamic limits of the kinetic theory of gases that can be formulated in terms of global solutions, and for any initial data within a finite distance to some uniform equilibrium state, measured in terms of relative entropy.

The first lecture describes how the most important partial differential equations of fluid dynamic (such as the Euler, Stokes or Navier-Stokes equations) can be derived as scaling limits of the Boltzmann equation. While this first lecture will review the basic mathematical properties of the Boltzmann equation, it leaves aside all the technicalities involved in either the proof of existence of global solutions of the Boltzmann equation, or in the proof of the fluid dynamic limits. This first lecture is concluded with an overview of some of the main mathematical tools and methods used in the proof of these limits.

Lecture 2 gives a rather detailed account of the proof of the incompressible Euler limit of (a model of) the Boltzmann equation, following [85]. Lecture 3 provides a much less detailed account of the derivation of the incompressible Navier-Stokes equation from the Boltzmann equation. This last lecture follows [50] rather closely. Since the Navier-Stokes limit involves a much heavier technical apparatus than the Euler limit, the presentation of the proof in Lecture 3 will be deliberately impressionistic. However, these lecture notes will give precise references to the main results in [50], and can therefore be used as a reader's guide for this last paper. Lectures 2 and 3 make a connection between three different notions of weak solutions of either the Boltzmann, or the Euler, or the Navier-Stokes equations: the Leray solutions of the Navier-Stokes equation, the DiPerna-Lions renormalized solutions of the Boltzmann equation, and the more recent notion of "dissipative solutions" of the Euler equation proposed by P.-L. Lions.

There are several other introductions to the material contained in these notes, including C. Villani's report at the Bourbaki seminar [97], which is less focused on the Euler and Navier-Stokes limits, and gives the main ideas used in the proofs of these limits with less many details as in the present notes. The lecture notes by C.D. Levermore and the author [46] leave aside the material presented in Lecture 2 (the incompressible Euler limit), and give a more detailed account of the material presented in Lecture 1. The various sets of lecture notes or monographs by L. Saint-Raymond and the author [39, 49, 88] are much more detailed and give a more comprehensive picture of the Boltzmann equation and its various fluid dynamic limits.

# 2 Lecture 1: Formal Derivations

This first lecture is a slightly expanded version of the author's Harold Grad Lecture [42], with an emphasis on mathematical tools and methods used in the theory of the Boltzmann equation and of its fluid dynamic limits.

For the sake of simplicity, the exposition is limited to the case of a (monatomic) hard sphere gas. More general collision processes, involving radial, binary intermolecular potentials satisfying Grad's angular cutoff assumption [56] can also be considered. The interested reader is referred to the original articles for a more complete account of these results.

## *2.1 The Boltzmann Equation*

### 2.1.1 Formal Structure

In the kinetic theory of gases (proposed by J.C. Maxwell and L. Boltzmann), the state at time $t$ of a monatomic gas is defined by its *distribution function* $F \equiv F(t,x,v) \geq 0$, which is the density (with respect to the Lebesgue measure $dx\,dv$) of gas molecules with velocity $v \in \mathbf{R}^3$ to be found at the position $x \in \mathbf{R}^3$ at time $t$. The evolution of the distribution function is governed by the Boltzmann equation.

If the effect of external forces (such as gravity) is negligible, the Boltzmann equation for the distribution function $F$ takes the form

$$\partial_t F + v \cdot \nabla_x F = \mathscr{C}(F)\,,$$

where the right-hand side is known as "the collision integral".

Assuming that all gas molecules are identical and that collisions between gas molecules are elastic, hard sphere binary collisions, the collision integral is defined on functions of the velocity variable $v$ that are rapidly decaying at infinity by the formula

$$\mathscr{C}(f)(v) := \tfrac{d^2}{2} \iint_{\mathbf{R}^3 \times \mathbf{S}^2} (f(v')f(v'_*) - f(v)f(v_*))|(v-v_*)\cdot\omega|dv_*d\omega\,,$$

where $d/2$ is the molecular radius, and where

$$\begin{cases} v' \equiv v'(v,v_*,\omega) := v\ - (v-v_*)\cdot\omega\omega\,, \\ v'_* \equiv v'_*(v,v_*,\omega) := v_* + (v-v_*)\cdot\omega\omega\,. \end{cases} \tag{1}$$

(See Fig. 1 for the geometric interpretation of the unit vector $\omega$.) The notation $d\omega$ in the collision integral designates the uniform measure on the unit sphere $\mathbf{S}^2$.

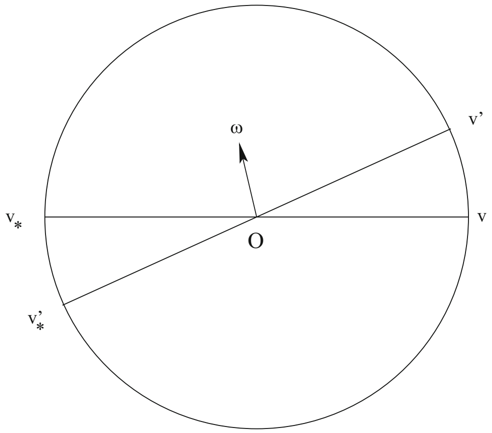


**Fig. 1** The velocities $v, v_*, v', v'_*$ in the center of mass reference frame, and the geometrical meaning of the unit vector $\omega$. The relative velocities $v - v_*$ and $v' - v'_*$ are exchanged by the reflection with respect to the plane orthogonal to $\omega$

This collision integral is extended to distribution functions (depending also on the time and position variables $t$ and $x$) by the formula

$$\mathscr{C}(F)(t,x,v) := \mathscr{C}(F(t,x,\cdot))(v)\,.$$

The physical meaning of this definition is that, except for the molecular radius appearing in front of the collision integral $\mathscr{C}(F)$, gas molecules are considered as point particles in kinetic theory, so that collisions are purely local and instantaneous. Besides, the collision integral is quadratic in the distribution function, because the Boltzmann equation is valid in a scaling regime where collisions other than binary can be neglected.

With the definition above of $v' \equiv v'(v, v_*, \omega)$ and $v'_* \equiv v'_*(v, v_*, \omega)$, for each $v, v_* \in \mathbf{R}^3$ and $\omega \in \mathbf{S}^2$, one has the following conservation laws, whose physical interpretation is obvious (since all the gas molecules are identical and therefore have the same mass):

$$\begin{aligned} v' + v'_* &= v + v_*\,, && \text{conservation of momentum,} \\ |v'|^2 + |v'_*|^2 &= |v|^2 + |v_*|^2\,, && \text{conservation of energy.} \end{aligned}$$

**Definition 1.** A collision invariant is a function $\phi : \mathbf{R}^3 \to \mathbf{R}$ satisfying

$$\phi(v'(v, v_*, \omega)) + \phi(v'_*(v, v_*, \omega)) = \phi(v) + \phi(v_*)\,, \quad \text{for all } v, v_* \in \mathbf{R}^3,\ \omega \in \mathbf{S}^2\,.$$

Obviously $\phi(v) \equiv 1$, $\phi(v) \equiv v_j$ for $j = 1, 2, 3$ and $\phi(v) = |v|^2$ are collision invariants (because elastic hard sphere collisions preserve the number of particles, together with the total momentum and energy of each colliding particle pair). A remarkable feature of the Boltzmann equation is that the converse is true (under some regularity assumption on $\phi$).

**Theorem 1.** *Let $\phi \in C(\mathbf{R}^3)$; then $\phi$ is a collision invariant if and only if there exist $a, c \in \mathbf{R}$ and $b \in \mathbf{R}^3$ such that*

$$\phi(v) = a + b \cdot v + c|v|^2 .$$

The proof of this result is rather involved; it is an extension of the well known proof that the only function $\psi \in C(\mathbf{R})$ such that

$$\psi(x+y) = \psi(x) + \psi(y) \text{ for all } x, y \in \mathbf{R}, \quad \text{and } \psi(1) = 1$$

is the identity, i.e.

$$\psi(x) = x \quad \text{for each } x \in \mathbf{R}^3 .$$

See for instance [28], Chap. II.6, especially pp. 74–77.

**Theorem 2.** *For each measurable $f \equiv f(v)$ rapidly decaying as $|v| \to \infty$ and each collision invariant $\phi \in C(\mathbf{R}^3)$ with at most polynomial growth as $|v| \to \infty$, one has*

$$\int_{\mathbf{R}^3} \mathscr{C}(f)\phi(v)dv = 0 .$$

*Proof.* Denoting $f = f(v)$, $f' = f(v')$, $f_* = f(v_*)$ and $f'_* = f(v'_*)$, one has

$$\begin{aligned}\int_{\mathbf{R}^3} \mathscr{C}(f)\phi dv &= \tfrac{d^2}{2} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \phi(f'f'_* - ff_*)|(v-v_*)\cdot\omega| dv dv_* d\omega \\ &= \tfrac{d^2}{2} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \tfrac{1}{2}(\phi+\phi_*)(f'f'_* - ff_*)|(v-v_*)\cdot\omega| dv dv_* d\omega ,\end{aligned}$$

since the collision integrand is symmetric in $v, v_*$.

Since $(v - v_*)\cdot\omega = -(v' - v'_*)\cdot\omega$ and $(v, v_*) \mapsto (v', v'_*)(v, v_*, \omega)$ is a linear isometry of $\mathbf{R}^6$ for each $\omega \in \mathbf{S}^2$ (by the conservation of energy), the Lebesgue measure is invariant under the change of variables $(v, v_*) \mapsto (v', v'_*)(v, v_*, \omega)$, which is an involution. Therefore

$$\begin{aligned}&\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \tfrac{1}{2}(\phi+\phi_*)(f'f'_* - ff_*)|(v-v_*)\cdot\omega| dv dv_* d\omega \\ &= \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \tfrac{1}{2}(\phi'+\phi'_*)(ff_* - f'f'_*)|(v-v_*)\cdot\omega| dv dv_* d\omega ,\end{aligned}$$

which implies the following important formula.

**Formula of collision observables**

For each $\phi \in C(\mathbf{R}^3)$ with at most polynomial growth as $|v| \to \infty$, and each $f \in C(\mathbf{R}^3)$ rapidly decaying as $|v|to\infty$,

$$\int_{\mathbf{R}^3} \mathscr{C}(f)\phi dv = \frac{d^2}{2} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \tfrac{1}{4}(\phi+\phi_*-\phi'-\phi'_*)(f' f'_*-f f_*)|(v-v_*)\cdot\omega|dv\, dv_* d\omega\,.$$

The conclusion of Theorem 2 follows from the definition of collision invariants.

Specializing the identity in the theorem above to $\phi(v) \equiv 1, v_k$ for $k = 1, 2, 3$ or $\phi(v) \equiv |v|^2$, for each $f \equiv f(v)$ rapidly decaying as $|v| \to \infty$, one has

$$\int_{\mathbf{R}^3} \mathscr{C}(f)dv = \int_{\mathbf{R}^3} \mathscr{C}(f)v_k dv = \int_{\mathbf{R}^3} \mathscr{C}(f)|v|^2 dv = 0\,, \quad k = 1, 2, 3\,.$$

Thus, solutions $F$ of the Boltzmann equation that are rapidly decaying together with their first order derivatives in $t$ and $x$ as $|v| \to \infty$ satisfy the local conservation laws

$$\begin{cases} \partial_t \int_{\mathbf{R}^3} F dv + \operatorname{div}_x \int_{\mathbf{R}^3} vF dv = 0\,, & \text{(mass)} \\ \partial_t \int_{\mathbf{R}^3} vF dv + \operatorname{div}_x \int_{\mathbf{R}^3} v\otimes vF dv = 0\,, & \text{(momentum)} \\ \partial_t \int_{\mathbf{R}^3} \frac{1}{2}|v|^2 F dv + \operatorname{div}_x \int_{\mathbf{R}^3} v\frac{1}{2}|v|^2 F dv = 0\,. & \text{(energy)} \end{cases}$$

The next most important property of the Boltzmann equation is Boltzmann's H Theorem. This is a rigorous mathematical result bearing on solutions of the Boltzmann equation, which corresponds to the second principle of thermodynamics. The second principle of thermodynamics states that the entropy of an isolated system can only increase until the system reaches an equilibrium state. However the second principle of thermodynamics does not provide any general formula for the entropy production. In the context of the kinetic theory of gases, Boltzmann's H Theorem gives an explicit formula for the entropy production in terms of the distribution function.

**Theorem 3 (Boltzmann's H Theorem).** *If* $f \equiv f(v)$ *is a measurable function on* $\mathbf{R}^3$ *such that* $0 < f = O(|v|^{-m})$ *for all* $m > 0$ *and* $\ln f = O(|v|^n)$ *for some* $n > 0$ *as* $|v| \to \infty$*, then*

$$\int_{\mathbf{R}^3} \mathscr{C}(f) \ln f dv \le 0\,.$$

*Moreover*

$$\int_{\mathbf{R}^3} \mathscr{C}(f) \ln f dv = 0 \Leftrightarrow \mathscr{C}(f) = 0 \Leftrightarrow f \text{ is a Maxwellian,}$$

*i.e. there exist* $\rho, \theta > 0$ *and* $u \in \mathbf{R}^3$ *s.t.*

$$f(v) = \mathscr{M}_{(\rho,u,\theta)}(v) := \frac{\rho}{(2\pi\theta)^{3/2}} \exp\left(-\frac{|v-u|^2}{2\theta}\right).$$

*Proof.* Applying the formula of collision observables with $\phi = \ln f$ shows that

$$\int_{\mathbf{R}^3} \mathscr{C}(f) \ln f dv = \tfrac{d^2}{2} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \tfrac{1}{4}(f' f'_* - f f_*) \ln\left(\frac{f f_*}{f' f'_*}\right) |(v-v_*)\cdot\omega| dv\, dv_* d\omega\,.$$

Since $z \mapsto \ln z$ is increasing on $\mathbf{R}^*_+$

$$(f' f'_* - f f_*) \ln\left(\frac{f f_*}{f' f'_*}\right) = (f' f'_* - f f_*)(\ln(f f_*) - \ln(f' f'_*)) \le 0\,,$$

so that

$$\int_{\mathbf{R}^3} \mathscr{C}(f) \ln f dv \le 0\,.$$

Now for the equality case:

$$\int_{\mathbf{R}^3} \mathscr{C}(f) \ln f dv = 0 \Leftrightarrow f' f'_* = f f_* \Leftrightarrow \ln f \text{ is a collision invariant}$$
$$\Leftrightarrow \mathscr{C}(f) = 0\,.$$

If $\ln f$ is a collision invariant and $f \to 0$ as $|v| \to \infty$, then

$$\ln f(v) = a + b\cdot v + c|v|^2 \quad \text{with } c < 0\,,$$

so that $f(v) = \mathscr{M}_{(\rho,u,\theta)}(v)$ with

$$\theta = -\tfrac{1}{2c}\,, \quad u = -\tfrac{b}{2c}\,, \quad \text{and } \rho = \left(\tfrac{\pi}{|c|}\right)^{3/2} e^{a+|b|^2/4c}\,.$$

Thus

$$\int_{\mathbf{R}^3} \mathscr{C}(f) \ln f dv = 0 \Leftrightarrow f \text{ is a Maxwellian.}$$

In particular, positive solutions $F$ of the Boltzmann equation that are rapidly decaying together with their first order derivatives in $t$ and $x$ as $|v| \to \infty$, and such that $\ln F$ has at most polynomial growth in $|v|$ satisfy the local entropy inequality

$$\partial_t \int_{\mathbf{R}^3} F \ln F dv + \operatorname{div}_x \int_{\mathbf{R}^3} vF \ln F dv = \int_{\mathbf{R}^3} \mathscr{C}(F) \ln F dv \le 0\,.$$

### 2.1.2 Global Existence Theory for the Boltzmann Equation

All the hydrodynamic limits that we consider below bear on the Boltzmann equation posed in the whole Euclidean space $\mathbf{R}^3$. Specifically, we are concerned with solutions of the Boltzmann equation which converge to some uniform Maxwellian equilibrium as $|x| \to \infty$. Without loss of generality, by Galilean invariance of the Boltzmann equation and with an appropriate choice of units of time and length, one can assume that this Maxwellian equilibrium is $\mathscr{M}_{(1,0,1)}$.

For simplicity, we shall henceforth use the notation

$$M := \mathscr{M}_{(1,0,1)}\,.$$

There are various ways of imposing the condition on the solution of the Boltzmann equation as $|x| \to \infty$. In the sequel, we retain the weakest possible notion of convergence to equilibrium at infinity. Perhaps the best reason for this choice is that this notion of "convergence to equilibrium at infinity" is conveniently expressed in terms of Boltzmann's H Theorem.

Specifically, we use the notion of *relative entropy* (of the distribution function $F$ with respect to the Maxwellian equilibrium $M$):

$$H(F|M) := \iint_{\mathbf{R}^3\times\mathbf{R}^3} \left[ F \ln\left(\frac{F}{M}\right) - F + M \right] dx\,dv\,.$$

Notice that the integrand is a nonnegative measurable function defined a.e. on $\mathbf{R}^3 \times \mathbf{R}^3$, so that $H(F|M)$ is a well defined element of $[0, \infty]$ for each nonnegative measurable function $F$ defined a.e. on $\mathbf{R}^3 \times \mathbf{R}^3$.

We are interested in the Cauchy problem

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = \mathscr{C}(F)\,, \quad (t,x,v) \in \mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3\,, \\ F(t,x,v) \to M \quad \text{as } |x| \to +\infty\,, \\ F\big|_{t=0} = F^{in}\,. \end{cases}$$

The convergence of the distribution function $F$ to the Maxwellian equilibrium $M$ as $|x| \to \infty$ is replaced with the condition

$$H(F|M)(t) < +\infty$$

for all $t \geq 0$. Because of Boltzmann's H theorem and the local conservation laws of mass momentum and energy, rapidly decaying solutions $F$ of the Boltzmann equation satisfy

$$H(F|M)(t) \leq H(F|M)(0)\,.$$

In other words, our substitute for the convergence of the distribution function to the uniform Maxwellian equilibrium $M$ as $|x| \to \infty$ is stable under the time evolution of the Boltzmann equation.

R. DiPerna and P.-L. Lions [34, 70] made the following observation: for each $r > 0$, one has

$$\iint_{|x|+|v|\le r} \frac{\mathscr{C}(F)}{\sqrt{1+F}} dv\, dx \le C \iint_{|x|\le r} (-\mathscr{C}(F) \ln F + (1+|v|^2)F) dx\, dv\,.$$

This is important for the following reason: the Boltzmann collision integral $\mathscr{C}(F)$ acts as a *nonlocal* quadratic operator on the $v$ variable in $F$, roughly equivalent to a convolution product, and as a *pointwise* product in $t$ and $x$. Now, the relative entropy bound $H(F|M)(t)$ obviously control a nonlocal product such as $F \star_v F$, where $\star_v$ denotes the convolution in the $v$ variable. But it cannot control a pointwise product in the $x$ variable, such as

$$F(t,x,v) \int_{\mathbf{R}^3} F(t,x,w) dw\,.$$

However, the observation of R. DiPerna and P.-L. Lions implies that $\mathscr{C}(F)/\sqrt{1+F}$ is well defined as a $L^1_{loc}$ function, while $\mathscr{C}(F)$ is only a measurable function, and in general not an element of any Lebesgue space. That $\mathscr{C}(F)/\sqrt{1+F}$ defines a $L^1_{loc}$ function is of course fundamental in order to formulate the Boltzmann equation in the sense of distributions.

More precisely, the observation above suggests considering the following (very weak) notion of solution of the Boltzmann equation.

**Definition 2 (Renormalized solutions of the Boltzmann equation).** A renorma lized solution relative to $M$ of the Boltzmann equation is a nonnegative function $F \in C(\mathbf{R}_+, L^1_{loc}(\mathbf{R}^3 \times \mathbf{R}^3))$ satisfying $H(F(t)|M) < +\infty$ and

$$M(\partial_t + v \cdot \nabla_x)\Gamma(F/M) = \Gamma'(F/M)\mathscr{C}(F)$$

in the sense of distributions on $\mathbf{R}^*_+ \times \mathbf{R}^3 \times \mathbf{R}^3$, for each $\Gamma \in C^1(\mathbf{R}_+)$ s.t.

$$\Gamma'(Z) \le \frac{C}{\sqrt{1+Z}}\,.$$

The main advantage of this notion of solution is the following global existence theorem, which holds for any initial distribution function with finite relative entropy with respect to the Maxwellian equilibrium $M$. The following theorem summarizes several results by R. DiPerna-P.-L. Lions [34], P.-L. Lions [70] and P.-L. Lions-N. Masmoudi [73].

**Theorem 4 (R. DiPerna-P.-L. Lions-N. Masmoudi).** *For each measurable initial data $F^{in} \geq 0$ a.e. such that $H(F^{in}|M) < +\infty$, there exists a renormalized solution relative to $M$ of the Boltzmann equation with initial data $F^{in}$. It satisfies*

$$\left\{\begin{aligned} &\partial_t \int_{\mathbf{R}^3} F dv + \mathrm{div}_x \int_{\mathbf{R}^3} vF dv = 0\,, \\ &\partial_t \int_{\mathbf{R}^3} vF dv + \mathrm{div}_x \int_{\mathbf{R}^3} v \otimes vF dv + \mathrm{div}_x\, m = 0\,, \end{aligned}\right.$$

*where $m = m^T \geq 0$ is a matrix-valued Radon measure on $\mathbf{R}_+ \times \mathbf{R}^3$, and the entropy inequality*

***DiPerna-Lions entropy inequality***

$$H(F(t)|M) + \int_{\mathbf{R}^3} \mathrm{trace}\, m(t) - \int_0^t \iint_{\mathbf{R}^3\times\mathbf{R}^3} \mathscr{C}(F) \ln F\, ds\, dx\, dv \leq H(F^{in}|M)\,,$$

*for all $t \geq 0$.*

Notice that, if $F$ is a classical solution of the Boltzmann equation that is rapidly decaying as $|v| \to \infty$ while $\ln F$ has at most polynomial growth as $|v| \to \infty$, the Boltzmann H Theorem and the local conservation laws of mass, momentum and energy imply that

$$\begin{aligned} \partial_t \int_{\mathbf{R}^3} (F \ln(F/M) - F + M) dv + \mathrm{div}_x \int_{\mathbf{R}^3} (F \ln(F/M) - F + M) dv \\ = \int_{\mathbf{R}^3} \mathscr{C}(F) \ln F dv \leq 0\,. \end{aligned}$$

Hence, assuming for instance that $F/M \to 1$ as $|x| \to \infty$, and integrating with respect to the $x$ variable, one finds that

$$H(F(t)|M) - \int_0^t \iint_{\mathbf{R}^3\times\mathbf{R}^3} \mathscr{C}(F) \ln F\, ds\, dx\, dv = H(F^{in}|M)\,,$$

for all $t \geq 0$. In other words, the DiPerna-Lions entropy inequality is a weak form of the Boltzmann H Theorem valid for the class of renormalized solutions of the Boltzmann equations. Whether renormalized solutions of the Boltzmann equation satisfy the equality in the DiPerna-Lions inequality remains an open problem at the time of this writing.

Another shortcoming of the DiPerna-Lions theory is the uniqueness theory. DiPerna-Lions solutions of the Boltzmann equation are not known to be uniquely determined by their initial data. However, there is a weaker uniqueness property that is satisfied by (a variant of) these solutions: assume that the Boltzmann equation has a classical solution defined on some time interval $[0, T]$ with $T > 0$. Then all DiPerna-Lions solutions of the Boltzmann equation with the same initial data as that classical solution must coincide with it on $[0, T]$ (see [71], Sect. V).

With this notion of solution of the Boltzmann equation, we shall establish the various hydrodynamic limits of the Boltzmann equation where the distribution function is in a weakly nonlinear regime about some uniform Maxwellian equilibrium.

## 2.2 *Hydrodynamic Limits of the Boltzmann Equation*

### 2.2.1 The Compressible Euler Limit

We shall study solutions of the Boltzmann equation that are slowly varying in both the time and space variables.

In other words, we want to study solutions $F$ of the Boltzmann equation of the form

$$F(t, x, v) = F_\epsilon(\epsilon t, \epsilon x, v)\,,$$

assuming

$$\partial_{\hat{t}} F_\epsilon\,,\ \nabla_{\hat{x}} F_\epsilon = O(1)\,, \qquad \text{with } (\hat{t}, \hat{x}) = (\epsilon t, \epsilon x)\,.$$

Since $F$ is a solution of the Boltzmann equation, one has

$$\partial_{\hat{t}} F_\epsilon + v \cdot \nabla_{\hat{x}} F_\epsilon = \frac{1}{\epsilon}\mathscr{C}(F_\epsilon)\,.$$

Hilbert [59] proposed to seek $F_\epsilon$ as a formal power series in $\epsilon$ with smooth coefficients:

$$F_\epsilon(\hat{t}, \hat{x}, v) = \sum_{n \geq 0} \epsilon^n F_n(\hat{t}, \hat{x}, v)\,.$$

In the literature on kinetic theory, this expansion bears the name of *Hilbert's expansion*. It is the most systematic method used to investigate all fluid dynamic limits of the Boltzmann equation (see [93, 94]). Other asymptotic expansions are also used in the context of kinetic models, such as the Chapman-Enskog equation (see below), and variants thereof proposed in [18].

The leading order term in Hilbert's expansion is of the form

$$F_0(\hat{t}, \hat{x}, v) = \mathscr{M}_{(\rho,u,\theta)(\hat{t},\hat{x})}(v)\,,$$

where $(\rho, u, \theta)$ is a solution of the compressible Euler system

$$\begin{cases} \partial_{\hat{t}}\rho + \operatorname{div}_{\hat{x}}(\rho u) = 0\,, \\ \rho(\partial_{\hat{t}} u + u \cdot \nabla_{\hat{x}} u) + \nabla_{\hat{x}}(\rho\theta) = 0\,, \\ \partial_{\hat{t}}\theta + u \cdot \nabla_{\hat{x}}\theta + \frac{2}{3}\theta \operatorname{div}_{\hat{x}} u = 0\,. \end{cases} \tag{2}$$

The Hilbert series is a formal object—in particular, its radius of convergence in $\epsilon$ may be, and often is 0. A mathematical proof of the compressible Euler limit based on a variant of Hilbert's expansion truncated at some finite order in $\epsilon$ was proposed by R. Caflisch [24].

While fairly direct and natural, Caflisch's approach to the compressible Euler limit meets with the following difficulties:

(a) The truncated Hilbert expansion may be negative for some $\hat{t}, \hat{x}, v$;
(b) The $k$-th term $F_k$ in Hilbert's expansion is of order $O(|\nabla_{\hat{x}}^k F_0|)$;
(c) Generic solutions of Euler's equations lose regularity in finite time (see [91]).

Statement (a) follows from a close inspection of Caflisch's asymptotic solution at time $t = 0$; statement (b) implies that the Hilbert expansion method can be used in the case of smooth solutions of the compressible Euler system, while statements (b–c) suggest that the Hilbert expansion breaks down in finite time for generic smooth solutions of the compressible Euler system. (This is obviously related to the onset of shock waves in compressible, inviscid fluid flows.)

There is another approach to the compressible Euler limit. T. Nishida studied the Cauchy problem for the scaled Boltzmann equation in [81]:

$$\begin{cases} \partial_{\hat{t}} F_\epsilon + v \cdot \nabla_{\hat{x}} F_\epsilon = \dfrac{1}{\epsilon}\mathscr{C}(F_\epsilon)\,, \\ F_\epsilon(0, \hat{x}, \hat{v}) = \mathscr{M}_{(\rho^{in}, u^{in}, \theta^{in})(\hat{x})}(v)\,, \end{cases} \tag{3}$$

for analytic $(\rho^{in}, u^{in}, \theta^{in})$. Nishida's key idea was to apply the Nirenberg-Ovsyannikov [79, 80] abstract variant of the Cauchy-Kovalevska theorem. He proved that the Cauchy problem (3) has a unique solution on a time interval $[0, T^*]$ with $T^* > 0$ independent of $\epsilon$, and that

$$F_\epsilon(\hat{t}, \hat{x}, v) \to \mathscr{M}_{(\rho, u, \theta)(\hat{t}, \hat{x})}(v)$$

as $\epsilon \to 0$, where $(\rho, u, \theta)$ is the solution of the compressible Euler system with initial data $(\rho^{in}, u^{in}, \theta^{in})$.

It is interesting to compare the Hilbert expansion method and the Caflisch proof with Nishida's.

Caflisch's method leads to a family $F_\epsilon$ of solutions of the scaled Boltzmann equation that converges to a Maxwellian whose parameters satisfy the compressible Euler system on the same time interval as that on which the Euler solution remains smooth.

However, these solutions fail to be everywhere nonnegative; besides the choice of the initial condition $F_\epsilon\big|_{\hat{t}=0}$ is seriously constrained to "well prepared data". This difficulty was later alleviated by M. Lachowicz [64].

In Nishida's method, we can choose $F_\epsilon\big|_{\hat{t}=0}$ to be any local Maxwellian with analytic parameters, and $F_\epsilon$ remains everywhere nonnegative.

However the uniform existence time $T^*$ can be a priori smaller than the time during which the Euler solution remains smooth. Besides, analytic regularity is physically unsatisfying.

The works of Caflisch and Nishida obviously raise the question of what happens to the family of solutions of the Boltzmann equation in the vanishing $\epsilon$ limit after the onset of shock waves in the solution of the Euler system. For instance the Cauchy problem for the Euler equations of gas dynamics is known to have global solutions defined for all initial data with small enough total variation, in space dimension 1. These solutions are constructed by Glimm's method [38,75]. Whether these solutions are somehow related to the Boltzmann equation after the onset of shock waves is therefore a very natural question.

Of course, weak solutions of a hyperbolic system of conservation laws such as the Euler equations of gas dynamics may fail to be uniquely determined by their initial data. For instance, weak solutions can include unphysical shock waves. In the case of gas dynamics, the notion of entropy provides precisely the criterion used to eliminate the possibility of unphysical shock waves. The following elementary observation shows that, under rather weak assumptions, weak solutions of the Euler equations of gas dynamics originating from solutions of the Boltzmann equation satisfy the entropy criterion.

**Theorem 5 (C. Bardos-F. Golse [5]).** *Let $\rho^{in} \geq 0$, $\theta^{in} > 0$ (resp. $u^{in}$) be measurable functions (resp. a measurable vector field) defined a.e. on $\mathbf{R}^3$ such that*

$$\int_{\mathbf{R}^3} (1 + |u^{in}|)(|u^{in}|^2 + \theta^{in} + |\ln \rho^{in}| + |\ln \theta^{in}|)d\hat{x} < \infty .$$

*For each $\epsilon > 0$, let $F_\epsilon$ be a solution of the Cauchy problem (3) satisfying the local conservation laws of mass momentum and energy. Assume that*

$$F_\epsilon \to F \text{ a.e. on } \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3 ,$$

*and that*

$$\int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} F_\epsilon(\hat{t}, \hat{x}, v)dv \to \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} F(\hat{t}, \hat{x}, v)dv$$

*in the sense of distributions on $\mathbf{R}^3$, uniformly on $[0, T]$ for each $T > 0$, while*

$$\int_{\mathbf{R}^3} \begin{pmatrix} v \otimes v \\ v|v|^2 \end{pmatrix} F_\epsilon dv \to \int_{\mathbf{R}^3} \begin{pmatrix} v \otimes v \\ v|v|^2 \end{pmatrix} F dv$$

*and*

$$\int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \end{pmatrix} F_\epsilon \ln F_\epsilon dv \to \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \end{pmatrix} F \ln F dv$$

*in the sense of distributions on* $\mathbf{R}_+^* \times \mathbf{R}^3$. *Then*

- *The limit* $F$ *is of the form*

$$F = \mathscr{M}_{(\rho,u,\theta)}$$

*where* $(\rho, u, \theta)$ *is a weak solution of the system of Euler equations of gas dynamics (2) (with perfect gas equation of state), with initial data*

$$(\rho, u, \theta)\big|_{t=0} = (\rho^{in}, u^{in}, \theta^{in}),$$

- *The solution* $(\rho, u, \theta)$ *of the system of Euler equations so obtained satisfies the entropy condition*

$$\partial_{\hat{t}} \left(\rho \ln \left(\frac{\rho}{\theta^{3/2}}\right)\right) + \mathrm{div}_{\hat{x}} \left(\rho u \ln \left(\frac{\rho}{\theta^{3/2}}\right)\right) \leq 0\,.$$

The key observation in this result is that

$$\begin{aligned} 0 \geq \partial_{\hat{t}} \int_{\mathbf{R}^3} F_\epsilon \ln F_\epsilon dv + \mathrm{div}_{\hat{x}} \int_{\mathbf{R}^3} v F_\epsilon \ln F_\epsilon dv \\ \to \partial_{\hat{t}} \int_{\mathbf{R}^3} F \ln F dv + \mathrm{div}_{\hat{x}} \int_{\mathbf{R}^3} v F \ln F dv \end{aligned}$$

in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3$ as $\epsilon \to 0$, while

$$\begin{aligned} \int_{\mathbf{R}^3} \mathscr{M}_{(\rho,u,\theta)} \ln \mathscr{M}_{(\rho,u,\theta)} dv &= \rho \ln \left(\frac{\rho}{(2\pi\theta)^{3/2}}\right) - \tfrac{3}{2}\rho\,, \\ \int_{\mathbf{R}^3} v \mathscr{M}_{(\rho,u,\theta)} \ln \mathscr{M}_{(\rho,u,\theta)} dv &= \rho u \ln \left(\frac{\rho}{(2\pi\theta)^{3/2}}\right) - \tfrac{3}{2}\rho u\,. \end{aligned}$$

(In other words, Boltzmann's H function specialized to Maxwellian distribution functions coincides with the entropy density for a perfect monatomic gas).

Of course, the assumption that $F_\epsilon \to F$ a.e. is an extremely strong one, and verifying it remains a major open problem. However, the purpose of this theorem is *not* the convergence itself to some solution of the Euler equations, but the fact that all solutions of the Euler equations obtained in this way satisfy the entropy condition.

In addition to the system of Euler's equations of gas dynamics, several other fluid dynamic equations can be derived from the Boltzmann equation. We shall review the most important such equations and their derivations from kinetic theory in the next sections.

### 2.2.2 From Boltzmann to Compressible Navier-Stokes

First we seek to derive viscous corrections to the Euler system from the Boltzmann equation. In order to do so, we use the Chapman-Enskog expansion—a variant of Hilbert's. (See [55] and especially Chap. V.3 in [28].) This asymptotic expansion in powers of $\epsilon$ takes the form

$$F_\epsilon(\hat{t},\hat{x},v) \simeq \sum_{n=0}^{N} \epsilon^n \Phi_n[\mathbf{P}_\epsilon^N(\hat{t},\hat{x})](v) =: F_\epsilon^N(\hat{t},\hat{x},v)\,,$$

where $\Phi_n[\mathbf{P}(\hat{t},\hat{x})]$ is a local functional of $\mathbf{P}$ evaluated at $(\hat{t},\hat{x})$ with values in the set of functions of $v \in \mathbf{R}^3$. The coefficients $\Phi_n[\mathbf{P}]$ are determined by the condition

$$\int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \Phi_n[\mathbf{P}](v)dv = \begin{cases} \mathbf{P} & \text{if } n = 0\,, \\ 0 & \text{if } n > 0\,, \end{cases} \tag{4}$$

and by the fact that the asymptotic expansion $F_\epsilon^N$ above satisfies

$$\partial_{\hat{t}} F_\epsilon^N + v \cdot \nabla_{\hat{x}} F_\epsilon^N = \frac{1}{\epsilon}\mathscr{C}(F_\epsilon^N) + O(\epsilon^N)\,. \tag{5}$$

At variance with Hilbert's expansion, the coefficients of the successive powers of $\epsilon$ in the Chapman-Enskog expansion depend on $\epsilon$ (except for the 0th order term, which is the local Maxwellian with parameters governed by the compressible Euler system, and therefore coincides with the 0th order term in the Hilbert expansion). As explained above, these coefficients are completely determined by the moments of order $\leq 2$ in the velocity variable (4) of the truncated Chapman-Enskog expansion $F_\epsilon^N$, and by the fact that $F_\epsilon^N$ is an asymptotic solution of the Boltzmann equation (5) to within order $O(\epsilon^N)$ (in the formal sense).

In particular, for $N = 2$, one finds that

$$\begin{aligned} F_\epsilon(\hat{t},\hat{x},v) \simeq \mathscr{M}_{(\rho_\epsilon,u_\epsilon,\theta_\epsilon)} &- \epsilon \mathscr{M}_{(1,u_\epsilon,\theta_\epsilon)}\alpha(|V_\epsilon|,\theta_\epsilon)A(V_\epsilon) : \nabla_x u_\epsilon \\ &- 2\epsilon \mathscr{M}_{(1,u_\epsilon,\theta_\epsilon)}\beta(|V_\epsilon|,\theta_\epsilon)B(V_\epsilon)\cdot\nabla_x\sqrt{\theta_\epsilon} \\ &+ O(\epsilon^2)\,, \end{aligned}$$

where

$$V_\epsilon := \frac{v - u_\epsilon}{\sqrt{\theta_\epsilon}}, \quad A(z) = z^{\otimes 2} - \tfrac{1}{3}|z|^2, \quad B(z) = \tfrac{1}{2}(|z|^2 - 5)z.$$

The functions $\alpha(\theta, r)$ and $\beta(\theta, r)$ are obtained by solving two integral equations involving the Boltzmann collision integral linearized about the Maxwellian state $\mathscr{M}_{(1,u,\theta)}$. We refer to Appendix 2 for more details on this matter. Thus

$$\begin{aligned}\Phi_0[\mathbf{P}](v) &= \mathscr{M}_{(\rho,u,\theta)} \qquad \text{with } \mathbf{P} = (\rho, u, \theta)^T, \\ \Phi_1[\mathbf{P}](v) &= -\mathscr{M}_{(1,u,\theta)}\alpha(|\tfrac{v-u}{\sqrt{\theta}}|, \theta)A(\tfrac{v-u}{\sqrt{\theta}}) : \nabla_x u \\ &\quad - 2\mathscr{M}_{(1,u,\theta)}\beta(|\tfrac{v-u}{\sqrt{\theta}}|, \theta)B(\tfrac{v-u}{\sqrt{\theta}}) \cdot \nabla_x \sqrt{\theta}.\end{aligned}$$

The compressible Navier-Stokes equations take the form

$$\left\{\begin{aligned} &\partial_{\hat{t}}\rho_\epsilon + \operatorname{div}_{\hat{x}}(\rho_\epsilon u_\epsilon) = 0, \\ &\partial_{\hat{t}}(\rho_\epsilon u_\epsilon) + \operatorname{div}_{\hat{x}}(\rho_\epsilon u_\epsilon^{\otimes 2}) + \nabla_{\hat{x}}(\rho_\epsilon \theta_\epsilon) \\ &\qquad = \epsilon \operatorname{div}(\mu(\theta_\epsilon)D(u_\epsilon)), \\ &\partial_{\hat{t}}(\rho_\epsilon(\tfrac{1}{2}|u_\epsilon|^2 + \tfrac{3}{2}\theta_\epsilon)) + \operatorname{div}_{\hat{x}}(\rho_\epsilon u_\epsilon(\tfrac{1}{2}|u_\epsilon|^2 + \tfrac{5}{2}\theta_\epsilon)) \\ &\qquad = \epsilon \operatorname{div}_{\hat{x}}(\kappa(\theta_\epsilon)\nabla_{\hat{x}}\theta_\epsilon) + \epsilon \operatorname{div}_{\hat{x}}(\mu(\theta_\epsilon)D(u_\epsilon)u_\epsilon), \end{aligned}\right.$$

where

$$D(u) = \nabla_{\hat{x}} u + (\nabla_{\hat{x}} u)^T - \tfrac{2}{3}(\operatorname{div}_{\hat{x}} u)\, I.$$

These equations are obtained from the local conservation laws of mass, momentum and energy for the Chapman-Enskog expansion of $F_\epsilon$ truncated at order 2.

Notice that the viscosity and heat diffusion terms are of order $O(\epsilon)$ in this scaling. In other words, compressible Navier-Stokes equations are *not a limit* of the Boltzmann equation, but a correction of the compressible Euler equations at the first order in $\epsilon$.

The formulas giving the viscosity and heat diffusion coefficients are worth a few comments. They are

$$\begin{aligned}\mu(\theta) &= \tfrac{2}{15}\theta \int_0^\infty \alpha(\theta, r) r^6 e^{-r^2/2} \tfrac{dr}{\sqrt{2\pi}}, \\ \kappa(\theta) &= \tfrac{1}{6}\theta \int_0^\infty \beta(\theta, r) r^4 (r^2 - 5)^2 e^{-r^2/2} \tfrac{dr}{\sqrt{2\pi}}.\end{aligned} \tag{6}$$

In the hard sphere case (which is the only case considered in these lectures), one finds

$$\mu(\theta) = \mu(1)\sqrt{\theta}\,, \qquad \kappa(\theta) = \kappa(1)\sqrt{\theta}\,. \tag{7}$$

(See Appendix 2 for the details.)

Some comments on the Chapman-Enskog and the Hilbert expansion are in order.

First, it is interesting to compare both expansions, at least up to order 1 in $\epsilon$. The first two terms of the Hilbert expansion are of the form

$$\begin{aligned}
F_0(t,x,v) &= \mathscr{M}_{(\rho_0,u_0,\theta_0)(t,x)}(v)\,,\\
F_1(t,x,v) &= \mathscr{M}_{(\rho_0,u_0,\theta_0)(t,x)}(v)\left(\frac{\rho_1(t,x)}{\rho_0(t,x)} + \frac{u_1(t,x)\cdot(v-u_0(t,x))}{\theta_0(t,x)}\right.\\
&\quad + \frac{\theta_1(t,x)}{2\theta_0(t,x)}\left(\frac{|v-u_0(t,x)|^2}{\theta_0(t,x)} - 3\right)\\
&\quad - \frac{1}{\rho_0(t,x)}\alpha\left(\frac{|v-u_0(t,x)|}{\sqrt{\theta_0(t,x)}}, \theta_0(t,x)\right) A\left(\frac{v-u_0(t,x)}{\sqrt{\theta_0(t,x)}}\right) : \nabla_x u_0(t,x)\\
&\quad \left. - \frac{2}{\rho_0(t,x)}\beta\left(\frac{|v-u_0(t,x)|}{\sqrt{\theta_0(t,x)}}, \theta_0(t,x)\right) B\left(\frac{v-u_0(t,x)}{\sqrt{\theta_0(t,x)}}\right)\cdot \nabla_x\sqrt{\theta_0(t,x)}\right).
\end{aligned}$$

This first order term is decomposed into a term, henceforth denoted $F_1^h$ which is a linear combination of collision invariants with coefficients that are functions of $t, x$, and a term, henceforth denoted $F_1^0$ that satisfies the condition

$$\int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} F_1^0(t,x,v)dv = 0\,,$$

i.e.

$$F_1 = F_1^h + F_h^0$$

where

$$\begin{aligned}
F_1^h(t,x,v) = F_1(t,x,v) = \mathscr{M}_{(\rho_0,u_0,\theta_0)(t,x)}(v)&\left(\frac{\rho_1(t,x)}{\rho_0(t,x)} + \frac{u_1(t,x)\cdot(v-u_0(t,x))}{\theta_0(t,x)}\right.\\
&\left. + \frac{\theta_1(t,x)}{2\theta_0(t,x)}\left(\frac{|v-u_0(t,x)|^2}{\theta_0(t,x)} - 3\right)\right),
\end{aligned}$$

while

$$F_1^0(t,x,v) =$$
$$-\mathscr{M}_{(1,u_0,\theta_0)(t,x)}(v)\left(\alpha\left(\frac{|v-u_0(t,x)|}{\sqrt{\theta_0(t,x)}},\theta_0(t,x)\right)A\left(\frac{v-u_0(t,x)}{\sqrt{\theta_0(t,x)}}\right):\nabla_x u_0(t,x)\right.$$
$$\left.+2\beta\left(\frac{|v-u_0(t,x)|}{\sqrt{\theta_0(t,x)}},\theta_0(t,x)\right)B\left(\frac{v-u_0(t,x)}{\sqrt{\theta_0(t,x)}}\right)\cdot\nabla_x\sqrt{\theta_0(t,x)}\right).$$

Elementary computations show that

$$\begin{aligned}\Phi_0[\rho_0+\epsilon\rho_1,u_0+\epsilon u_1,\theta_0+\epsilon\theta_1] &= \mathscr{M}_{(\rho_0+\epsilon\rho_1,u_0+\epsilon u_1,\theta_0+\epsilon\theta_1)}\\ &= F_0+\epsilon F_1^h + O(\epsilon^2)\,,\end{aligned}$$

while

$$\Phi_1[\rho_0,u_0,\theta_0] = F_1^0\,.$$

Therefore

$$\begin{aligned}\Phi_0[\rho_0+\epsilon\rho_1,u_0+\epsilon u_1,\theta_0+\epsilon\theta_1]+\epsilon\Phi_1[\rho_0+\epsilon\rho_1,u_0+\epsilon u_1,\theta_0+\epsilon\theta_1]&\\ = F_0+\epsilon F_1+O(\epsilon^2)&\end{aligned}$$

and the Hilbert and Chapman-Enskog agree up to order $O(\epsilon^2)$. Notice however the following subtle difference: the term $F_1^h$, which appears at order 1 in the Hilbert expansion, is combined with $F_0$ in the leading order term $\Phi_0$ of the Chapman-Enskog expansion. The remaining part of the first order term in Hilbert's expansion, i.e. $F_1^0$ is the leading order part of the next order term $\Phi_1$ in the Chapman-Enskog expansion. This re-ordering is a consequence of (4). One advantage of the Chapman-Enskog expansion over Hilbert's is that the condition (4) guarantees that the equation defining the $n+1$-st order term $\Phi_{n+1}$ in terms of $\Phi_k$ for all $k=0,\dots,n$ has exactly one solution. Indeed, this equation involves the linearization at $\Phi_0$ of the Boltzmann collision integral. An important theorem due to Hilbert [59] in the case of the hard sphere gas shows that the linearized collision operator satisfies the Fredholm alternative. The condition (4) is precisely the unique solvability condition for that operator. (See Theorem 16 below.)

We have deliberately limited our presentation of the Chapman-Enskog method to the expansion at order 1 in the small parameter $\epsilon$. Of course, the Chapman-Enskog expansion can be pushed to higher orders, and leads to further corrections of the compressible Navier-Stokes equations, known as the Burnett system (the correction of order 2 in $\epsilon$ of the compressible Navier-Stokes equations) and the super-Burnett system (the correction at order

3 in $\epsilon$ of the compressible Navier-Stokes equations). The Burnett system is known to be ill-posed and is therefore of limited practical interest. However, these higher order hydrodynamic theories *à la* Burnett have been recently revisited in a series of remarkable contributions. In [62], the Boltzmann equation is replaced with an approximate system of conservation laws with relaxation terms, and the analogue of the classical Chapman-Enskog method for that relaxation system leads to a variant of the Burnett system that is hyperbolic and therefore well posed. (Notice that the Chapman-Enskog method applies not only to the Boltzmann equation, but also to systems of conservation laws with relaxation terms: see for instance [29].)

The same issue is addressed by Bobylev [18, 19] in yet another manner, by working directly on the original Boltzmann equation. Bobylev's idea is that the ill-posedness in the Burnett system is caused by the particular way in which the Chapman-Enskog expansion is truncated at order 2. By analogy with the classical theory of asymptotic expansions in the context of celestial mechanics, Bobylev proposed another truncation method for the Hilbert-Chapman-Enskog expansion of the solution of the Boltzmann equation which leads to a well posed analogue of the Burnett system.

### 2.2.3 The Acoustic Limit

The first result on the acoustic limit of the Boltzmann equation in the regime of renormalized solutions can be found in [12]. This early result, valid only in the case of bounded collision kernels, was shortly thereafter extended to more general collision kernels including all hard potentials satisfying Grad's cutoff assumption [56], and in particular the hard sphere case.

**Theorem 6 (F. Golse-C.D. Levermore [45]).** *Let $F_\epsilon$ be a family of renormalized solutions of the Cauchy problem for the Boltzmann equation with initial data*

$$F_\epsilon\big|_{t=0} = \mathscr{M}_{(1+\delta_\epsilon\rho^{in}(\epsilon x),\delta_\epsilon u^{in}(\epsilon x),1+\delta_\epsilon\theta^{in}(\epsilon x))}\,,$$

*for $\rho^{in}, u^{in}, \theta^{in} \in L^2(\mathbf{R}^3)$ and $\delta_\epsilon|\ln\delta_\epsilon|^{1/2} = o(\sqrt{\epsilon})$. As $\epsilon \to 0$,*

$$\frac{1}{\delta_\epsilon}\int_{\mathbf{R}^3}\left(F_\epsilon\left(\frac{t}{\epsilon},\frac{x}{\epsilon},v\right) - M\right)(1, v, \tfrac{1}{3}|v|^2 - 1)dv \to (\rho, u, \theta)(t, x)$$

*in $L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)$ for all $t \geq 0$, where $\rho, u, \theta \in C(\mathbf{R}_+; L^2(\mathbf{R}^3))$ satisfy the acoustic system*

$$\begin{cases} \partial_t\rho + \operatorname{div}_x u = 0\,, & \rho\big|_{t=0} = \rho^{in}\,,\\ \partial_t u + \nabla_x(\rho+\theta) = 0\,, & u\big|_{t=0} = u^{in}\,,\\ \frac{3}{2}\partial_t\theta + \operatorname{div}_x u = 0\,, & \theta\big|_{t=0} = \theta^{in}\,. \end{cases}$$

### 2.2.4 The Incompressible Euler Limit

Steady solutions $(\rho, u, \theta)$ of the acoustic system are obviously triples $(\rho, u, \theta) \equiv (\rho(x), u(x), \theta(x))$ satisfying the conditions

$$\operatorname{div} u = 0\,, \qquad \text{and } \nabla(\rho + \theta) = 0\,.$$

The second constraint implies that $\rho + \theta \ = \text{Const}$. In fact, with the additional assumption that $\rho, \theta \in L^2(\mathbf{R}^3)$, one has

$$\rho + \theta = 0\,.$$

This observation suggests that, if the fluctuations around the equilibrium $(1, 0, 1)$ of density, velocity field and temperature satisfy the conditions above, the acoustic and vortical modes in the moments of the distribution function should decouple in the long time limit, and lead to some incompressible flow.

Of course, this does not mean that the gas is incompressible, but only that its motion is the same as that of an incompressible fluid with constant density. This observation is made rigorous by the following theorem.

**Theorem 7 (L. Saint-Raymond [87]).** *Let $u^{in} \in H^3(\mathbf{R}^3)$ s.t. $\operatorname{div} u^{in} = 0$ and let $u \in C([0,T]; H^3(\mathbf{R}^3))$ satisfy*

$$\begin{cases} \partial_t u + u \cdot \nabla_x u + \nabla_x p = 0\,, \quad \operatorname{div}_x u = 0\,, \\ u\big|_{t=0} = u^{in}\,, \end{cases}$$

*for some $p \in \mathscr{D}'((0,T) \times \mathbf{R}^3)$. Let $F_\epsilon$ be a family of renormalized solutions of the Cauchy problem for the Boltzmann equation with initial data*

$$F_\epsilon\big|_{t=0} = \mathscr{M}_{(1,\delta_\epsilon u^{in}(\epsilon x),1)}$$

*for $\delta_\epsilon = \epsilon^\alpha$ with $0 < \alpha < 1$. Then, in the limit as $\epsilon \to 0$, one has*

$$\frac{1}{\delta_\epsilon}\int_{\mathbf{R}^3} vF_\epsilon\left(\frac{t}{\epsilon\delta_\epsilon}, \frac{x}{\epsilon}, v\right) dv \to u(t,x) \text{ in } L^\infty([0,T]; L^1_{loc}(\mathbf{R}^3))\,.$$

### 2.2.5 The (Time-Dependent) Stokes Limit

The previous limit neglects viscous dissipation in the gas. Viscous dissipation and heat diffusion are observed on a longer time scale. We first treat the case where the nonlinearity is weak even after taking the fluid dynamic limit. This limit is described by the following theorem. Observe that the time scale in this result is $1/\epsilon^2$, which is large compared to the time scale $1/\epsilon\delta_\epsilon$ used in the incompressible Euler limit.

On the other hand, the size $\delta_\epsilon$ of the fluctuations is $o(\epsilon)$, i.e. much smaller than in the case of the incompressible Euler limit, where it is $\gg\epsilon$. Thus the nonlinearity is so weak in this case that it vanishes in the fluid dynamic limit.

**Theorem 8 (F. Golse-C.D. Levermore [45]).** *Let $F_\epsilon$ be a family of renormalized solutions of the Cauchy problem for the Boltzmann equation with initial data*

$$F_\epsilon\big|_{t=0} = \mathscr{M}_{(1-\delta_\epsilon\theta^{in}(\epsilon x),\delta_\epsilon u^{in}(\epsilon x),1+\delta_\epsilon\theta^{in}(\epsilon x))} ,$$

*where $\delta_\epsilon|\ln\delta_\epsilon| = o(\epsilon)$ and $(u^{in},\theta^{in}) \in L^2 \times L^\infty(\mathbf{R}^3)$ s.t. $\operatorname{div}_x u^{in} = 0$. Then, in the limit as $\epsilon \to 0$, one has*

$$\frac{1}{\delta_\epsilon}\int_{\mathbf{R}^3}\left(F_\epsilon\left(\frac{t}{\epsilon^2},\frac{x}{\epsilon},v\right) - M\right)(v,\tfrac{1}{3}|v|^2-1)dv \to (u,\theta)(t,x) \text{ in } L^1_{loc}(\mathbf{R}_+\times\mathbf{R}^3) ,$$

*where*

$$\begin{cases} \partial_t u + \nabla_x p = \mu\Delta_x u , \quad \operatorname{div}_x u = 0 , & u\big|_{t=0} = u^{in} , \\ \tfrac{5}{2}\partial_t\theta = \kappa\Delta_x\theta , & \theta\big|_{t=0} = \theta^{in} , \end{cases}$$

*for some $p \in \mathscr{D}'(\mathbf{R}_+^*\times\mathbf{R}^3)$.*

The viscosity and heat conductivity are given by the formulas

$$\mu = \tfrac{1}{5}\mathscr{D}^*(v\otimes v - \tfrac{1}{3}|v|^2 I) , \qquad \kappa = \tfrac{2}{3}\mathscr{D}^*(\tfrac{1}{2}(|v|^2-5)v) , \tag{8}$$

where $\mathscr{D}$ is the Dirichlet form of the linearized collision operator

$$\mathscr{D}(\Phi) = \tfrac{1}{8}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} |\Phi + \Phi_* - \Phi' - \Phi'_*|^2|(v-v_*)\cdot\omega|MM_*dv\,dv_*d\omega ,$$

and $\mathscr{D}^*$ is its Legendre dual. (If $\Phi$ is vector valued, $|\cdot|$ designates the canonical Euclidean norm in $\mathbf{R}^3$; if $\Phi$ is matrix valued, $|\cdot|$ designates the Frobenius norm on $M_3(\mathbf{R})$, i.e. $|A| := \sqrt{\operatorname{trace}(A^TA)}$.)

It should be noticed that P.-L. Lions and N. Masmoudi [73] had independently obtained a version of the above theorem with the motion equation only, i.e. without deriving the heat equation for $\theta$.

### 2.2.6 Incompressible Navier-Stokes Limit

Finally, we discuss the case where viscous dissipation and heat diffusion are observed in the fluid dynamic limit, together with the nonlinear convection term. This follows from a scaling assumption where the length and time scale are respectively $1/\epsilon$ and $1/\epsilon^2$ (corresponding to the invariance scaling for the heat equation), while the size of the fluctuation is precisely of order $\epsilon$. Thus the asymptotic regime under consideration is *weakly nonlinear at the level of the kinetic theory of gases, but fully nonlinear at the level of fluid dynamics*. These scaling assumptions correspond exactly to the invariance scaling for the incompressible Navier-Stokes motion equation.

**Theorem 9 (F. Golse-L. Saint-Raymond [48,50]).** *Let $F_\epsilon$ be a family of renormalized solutions of the Cauchy problem for the Boltzmann equation with initial data*

$$F_\epsilon\big|_{t=0} = \mathscr{M}_{(1-\epsilon\theta^{in}(\epsilon x),\epsilon u^{in}(\epsilon x),1+\epsilon\theta^{in}(\epsilon x))} ,$$

*where $(u^{in}, \theta^{in}) \in L^2 \times L^\infty(\mathbf{R}^3)$ s.t. $\operatorname{div}_x u^{in} = 0$. For some subsequence $\epsilon_n \to 0$, one has*

$$\frac{1}{\epsilon_n}\int_{\mathbf{R}^3}\left(F_{\epsilon_n}\left(\frac{t}{\epsilon_n^2},\frac{x}{\epsilon_n},v\right)-M\right)(v, \tfrac{1}{3}|v|^2-1)dv \to (u,\theta)(t,x)$$

*weakly in $L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)$, where $(u,\theta)$ is a "Leray solution" with initial data $(u^{in},\theta^{in})$ of*

$$\begin{cases} \partial_t u + \operatorname{div}_x(u\otimes u) + \nabla_x p = \mu\Delta_x u\,, \quad \operatorname{div}_x u = 0\,, \\ \tfrac{5}{2}(\partial_t\theta + \operatorname{div}_x(u\theta)) = \kappa\Delta_x\theta\,, \end{cases}$$

*for some $p \in \mathscr{D}'(\mathbf{R}_+^* \times \mathbf{R}^3)$.*

The viscosity $\mu$ and the heat diffusion $\kappa$ in this theorem are given by the same formulas (8) as in the case of the time dependent Stokes limit.

We recall the notion of "Leray solution" of the Navier-Stokes-Fourier system. A Leray solution of the Navier-Stokes-Fourier system above is a couple $(u,\theta)$ of elements of[1] $C(\mathbf{R}_+; w-L^2(\mathbf{R}^3)) \cap L^2(\mathbf{R}_+; H^1(\mathbf{R}^3))$ that is a solution in the sense of distributions and satisfies the Leray inequality below:

**Leray inequality**

$$\begin{aligned}\tfrac{1}{2}\int_{\mathbf{R}^3}(|u|^2+\tfrac{5}{2}|\theta|^2)(t,x)dx + \int_0^t\int_{\mathbf{R}^3}(\mu|\nabla_x u|^2+\kappa|\nabla_x\theta|^2)dx\,ds \\ \le \tfrac{1}{2}\int_{\mathbf{R}^3}(|u^{in}|^2+\tfrac{5}{2}|\theta^{in}|^2)(t,x)dx\,.\end{aligned}$$

[1]If $X$ is a topological space and $E$ is a Banach space, the notation $C(X, w-E)$ designates the set of continuous maps from $X$ to $E$ equipped with its weak topology.

This notion of "Leray solution" of the Navier-Stokes-Fourier system finds its origin in the pioneering work of J. Leray [67] on the incompressible Navier-Stokes equations. These solutions bear considerable resemblance with DiPerna-Lions solutions of the Boltzmann equation. There is indeed an obvious analogy between the Leray energy inequality which reduces to an equality for classical solutions of the Navier-Stokes equations, and the DiPerna-Lions entropy inequality, which also reduces to an equality for classical solutions of the Boltzmann with appropriate decay as $|x|, |v| \to \infty$. Moreover, Leray solutions of the Navier-Stokes equations are not known to be uniquely determined by their initial data, exactly as DiPerna-Lions solutions of the Boltzmann equation are not uniquely determined by their initial data. However, if there exists a classical solution of the Navier-Stokes equation defined on some time interval $[0, T]$ with $T > 0$, each Leray solution with the same initial data as that classical solution must coincide with it on $[0, T]$. (See [67] for a detailed account of all these properties of Leray solutions of the Navier-Stokes equations.)

The reader should be aware that the terminology of "incompressible Navier-Stokes limit" is misleading from the physical viewpoint. It is true that the motion equation satisfied by the velocity field $u$ coincides with the Navier-Stokes equation for an incompressible fluid with constant density. However, the diffusion coefficient in the temperature equation is $3/5$ of its value for an incompressible fluid with the same heat capacity and heat conductivity. The difference comes from the work of the pressure: see the detailed discussion of this subtle point in [42] on pp. 22–23, and especially in [93] (footnote 6 on p. 93) and [94] (footnote 43 on p. 107, together with Sect. 3.7.2). However, the system obtained in the limit has the same mathematical structure as the Navier-Stokes-Fourier system for incompressible fluids, and we shall therefore abuse the terminology of incompressible limit in that case—although it is improper on physical grounds. Of course, the Navier-Stokes limit theorem above does not mean that the gas under consideration is an incompressible fluid. Furthermore, the Navier-Stokes equation is a fundamental equation of hydrodynamics. It applies to a wide variety of fluids (such as liquids, for instance) and is therefore a much more universal model than the Boltzmann equation.

The derivation of the acoustic, incompressible Euler, Stokes and Navier-Stokes equations from global (renormalized) solutions of the Boltzmann equation is a program started by Bardos-Golse-Levermore [11].

As for the incompressible Navier-Stokes limit, partial results were obtained by Bardos-Golse-Levermore [9–11], P.-L. Lions-N. Masmoudi [73] before the complete proof by F. Golse-L. Saint-Raymond appeared in [48, 50]. Subsequently, the validity of this limit was extended to the case of weak cutoff potentials (hard and soft), by C.D. Levermore-N. Masmoudi [68].

In the regime of smooth solutions, the incompressible Navier-Stokes limit for small initial data (a case where Leray solutions are known to be smooth globally in time) had been obtained by C. Bardos-S. Ukai [6]. In the same regime, short time convergence was obtained by A. DeMasi-R. Esposito-J. Lebowitz [33] by an argument similar to Caflisch's for the compressible limit, i.e. by means of a truncated Hilbert expansion.

The various scalings on the Boltzmann equation and the corresponding fluid dynamic limits are summarized in the table below. In all the scaling limits presented above, the small parameter $\epsilon$ is the ratio of the molecular mean free path to some characteristic, macroscopic length scale in the flow, known as the Knudsen number and denoted Kn. The parameter $\delta_\epsilon$ entering the initial condition, as in $\mathcal{M}_{(1,\delta_\epsilon u^{in},1)}$ measures the scale of fluctuations of the velocity field in terms of the velocity scale defined by the background temperature 1, i.e. the speed of sound $\sqrt{\frac{5}{3}}$. Therefore $\delta_\epsilon$ can be regarded as the Mach number (denoted Ma) associated to the initial state of the gas. Finally, the fluid dynamic limits described above may involve a different scaling of the time and space variables. Whenever one considers the distribution function $F$ scaled as $F(t/\epsilon\lambda_\epsilon, x/\epsilon, v)$, the additional scaling parameter $\lambda_\epsilon$ acting on the time variable can be thought of as the Strouhal number (denoted Sh), following the terminology introduced by Y. Sone [94].

The ratio of viscous dissipation to the strength of nonlinear advection in a fluid is measured by a dimensionless parameter called the Reynolds number, denoted Re. Specifically, $\mathrm{Re} = UL/\mu$, where $U$ and $L$ are respectively the typical velocity and length scales in the fluid flow, while $\mu$ is the kinematic viscosity of the fluid. The Reynolds, Mach and Knudsen numbers are related by the following relation:

**Von Karman relation**

$$\mathrm{Kn} = a\frac{\mathrm{Ma}}{\mathrm{Re}}$$

where $a$ is some "absolute number" (such as $\sqrt{\pi}\ldots$).

This important observation explains why the compressible Navier-Stokes equation cannot be obtained as a hydrodynamic *limit* of the Boltzmann equation, but just as a first order correction of the compressible Euler limit. Indeed, the hydrodynamic limit always assumes that $\mathrm{Kn} \to 0$; if one seeks a regime where the viscosity coefficient remains positive uniformly as $\mathrm{Kn} \to 0$, then $\mathrm{Re} = O(1)$. This implies that $\mathrm{Ma} \to 0$, so that the limiting velocity field is necessarily divergence-free. In other words, one can only obtain in this way the incompressible Navier-Stokes equations, and not the compressible Navier-Stokes system.

The fluid dynamic regimes presented above are summarized in the following Table 1.

The presentation of hydrodynamic limits above does not exhaust all possible limits of the Boltzmann equation leading to fluid dynamic equations. Other scalings lead to Navier-Stokes equations involving viscous heating terms as in [14], or fluid dynamic equations in thin layers [43]. An important class of hydrodynamic equations, for which Y. Sone coined the term of "ghost effect" corresponds to the

**Table 1** The various incompressible fluid dynamic regimes of the Boltzmann equation in terms of the dimensionless parameters Kn (Knudsen number), Ma (Mach number), Re (Reynolds number) and Sh (Strouhal number)

| Boltzmann equation $Kn = \epsilon \ll 1$ | | |
|---|---|---|
| von Karman relation $Ma/Kn = Re$ | | |
| Ma | Sh | Hydrodynamic limit |
| $\delta_\epsilon \ll 1$ | 1 | Acoustic system |
| $\delta_\epsilon \ll \epsilon$ | $\epsilon$ | Stokes system |
| $\delta_\epsilon \gg \epsilon$ | $\delta_\epsilon$ | Incompressible Euler equations |
| $\epsilon$ | $\epsilon$ | Incompressible Navier-Stokes equations |

persistence at macroscopic scale of effects caused by quantities of the order of Kn (or smaller), which are therefore negligible in the hydrodynamic limit. There is an important literature on ghost effects: see for instance [17, 63, 95], Y. Sone's Harold Grad lecture [92] or Chap. 3.3 in [94] for a more detailed presentation of this class of problems.

In the next two lectures, we shall discuss in more detail the incompressible Euler and the incompressible Navier-Stokes-Fourier limits.

## 2.3 *Mathematical Tools: An Overview*

We conclude this first lecture with a quick overview of the mathematical notions and methods used in the proof of these limits.

### 2.3.1 Local Conservation Laws

At the formal level, an important step in deriving fluid dynamic models from the Boltzmann equation is to start from the local conservation laws implied by the Boltzmann equation, which are recalled below for the reader's convenience:

$$\partial_t \int_{\mathbf{R}^3} F_\epsilon \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} dv + \operatorname{div}_x \int_{\mathbf{R}^3} F_\epsilon \begin{pmatrix} v \\ v \otimes v \\ v\frac{1}{2}|v|^2 \end{pmatrix} dv = 0\,.$$

For instance, if one knows that

$$F_\epsilon \to F \text{ a.e. pointwise}$$

as $\epsilon \to 0^+$, Boltzmann's H Theorem and Fatou's lemma imply that

$$\int_0^\infty \iint \mathscr{C}(F) \ln F \, dx \, dv \, dt = 0 \,,$$

and thus

$$F \equiv \mathscr{M}_{(\rho,u,\theta)(t,x)}(v) \,.$$

This implies the following "closure relations": in other words, one expresses

$$\int_{\mathbf{R}^3} F \begin{pmatrix} v \otimes v \\ v\frac{1}{2}|v|^2 \end{pmatrix} dv \quad \text{in terms of} \int_{\mathbf{R}^3} F \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} dv \,.$$

Thus, passing to the limit in the local conservation laws (at the formal level) results in a system of PDEs on $\rho$, $u$ and $\theta$.

Because the renormalization procedure is a purely local change of unknown function, it destroys the delicate, nonlocal symmetries in the Boltzmann collision integral. For this reason, it is yet unknown at the time of this writing whether renormalized solutions of the Boltzmann equation satisfy all the local conservation laws above. They are only known to satisfy the local conservation of mass

$$\partial_t \int_{\mathbf{R}^3} F + \operatorname{div}_x \int_{\mathbf{R}^3} vF dv = 0 \,.$$

Instead of the usual local conservation laws of momentum and energy, renormalized solutions of the Boltzmann equation satisfy

$$\begin{aligned} \partial_t \int_{\mathbf{R}^3} \Gamma\left(\frac{F_\epsilon}{M}\right) \begin{pmatrix} v \\ \frac{1}{2}|v|^2 \end{pmatrix} M dv + \operatorname{div}_x \int_{\mathbf{R}^3} \Gamma\left(\frac{F_\epsilon}{M}\right) \begin{pmatrix} v \otimes v \\ \frac{1}{2}|v|^2 v \end{pmatrix} M dv \\ = \int_{\mathbf{R}^3} \Gamma'\left(\frac{F_\epsilon}{M}\right) \mathscr{C}(F_\epsilon) \begin{pmatrix} v \\ \frac{1}{2}|v|^2 \end{pmatrix} dv \,. \end{aligned}$$

An important step in the proof of all the hydrodynamic limits described above will be (a) to prove that the r.h.s. of the equalities above vanishes as $\epsilon \to 0$ and (b) that one recovers the usual conservation laws of momentum and energy *in the hydrodynamic limit*, i.e. as $\epsilon \to 0$.

### 2.3.2 Strong Compactness Tools

Since the Navier-Stokes equations are nonlinear, strong compactness (in the Lebesgue $L^1_{loc}$ space) of number density fluctuations is needed in order to pass to the limit in nonlinearities.

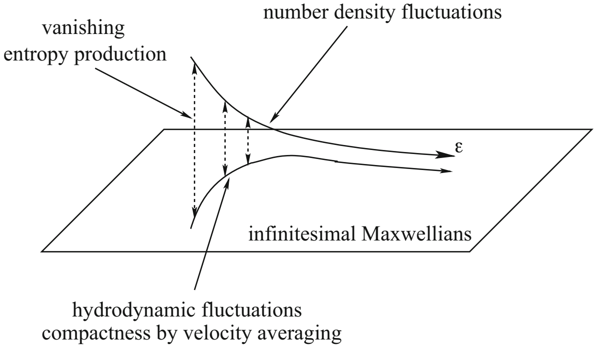


**Fig. 2** The family of number density fluctuations approaching the linear manifold of infinitesimal Maxwellian equilibria

The tool for obtaining this compactness is the method of velocity averaging (V. Agoshkov [1], F. Golse-B. Perthame-R. Sentis [51], F. Golse-P.-L. Lions-B. Perthame-R. Sentis [52]), adapted to the $L^1$ setting. The main statement needed for our purposes is essentially the theorem below.

**Theorem 10 (F. Golse-L. Saint-Raymond [47]).** *Assume that* $f_n \equiv f_n(x,v)$ *and* $v \cdot \nabla_x f_n$ *are bounded in* $L^1(\mathbf{R}_x^N \times \mathbf{R}_v^N)$*, while* $f_n$ *is bounded in* $L^1(\mathbf{R}_x^N; L^p(\mathbf{R}_v^N))$ *for some* $p > 1$*. Then*

*(a)* $f_n$ *is weakly relatively compact in* $L^1_{loc}(\mathbf{R}_x^N \times \mathbf{R}_v^N)$*; and*
*(b) For each* $\phi \in C_c(\mathbf{R}^N)$*, the sequence of velocity averages*

$$\int_{\mathbf{R}^N} f_n(x,v)\phi(v)dv$$

*is strongly relatively compact in* $L^1_{loc}(\mathbf{R}^N)$.

Observe that the velocity averaging theorem above only gives the strong compactness in $L^1_{loc}$ of *moments* of the sequence of distribution functions $f_n$, and not of the distribution functions themselves.

However, the bound on the entropy production coming from Boltzmann's H Theorem shows that the fluctuations of number densities approach the manifold of infinitesimal Maxwellians (i.e. the tangent linear space of the manifold of Maxwellian equilibrium distribution functions at $M := \mathscr{M}_{(1,0,1)}$). Infinitesimal Maxwellians are—exactly like Maxwellian distribution functions—parametrized by their moments of order $\leq 2$ in the $v$ variables, and this explains why strong compactness of the moments of the fluctuations of number density about the uniform Maxwellian equilibrium $M$ is enough for the Navier-Stokes limit (Fig. 2).

This will be discussed in a more detailed manner in Lecture 3.

### 2.3.3 The Relative Entropy Method: General Principle

In the regime of *inviscid* hydrodynamic limits, entropy production does **not** balance streaming in the Boltzmann equation. Therefore, the velocity averaging method cannot be applied in the case of inviscid limits, in general.[2]

For this reason, we choose another approach, namely to use the regularity of the solution of the target equation together with the relaxation towards local equilibrium in order to prove the compactness of fluctuations.

Our starting point is to pick $u$, a smooth solution of the target equations—say, in the case of the incompressible Euler equations—and to study the evolution of the quantity

$$Z_\epsilon(t) := \frac{1}{\delta_\epsilon^2} H(F_\epsilon(t/\epsilon\delta\epsilon, x/\epsilon, \cdot) | \mathscr{M}_{(1,\delta_\epsilon u(t,x),1)}) .$$

Notice the subtle difference with the usual Boltzmann H Theorem used in the DiPerna-Lions existence theorem of renormalized solutions described above. In the present case, the relative entropy is computed with respect to the *local* Maxwellian equilibrium whose parameters are defined in terms of the solution of the target equation. In the work of DiPerna-Lions, the relative entropy is defined with respect to the *global* Maxwellian equilibrium $M$.

The idea of studying the evolution of this quantity goes back to the work of H.T. Yau (for Ginzburg-Landau lattice models [99]). It was later adapted to the case of the Boltzmann equation (see Chap. 2 in [20] and [73]).

At the formal level, assuming the incompressible Euler scaling, one finds that

$$\begin{aligned}\frac{dZ_\epsilon}{dt}(t) \le &- \frac{1}{\delta_\epsilon^2} \int_{\mathbf{R}^3} \nabla_x u(t,x) : \int_{\mathbf{R}^3} (v - \delta_\epsilon u(t,x))^{\otimes 2} F_\epsilon \left( \frac{t}{\epsilon\delta_\epsilon}, \frac{x}{\epsilon}, v \right) dv\, dx \\ &+ \frac{1}{\delta_\epsilon} \int_{\mathbf{T}^3} \nabla_x p(t,x) \cdot \int_{\mathbf{R}^3} (v - \delta_\epsilon u(t,x)) F_\epsilon \left( \frac{t}{\epsilon\delta_\epsilon}, \frac{x}{\epsilon}, v \right) dv\, dx .\end{aligned}$$

The second term on the right hand side vanishes with $\epsilon$ since one expects that

$$\frac{1}{\delta_\epsilon} \int_{\mathbf{R}^3} v F_\epsilon \left( \frac{t}{\delta_\epsilon \epsilon}, \frac{x}{\epsilon}, v \right) dv \to \text{ divergence free field.}$$

The key step in the relative entropy method is to estimate the first term in the right hand side by $Z_\epsilon$ plus $o(1)$, at least locally in time. In other words, for all $T > 0$, there exists $C_T > 0$ such that

[2] This is not completely true, however, since the velocity averaging method is at the heart of the kinetic formulation of hyperbolic conservation laws. Unfortunately, while this approach is rather successful in the case of scalar conservation laws, it seems so far limited to some very special kind of hyperbolic systems: see P.-L. Lions-B. Perthame-E. Tadmor [74], P.-E. Jabin-B. Perthame [61], B. Perthame [83].

$$\frac{1}{\delta_\epsilon^2}\iint_{\mathbf{R}^3\times\mathbf{R}^3}\left|\nabla_x u(t,x):(v-\delta_\epsilon u(t,x))^{\otimes 2}F_\epsilon\left(\frac{t}{\epsilon\delta_\epsilon},\frac{x}{\epsilon},v\right)\right|dv\,dx\le C_T Z_\epsilon(t)+o(1)$$

for each $t\in[0,T]$.

Applying Gronwall's lemma, we conclude that

$$Z_\epsilon(t)\le e^{C_T t}(Z_\epsilon(0)+o(1))$$

for all $t\in[0,T]$.

By choosing appropriately the initial distribution function $F_\epsilon\big|_{t=0}$, the right hand side of this inequality vanishes as $\epsilon\to 0$, and this shows that $Z_\epsilon(t)\to 0$ as $\epsilon\to 0$ for all $t>0$. Since the relative entropy $H(F|G)$ somehow measures the "distance" between the distribution functions $F$ and $G$, this last estimate is exactly what is needed to conclude that the fluctuations of velocity field appropriately scaled, i.e.

$$\frac{1}{\delta_\epsilon}\int_{\mathbf{R}^3} vF_\epsilon\left(\frac{t}{\delta_\epsilon\epsilon},\frac{x}{\epsilon},v\right)dv$$

converge strongly to the solution $u$ of the incompressible Euler equations as $\epsilon\to 0$.

As we shall see, the constant $C_T$ is (essentially) given by the formula

$$C_T=\|\nabla_x u\|_{L^\infty([0,T]\times\mathbf{R}^3)}$$

and this is precisely why the regularity of solution of the target equation—of the incompressible Euler equation in the present case—is essential for this method.

More precisely, a distinctive feature of the relative entropy method is that it is particularly well adapted to study hydrodynamic limits of *weak* (or even renormalized) solutions of kinetic models when the target solution is *smooth*—or at least satisfies some stability property.

## 3 Lecture 2: The Incompressible Euler Limit

This lecture is devoted to a simplified variant of L. Saint-Raymond's theorem (Theorem 7). In order to alleviate the technicalities in the proof, we have chosen to discuss the incompressible Euler limit of the BGK, instead of the Boltzmann equation. As we shall explain below, the BGK equation is a much simplified analogue of the Boltzmann equation.

### *3.1 The Incompressible Euler Equations*

Since the stability of the target solution of the incompressible Euler equation is essential for applying the relative entropy method, we first briefly review the existence, uniqueness and regularity theory for that equation.

The incompressible Euler equation considered here describes the motion of an incompressible fluid, with constant density 1, in space dimension $N = 2$ or $N = 3$. The state of the fluid at time $t$ is defined by the velocity field $u \equiv u(t,x) \in \mathbf{R}^N$ and the pressure $p \equiv p(t,x) \in \mathbf{R}$. They satisfy the system of partial differential equations (see for instance [72])

$$\begin{aligned} \operatorname{div}_x u = 0\,, &\qquad \text{(continuity equation)}\\ \partial_t u + (u\cdot\nabla_x)u + \nabla_x p = 0\,. &\qquad \text{(momentum equation)} \end{aligned}$$

In the case of an incompressible fluid without external force (such as gravity), the kinetic energy is a locally conserved quantity. Taking the inner product of both sides of the momentum equation above with $u$ leads to the identity:

$$\partial_t(\tfrac{1}{2}|u|^2) + \operatorname{div}_x\left(u(\tfrac{1}{2}|u|^2 + p)\right) = 0\,.$$

(Indeed, one has

$$(u\cdot\nabla_x u + \nabla_x p)\cdot u = u\cdot\nabla_x(\tfrac{1}{2}|u|^2) + u\cdot\nabla_x p = \operatorname{div}_x\left(u(\tfrac{1}{2}|u|^2 + p)\right)$$

because $\operatorname{div}_x u = 0$.)

Another quantity of paramount importance in the theory of inviscid incompressible fluids with constant density is the vorticity field, denoted by $\Omega$, whose evolution is described as follows:

- If $N = 2$, the vorticity field is defined as $\Omega := \partial_1 u_2 - \partial_2 u_1 \in \mathbf{R}$ and one easily checks that

$$\partial_t \Omega + u\cdot\nabla_x\Omega = 0\,;$$

- If $N = 3$, the vorticity field is defined as $\Omega := \operatorname{curl}_x u \in \mathbf{R}^3$ and one has

$$\partial_t\Omega + (u\cdot\nabla_x)\Omega - (\Omega\cdot\nabla_x)u = 0\,.$$

### 3.1.1 Existence and Uniqueness Theory for the Incompressible Euler Equation

Consider the Cauchy problem for the incompressible Euler equations:

$$\begin{cases} \operatorname{div}_x u = 0\,,\\ \partial_t u + (u\cdot\nabla_x)u + \nabla_x p = 0\,, \quad x\in\mathbf{R}^N\,,\\ u\big|_{t=0} = u^{in}\,. \end{cases}$$

**Theorem 11 (V. Yudovich, T. Kato).** *Consider the Cauchy problem for the incompressible Euler equations in space dimension $N = 2$ or $3$. Then*

- $N = 2$*: if* $u^{in} \in L^2 \cap C^{1,\alpha}(\mathbf{R}^2)$ *for* $\alpha \in (0,1)$ *and* $\Omega^{in} \in L^\infty(\mathbf{R}^2)$*, then there exists a unique solution* $u \in C(\mathbf{R}_+^*; L^2 \cap C^{1,\alpha}(\mathbf{R}^2))$ *of the Cauchy problem for the incompressible Euler equation with initial velocity field* $u^{in}$*, and* $\Omega \in L^\infty(\mathbf{R}_+ \times \mathbf{R}^2)$*;*
- $N = 3$*: if* $u^{in} \in L^2 \cap C^{1,\alpha}(\mathbf{R}^3)$ *for* $\alpha \in (0,1)$*, then there exist* $T^* > 0$ *and a unique maximal solution* $u \in C([0,T^*); L^2 \cap C^{1,\alpha}(\mathbf{R}^3))$ *of the Cauchy problem for the incompressible Euler equation with initial velocity field* $u^{in}$*.*

See Theorem 4.1 in [72] for the case $N = 2$, and Sect. 4.3 in the same references for the case $N = 3$. Whether $T^* = +\infty$ in the case where $N = 3$ remains an outstanding open question at the time of this writing.

### 3.1.2 Dissipative Solutions of the Incompressible Euler Equation

Since little is known about the global existence of classical solutions of the incompressible Euler equation in space dimension $N = 3$, there have been several attempts at defining a convenient notion of weak solutions of this equation. Weak solutions of the Euler equation in the sense of distributions are not expected to be unique—in fact, these solutions have some rather paradoxical features. For instance, there exist solutions of the incompressible Euler equations corresponding to a fluid at rest, which starts to move in the absence of any external force, and stops moving after some finite time (see [32, 89, 90]). On the other hand, a global existence result for weak solutions of the Euler equation in the sense of distributions has been recently obtained by E. Wiedemann [98]. Other notions of generalized solutions of the Euler equation had been proposed earlier, such as the notion of measure-valued solutions [35].

While not much can be said of these solutions, returning to the variational formulation of the incompressible Euler equations viewed as defining a geodesic flow in infinite dimension [3, 4] leads to well-posed problems for these equations [22]—but unfortunately, these problems, although interesting in their own right, are different from the Cauchy problem.

In view of all these difficulties, P.-L. Lions proposed a very weak notion of solution of the incompressible Euler equation, which he called "dissipative solutions", and whose definition is recalled below (see Sect. 4.4 in [72]).

Set

$$\mathscr{X}_T := \{v \in C([0,T]; L^2(\mathbf{R}^3)) \text{ s.t. } \operatorname{div}_x v = 0\,,\ \ \Sigma(v) \in L^1([0,T]; L^\infty(\mathbf{R}^3)) \\ \text{and } E(v) \in L^1([0,T]; L^2(\mathbf{R}^3))\}\,,$$

where

$$\Sigma(v) := \tfrac{1}{2}(\nabla_x v + (\nabla_x v)^T)\,, \qquad \text{and } E(v) := \partial_t v + (v \cdot \nabla_x)v\,.$$

**Definition 3 (P.-L. Lions).** A vector field[3] $u \in C_b(\mathbf{R}_+; w-L^2(\mathbf{R}^3))$ is a dissipative solution of the Cauchy problem for the incompressible Euler equation with initial velocity field $u^{in}$ if $\operatorname{div}_x u = 0$ and, for each $T > 0$, each $v \in \mathscr{X}_T$ and each $t \in [0, T]$, one has

$$\tfrac{1}{2}\|u-v\|_{L^2}^2(t) \le \exp\left(\int_0^t 2\|\Sigma(v)\|_{L^\infty}(s)ds\right)\tfrac{1}{2}\|u^{in}-v\big|_{t=0}\|_{L^2}^2$$
$$+\int_0^t \exp\left(\int_\tau^t 2\|\Sigma(v)\|_{L^\infty}(s)ds\right)\int E(v)\cdot(u-v)(\tau,x)dx\,d\tau\,.$$

The nicest features of this notion of dissipative solution is that the Cauchy problem for the incompressible Euler equation always has at least one dissipative solution, and also the fact that classical solutions of the incompressible Euler equation are uniquely determined by their initial data within the class of dissipative solutions.

**Theorem 12 (P.-L. Lions [72]).** *For each $u^{in} \in L^2(\mathbf{R}^N)$ s.t. $\operatorname{div}_x u^{in} = 0$, there exists a dissipative solution of the Cauchy problem for the incompressible Euler equation defined for all $t \ge 0$. Besides*

- *If $u \in C_b^1([0,T]\times\mathbf{R}^N)$ is a classical solution of the Cauchy problem for the Euler equation with initial velocity field $u^{in}$, then $u$ is a dissipative solution.*
- *If the Cauchy problem for the incompressible Euler equation with initial velocity field $u^{in}$ has a solution $\overline{u} \in \mathscr{X}_T$ for some $T > 0$, any dissipative solution $u$ of the incompressible Euler equation with initial velocity field $u^{in}$ satisfies*

$$u(t,x) = \overline{u}(t,x) \text{ for a.e. } x \in \mathbf{R}^N\,, \quad \text{for all } t \in [0,T]$$

*Proof.* Observe that limit points of Leray solutions of the incompressible Navier-Stokes equation in the vanishing viscosity limit are dissipative solutions of the incompressible Euler equations. This implies the global existence of dissipative solutions of the Cauchy problem for the incompressible Euler equation for all initial square integrable, divergence free velocity field $u^{in}$.

Observe next that, if $u$ is a $C^1$ solution of the Euler equation

$$E(u) - E(v) = (\partial_t + u\cdot\nabla_x)(u-v) + (u-v)\cdot\nabla_x v\,,$$

which implies that

$$(\partial_t + u\cdot\nabla_x)\tfrac{1}{2}|u-v|^2 + \Sigma(v) : (u-v)^{\otimes 2} = (E(u)-E(v))\cdot(u-v)\,.$$

[3]Let $X$ be a topological space and $E$ a topological vector space. The notation $C_b(X,E)$ designates the set of bounded continuous maps from $X$ to $E$.

Since $\operatorname{div}_x u = 0$, integrating in $x$ both sides of the identity above shows that

$$\frac{d}{dt}\tfrac{1}{2}\|u-v\|_{L^2}^2 \leq \|\Sigma(v)\|_{L^\infty}\|u-v\|_{L^2}^2 + (E(v)|u-v)_{L^2}\,,$$

since

$$\int_{\mathbf{R}^3} E(u)\cdot(u-v)dx = -\int_{\mathbf{R}^3} \nabla_x p\cdot(u-v)dx = \int_{\mathbf{R}^3} p\operatorname{div}_x(u-v)dx = 0\,.$$

Applying Gronwall's lemma shows that $u$ is a dissipative solution of Euler's equation.

Finally the last property, usually referred to as the "weak-strong uniqueness" property of dissipative solutions of the incompressible Euler equation is obtained by the observation below. If one choose $v = \overline{u}$ in the defining inequality for dissipative solutions, one finds that

$$\int_{\mathbf{R}^3} E(v)\cdot(u-v)(\tau,x)dx = -\int_{\mathbf{R}^3} \nabla_x\overline{p}\cdot(u-\overline{u})(\tau,x)dx = 0$$

because $\operatorname{div}_x u = \operatorname{div}_x \overline{u} = 0$. Therefore

$$\tfrac{1}{2}\|u-\overline{u}\|_{L^2}^2(t) \leq \exp\left(\int_0^t 2\|\Sigma(\overline{u})\|_{L^\infty}(s)ds\right)\tfrac{1}{2}\|u^{in}-\overline{u}\big|_{t=0}\|_{L^2}^2 = 0$$

for all $t \in [0,T]$.

Of course, it is unknown whether two dissipative solutions of the incompressible Euler equation with the same initial condition coincide on the time interval on which they are both defined.

Any dissipative solution of the Euler equation that is obtained as limits points of Leray solutions of the Navier-Stokes equation in the vanishing viscosity limit satisfies the following variant of the motion equation:

$$\partial_t u + \operatorname{div}_x(u\otimes u + \sigma) + \nabla_x p = 0\,,$$

where $\sigma \equiv \sigma(t,x) \in M_3(\mathbf{R})$ is a matrix field satisfying

$$\sigma = \sigma^T \geq 0\,.$$

Whether $\sigma = 0$—in other words, whether $u$ is a solution of the Euler equation in the sense of distributions—remains unknown at the time of this writing.

There are other notions of weak solutions of the incompressible Euler equation, in particular the measure-valued solutions defined by R. DiPerna and A. Majda [35]. An important observation on this class of solutions is that they satisfy the weak-strong uniqueness property, exactly as dissipative solutions: see [23]. This is at variance with other classes of weak solutions of the Euler equation, such as the general weak solutions in the sense of distributions. As mentioned above, nonzero weak solutions in the sense of distributions of the Euler equation with compact support have been constructed by V. Scheffer (see [32, 89, 90]). Therefore, weak solutions of the Euler equation in the sense of distributions cannot satisfy the weak-strong uniqueness property—otherwise, the only such solution with compact support in the time variable would be $u = 0$.

## *3.2 The BGK Model with Constant Relaxation Time*

In order to alleviate some technical steps in the proof of the incompressible Euler limit of the Boltzmann equation, we shall consider as our starting point the BGK model with constant relaxation time instead of the Boltzmann equation itself. Some of the unpleasant features of the theory of renormalized solutions of the Boltzmann equation, especially regarding the local conservation laws either disappear or become significantly simpler with the BGK model.

The idea is therefore to replace the Boltzmann equation with the simplest imaginable relaxation model with constant relaxation time $\tau > 0$

$$(\partial_t + v \cdot \nabla_x)F = \frac{1}{\tau}(M_F - F)\,, \quad x \in \mathbf{T}^3\,,\ v \in \mathbf{R}^3\,,$$

where

$$M_F \equiv M_F(t,x,v) := \mathscr{M}_{(\rho_F,u_F,\theta_F)(t,x)}(v)\,,$$

with

$$\int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} M_F(t,x,v)dv = \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} F(t,x,v)dv\,.$$

In other words, $(\rho_F, u_F, \theta_F)$ are defined as follows:

$$\rho_F = \int_{\mathbf{R}^3} F dv\,, \quad u_F = \frac{1}{\rho_F}\int_{\mathbf{R}^3} vF dv\,, \quad \theta_F = \frac{1}{\rho_F}\int_{\mathbf{R}^3} \tfrac{1}{3}|v-u_F|^2 F dv\,.$$

This model Boltzmann equation is called the "BGK model", after Bhatnagar, Gross and Krook, who proposed (a more complicated variant of) this model for the first time in 1954 [16].

We recall below the notation already adopted above for Maxwellians: in space dimension 3, for $\rho \geq 0$, $u \in \mathbf{R}^3$ and $\theta > 0$,

$$\mathscr{M}_{(\rho,u,\theta)}(v) := \frac{\rho}{(2\pi\theta)^{3/2}} e^{-|v-u|^2/2\theta}\,.$$

In the limit as $\theta \to 0^+$, one has $\mathscr{M}_{(\rho,u,\theta)} \to \mathscr{M}_{(\rho,u,0)}$, where

$$\mathscr{M}_{(\rho,u,0)} := \rho\delta(v-u)\,.$$

In the particular case $\rho = \theta = 1$ and $u = 0$, we denote as above

$$M(v) := \mathscr{M}_{(1,0,1)}(v) = \tfrac{1}{(2\pi)^{3/2}} e^{-|v|^2/2}\,.$$

### 3.2.1 Formal Properties of the BGK Model

Classical solutions of the BGK model satisfy exactly the same local conservation laws of mass, momentum and energy as classical solutions of the Boltzmann equation, under appropriate decay assumptions as $|v| \to \infty$.

**Proposition 1.** *Let $F \in C(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$ such that $\nabla_{t,x} F \in C(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$ satisfy*

$$F \ge 0 \textit{ and } \sup_{t+|x|\le R} F(t,x,v) + |\nabla_{t,x} F(t,x,v)| \le \frac{C_R}{(1+|v|)^7}$$

*for each $R > 0$. Then, the following conservation laws are satisfied:*

$$\begin{aligned}
\partial_t \int_{\mathbf{R}^3} F dv + \operatorname{div}_x \int_{\mathbf{R}^3} vF dv &= 0\,, \quad \textit{(mass)}\\
\partial_t \int_{\mathbf{R}^3} vF dv + \operatorname{div}_x \int_{\mathbf{R}^3} v^{\otimes 2} F dv &= 0\,, \quad \textit{(momentum)}\\
\partial_t \int_{\mathbf{R}^3} \tfrac{1}{2}|v|^2 F dv + \operatorname{div}_x \int_{\mathbf{R}^3} v\tfrac{1}{2}|v|^2 F dv &= 0\,. \quad \textit{(energy)}
\end{aligned}$$

*Proof.* The assumptions on the decay of $F$ and $\nabla_{t,x} F$ as $|v| \to \infty$ imply that

$$\begin{aligned}
\partial_t \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F dv + \operatorname{div}_x \int_{\mathbf{R}^3} \begin{pmatrix} v \\ v \otimes v \\ \frac{1}{2} v|v|^2 \end{pmatrix} F dv
&= \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} (\partial_t + v \cdot \nabla_x) F dv \\
&= \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} (M_F - F) dv = 0\,,
\end{aligned}$$

by definition of $M_F$.

They also satisfy the following local variant of Boltzmann's H Theorem.

**Proposition 2.** *Let* $F \in C(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$ *such that* $\nabla_{t,x} F \in C(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$ *satisfy*

$$F \geq 0 \quad \textit{and} \quad \sup_{t+|x| \leq R} (F \ln F(t,x,v) + |\nabla_{t,x}(F \ln F)(t,x,v)|) \leq \frac{C_R}{(1+|v|)^4}$$

*for each* $R > 0$*. Then*

$$\partial_t \int_{\mathbf{R}^3} F \ln F dv + \operatorname{div}_x \int_{\mathbf{R}^3} vF \ln F dv = \frac{1}{\tau} \int_{\mathbf{R}^3} (M_F - F) \ln \frac{F}{M_F} dv \leq 0\,.$$

*Proof.* Indeed $\ln M_F$ is a linear combination of $1, v_1, v_2, v_3, |v|^2$ so that

$$\int_{\mathbf{R}^3} (F - M_F) \ln M_F dv = 0\,,$$

again by definition of $M_F$.

### 3.2.2 The Cauchy Problem for the BGK Model with Constant Relaxation Time

Consider the Cauchy problem

$$\begin{cases} (\partial_t + v \cdot \nabla_x) F = \dfrac{1}{\tau}(M_F - F)\,, \quad x \in \mathbf{T}^3\,,\ v \in \mathbf{R}^3\,, \\ F\big|_{t=0} = F^{in}\,. \end{cases} \tag{9}$$

**Theorem 13 (B. Perthame-M. Pulvirenti).** *Assume that there exist* $\rho_2 > \rho_1 > 0$ *and* $\theta_2 > \theta_1 > 0$ *such that the initial distribution function* $F^{in}$ *satisfies the inequalities*

$$\mathscr{M}_{(\rho_1,0,\theta_1)} \leq F^{in} \leq \mathscr{M}_{(\rho_2,0,\theta_2)}\,.$$

*Then there exists a unique solution of the Cauchy problem (9), which satisfies*

$$C_1(t,\tau) \leq \rho_F(t,x),\ \theta_F(t,x) \leq C_2(t,\tau)\,, \quad |u_F(t,x)| \leq C_2(t,\tau)\,,$$
$$\textit{and} \ \sup_{x,v} |v|^m F(t,x,v) < C_3(t,\tau,m)\,.$$

See [84] for a proof of this result.

Since the relaxation time in the model above is a constant, the collision term $M_F - F$ is homogeneous of degree 1 in the distribution function $F$. In other words, one has $M_{\lambda F} - \lambda F = \lambda(M_F - F)$. This is precisely the reason why there is no need for the renormalization procedure used for the Boltzmann equation. Thus the existence theory is significantly simpler for this model than for the Boltzmann equation itself.

In fact, the genuine BGK model involves a relaxation time that is proportional to the reciprocal local macroscopic density. In other words, this model is of the form

$$(\partial_t + v \cdot \nabla_x)F = \frac{1}{\tau_0}\rho_F(M_F - F)\,,$$

with

$$\rho_F(t,x) := \int_{\mathbf{R}^3} F(t,x,v)dv\,.$$

The collision term $\frac{1}{\tau_0}\rho_F(M_F - F)$ is now homogeneous of degree 2, meaning that

$$\rho_{\lambda F}(M_{\lambda F} - \lambda F) = \lambda^2 \rho_F(M_F - F)\,,$$

just like the Boltzmann collision integral which is a quadratic operator. This model is obviously more natural than the one with constant relaxation time, since the higher the local density $\rho_F$, the smaller the local particle mean free path, i.e. $\tau_0/\rho_F$. This BGK model is used as a toy model in rarefied gas dynamics. Unfortunately, even though the numerical analysis of the BGK model is significantly simpler than that of the Boltzmann equation, much less is known on the mathematical analysis of this model than on the Boltzmann equation itself. For instance, the renormalization procedure is rather uneffective on the BGK model, so that there is no analogue of the DiPerna-Lions theory on that model.

### 3.2.3 The BGK Equation in the Incompressible Euler Scaling

Set the relaxation time $\tau = \epsilon^q$ with $q > 1$ for $\epsilon > 0$ small enough, and rescale the time variable as $\hat{t} = t/\epsilon$. The Cauchy problem for the BGK equation with constant relaxation time takes the form

$$\begin{cases} (\epsilon\partial_{\hat{t}} + v \cdot \nabla_x)F_\epsilon = \dfrac{1}{\epsilon^q}(M_{F_\epsilon} - F_\epsilon)\,, \quad x \in \mathbf{T}^3\,,\ v \in \mathbf{R}^3\,, \\ F\big|_{t=0} = \mathscr{M}_{(1,\epsilon u^{in},1)}\,. \end{cases} \tag{10}$$

Henceforth, we assume that

$$u^{in} \in C(\mathbf{T}^3)\,, \quad \text{with } \operatorname{div} u^{in} = 0\,.$$

The incompressible Euler limit of the BGK model with constant relaxation time is described in the following theorem.

**Theorem 14 (L. Saint-Raymond).** *Let* $u^{in} \in C^{1,\alpha}(\mathbf{T}^3)$ *be s.t.* $\operatorname{div} u^{in} = 0$ *and let* $u$ *be the maximal solution of the incompressible Euler equation with initial data* $u^{in}$ *defined on* $[0, T^*)$. *Let* $F_\epsilon$ *be the solution of the scaled BGK equation with initial data* $\mathscr{M}_{(1,\epsilon u^{in},1)}$. *Then*

$$\frac{1}{\epsilon}\int_{\mathbf{R}^3} vF_\epsilon(t,\cdot,v)dv \to u(t,\cdot) \text{ in weak } L^1(\mathbf{T}^3)\,,$$

*uniformly on* $[0, T]$ *for each* $0 \le T < T^*$ *as* $\epsilon \to 0$.

The proof of this theorem can be found in [85]. This result was later extended to renormalized solutions of the Boltzmann equation [87]. Earlier partial results were obtained by Golse [20], and by P.-L. Lions and N. Masmoudi [73].

This result is based on the relative entropy method, which is a very important tool in the rigorous asymptotic analysis of partial differential equations. For that reason, we have given a rather detailed account of the proof in the case of the BGK model. Proving the same result for the Boltzmann equation involves additional technicalities that are special to the theory of renormalized solutions.

## 3.3 *Proof of the Incompressible Euler Limit*

This section is devoted to L. Saint-Raymond's proof of the incompressible Euler limit of the BGK equation.

### 3.3.1 Step 1: Uniform Estimates

All the uniform estimates on this problem come from (the analogue of) Boltzmann's H theorem. Specifically, we compute the evolution of the relative entropy $H(F_\epsilon|M)$; one has

$$\epsilon\partial_t \int_{\mathbf{R}^3}\left(F_\epsilon \ln\left(\frac{F_\epsilon}{M}\right) - F_\epsilon + M\right)dv + \operatorname{div}_x \int_{\mathbf{R}^3} v\left(F_\epsilon \ln\left(\frac{F_\epsilon}{M}\right) - F_\epsilon + M\right)dv$$
$$= \frac{1}{\epsilon^q}\int_{\mathbf{R}^3}(M_{F_\epsilon} - F_\epsilon)\ln\left(\frac{F}{M_{F_\epsilon}}\right)dv \le 0\,,$$

in view of the decay (in $|v|$) estimate in the Perthame-Pulvirenti theorem. Integrating further in $t$ and $x$, one finds that

$$H(F_\epsilon|M)(t) + \frac{1}{\epsilon^{q+1}}\int_0^\infty D(F_\epsilon)dt = H(\mathscr{M}_{(1,\epsilon u^{in},1)}|M) = \tfrac{1}{2}\epsilon^2\|u^{in}\|_{L^2}^2\,,$$

where

$$D(F) := \int_{\mathbf{T}^3}\int_{\mathbf{R}^3} (F - M_F) \ln\left(\frac{F}{M_F}\right) dv\, dx\,.$$

Hence

$$H(F_\epsilon|M)(t) \le C^{in}\epsilon^2\,, \quad \text{and} \int_0^\infty D(F_\epsilon)dt \le C^{in}\epsilon^{q+3}$$

with $C^{in} = \frac{1}{2}\|u^{in}\|_{L^2}^2$.

Instead of the distribution function $F_\epsilon$ itself, it will be more convenient to work with the relative fluctuation thereof, denoted

$$g_\epsilon := \frac{F_\epsilon - M}{\epsilon M}\,.$$

Consider the function $h$ defined on $(-1,\infty)$ by the formula

$$h(z) := (1+z)\ln(1+z) - z\,.$$

Its Legendre dual,[4] henceforth denoted $h^*$, is given by the formula

$$h^*(y) := e^y - y - 1\,, \qquad y \ge 0\,.$$

The Young inequality for the convex function $h$ implies that, for all $\alpha > \epsilon$,

$$\begin{aligned}\tfrac{1}{4}(1+|v|^2)|g_\epsilon| &= \frac{\alpha}{\epsilon^2}\cdot\frac{\epsilon}{\alpha}\frac{1+|v|^2}{4}\epsilon|g_\epsilon| \\ &\le \frac{\alpha}{\epsilon^2}h(\epsilon|g_\epsilon|) + \frac{\alpha}{\epsilon^2}h^*\left(\frac{\epsilon}{\alpha}\frac{1+|v|^2}{4}\right)\,.\end{aligned}$$

Using the elementary inequalities

$$h(|z|) \le h(z)\,, \text{ and } h^*(\theta y) = \sum_{k\ge 2}\frac{\theta^k z^k}{k!} \le \theta^2\sum_{k\ge 2}\frac{z^k}{k!} = \theta^2 h^*(y)$$

[4]The Legendre dual $f^*$ of a function $f : I \to \mathbf{R}$, where $I$ is an interval of $\mathbf{R}$, is defined for all $p \in \mathbf{R}$ by the formula

$$f^*(p) = \sup_{x\in I}(px - f(x))\,.$$

for all $z > -1$ and all $y \geq 0$ whenever $0 \leq \theta \leq 1$, we conclude that

$$\begin{aligned}\tfrac{1}{4}(1+|v|^2)|g_\epsilon| &= \frac{\alpha}{\epsilon^2}\tfrac{1}{4}\frac{\epsilon}{\alpha}(1+|v|^2)\epsilon|g_\epsilon| \\ &\leq \frac{\alpha}{\epsilon^2}h(\epsilon g_\epsilon) + \frac{1}{\alpha}h^*\left(\tfrac{1}{4}(1+|v|^2)\right).\end{aligned}$$

A first major consequence of the uniform bounds obtained above is the next proposition.

**Proposition 3.** *The family $(1+|v|^2)g_\epsilon$ is weakly relatively compact in the space $L^1([0,T];L^2(\mathbf{T}^3\times\mathbf{R}^3, M\,dx\,dv))$ for all $T>0$. If $(1+|v|^2)g$ is a limit point of this family (along a sequence $\epsilon_n\to 0$), then*

$$\iint_{\mathbf{T}^3\times\mathbf{R}^3} g^2 M\,dv\,dx \leq \underline{\lim}\frac{1}{\epsilon_n^2}H(F_{\epsilon_n}|M).$$

Another important observation is the following lemma, which follows from the elementary inequality

$$(1+z)\ln(1+z) - z \leq z\ln(1+z), \qquad z > -1.$$

**Lemma 1.** *For each $\epsilon > 0$ and all $t \geq 0$,*

$$H(F_\epsilon|M_{F_\epsilon})(t) \leq D(F_\epsilon)(t).$$

### 3.3.2 Step 2: The Modulated Relative Entropy

First observe that, for each vector field $u \in L^2(\mathbf{T}^3)$, one has

$$H(F_\epsilon|\mathscr{M}_{1,u,1}) = H(F_\epsilon|M_{F_\epsilon}) + H(M_{F_\epsilon}|\mathscr{M}_{1,u,\theta}),$$

since $M_{F_\epsilon}$ and $M$ have the same total mass.

Let $w \equiv w(t,x) \in \mathbf{R}^3$ be a vector field of class $C^1$ on $[0,T]\times\mathbf{T}^3$ satisfying $\operatorname{div}_x w = 0$, but not necessarily the Euler motion equation. Then

$$\begin{aligned}H(F_\epsilon|\mathscr{M}_{(1,\epsilon w,1)}) &= H(F_\epsilon|M) + \iint_{\mathbf{T}^3\times\mathbf{R}^3} F_\epsilon \ln\left(\frac{M}{\mathscr{M}_{(1,\epsilon w,1)}}\right)dx\,dv \\ &= H(F_\epsilon|M) + \iint_{\mathbf{T}^3\times\mathbf{R}^3}\tfrac{1}{2}(|v-\epsilon w|^2 - |v|^2)F_\epsilon dx\,dv \\ &= H(F_\epsilon|M) + \iint_{\mathbf{T}^3\times\mathbf{R}^3}(\tfrac{1}{2}\epsilon^2|w|^2 - \epsilon v\cdot w)F_\epsilon dx\,dv \\ &= H(F_\epsilon|M) + \int_{\mathbf{T}^3}\rho_{F_\epsilon}(\tfrac{1}{2}\epsilon^2|w|^2 - \epsilon u_{F_\epsilon}\cdot w)dx.\end{aligned}$$

Apply first the local conservation laws implied by the BGK equation

$$\epsilon\partial_t\rho_{F_\epsilon} + \operatorname{div}_x(\rho_{F_\epsilon}u_{F_\epsilon}) = 0\,,$$
$$\epsilon\partial_t(\rho_{F_\epsilon}u_{F_\epsilon}) + \operatorname{div}_x\int_{\mathbf{R}^3} v^{\otimes 2}F_\epsilon dv = 0\,.$$

Using the operator $E$ entering the definition of dissipative solutions, one has

$$\partial_t w = E(w) - (w\cdot\nabla_x)w\,,$$

and therefore

$$\begin{aligned}\frac{d}{dt}\int_{\mathbf{T}^3}\rho_{F_\epsilon}(\tfrac{1}{2}\epsilon^2|w|^2 - \epsilon u_{F_\epsilon}\cdot w)dx\\ = \iint_{\mathbf{T}^3\times\mathbf{R}^3}\big(\nabla_x w : (v-\epsilon w)^{\otimes 2} - \epsilon E(w)\cdot(v-\epsilon w)\big)F_\epsilon dx\,dv\\ = \iint_{\mathbf{T}^3\times\mathbf{R}^3}\big(\Sigma(w) : (v-\epsilon w)^{\otimes 2} - \epsilon E(w)\cdot(v-\epsilon w)\big)F_\epsilon dx\,dv\,,\end{aligned}$$

where we recall that

$$\Sigma(w) = \tfrac{1}{2}(\nabla_x w + (\nabla_x w)^T)\,.$$

The core of the proof is the inequality stated in the next proposition.

**Proposition 4.** *Let* $u^{in}\in C(\mathbf{T}^3)$ *satisfy* $\operatorname{div}u^{in} = 0$*; then for each test vector field* $w\in C^1([0,T]\times\mathbf{T}^3;\mathbf{R}^3)$ *such that* $\operatorname{div}_x w = 0$*, one has*

$$\begin{aligned}\frac{1}{\epsilon^2}H(F_\epsilon|\mathscr{M}_{(1,\epsilon w,1)})(t) + \frac{1}{\epsilon^{3+q}}\int_0^t D(F_\epsilon)ds \le \tfrac{1}{2}\|u^{in} - w\big|_{t=0}\|_{L^2}^2\\ -\frac{1}{\epsilon^2}\int_0^t\iint_{\mathbf{T}^3\times\mathbf{R}^3}\Sigma(w) : (v-\epsilon w)^{\otimes 2}F_\epsilon dx\,dv\\ -\frac{1}{\epsilon}\int_0^t\iint_{\mathbf{T}^3\times\mathbf{R}^3}E(w)\cdot(v-\epsilon w)F_\epsilon dx\,dv\,.\end{aligned}$$

This inequality is the analogue for the BGK equation of the weak-strong uniqueness inequality for the Euler equation, i.e.

$$\begin{aligned}\tfrac{1}{2}\|u-w\|_{L^2}^2(t) \le \tfrac{1}{2}\|u^{in} - w\big|_{t=0}\|_{L^2}^2\\ + \int_0^t\|\Sigma(w)\|_{L^\infty}\|u-w\|_{L^2}^2(s)ds + \int_0^t(E(v)|u-v)_{L^2}(s)ds\,,\end{aligned}$$

leading to the notion of dissipative solution (after applying Gronwall's inequality).

More precisely, one has the following correspondences

- Velocity field

$$\frac{1}{\epsilon}\int_{\mathbf{R}^3} vF_\epsilon dv \leftrightarrow u\,,$$

- Modulated energy

$$\frac{1}{\epsilon^2}H(F_\epsilon|\mathscr{M}_{(1,\epsilon w,1)})(t) \leftrightarrow \tfrac{1}{2}\|u-w\|^2_{L^2}(t)\,,$$

- Modulated inertial term

$$\frac{1}{\epsilon^2}\int_0^t\iint_{\mathbf{T}^3\times\mathbf{R}^3}\Sigma(w):(v-\epsilon w)^{\otimes 2}F_\epsilon dx\,dv$$
$$\leftrightarrow \int_0^t \|\Sigma(w)\|_{L^\infty}\|u-w\|^2_{L^2}(s)ds\,.$$

It remains to control both terms on the right hand side of the inequality in the proposition above in terms of the relative entropy and to conclude by Gronwall's lemma.

The last such term is disposed of without difficulty. We already know that

$$(1+|v|^2)g_\epsilon \to (1+|v|^2)g \quad \text{weakly in } L^1([0,T];L^1(\mathbf{T}^3\times\mathbf{R}^3;M\,dx\,dv))\,,$$

with

$$g\in L^\infty([0,T];L^2(\mathbf{T}^3\times\mathbf{R}^3;M\,dx\,dv))\,.$$

Therefore, one has the following limit.

**Lemma 2.** *Let* $U:=\langle vg\rangle$*; then* $\operatorname{div}_x U=0$ *and*

$$\frac{1}{\epsilon}\int_0^t\iint_{\mathbf{T}^3\times\mathbf{R}^3}E(w)\cdot(v-\epsilon w)F_\epsilon dx\,dv \to \int_{\mathbf{T}^3}E(w)\cdot(U-w)dx$$

*weakly in* $L^1([0,T])$ *for all* $T>0$.

### 3.3.3 Step 3: Controling the Modulated Inertial Term

In the case of the Euler equation, the contribution of the inertial term to the energy balance, i.e. $\Sigma(v) : (u - v)^{\otimes 2}$, is obviously controlled as follows:

$$|\Sigma(v) : (u - v)^{\otimes 2}| \leq \|\Sigma(v)\|_{L^\infty} \|u - v\|_{L^2}^2 .$$

Whether the analogue of the modulated inertial term in the context of the BGK equation can be controlled by the modulated relative entropy is more subtle. A major difficulty in obtaining this type of control is the fact that the relative entropy is subquadratic, unless the fluctuations of distribution function are already known to be small (of order $\epsilon$).

However, this difficulty can be solved by using the entropy production as well as the relative entropy. This control is explained in the next lemma, which is the key argument in the proof.

**Lemma 3.** *Under the same assumptions as in Theorem 14,*

$$\begin{aligned}\frac{1}{\epsilon^2}\int_0^t \iint_{\mathbf{T}^3\times\mathbf{R}^3} & \Sigma(w) : (v - \epsilon w)^{\otimes 2} F_\epsilon dx\, dv\, ds \\ & \leq C\|\Sigma(w)\|_{L^\infty} \int_0^t \frac{1}{\epsilon^2} H(F_\epsilon | \mathscr{M}_{(1,\epsilon w,1)}) ds \\ & + \epsilon^{(q-1)/2} \|\Sigma(w)\|_{L^\infty} \frac{1}{\epsilon^{q+3}} \int_0^t D(F_\epsilon) ds \\ & + C\epsilon^{(q-1)/2} \|\Sigma(w)\|_{L^1} .\end{aligned}$$

The idea is to split the distribution function $F_\epsilon$ as

$$F_\epsilon = M_{F_\epsilon} + (F_\epsilon - M_{F_\epsilon}) ,$$

and use both the entropy and entropy production bounds.

*Proof.* By definition of $M_{F_\epsilon}$, one has

$$\begin{aligned}&\iint_{\mathbf{T}^3\times\mathbf{R}^3} \Sigma(w) : (v - \epsilon w)^{\otimes 2} M_{F_\epsilon} dx\, dv \\ &= \int_{\mathbf{T}^3} \Sigma(w) : \left((u_{F_\epsilon} - \epsilon w)^{\otimes 2} + 3\theta_{F_\epsilon} I\right) \rho_{F_\epsilon} dx \\ &= \int_{\mathbf{T}^3} \Sigma(w) : (u_{F_\epsilon} - \epsilon w)^{\otimes 2} \rho_{F_\epsilon} dx .\end{aligned}$$

Notice that the 2nd equality follows from $\operatorname{div}_x w = 0$ so that

$$\operatorname{trace}(\Sigma(w)) = \operatorname{div}_x w = 0\,.$$

This term is compared with

$$\begin{aligned} H(M_{F_\epsilon}|\mathscr{M}_{(1,\epsilon w,1)}) := &\int_{\mathbf{T}^3}(\rho_{F_\epsilon}\ln\rho_{F_\epsilon} - \rho_{F_\epsilon} + 1)dx \\ &+ \tfrac{1}{2}\int_{\mathbf{T}^3}\rho_{F_\epsilon}|u_{F_\epsilon} - \epsilon w|^2 dx \\ &+ \tfrac{3}{2}\int_{\mathbf{T}^3}\rho_{F_\epsilon}(\theta_{F_\epsilon} - \ln\theta_{F_\epsilon} - 1)dx \\ &\le H(F_\epsilon|\mathscr{M}_{(1,\epsilon w,1)})\,, \end{aligned} \tag{11}$$

so that

$$\begin{aligned} \iint_{\mathbf{T}^3\times\mathbf{R}^3} &\Sigma(w) : (v-\epsilon w)^{\otimes 2} M_{F_\epsilon}\,dx\,dv \\ &\le 2\|\Sigma(w)\|_{L^\infty} H(F_\epsilon|\mathscr{M}_{(1,\epsilon w,1)})\,. \end{aligned}$$

At this point, we seek to decompose the space of positions according to whether or not the local hydrodynamic moments are $O(1)$ fluctuations of equilibrium. Specifically

$$\begin{aligned} &\iint_{\mathbf{T}^3\times\mathbf{R}^3} \Sigma(w) : (v-\epsilon w)^{\otimes 2}(F_\epsilon - M_{F_\epsilon})dx\,dv \\ &= \iint_{\mathscr{A}_\epsilon(t)\times\mathbf{R}^3} \Sigma(w) : (v-\epsilon w)^{\otimes 2}(F_\epsilon - M_{F_\epsilon})dx\,dv \\ &+ \iint_{\mathscr{B}_\epsilon(t)\times\mathbf{R}^3} \Sigma(w) : (v-\epsilon w)^{\otimes 2}(F_\epsilon - M_{F_\epsilon})dx\,dv\,, \end{aligned}$$

where $\mathscr{A}_\epsilon(t) \subset \mathbf{T}^3$ is defined as the set of $x$ such that

$$\max(|\rho_{F_\epsilon}(t,x) - 1|, |u_{F_\epsilon} - \epsilon w|(t,x), |\theta_{F_\epsilon}(t,x) - 1|) \le \tfrac{1}{2}\,,$$

while $\mathscr{B}_\epsilon(t) := \mathbf{T}^3 \setminus \mathscr{A}_\epsilon(t)$.

Using the elementary inequality $h(|z|) \le h(z)$ for all $z > -1$, we see that, on $\mathscr{A}_\epsilon(t)$

$$\begin{aligned}
&\frac{1}{4\epsilon^2}|v-\epsilon w|^2\left|\frac{F_\epsilon}{M_{F_\epsilon}}-1\right| \\
&\le \frac{1}{\epsilon^{(q+7)/2}}h\left(\frac{F_\epsilon}{M_{F_\epsilon}}-1\right)+\frac{1}{\epsilon^{(q+7)/2}}h^*(\epsilon^{(q+3)/2}\tfrac{1}{4}|v-\epsilon w|^2) \\
&\le \frac{1}{\epsilon^{(q+7)/2}}\left(\frac{F_\epsilon}{M_{F_\epsilon}}-1\right)\ln\left(\frac{F_\epsilon}{M_{F_\epsilon}}\right)+\epsilon^{(q-1)/2}h^*(\tfrac{1}{4}|v-\epsilon w|^2) \\
&\le \epsilon^{(q-1)/2}\left(\frac{1}{\epsilon^{q+3}}\left(\frac{F_\epsilon}{M_{F_\epsilon}}-1\right)\ln\left(\frac{F_\epsilon}{M_{F_\epsilon}}\right)+h^*(\tfrac{1}{4}|v-\epsilon w|^2)\right),
\end{aligned}$$

and

$$M_{F_\epsilon}(t,x,v)\le\frac{3}{2\pi^{3/2}}e^{-(|v-\epsilon w|-\frac{1}{2})^2/3},$$

so that

$$\begin{aligned}
&\frac{1}{\epsilon^2}\iint_{\mathscr{A}_\epsilon(t)\times\mathbf{R}^3}\Sigma(w):(v-\epsilon w)^{\otimes 2}(F_\epsilon-M_{F_\epsilon})dx\,dv \\
&\le 4\epsilon^{(q-1)/2}\left(\frac{1}{\epsilon^{q+3}}\|\Sigma(w)\|_{L^\infty}D(F_\epsilon)+C_1\|\Sigma(w\|_{L^1}\right).
\end{aligned}$$

On $\mathscr{B}_\epsilon(t)$

$$\begin{aligned}
\int_{\mathbf{R}^3}|v-\epsilon w|^2F_\epsilon dv &= \int_{\mathbf{R}^3}|v-\epsilon w|^2M_{F_\epsilon}dv \\
&=\rho_{F_\epsilon}(|u_{F_\epsilon}-\epsilon w|^2+3\theta_{F_\epsilon}) \\
&\le C_2\rho_{F_\epsilon}(|u_{F_\epsilon}-\epsilon w|^2+3(\theta_{F_\epsilon}-\ln\theta_{F_\epsilon}-1)),
\end{aligned}$$

so that

$$\begin{aligned}
&\iint_{\mathscr{B}_\epsilon(t)\times\mathbf{R}^3}\Sigma(w):(v-\epsilon w)^{\otimes 2}(F_\epsilon-M_{F_\epsilon})dx\,dv \\
&\le C_2H(M_{F_\epsilon}|\mathscr{M}_{(1,\epsilon w,1)})\le C_2H(F_\epsilon|\mathscr{M}_{(1,\epsilon w,1)}),
\end{aligned}$$

in view of (11).

### 3.3.4 Step 4: Applying Gronwall's Inequality

We start from the identity

$$\frac{1}{\epsilon^2}H(F_\epsilon|\mathscr{M}_{(1,\epsilon w,1)})(t) = \frac{1}{\epsilon^2}H(F_\epsilon|M) + \frac{1}{\epsilon^2}\iint_{\mathbf{T}^3\times\mathbf{R}^3} F_\epsilon \ln\left(\frac{M}{\mathscr{M}_{(1,\epsilon w,1)}}\right)dx\,dv \leq C^{in} + \tfrac{1}{2}\int_{\mathbf{T}^3}\rho_{F_\epsilon}|w|^2dx - \frac{1}{\epsilon}\int_{\mathbf{T}^3}\rho_{F_\epsilon}u_{F_\epsilon}\cdot w\,dx\,,$$

and use the conservation of mass to check that

$$\tfrac{1}{2}\int_{\mathbf{T}^3}\rho_{F_\epsilon}|w|^2dx \leq \|w\|^2_{L^\infty}\int_{\mathbf{T}^3}\rho_{F_\epsilon}dx = \|w\|^2_{L^\infty}\,.$$

The entropy control and Proposition 3 imply that

$$\frac{1}{\epsilon}\rho_{F_\epsilon}u_{F_\epsilon} = \frac{1}{\epsilon}\int vF_\epsilon dv \text{ is bounded in } L^\infty(\mathbf{R}_+;L^1(\mathbf{T}^3))\,.$$

Hence there exists a positive constant $C$ such that

$$\frac{1}{\epsilon^2}H(F_\epsilon|\mathscr{M}_{(1,\epsilon w,1)}) \leq C\,.$$

Therefore, up to extracting a subsequence if needed, one has

$$\frac{1}{\epsilon^2}H(F_\epsilon|\mathscr{M}_{(1,\epsilon w,1)}) \to H_w \text{ in } L^\infty([0,T]) \text{ weak-*}\,,$$

for each $T > 0$.

Applying Proposition 4 together with Lemmas 2 and 3 above, one finds that

$$H_w(t) \leq H_w(0) + C\|\Sigma(w)\|_{L^\infty}\int_0^t H_w ds - \int_0^t\int_{\mathbf{T}^3} E(w)\cdot(U-w)dx\,ds\,.$$

(Notice that the entropy production $\epsilon^{-(q+3)}D(F_\epsilon)$ disappears in this scaling. It enters only in the contribution of the set $\mathscr{A}_\epsilon(t)$ in step 3, and is multiplied by the vanishing scaling factor $\epsilon^{(q-1)/2}$ as explained above.) Gronwall's inequality implies that

$$H_w(t) \leq H_w(0)\exp\left(C\int_0^t \|\Sigma(w)\|_{L^\infty}(s)ds\right) - \int_0^t \exp\left(C\int_s^t \|\Sigma(w)\|_{L^\infty}(\tau)d\tau\right)\int_{\mathbf{T}^3} E(w)\cdot(U-w)(s,x)dx\,ds\,.$$

Set

$$\begin{aligned}h_\epsilon[w](t) :&= \frac{1}{\epsilon^2}\int_{\mathbf{T}^3} \tfrac{1}{2}\rho_{F_\epsilon}|u_{F_\epsilon} - \epsilon w|^2(t,x)dx \\ &= \sup_{b\in C_b(\mathbf{T}^3;\mathbf{R}^3)} \int_{\mathbf{T}^3} \left(\frac{1}{\epsilon}(u_{F_\epsilon} - \epsilon w)\cdot b - \tfrac{1}{2}|b|^2\right)\rho_{F_\epsilon} dx \\ &= \mathscr{F}\left[\rho_{F_\epsilon}(t,\cdot), \rho_{F_\epsilon}\frac{1}{\epsilon}(u_{F_\epsilon} - \epsilon w)(t,\cdot)\right].\end{aligned}$$

Observe that $\mathscr{F}$ is a jointly weakly lower semicontinuous and convex functional on the class of bounded, vector valued Radon measures on $\mathbf{T}^3$. Besides

$$\begin{aligned}h_\epsilon[w](t) :&= \frac{1}{\epsilon^2}\int_{\mathbf{T}^3} \tfrac{1}{2}\rho_{F_\epsilon}|u_{F_\epsilon} - \epsilon w|^2(t,x)dx \\ &\le \frac{1}{\epsilon^2}H(M_{F_\epsilon}|\mathscr{M}_{(1,\epsilon w,1)})(t) \le \frac{1}{\epsilon^2}H(F_\epsilon|\mathscr{M}_{(1,\epsilon w,1)})(t) \le C^{in}.\end{aligned}$$

By the Banach-Alaoglu theorem, possibly after extracting subsequences, one has

$$\rho_{F_\epsilon}(t,\cdot) \rightharpoonup 1\,, \quad \rho_{F_\epsilon}\frac{1}{\epsilon}(u_{F_\epsilon} - \epsilon w)(t,\cdot) \rightharpoonup (U - w)(t,\cdot)$$

in the weak topology of measures on $\mathbf{T}^3$, while

$$h_\epsilon[w](t) \rightharpoonup h_w(t) \le H_w(t)$$

in $L^\infty([0,T])$ weak-*. Therefore

$$\begin{aligned}\mathscr{F}[1,(U-w)(t,\cdot)] \le h_w(t) \le H_w(0)\exp\left(C\int_0^t \|\Sigma(w)\|_{L^\infty}(s)ds\right) \\ - \int_0^t \exp\left(C\int_s^t \|\Sigma(w)\|_{L^\infty}(\tau)d\tau\right)\int_{\mathbf{T}^3} E(w)\cdot(U-w)(s,x)dx\,ds.\end{aligned}$$

Observing that

$$\mathscr{F}[1,(U-w)(t,\cdot)] = \tfrac{1}{2}\int_{\mathbf{T}^3} |U-w|^2(t,x)dx$$

while

$$\begin{aligned}&\frac{1}{\epsilon^2}H(\mathscr{M}_{(1,\epsilon u^{in},1)}|\mathscr{M}_{(1,\epsilon w(0,\cdot),1)}) \\ &\qquad = \tfrac{1}{2}\int_{\mathbf{T}^3} |u^{in}(x) - w(0,x)|^2 dx = H_w(0)\,,\end{aligned}$$

we conclude that

$$\begin{aligned}&\tfrac{1}{2}\int_{\mathbf{T}^3}|U-w|^2(t,x)dx\\ &\le \tfrac{1}{2}\int_{\mathbf{T}^3}|u^{in}(x)-w(0,x)|^2dx\,\exp\left(\int_0^t C\|\Sigma(w)\|_{L^\infty}(s)ds\right)\\ &+\int_0^t \exp\left(\int_s^t C\|\Sigma(w)\|_{L^\infty}(\tau)d\tau\right)\int_{\mathbf{T}^3}E(w)\cdot(U-w)(s,x)dx\,ds\,.\end{aligned}$$

In other words, $U$ satisfies an inequality analogous to the one defining the notion of dissipative solution—up to replacing the constant $C$ with 2.

By the same argument as the one proving the uniqueness of classical solutions of Euler's equation within the class of dissipative solutions, setting $w=u$ (the solution of the Cauchy problem for the Euler equation with initial data $u^{in}$ defined on $[0,T^*)$ for each $T<T^*$), one has

$$\begin{aligned}&\tfrac{1}{2}\int_{\mathbf{T}^3}|U-u|^2(t,x)dx\\ &\le \int_0^t \exp\int_s^t C\|\Sigma(w)\|_{L^\infty}(\tau)d\tau\int_{\mathbf{T}^3}E(u)\cdot(U-u)(s,x)dx\,ds=0\,,\end{aligned}$$

since

$$\begin{aligned}\int_{\mathbf{T}^3}E(u)\cdot(U-u)(s,x)dx&=\int_{\mathbf{T}^3}-\nabla_x p\cdot(U-u)(s,x)dx\\ &=\int_{\mathbf{T}^3}p\,\mathrm{div}_x(U-u)(s,x)dx=0\,.\end{aligned}$$

This completes the proof of Theorem 14.

## 4 Lecture 3: The Incompressible Navier-Stokes Limit

The incompressible Navier-Stokes limit is the only nonlinear regime where the fluid dynamic limit of the Boltzmann equation is known to hold without any restriction on the time interval on which the limit is valid, or on the size and regularity of the initial distribution function. It connects two analogous theories of global weak solutions, the Leray existence theory of weak solutions of the incompressible Navier-Stokes equation, and the DiPerna-Lions theory of renormalized solutions of the Boltzmann equation. This last lecture will give an idea of the proof of the fluid dynamic limit in this regime.

For the sake of simplicity, we consider only the Navier-Stokes motion equation, without the drift-diffusion equation for the temperature. In other words, this lecture is focussed on the following theorem, that is a slightly simpler variant of the incompressible Navier-Stokes limit theorem presented in Lecture 1.

**Theorem 15 (F. Golse-L. Saint-Raymond [48, 50]).** *Let $F_\epsilon$ be a family of renormalized solutions of the Cauchy problem for the Boltzmann equation with initial data*

$$F_\epsilon\Big|_{t=0} = \mathscr{M}_{(1,\epsilon u^{in}(\epsilon x),1)}\,,$$

*where $u^{in} \in L^2(\mathbf{R}^3)$ satisfies $\operatorname{div}_x u^{in} = 0$. For some subsequence $\epsilon_n \to 0$, one has*

$$\frac{1}{\epsilon_n}\int_{\mathbf{R}^3} vF_{\epsilon_n}\left(\frac{t}{\epsilon_n^2}, \frac{x}{\epsilon_n}, v\right) dv \to u(t,x) \text{ weakly in } L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)\,,$$

*where $u$ is a Leray solution with initial data $u^{in}$ of*

$$\partial_t u + \operatorname{div}_x(u\otimes u) + \nabla_x p = \mu \Delta_x u\,, \quad \operatorname{div}_x u = 0\,.$$

*The viscosity $\mu$ is given by the same formula as in (8), recalled below:*

$$\mu = \tfrac{1}{5}\mathscr{D}^*(v\otimes v - \tfrac{1}{3}|v|^2 I)\,,$$

*where $\mathscr{D}$ is the quadratic functional*

$$\mathscr{D}(\Phi) = \tfrac{1}{8}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} |\Phi + \Phi_* - \Phi' - \Phi'_*|^2 |(v-v_*)\cdot\omega| MM_* dv\, dv_* d\omega\,,$$

*and $\mathscr{D}^*$ its Legendre dual.*

We also recall that a Leray solution of the incompressible Navier-Stokes equation is a divergence free vector field

$$u \in C(\mathbf{R}_+; w - L^2(\mathbf{R}^3)) \cap L^2(\mathbf{R}_+; H^1(\mathbf{R}^3))$$

such that

$$\frac{d}{dt}\int_{\mathbf{R}^3} u(t,x)\cdot w(x)dx + \mu\int_{\mathbf{R}^3} \nabla_x u(t,x) : \nabla w(x) dx = \int_{\mathbf{R}^3} \nabla w(x) : u(t,x)\otimes u(t,x) dx$$

in the sense of distributions on $\mathbf{R}_+^*$ for each divergence free vector field $w$ in the Sobolev space $H^1(\mathbf{R}^3)$, together with the energy inequality

$$\tfrac{1}{2}\int_{\mathbf{R}^3} |u(t,x)|^2 dx + \mu\int_{\mathbf{R}^3} |\nabla_x u|^2 dx \le \tfrac{1}{2}\int_{\mathbf{R}^3} |u(0,x)|^2 dx$$

for all $t \geq 0$. The reader is referred to the original work of J. Leray [67] for more details on this notion, together with [30] or Chap. 3 in [72].

## *4.1 Formal Derivation of the Incompressible Navier-Stokes Equations from the Boltzmann Equation*

### 4.1.1 The Rescaled Boltzmann Equation

The incompressible Navier-Stokes scaling for the Boltzmann equation assumes that the Knudsen, Mach and Strouhal numbers satisfy $\mathrm{Kn} = \mathrm{Ma} = \mathrm{Sh} = \epsilon$ (in the terminology introduced at the end of the Lecture 1) so that $\mathrm{Re} = 1$ (by the von Karman relation).

In other words, the assumption $\mathrm{Kn} = \mathrm{Sh} = \epsilon$ means that, if $F$ is the solution of the Boltzmann equation, the incompressible Navier-Stokes limit involves the rescaled distribution function $F_\epsilon(t,x,v) := F(t/\epsilon^2, x/\epsilon, v)$. This rescaled distribution function satisfies the rescaled Boltzmann equation

$$\epsilon\partial_t F_\epsilon + v\cdot\nabla_x F_\epsilon = \frac{1}{\epsilon}\mathscr{C}(F_\epsilon)\,.$$

Henceforth we set $d^2/2 = 1$ (where $d/2$ is the molecular radius) in the definition of the collision integral—in other words, the molecular radius is absorlbed in the scaling throughout the present section.

On the other hand, the assumption $\mathrm{Ma} = \epsilon$ indicates that $F_\epsilon$ is sought as an $O(\epsilon)$ perturbation of the uniform Maxwellian equilibrium $M := \mathscr{M}_{(1,0,1)}$, i.e. that one has

$$F_\epsilon(t,x,v) = M(v)G_\epsilon(t,x,v)\,,\quad G_\epsilon(t,x,v) = 1 + \epsilon g_\epsilon(t,x,v)\,,$$

with $g_\epsilon = O(1)$ as $\epsilon \to 0$.

The proof of the incompressible Navier-Stokes limit of the Boltzmann equation that we discuss below is not based on Hilbert's expansion. As explained in Lecture 1, Hilbert's expansion truncated as in [24, 33] may fail to guarantee the positivity of the distribution function, and may break down if the solution of the Navier-Stokes equations loses regularity in finite time—a problem still open in the 3-dimensional case at the time of this writing.

For that reason, a more robust moment method was proposed by Bardos-Golse-Levermore in [9]. This method leads to a formal argument for the incompressible Navier-Stokes limit that is very close to the structure of the complete proof. For that reason, we first present this formal argument before sketching the proof itself.

In terms of the relative number density fluctuation $g_\epsilon$, the scaled Boltzmann equation becomes

$$\epsilon\partial_t g_\epsilon + v\cdot\nabla_x g_\epsilon + \frac{1}{\epsilon}\mathscr{L}g_\epsilon = \mathscr{Q}(g_\epsilon, g_\epsilon)\,.$$

This form of the rescaled Boltzmann equation involves the collision integral linearized at $M$ and intertwined with the multiplication by $M$, denoted

$$\mathscr{L}g := -M^{-1}D\mathscr{C}(M)\cdot(Mg)\,,$$

together with the Hessian of the collision integral (also intertwined with the multiplication by $M$), denoted

$$\mathscr{Q}(g,g) := \tfrac{1}{2}M^{-1}\mathscr{C}(Mg)\,.$$

### 4.1.2 The Linearized Collision Integral

The explicit form of $\mathscr{L}$ is as follows:

$$\mathscr{L}g(v) := \iint_{\mathbf{R}^3\times\mathbf{S}^2}(g(v)+g(v_*)-g(v')-g(v'_*))|(v-v_*)\cdot\omega|M(v_*)dv_*d\omega\,.$$

**Theorem 16 (D. Hilbert [59]).** *The linearized collision integral operator $\mathscr{L}$ is a self-adjoint, nonnegative, Fredholm, unbounded operator on $L^2(\mathbf{R}^3;M\,dv)$ with domain*

$$\operatorname{Dom}\mathscr{L} = L^2(\mathbf{R}^3;(1+|v|)M\,dv)$$

*and nullspace*

$$\operatorname{Ker}\mathscr{L} = \operatorname{span}\{1,v_1,v_2,v_3,|v|^2\}\,.$$

### 4.1.3 Asymptotic Fluctuations

Multiplying the Boltzmann equation by $\epsilon$ and letting $\epsilon\to 0$ suggests that

$$g_\epsilon \to g \text{ as } \epsilon\to 0\,,\qquad \text{with } \mathscr{L}g = 0\,.$$

By Hilbert's theorem, $g$ is an *infinitesimal Maxwellian*, meaning that $g(t,x,v)$ is of the form

$$g(t,x,v) = \rho(t,x) + u(t,x)\cdot v + \tfrac{1}{2}\theta(t,x)(|v|^2-3)\,. \tag{12}$$

Notice that, in this case, $g$ is parametrized by its own moments in the $v$ variable, since

$$\rho = \langle g\rangle\,,\quad u = \langle vg\rangle\,,\quad \text{and } \theta = \langle(\tfrac{1}{3}|v|^2-1)g\rangle\,. \tag{13}$$

This observation is important in the rigorous derivation of the incompressible Navier-Stokes equations from the Boltzmann equation.

Henceforth, we systematically use the following notation.

**Notation:** for all $\phi \in L^1(\mathbf{R}^3; M\,dv)$, one denotes

$$\langle \phi \rangle := \int_{\mathbf{R}^3} \phi(v) M(v) dv \,.$$

### 4.1.4 The Incompressibility and Boussinesq Relations

The continuity equation (local conservation of mass) reads

$$\epsilon \partial_t \langle g_\epsilon \rangle + \operatorname{div}_x \langle v g_\epsilon \rangle = 0 \,,$$

and passing to the limit in the sense of distributions, we expect that

$$\langle v g_\epsilon \rangle \to \langle v g \rangle = u \,, \qquad \text{and thus } \operatorname{div}_x \langle v g \rangle = \operatorname{div}_x u = 0 \,.$$

This is incompressibility condition in the Navier-Stokes equations.

Likewise, the local conservation of momentum takes the form

$$\epsilon \partial_t \langle v g_\epsilon \rangle + \operatorname{div}_x \langle v \otimes v g_\epsilon \rangle = 0 \,.$$

Passing to the limit in the sense of distributions on both sides of the equality above, we expect that

$$\langle v \otimes v g_\epsilon \rangle \to \langle v \otimes v g \rangle = (\rho + \theta) I \,,$$

(where the last equality follows from straightforward computations based on formulas (12) and (13)) so that

$$\operatorname{div}_x ((\rho + \theta) I) = \nabla_x (\rho + \theta) = 0 \,.$$

The following slight variant of this argument provides insight into the next step of this proof, namely the derivation of the Navier-Stokes motion equation.

Recall that the incompressible Navier-Stokes motion equation is

$$\partial_t u + u \cdot \nabla_x u - \mu \Delta_x u = -\nabla_x p \,,$$

and that it involves the term $\nabla_x p$ as the Lagrange multiplier associated to the constraint $\operatorname{div}_x u = 0$. Accordingly, we split the tensor $v \otimes v$ into its traceless and scalar component:

$$v \otimes v = \left( v \otimes v - \tfrac{1}{3} |v|^2 I \right) + \tfrac{1}{3} |v|^2 I \,,$$

so that the local conservation of momentum becomes

$$\epsilon\partial_t\langle vg_\epsilon\rangle + \operatorname{div}_x\langle Ag_\epsilon\rangle + \nabla_x\langle \tfrac{1}{3}|v|^2 g_\epsilon\rangle = 0\,,$$

where

$$A(v) = v\otimes v - \tfrac{1}{3}|v|^2 I\,.$$

The key observation here is that

$$A \perp \operatorname{Ker}\mathscr{L} \quad \text{ in } L^2(M\,dv)\,;$$

see Appendix 2 (and especially Lemma 13).

Passing to the limit in the local conservation of momentum above in the sense of distributions, we expect that

$$\langle Ag_\epsilon\rangle \to \langle Ag\rangle = 0 \text{ since } g(t,x,\cdot)\in\operatorname{Ker}\mathscr{L} \text{ for a.e. } (t,x)\in\mathbf{R}_+\times\mathbf{R}^3\,.$$

On the other hand, by (13),

$$\langle \tfrac{1}{3}|v|^2 g_\epsilon\rangle \to \langle \tfrac{1}{3}|v|^2 g\rangle = \rho+\theta\,.$$

Thus

$$\operatorname{div}_x\langle Ag\rangle + \nabla_x\langle \tfrac{1}{3}|v|^2 g\rangle = \nabla_x(\rho+\theta) = 0\,.$$

If $g\in L^\infty(\mathbf{R}_+;L^2(\mathbf{R}^3;M\,dv\,dx))$, this implies the Boussinesq relation

$$\rho+\theta=0\,,\qquad \text{so that } g(t,x,v) = u(t,x)\cdot v + \theta(t,x)\tfrac{1}{2}(|v|^2-5)\,.$$

### 4.1.5 The Motion Equation

It remains to derive the Navier-Stokes motion equation. Following the last argument in the previous section, we start from the local conservation of momentum in the form

$$\partial_t\langle vg_\epsilon\rangle + \operatorname{div}_x\frac{1}{\epsilon}\langle Ag_\epsilon\rangle + \nabla_x\frac{1}{\epsilon}\langle \tfrac{1}{3}|v|^2 g_\epsilon\rangle = 0\,.$$

As mentioned above, $A_{kl}\perp\operatorname{Ker}\mathscr{L}$ for all $k,l = 1,2,3$ (see Lemma 13). Applying the Fredholm alternative to the linearized collision integral $\mathscr{L}$ shows the existence of a unique tensor field $\hat{A}\in\operatorname{Dom}(\mathscr{L})$ such that

$$A_{kl} = \mathscr{L}\hat{A}_{kl}\,,\qquad \text{and } \hat{A}_{kl}\perp\operatorname{Ker}\mathscr{L} \text{ for all } k,l=1,2,3\,.$$

Since $\mathscr{L}$ is self-adjoint, one has

$$\frac{1}{\epsilon}\langle A g_\epsilon\rangle = \left\langle \frac{1}{\epsilon} g_\epsilon \mathscr{L}\hat{A}\right\rangle = \left\langle \hat{A}\frac{1}{\epsilon}\mathscr{L} g_\epsilon\right\rangle = \langle \hat{A}\mathscr{Q}(g_\epsilon, g_\epsilon)\rangle - \langle \hat{A}(\epsilon\partial_t + v\cdot\nabla_x)g_\epsilon\rangle$$
$$\to \langle \hat{A}\mathscr{Q}(g,g)\rangle - \langle \hat{A}v\cdot\nabla_x g\rangle$$

as $\epsilon \to 0$.

Since $g$ is an infinitesimal Maxwellian and $\rho, \theta$ satisfy the Boussinesq relation, one has

$$g = u\cdot v + \theta\tfrac{1}{2}(|v|^2 - 5)\,,$$

so that

$$\langle \hat{A}v\cdot\nabla_x g\rangle = \tfrac{1}{2}\langle \hat{A}\otimes A\rangle : D(u) + \langle \hat{A}\otimes \tfrac{1}{2}(|v|^2-5)v\rangle\cdot\nabla_x\theta$$
$$= \tfrac{1}{2}\langle \hat{A}\otimes A\rangle : D(u) \text{ since } \hat{A} \text{ is even,}$$

where

$$D(u) := \nabla_x u + (\nabla_x u)^T - \tfrac{2}{3}\operatorname{div}_x u I$$

is the traceless deformation tensor of $u$. Notice that $\langle \hat{A}|v|^2\rangle = 0$ since $\hat{A}_{kl}\perp \operatorname{Ker}\mathscr{L}$ for all $k, l = 1, 2, 3$, so that

$$\langle \hat{A}\otimes(v\otimes v)\rangle = \langle \hat{A}\otimes A\rangle\,.$$

It remains to compute the term $\langle \hat{A}\mathscr{Q}(g,g)\rangle$. This is done with the next lemma.

**Lemma 4 (C. Cercignani [27], C. Bardos-F. Golse-C.D. Levermore [10]).** *Each infinitesimal Maxwellian $g \in \operatorname{Ker}\mathscr{L}$ satisfies the relation*

$$\mathscr{Q}(g,g) = \tfrac{1}{2}\mathscr{L}(g^2)\,.$$

*Proof.* Differentiate twice the relation $\mathscr{C}(\mathscr{M}_{(\rho,u,\theta)}) = 0$ with respect to $\rho, u, \theta$, and observe that the range of the differential $d\mathscr{M}_{(\rho,u,\theta)}$ is equal to $\operatorname{Ker}\mathscr{L}$.

With this observation, one has

$$\langle \hat{A}\mathscr{Q}(g,g)\rangle = \tfrac{1}{2}\langle \hat{A}\mathscr{L}(g^2)\rangle = \frac{\langle}{g}^{2}\mathscr{L}\hat{A}\rangle = \tfrac{1}{2}\langle A g^2\rangle$$
$$= \tfrac{1}{2}\langle A\otimes A\rangle : \left(u\otimes u - \tfrac{1}{3}|u|^2 I\right)\,,$$

using again the fact that $\mathscr{L}$ is self-adjoint and that

$$\langle A \otimes v \otimes v \rangle = \langle A \otimes A \rangle .$$

(Indeed $A_{ij} \perp \operatorname{Ker} \mathscr{L}$ and therefore $\langle A|v|^2 \rangle = 0$.)

Therefore

$$\frac{1}{\epsilon}\langle A g_\epsilon \rangle \to \tfrac{1}{2}\langle A \otimes A \rangle : \left(u \otimes u - \tfrac{1}{3}|u|^2 I\right) - \tfrac{1}{2}\langle \hat{A} \otimes A \rangle : D(u) .$$

**Lemma 5.** *For all $i, j, k, l \in \{1, 2, 3\}$, one has*

$$\langle A_{ij} A_{kl} \rangle = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl} ,$$
$$\langle \hat{A}_{ij} A_{kl} \rangle = \mu \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}\right) ,$$

*where*

$$\mu = \tfrac{1}{10}\langle \hat{A} : A \rangle > 0$$

*is the viscosity.*

The proof of this Lemma will be given in Appendix 2.

Thus

$$\frac{1}{\epsilon}\langle A g_\epsilon \rangle \to \left(u \otimes u - \tfrac{1}{3}|u|^2 I\right) - \mu D(u) .$$

Substituting this expression in the momentum conservation laws shows that, for each $\xi \in C_c^1(\mathbf{R}^3)$ such that $\operatorname{div} \xi = 0$

$$\frac{d}{dt}\int_{\mathbf{R}^3} u(t,x)\cdot \xi(x)dx + \mu \int D(u)(t,x) : \nabla_x \xi(x)dx = \int \nabla \xi(x) : u \otimes u(t,x)dx$$

The divergence free condition $\operatorname{div}_x u = 0$ implies that

$$\operatorname{div}_x D(u) = \Delta_x u + \nabla_x(\operatorname{div}_x u) - \tfrac{2}{3}\nabla_x(\operatorname{div}_x u) = \Delta_x u .$$

Equivalently,

$$\partial_t u + \operatorname{div}_x(u \otimes u) - \mu \Delta_x u = 0 \text{ modulo gradient fields.}$$

We conclude this section with the formula for $\mu$ in the statement of Theorem 15. Let the Dirichlet form for the linearized collision integral $\mathscr{L}$ be defined as follows:

$$\mathscr{D}(\Phi) := \tfrac{1}{2}\langle \Phi : \mathscr{L}\Phi \rangle .$$

As explained in Lecture 1, the formula for the viscosity can be put in the form

$$\mu = \tfrac{1}{5}\mathscr{D}^*(A)\,,$$

where $\mathscr{D}^*$ designates the Legendre dual of $\mathscr{D}$. Indeed, since $\mathscr{D}$ is a quadratic functional defined on $\operatorname{Dom}\mathscr{L} \otimes M_3(\mathbf{R}) \simeq (\operatorname{Dom}\mathscr{L})^9$, one has

$$\mathscr{D}^*(\Phi) = \tfrac{1}{2}\langle \Phi : \mathscr{L}^{-1}\Phi\rangle$$

for all $\Phi \in (\operatorname{Ker}\mathscr{L})^\perp$. Applying this to $\Phi = A$ gives back the formula in Lemma 5.

## 4.2 *Sketch of the Proof of the Incompressible Navier-Stokes Limit of the Boltzmann Equation*

The complete proof of the incompressible Navier-Stokes limit of the Boltzmann equation is quite involved (see [48, 50]). Therefore we only sketch the main steps in the argument.

### 4.2.1 The Strategy

First we choose a convenient normalizing nonlinearity for the Boltzmann equation. Pick $\gamma \in C^\infty(\mathbf{R}_+)$, a nonincreasing function such that

$$\gamma\big|_{[0,3/2]} \equiv 1\,, \quad \gamma\big|_{[2,+\infty)} \equiv 0\,; \qquad \text{and set } \hat{\gamma}(z) := \frac{d}{dz}((z-1)\gamma(z))\,.$$

The Boltzmann equation is renormalized relatively to $M$ as follows:

$$\partial_t(g_\epsilon\gamma_\epsilon) + \frac{1}{\epsilon}v\cdot\nabla_x(g_\epsilon\gamma_\epsilon) = \frac{1}{\epsilon^3}\hat{\gamma}_\epsilon\mathscr{Q}(G_\epsilon, G_\epsilon)\,,$$

where

$$\gamma_\epsilon := \gamma(G_\epsilon) \quad \text{while } \hat{\gamma}_\epsilon := \hat{\gamma}(G_\epsilon)\,.$$

We recall the notation $\mathscr{Q}(G,G) = M^{-1}\mathscr{C}(MG)$.

Renormalized solutions of the Boltzmann equation satisfy the local conservation law of mass:

$$\epsilon\partial_t\langle g_\epsilon\rangle + \operatorname{div}_x\langle v g_\epsilon\rangle = 0\,.$$

The entropy bound and Young's inequality imply that

$$(1+|v|^2)g_\epsilon \text{ is relatively compact in } w-L^1_{loc}(dt\,dx; L^1(M\,dv))\,.$$

Therefore, modulo extraction of a subsequence,

$$g_\epsilon \to g \text{ weakly in } L^1_{loc}(dt\,dx; L^1(M(1+|v|^2)dv))\,.$$

Hence

$$\langle v g_\epsilon \rangle \to \langle v g \rangle =: u \quad \text{ weakly in } L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)\,.$$

Passing to the limit in the continuity equation leads to the incompressibility condition:

$$\operatorname{div}_x u = 0\,.$$

Since high velocities are a source of difficulties in the hydrodynamic limit, we shall use a special truncation procedure, defined as follows. Pick $K > 6$ and set $K_\epsilon = K|\ln \epsilon|$; for each function $\xi \equiv \xi(v)$, define

$$\xi_{K_\epsilon}(v) := \xi(v)\mathbf{1}_{|v|^2 \le K_\epsilon}\,.$$

We recall that our goal is to pass to the limit in the local conservation law of momentum. Multiplying both sides of the scaled, renormalized Boltzmann equation by each component of $v_{K_\epsilon}$, one finds that

$$\partial_t \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle + \operatorname{div}_x \mathbf{F}_\epsilon(A) + \nabla_x \frac{1}{\epsilon} \langle \tfrac{1}{3}|v|^2_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle = \mathbf{D}_\epsilon(v)\,,$$

where

$$\begin{cases} \mathbf{F}_\epsilon(A) := \frac{1}{\epsilon}\langle A_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle\,, \\ \mathbf{D}_\epsilon(v) := \dfrac{1}{\epsilon^3}\langle\!\langle v_{K_\epsilon} \hat{\gamma}_\epsilon (G'_\epsilon G'_{\epsilon *} - G_\epsilon G_{\epsilon *})\rangle\!\rangle\,. \end{cases}$$

We recall that

$$\langle \phi \rangle = \int_{\mathbf{R}^3} \phi(v) M\,dv\,,$$

and introduce a new element of notation

$$\langle\!\langle \psi \rangle\!\rangle := \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \psi(v, v_*, \omega) dm(v, v_*, \omega)\,,$$

where

$$dm(v, v_*, \omega) := |(v - v_*) \cdot \omega| M\, dv\, M_* dv_* d\omega\,.$$

Our goal is to prove that, modulo extraction of a subsequence,

$$\begin{aligned} &\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \to \langle v g\rangle =: u && \text{weakly in } L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)\,,\\ &\mathbf{D}_\epsilon(v) \to 0 && \text{strongly in } L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)\,, \text{ and}\\ &P\,(\mathrm{div}_x\, \mathbf{F}_\epsilon(A)) \to P\,\mathrm{div}_x(u^{\otimes 2}) - \mu \Delta_x u && \text{weakly in } L^1_{loc}(\mathbf{R}_+, W^{-s,1}_{loc}(\mathbf{R}^3))\,,\end{aligned}$$

for $s > 1$ as $\epsilon \to 0$, where $P$ denotes the Leray projection, i.e. the orthogonal projection on divergence-free vector fields in $L^2(\mathbf{R}^3)$.

See Sect. 2.4 in [50] for the missing details.

### 4.2.2 Uniform A Priori Estimates

The only uniform a priori estimate satisfied by renormalized solutions of the Boltzmann equation comes from the DiPerna-Lions entropy inequality:

$$H(F_\epsilon|M)(t) + \frac{1}{\epsilon^2}\int_0^t \int_{\mathbf{R}^3} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} d(F_\epsilon)|(v - v_*)\cdot\omega|\,dv\,dv_* d\omega\,dx\,ds \\ \le H(F_\epsilon^{in}|M) = \tfrac{1}{2}\epsilon^2 \|u^{in}\|^2_{L^2}\,,$$

where the entropy production integrand is denoted

$$d(f) := \tfrac{1}{4}(f' f'_* - f f_*) \ln\left(\frac{f' f'_*}{f f_*}\right)\,.$$

We also recall the following elementary, pointwise inequalities:

$$(\sqrt{Z} - 1)^2 \le Z \ln Z - Z + 1\,, \quad 4(\sqrt{X} - \sqrt{Y})^2 \le (X - Y)\ln(X/Y)\,,$$

for all $X, Y, Z > 0$.

With the DiPerna-Lions entropy inequality, and the pointwise inequalities above, one gets the following bounds that are uniform in $\epsilon$:

$$\begin{aligned} \int_{\mathbf{R}^3} \langle (\sqrt{G_\epsilon} - 1)^2 \rangle dx \le C\epsilon^2\,,\\ \int_0^{+\infty} \int_{\mathbf{R}^3} \langle\!\langle \left(\sqrt{G'_\epsilon G'_{\epsilon *}} - \sqrt{G_\epsilon G_{\epsilon *}}\right)^2 \rangle\!\rangle dx\,dt \le C\epsilon^4\,. \end{aligned} \tag{14}$$

This is precisely Proposition 2.3 in [50].

### 4.2.3 Vanishing of Conservation Defects

Since renormalized solutions of the Boltzmann equation are not known to satisfy the local conservation laws of momentum and energy, one has to consider instead the local conservation laws of moments of renormalized distribution functions, truncated at high velocities, modulo conservation defects. The idea is to prove that the conservation defects vanish in the hydrodynamic limit. In other words, even if the local conservation of momentum and energy are not known to be satisfied by renormalized solutions of the Boltzmann equation, they are satisfied *after* passing to the hydrodynamic limit.

This approach was proposed for the first time in [12]. The procedure for proving the vanishing of conservation defects was formulated in essentially the most general possible setting in [45], and applied to the acoustic and Stokes-Fourier limits. The statement below is taken from [50], it is more general and slightly less technical than the analogous result in [48].

**Proposition 5.** *The conservation defect*

$$\mathbf{D}_\epsilon(v) := \frac{1}{\epsilon^3}\langle\!\langle v_{K_\epsilon}\hat{\gamma}_\epsilon(G'_\epsilon G'_{\epsilon *} - G_\epsilon G_{\epsilon *})\rangle\!\rangle$$

*satisfies*

$$\mathbf{D}_\epsilon(v) \to 0$$

*in* $L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)$ *as* $\epsilon \to 0$.

This is Proposition 5.1 in [50].

*Proof.* Split the conservation defect as $\mathbf{D}_\epsilon(v) = \mathbf{D}^1_\epsilon(v) + \mathbf{D}^2_\epsilon(v)$ with

$$\begin{aligned}
\mathbf{D}^1_\epsilon(v) &:= \frac{1}{\epsilon^3}\langle\!\langle v_{K_\epsilon}\hat{\gamma}_\epsilon\left(\sqrt{G'_\epsilon G'_{\epsilon *}} - \sqrt{G_\epsilon G_\epsilon}\right)^2\rangle\!\rangle,\\
\mathbf{D}^2_\epsilon(v) &:= \frac{2}{\epsilon^3}\langle\!\langle v_{K_\epsilon}\hat{\gamma}_\epsilon\left(\sqrt{G'_\epsilon G'_{\epsilon *}} - \sqrt{G_\epsilon G_\epsilon}\right)\sqrt{G_\epsilon G_\epsilon}\rangle\!\rangle.
\end{aligned}$$

That $\mathbf{D}^1_\epsilon(v) \to 0$ follows from the entropy production estimate.

Setting

$$\Xi_\epsilon := \frac{1}{\epsilon^2}\left(\sqrt{G'_\epsilon G'_{\epsilon *}} - \sqrt{G_\epsilon G_\epsilon}\right)\sqrt{G_\epsilon G_\epsilon}\,,$$

we further split $\mathbf{D}^2_\epsilon(v)$ as

$$\begin{aligned}
\mathbf{D}^2_\epsilon(v) = &-\frac{2}{\epsilon}\langle\!\langle v\mathbf{1}_{|v|^2>K_\epsilon}\hat{\gamma}_\epsilon\Xi_\epsilon\rangle\!\rangle + \frac{2}{\epsilon}\langle\!\langle v\hat{\gamma}_\epsilon(1-\hat{\gamma}_{\epsilon *}\hat{\gamma}'_\epsilon\hat{\gamma}_{\epsilon *})\Xi_\epsilon\rangle\!\rangle\\
&+\frac{1}{\epsilon}\langle\!\langle (v+v_1)\hat{\gamma}_\epsilon\hat{\gamma}_{\epsilon *}\hat{\gamma}'_\epsilon\hat{\gamma}_{\epsilon *}\Xi_\epsilon\rangle\!\rangle.
\end{aligned}$$

The first and third terms are mastered by the entropy production bound and classical estimates on the tail of Gaussian distributions. See Lemma 5.2 in [50] and the discussion on pp. 530–531.

Sending the second term to 0 requires knowing that

$$(1+|v|)\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2 \text{ is uniformly integrable on } [0,T]\times K\times\mathbf{R}^3$$

for the measure $dt\,dx\,M\,dv$, for each $T>0$ and each compact $K\subset\mathbf{R}^3$. See [50] on pp. 531–532 for the (rather involved) missing details.

### 4.2.4 Asymptotic Behavior of the Momentum Flux

We recall that the momentum flux is defined by the formula

$$\mathbf{F}_\epsilon(A) = \tfrac{1}{\epsilon}\langle A_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle\,.$$

**Proposition 6.** *Denoting by $\Pi$ the $L^2(M\,dv)$-orthogonal projection on* Ker $\mathscr{L}$*, one has*

$$\mathbf{F}_\epsilon(A) = 2\left\langle A\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right\rangle - 2\left\langle \hat{A}\frac{1}{\epsilon^2}\mathscr{Q}(\sqrt{G_\epsilon},\sqrt{G_\epsilon})\right\rangle + o(1)_{L^1_{loc}(dt\,dx)}\,.$$

This is Proposition 6.1 in [50].

The proof is based upon splitting $\mathbf{F}_\epsilon(A)$ as

$$\mathbf{F}_\epsilon(A) = \left\langle A_{K_\epsilon}\gamma_\epsilon\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right\rangle + \frac{2}{\epsilon}\left\langle A_{K_\epsilon}\gamma_\epsilon\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\rangle,$$

and the uniform integrability of $(1+|v|)\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2$, which implies in turn that

$$\lim_{\epsilon\to 0}\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon} - \Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|_{L^2_{loc}(dt\,dx;L^2((1+|v|)M\,dv))} = 0\,.$$

By the entropy production bound, up to extraction of a subsequence

$$\frac{1}{\epsilon^2}\left(\sqrt{G'_\epsilon G'_{\epsilon*}} - \sqrt{G_\epsilon G_\epsilon}\right) \to q \text{ weakly in } L^2(dt\,dx\,dm)\,.$$

Passing to the limit in the scaled, renormalized Boltzmann equation:

$$\iint_{\mathbf{R}^3\times\mathbf{S}^2} q|(v-v_*)\cdot\omega|M_* dv_* d\omega = \tfrac{1}{2}v\cdot\nabla_x g = \tfrac{1}{2}A:\nabla_x u + \text{ odd function of } v\,.$$

Since $\frac{\sqrt{G_\epsilon}-1}{\epsilon} \simeq \frac{1}{2} g_\epsilon \gamma_\epsilon$, one eventually obtains

$$\mathbf{F}_\epsilon(A) = A(\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle) - \mu(\nabla_x u + (\nabla_x u)^T) + o(1)_{w-L^1_{loc}(dt\,dx)} .$$

At this point, we recall that $A(u) := u \otimes u - \frac{1}{3}|u|^2 I$, while

$$\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \to u \text{ weakly in } L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3) .$$

## 4.3 Strong Compactness

Because the Navier-Stokes equation is nonlinear, weak compactness of truncated variants of the relative fluctuations of the distribution functions is not enough to prove the fluid dynamic limit. Proving that some appropriate quantities, such as $\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle$, defined in terms of renormalized solutions of the Boltzmann equation are relatively compact in the strong topology of $L^2$ is an essential step in order to pass to the limit in the quadratic term $A(\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle)$.

For that purpose, we appeal to "velocity averaging" theorems, a special class of regularity/compactness results on velocity averages of solutions of kinetic equations—see [1, 36, 51, 52].

Before discussing these results in detail, we recall the following elementary observations.

It is well known that, if $F \equiv F(x)$ and $R \equiv R(x)$ satisfy both $F, R \in L^2(\mathbf{R}^N)$ and $\Delta F = R$, then $F$ belongs to the Sobolev space $H^2(\mathbf{R}^N)$—in other words, knowing that

$$F \text{ and } \sum_{i=1}^{N} \partial^2_{x_i} F \in L^2(\mathbf{R}^N) \text{ implies that } \partial_{x_i}\partial_{x_j} F \in L^2(\mathbf{R}^N) \text{ for } i, j = 1, \dots, N .$$

The analogous question with the advection operator in the place of the Laplacian is as follows: given $G$ and $S \in L^p(\mathbf{R}^N \times \mathbf{R}^N)$ such that $v \cdot \nabla_x G = S$, what is the regularity of $G$ in the $x$-variable? For instance, does this imply that the function $G \in L^p(\mathbf{R}^N_v; W^{1,p}(\mathbf{R}^N_x))$?

This question is answered in the negative.

For instance, in space dimension $N = 2$, take $\gamma = \mathbf{1}_A$ with $A$ measurable and bounded, and set $G(x, v) = \gamma(x_1 v_2 - x_2 v_1)\mathbf{1}_{|v| \le 1}$. Obviously the function $G$ satisfies $v \cdot \nabla_x G = 0$ and $G \in L^\infty(\mathbf{R}^2 \times \mathbf{R}^2)$ so that $G \in L^p_{loc}(\mathbf{R}^N \times \mathbf{R}^N)$ for all $1 \le p < \infty$. Yet $G$ does not belong to $W^{s,p}(\mathbf{R}^2)$ for a.e. $v \in \mathbf{R}^2$.

Of course, the reason for the difference between both situations is explained by the fact that the Laplacian is an elliptic operator, while the advection operator is hyperbolic.

### 4.3.1 Velocity Averaging

The counterexample above suggests that the regularity of $G$ is not the interesting issue to be discussed in the first place. Instead of considering the regularity of $G$ itself, one should study the regularity of *velocity averages* of $G$, i.e. of quantities of the form

$$\int_{\mathbf{R}^3} G(x,v)\phi(v)dv$$

with smooth and compactly supported test function $\phi$.

The first result in this direction is the following theorem (see also [1, 51]).

**Theorem 17 (F. Golse-P.-L. Lions-B. Perthame-R. Sentis [52]).** *Assume that $G$ and $S$ both belong to $L^2(\mathbf{R}^N_x \times \mathbf{R}^N_v)$ and that $v\cdot\nabla_x G = S$. Then, for each $\phi \in C_c(\mathbf{R}^N)$, the velocity average*

$$\mathscr{A}_\phi[G] : x \mapsto \int_{\mathbf{R}^N} G(x,v)\phi(v)dv$$

*satisfies $\mathscr{A}_\phi[G] \in H^{1/2}(\mathbf{R}^N)$, with a bound of the form*

$$\|\mathscr{A}_\phi[G]\|_{\dot{H}^{1/2}(\mathbf{R}^N_x)} \le C\|G\|^{1/2}_{L^2(\mathbf{R}^N\times\mathbf{R}^N)}\|v\cdot\nabla_x G\|^{1/2}_{L^2(\mathbf{R}^N\times\mathbf{R}^N)}.$$

In this statement, the notation $\|\cdot\|_{\dot{H}^s}$ designates the homogeneous $H^s$ seminorm:

$$\|f\|_{\dot{H}^s(\mathbf{R}^N)} := \left(\iint_{\mathbf{R}^N\times\mathbf{R}^N} \frac{|f(x)-f(y)|^2}{|x-y|^{N+2s}}dx\,dy\right)^{1/2}, \quad 0<s<1.$$

In the context of the incompressible Navier-Stokes limit of the Boltzmann equation, the situation is slightly different from the one in the theorem above. Specifically, one has the following controls:

$$\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon}-1}{\epsilon}\right)^2 \text{ is locally uniformly integrable on } \mathbf{R}_+\times\mathbf{R}^3\times\mathbf{R}^3,$$

$$(\epsilon\partial_t + v\cdot\nabla_x)\frac{\sqrt{\epsilon^\alpha+G_\epsilon}-1}{\epsilon} \text{ is bounded in } L^1_{loc}(\mathbf{R}_+\times\mathbf{R}^3\times\mathbf{R}^3).$$

Mimicking the proof of the velocity averaging theorem above, one deduces from these assumptions that, for each $T>0$ and each compact $C\subset\mathbf{R}^3$,

$$\int_0^T \int_C |\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle(t, x+y) - \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle(t,x)|^2 dx\, dt \to 0 \quad \text{as } |y| \to 0\,, \text{ uniformly in } \epsilon > 0\,. \tag{15}$$

See Sect. 4 in [50], especially Proposition 4.4.

### 4.3.2 Filtering Acoustic Waves

The velocity averaging method presented in the previous section provides compactness of $\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle$ in the $x$ variable. It remains to prove compactness in the time variable. Observe that

$$\partial_t P \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle = P(\mathbf{D}_\epsilon(v) - \mathrm{div}_x\, \mathbf{F}_\epsilon(A)) \text{ is bounded in } L^1_{loc}(\mathbf{R}_+, W^{-s,1}_{loc}(\mathbf{R}^3))$$

(Indeed, we recall that $\mathbf{D}_\epsilon(v) \to 0$ while $\mathbf{F}_\epsilon(A)$ is bounded in $L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)$.).

Together with the compactness in the $x$-variable that follows from velocity averaging, this implies that

$$P \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \to u \text{ in } L^2_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)\,.$$

We also recall that

$$\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \to u \text{ weakly in } L^2_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)\,.$$

However, we *do not* seek to prove that

$$\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \to u \text{ strongly in } L^2_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)\,.$$

Instead, we prove that

$$P\, \mathrm{div}_x \left( \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle^{\otimes 2} \right) \to P\, \mathrm{div}_x \left( u^{\otimes 2} \right) \text{ in } \mathscr{D}'(\mathbf{R}^*_+ \times \mathbf{R}^3) \text{ as } \epsilon \to 0\,.$$

This is discussed in detail in Sect. 7.2.3 of [50]. Observe that

$$\begin{aligned}
\epsilon \partial_t \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle + \nabla_x \langle \tfrac{1}{3} |v|^2_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle &\to 0 \text{ in } L^1_{loc}(\mathbf{R}_+; W^{-1,1}_{loc}(\mathbf{R}^3))\,,\\
\epsilon \partial_t \langle \tfrac{1}{3} |v|^2_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle + \mathrm{div}_x \langle \tfrac{5}{3} v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle &\to 0 \text{ in } L^1_{loc}(\mathbf{R}_+; W^{-1,1}_{loc}(\mathbf{R}^3))\,,
\end{aligned}$$

as $\epsilon \to 0$. Setting $\nabla_x \pi_\epsilon = (I - P)\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle$, the system above becomes

$$\begin{aligned}
\epsilon \partial_t \nabla_x \pi_\epsilon + \nabla_x \langle \tfrac{1}{3} |v|^2_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle &\to 0 \text{ in } L^1_{loc}(\mathbf{R}_+; W^{-s,1}_{loc}(\mathbf{R}^3))\,, \quad s > 1\,,\\
\epsilon \partial_t \langle \tfrac{1}{3} |v|^2_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle + \tfrac{5}{3} \Delta_x \pi_\epsilon &\to 0 \text{ in } L^1_{loc}(\mathbf{R}_+; W^{-1,1}_{loc}(\mathbf{R}^3))\,.
\end{aligned}$$

At this point, we apply the following elegant observation.

**Lemma 6 (P.-L. Lions-N. Masmoudi [73]).** *Let $c \neq 0$ and let $\phi_\epsilon$ and $\nabla_x \psi_\epsilon$ be bounded families in $L^\infty_{loc}(\mathbf{R}_+; L^2_{loc}(\mathbf{R}^3))$ such that*

$$\begin{cases} \partial_t \phi_\epsilon + \frac{1}{\epsilon}\Delta_x \psi_\epsilon = \frac{1}{\epsilon}\Phi_\epsilon \,, \\ \partial_t \nabla_x \psi_\epsilon + \frac{c^2}{\epsilon}\nabla_x \phi_\epsilon = \frac{1}{\epsilon}\nabla \Psi_\epsilon \,, \end{cases}$$

*where*

$$\Phi_\epsilon \text{ and } \nabla\Psi_\epsilon \to 0 \text{ strongly in } L^1_{loc}(\mathbf{R}_+; L^2_{loc}(\mathbf{R}^3))$$

*as $\epsilon \to 0$. Then*

$$P\,\mathrm{div}_x((\nabla_x \psi_\epsilon)^{\otimes 2}) \text{ and } \mathrm{div}_x(\phi_\epsilon \nabla_x \psi_\epsilon) \to 0$$

*in the sense of distributions on $\mathbf{R}^*_+ \times \mathbf{R}^3$ as $\epsilon \to 0$.*

In view of the uniform in time modulus of $L^2$ continuity (15), the Lions-Masmoudi argument can be applied with $\pi_\epsilon$ in the place of $\psi_\epsilon$ after regularization in the $x$ variable. Eventually, one finds that

$$P\,\mathrm{div}_x((\nabla_x \pi_\epsilon)^{\otimes 2}) \to 0 \text{ in } \mathscr{D}'(\mathbf{R}^*_+ \times \mathbf{R}^3)\,.$$

On the other hand, the limiting velocity field is divergence-free and therefore

$$\nabla_x \pi_\epsilon \to 0 \text{ weakly in } L^2_{loc}(\mathbf{R}_+ \times \mathbf{R}^3) \text{ as } \epsilon \to 0\,.$$

Splitting

$$\begin{aligned} P\,\mathrm{div}_x\left(\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle^{\otimes 2}\right) &= P\,\mathrm{div}_x\left((P\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle)^{\otimes 2}\right) + P\,\mathrm{div}_x\left(P\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle \otimes \nabla_x \pi_\epsilon\right) \\ &\quad + P\left(\nabla_x \pi_\epsilon \otimes P\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle\right) + P\,\mathrm{div}_x\left((\nabla_x \pi_\epsilon)^{\otimes 2}\right) \end{aligned}$$

The last three terms vanish with $\epsilon$ while the first converges to $P\,\mathrm{div}_x(u^{\otimes 2})$ since $P\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle \to u$ strongly in $L^2_{loc}(dt\,dx)$.

The interested reader is referred to Sect. 7.3.2 of [50] for the missing details.

## *4.4 The Key Uniform Integrability Estimates*

Eventually, in view of the discussion above, everything is reduced to obtaining the uniform integrability of the family

$$\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2(1+|v|) \quad \text{on } [0,T]\times K\times\mathbf{R}^3\,,$$

for each compact $K\subset\mathbf{R}^3$ and each $T>0$. This is the main objective of the present section, stated in the proposition below. This is a (slightly easier) variant of some analogous control on the relative fluctuations of distribution function, identified but left unverified in [11].

Proving this uniform integrability statement remained the main obstruction in deriving Leray solutions of the Navier-Stokes equation from renormalized solutions of the Boltzmann equation, after a sequence of important steps in the understanding of the limit, such as [73] (which explained how to handle oscillations in the time variable), and [12, 45] which reduced the task of controlling conservation defects to the uniform integrability result stated below.

Therefore, obtaining this uniform integrability property remained the only missing step for a complete proof of the incompressible Navier-Stokes limit of the Boltzmann equation. The arguments leading to this uniform integrability property were eventually found in [48]. They involved a refinement of velocity averaging techniques adapted to the $L^1$ setting [47].

**Proposition 7 (F. Golse-L. Saint-Raymond [48, 50]).** *For each* $T>0$ *and each compact* $K\subset\mathbf{R}^3$*, the family* $\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2(1+|v|)$ *is uniformly integrable on the set* $[0,T]\times K\times\mathbf{R}^3$.

This proposition is really the core of the proof of the incompressible Navier-Stokes limit of the Boltzmann equation in [48, 50]. It involves two main ideas.

### 4.4.1 Idea No. 1: Uniform Integrability in the $v$ Variable

First we must define this notion of "uniform integrability in one variable" for functions of several variables.

**Definition 4.** A family of functions $\phi_\epsilon\equiv\phi_\epsilon(x,y)\in L^1_{x,y}(d\mu(x)d\nu(y))$ is uniformly integrable in the $y$-variable for the measure $\mu\otimes\nu$ if and only if

$$\int\left(\sup_{\nu(A)<\alpha}\int_A|\phi_\epsilon(x,y)|d\nu(y)\right)d\mu(x)\to 0 \text{ as } \alpha\to 0 \text{ uniformly in } \epsilon\,.$$

The following observation is a first step in the proof of the proposition above.

**Lemma 7.** *For each compact* $K\subset\mathbf{R}^3$*, each* $T>0$ *and each* $p\in[0,2)$*, the family*

$$\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2(1+|v|)^p$$

*is uniformly integrable in the $v$ variable on $[0,T]\times K\times\mathbf{R}^3$ for the measure $dt\,dx\,M\,dv$.*

This is Proposition 3.2 in [50] (see also Lemma 3.1 in that same reference).

*Proof.* Instead of the hard sphere collision kernel $|(v-v_*)\cdot\omega|$, consider the truncated kernel

$$\frac{|(v-v_*)\cdot\omega|}{1+|v-v_*|}\,.$$

Denote by $\tilde{\mathscr{L}}$ and $\tilde{\mathscr{Q}}$ the operators analogous to $\mathscr{L}$ and $\mathscr{Q}$ respectively with the truncated kernel instead of the original hard sphere kernel.

Expanding the quadratic collision integral as follows

$$\begin{aligned}\tilde{\mathscr{Q}}\left(\sqrt{G_\epsilon},\sqrt{G_\epsilon}\right)&=\tilde{\mathscr{Q}}\left(1+\epsilon\frac{\sqrt{G_\epsilon}-1}{\epsilon},1+\epsilon\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)\\&=-\epsilon\tilde{\mathscr{L}}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)+\epsilon^2\tilde{\mathscr{Q}}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon},\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right),\end{aligned}$$

we arrive at the formula

$$\tilde{\mathscr{L}}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)=\epsilon\tilde{\mathscr{Q}}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon},\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)-\frac{1}{\epsilon}\tilde{\mathscr{Q}}\left(\sqrt{G_\epsilon},\sqrt{G_\epsilon}\right).$$

One can prove that $\tilde{\mathscr{L}}$ is a bounded Fredholm self-adjoint operator on $L^2(M\,dv)$ with $\operatorname{Ker}\tilde{\mathscr{L}}=\operatorname{Ker}\mathscr{L}=\operatorname{span}\{1,v_1,v_2,v_3,|v|^2\}$ essentially by the same argument as Hilbert's in Theorem 16. Therefore, $\tilde{\mathscr{L}}$ satisfies the spectral gap estimate

$$\|\mathscr{L}\phi\|_{L^2(M\,dv)}\geq c\|\phi-\Pi\phi\|^2_{L^2(M\,dv)}$$

for some constant $c>0$ and for all $\phi\in L^2(\mathbf{R}^3;M\,dv)$, where we recall that $\Pi$ designates the $L^2(M\,dv)$-orthogonal projection on $\operatorname{Ker}\tilde{\mathscr{L}}$. On the other hand, the first term on the right hand side of this equality is estimated by using the following bound (see [53]):

$$\|\tilde{\mathscr{Q}}(f,f)\|_{L^2(M\,dv)}\leq C\|f\|^2_{L^2(M\,dv)}\,.$$

Putting all these inequalities together, we find that

$$\begin{aligned}c\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon}-\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|^2_{L^2(M\,dv)}&\leq\left\|\tilde{\mathscr{L}}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)\right\|_{L^2(M\,dv)}\\&\leq C\epsilon\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|^2_{L^2(M\,dv)}+\frac{1}{\epsilon}\|\tilde{\mathscr{Q}}\left(\sqrt{G_\epsilon},\sqrt{G_\epsilon}\right)\|_{L^2(M\,dv)}\,.\end{aligned}$$

The second term on the right hand side is mastered by the entropy production bound, i.e. the second inequality in (14). Eventually, one arrive at the estimate

$$c\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon}-\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|^2_{L^2(M\,dv)}\le O(\epsilon)_{L^2_{t,x}}+C\epsilon\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|^2_{L^2(M\,dv)}$$

This bound tells us that the quantity $\frac{\sqrt{G_\epsilon}-1}{\epsilon}$ stays close in $L^2(M\,dv)$ norm to its associated infinitesimal Maxwellian $\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}$, which is both smooth and rapidly decaying in the variable $v$.

Observe that the uniform integrability property stated in the lemma involves the weight $(1+|v|)^p$. The idea is to start from the decomposition

$$\begin{aligned}(1+|v|)^p\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2 &= (1+|v|)^p\frac{\sqrt{G_\epsilon}-1}{\epsilon}\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\\ &\quad+(1+|v|)^p\frac{\sqrt{G_\epsilon}-1}{\epsilon}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}-\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)\end{aligned}$$

The first term on the right hand side is easily controlled because $\Pi\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}$ grows at most quadratically as $|v|\to\infty$. Estimating the second term requires more technical arguments involving in particular the generalized Young inequality for the convex entropy integrand $h(z):=(1+z)\ln(1+z)-z$ already used in Lecture 2 (see Step 1 in the proof of the incompressible Euler limit of the BGK equation). The interested reader is invited to read the complete proof of Proposition 3.2 in [50].

### 4.4.2 Idea No. 2: A $L^1$ Variant of Velocity Averaging

The exact analogue of the velocity averaging theorem (Theorem 17) above would be the following statement:

"Let $G_n$ be a bounded sequence in $L^1(\mathbf{R}^N_x\times\mathbf{R}^N_v)$ such that $S_n:=v\cdot\nabla_x G_n$ is bounded in $L^1(\mathbf{R}^N_x\times\mathbf{R}^N_v)$. Then the sequence $\mathscr{A}_\phi[G_n]$ is strongly relatively compact in $L^1_{loc}(\mathbf{R}^N_x)$ for each $\phi\in C_b(\mathbf{R}^N)$."

Unfortunately, this statement is wrong, as shown by the following counterexample (see counterexample 1 in [52]).

Let $N>1$ and let $\psi\in C^\infty_c(\mathbf{R}^N)$ satisfy

$$\psi\ge 0\quad\text{on }\mathbf{R}^N\,,\qquad\text{and}\int_{\mathbf{R}^N}\psi(z)dz=1\,.$$

Let $v_0\neq 0$, and consider the sequence $\Psi_n(x,v)=n^{2N}\psi(nx)\psi(n(v-v_0))$. Obviously

$$\|\Psi_n\|_{L^1(\mathbf{R}^N\times\mathbf{R}^N)}=1\,,\qquad\text{and }\Psi_n\to\delta_{(0,v_0)}\text{ in }\mathscr{D}'(\mathbf{R}^N\times\mathbf{R}^N)$$

as $n \to \infty$. Let $\Phi_n \equiv \Phi_n(x,v)$ be defined by the formula

$$\Phi_n(x,v) := \int_0^\infty e^{-t}\Psi_n(x-tv,v)dt\,,$$

so that

$$\Phi_n + v\cdot\nabla_x\Phi_n = \Psi_n\,.$$

In particular, one has

$$\|\Phi_n\|_{L^1(\mathbf{R}^N\times\mathbf{R}^N)} \le 1\,,\ \text{ so that } \|v\cdot\nabla_x\Phi_n\|_{L^1(\mathbf{R}^N\times\mathbf{R}^N)} \le 2\,.$$

Yet the explicit formula above for $\Phi_n$ shows that $\mathscr{A}_1[\Phi_n] \to \lambda$ in $\mathscr{D}'(\mathbf{R}^N\times\mathbf{R}^N)$ as $n\to\infty$, where $\lambda$ is the Radon measure defined by the formula

$$\langle\lambda,\chi\rangle := \int_0^\infty e^{-t}\chi(-tv_0)dt\,.$$

In particular, $\lambda$ is a Borel probability measure concentrated on a half-line, which is therefore not absolutely continuous with respect to the Lebesgue measure if $N \ge 2$. This excludes the possibility that any subsequence of $\mathscr{A}_1[\Phi_n]$ might converge in $L^1_{loc}(\mathbf{R}^N)$ for the strong topology.

The appropriate generalization to the $L^1$ setting of the velocity averaging theorem is as follows.

**Theorem 18 (F. Golse-L. Saint-Raymond [47]).** *Let $f_n \equiv f_n(x,v)$ be a bounded sequence in $L^1_{loc}(\mathbf{R}^N\times\mathbf{R}^N)$ such that $v\cdot\nabla_x f_n$ is also bounded in $L^1_{loc}(\mathbf{R}^N\times\mathbf{R}^N)$. Assume that $\mathbf{1}_{|x|+|v|<R}f_n$ is uniformly integrable in $v$ for each $R>0$. Then*

- *$\mathbf{1}_{|x|+|v|<R}f_n$ is uniformly integrable (in $x,v$) for each $R>0$, and*
- *For each $R>0$, and for each test function $\phi\in L^\infty(\mathbf{R}^N_v)$ such that $\phi(v)=0$ for a.e. $v\in\mathbf{R}^N$ satisfying $|v|>R$, the sequence of averages*

$$\mathscr{A}_\phi[f_n]:\ x\mapsto\int f_n(x,v)\phi(v)dv$$

*is relatively compact in $L^1_{loc}(\mathbf{R}^N)$.*

*Proof.* Let us prove that the sequence of averages $\mathscr{A}_\phi[f_n]$ is locally uniformly integrable. Without loss of generality, one can assume that both $f_n \ge 0$ and $\phi \ge 0$.

Let $A$ be a measurable subset of $\mathbf{R}^N$ of finite Lebesgue measure. Let $\chi \equiv \chi(t,x,v)$ be the solution of the Cauchy problem

$$\partial_t\chi + v\cdot\nabla_x\chi = 0\,,\qquad \chi(0,x,v) = \mathbf{1}_A(x)\,.$$

Clearly the solution $\chi$ of this Cauchy problem is of the form $\chi(t,x,v) = \mathbf{1}_{A_x(t)}(v)$. (Indeed, $\chi$ takes the values 0 and 1 only.) On the other hand,

$$|A_x(t)| = \int_{\mathbf{R}^N} \chi(t,x,v)dv = \int_{\mathbf{R}^N} \mathbf{1}_A(x-tv)dv = \frac{|A|}{t^N}\,.$$

(This is the basic dispersion estimate for the free transport equation.)

Set

$$\begin{cases} g_n(x,v) := f_n(x,v)\phi(v)\,, \text{ and} \\ h_n(x,v) := v\cdot\nabla_x g_n(x,v) = \phi(v)(v\cdot\nabla_x f_n(x,v))\,. \end{cases}$$

Both $g_n$ and $h_n$ are bounded in $L^1(\mathbf{R}^N\times\mathbf{R}^N)$, while $g_n$ is uniformly integrable in $v$.

Observe next that

$$\int_A\int g_n\,dv\,dx = \int_{\mathbf{R}^N}\int_{A_x(t)} g_n\,dv\,dx - \int_0^t \iint_{\mathbf{R}^N\times\mathbf{R}^N} h_n(x,v)\chi(s,x,v)dx\,dv\,ds\,.$$

(To see this, integrate by parts in the second term on the right hand side.)

The second integral on the right hand side is $O(t)\sup\|h_n\|_{L^1(\mathbf{R}^N\times\mathbf{R}^N)}$ and can be made less than $\epsilon$ by choosing $t>0$ small enough. With $t>0$ chosen in this way, observe that $|A_x(t)|\to 0$ as $|A|\to 0$ by the dispersion estimate above. Hence the first integral on the right hand side vanishes by uniform integrability in $v$.

A preliminary result in this direction was obtained in [86]—see also Proposition 6 in [52] in the case where the assumption of uniform integrability in the $v$ variable is replaced with the assumption of the type

$$f_n + v\cdot\nabla_x f_n \text{ bounded in } L^1(\mathbf{R}^N_x;(L^p(\mathbf{R}^N_v)) \text{ with } p>1)\,.$$

## 5 Conclusion

There are several other problems on the fluid dynamic limits of the kinetic theory of gases which have not been discussed in these lectures.

Boundary value problems are one such class of problems. The theory of renormalized solutions of the boundary value problem for the Boltzmann equation involves significant additional difficulties not present in the case of the Cauchy problem in the whole Euclidean space or in the torus. These difficulties are due to the nonlocal character (in the $v$ variable) of most of the physically relevant boundary conditions in the kinetic theory of gases. The interaction of the renormalization procedure with the boundary condition was fully understood in a rather remarkable paper by S. Mischler [78]. The fluid dynamic limits of boundary value problems for the Boltzmann equation are reviewed in [88] (see also [77] for a thorough discussion of the Stokes limit of the Boltzmann equation in the presence of boundaries). See also [15, 44] for a discussion of the incompressible Euler limit, also in the presence of boundaries.

We also refer to [93] for a discussion of fluid dynamic limits of the Boltzmann equation in the presence of boundaries in terms of a modified analogue of the Hilbert expansion involving various kinds of boundary layer terms. These boundary layers include in particular Knudsen layers, matching the first terms in Hilbert's expansion with the boundary data—which may fail to be compatible with the dependence in the velocity variable of the various terms in Hilbert's expansion. The mathematical theory of Knudsen layers has been treated in a series of articles [8, 13, 31, 41, 96].

In these lectures, we have systematically considered the case of a hard sphere gas for simplicity. Of course this choice is very legitimate in view of Lanford's theorem [65], showing how the Boltzmann equation for a hard sphere gas can be derived rigorously from Newton's second law applied to each gas molecule. However, more realistic applications may involve polyatomic gases with several internal degrees of freedom, mixture of gases, possibly with species having very different molecular masses, reacting flows... Such problems obviously involve more scaling parameters (mass ratio, reaction speeds...) than the simple situations considered in the present notes. An interesting example is the case of the diffusion of a gas with very light molecules in another gas with very heavy molecules assumed to be at rest—in other words at zero temperature. This case, known as the *Lorentz gas*—see for instance Chap. 1, § 11 in [69]—leads to completely different descriptions even at the level of kinetic theory depending on whether the heavy particles are distributed at random, or are located at the vertices of a lattice in the Euclidean space (see [37] for the random case, and [21, 25, 26, 40, 76] for the periodic case). In the case where the heavy particles are distributed at random, the kinetic equation obtained in [37] is an example of a class of models known as the *linear Boltzmann equation*, and its hydrodynamic limit is governed by a linear diffusion equation (i.e. the heat equation): see [7, 66, 82].

There also remain several outstanding open problems in the context of fluid dynamic limits of the kinetic theory of gases.

First, it would be important to have a proof of the compressible Euler limit of the Boltzmann equation that would not be limited by the regularity of the solution of the target system as in the work of Caflisch or Nishida described in Lecture 1. Of course, this would require having an adequate existence theory of global weak solutions of the compressible Euler system. This is of course a formidable problem in itself, which may not necessarily be directly related to kinetic models. At the time of this writing, global existence of weak solutions of the compressible Euler system has been proved in space dimension 1, for all bounded initial data with small total variation, by using Glimm's scheme [38, 75]. Whether such solutions can be obtained as limits of solutions of the Boltzmann equation is a difficult open problem.

We have mentioned the case of "ghost effects" in Lecture 1. While the formal asymptotic methods leading to these fluid dynamic equations is well understood by now (see for instance Chap. 3.3 of [94]), complete mathematical justifications of these limits are still missing. Even the mathematical theory of the limiting PDE systems describing ghost effects, which involve strongly nonlinear terms, remains to be done.

Finally, we should mention that fluid dynamic limits of the Boltzmann equation should also be investigated in the regime of steady solutions. These are important for applications, since steady solutions describe flows in a permanent regime. Unfortunately the theory of steady solutions of the Boltzmann equation is much less well understood than that of the evolution problem—see [2, 57, 58]. Formal results on fluid dynamic limits of steady solutions of the Boltzmann equation are discussed in [93].

## Appendix 1: On Isotropic Tensor Fields

In this section, we have gathered several results bearing on isotropic tensor fields that are used in Lectures 1 and 3.

### *On the Structure of Isotropic Tensor Fields*

Let $T : \mathbf{R}^N \to (\mathbf{R}^N)^{\otimes m}$ be a tensor field on the $N$-dimensional Euclidean space $\mathbf{R}^N$, endowed with the canonical inner product (i.e. the one for which the canonical basis is orthonormal). The tensor field $T$ is said to be *isotropic* if

$$T(Qv) = Q \cdot T(v)\,, \qquad \text{for each } v \in \mathbf{R}^N \text{ and each } Q \in O_N(\mathbf{R})\,.$$

Here, the notation $A \cdot \tau$ designates the action of the matrix $A \in M_N(\mathbf{R})$ on the tensor $\tau \in (\mathbf{R}^N)^m$ defined by

$$A \cdot (v_1 \otimes \ldots \otimes v_m) = (Av_1) \otimes \ldots \otimes (Av_m)\,.$$

**Lemma 8.** *Let $T : \mathbf{R}^N \to (\mathbf{R}^N)^{\otimes m}$ be an isotropic tensor field on $\mathbf{R}^N$.*

- *If $m = 0$, then $T$ is a radial real-valued function, i.e. $T$ is of the form*

$$T(\xi) = \tau(|\xi|)\,, \quad \xi \in \mathbf{R}^N\,,$$

*where $\tau$ is a real-valued function defined on $\mathbf{R}_+$.*

- *If $m = 1$, then $T$ is of the form*

$$T(\xi) = \tau(|\xi|)\xi\,, \quad \xi \in \mathbf{R}^N\,,$$

*where $\tau$ is a real-valued function defined on $\mathbf{R}_+$.*

- *If $m = 2$ and $T(\xi)$ is symmetric*[5] *for each $\xi \in \mathbf{R}^N$, then $T$ is of the form*

$$T(\xi) = \lambda_1(|\xi|)I + \lambda_2(|\xi|)\xi^{\otimes 2}, \quad \xi \in \mathbf{R}^N .$$

*Proof.* We distinguish the cases corresponding to the different values of $m$.

*Case* $m = 0$. In that case $T : \mathbf{R}^N \to \mathbf{R}$ satisfies $T(Q\xi) = T(\xi)$ for all $Q \in O_N(\mathbf{R})$. Let $e_1$ be the first vector in the canonical basis of $\mathbf{R}^N$. For each $\xi \in \mathbf{R}^N$, there exists $Q \in O_N(\mathbf{R})$ such that $Q\xi = |\xi|e_1$. Thus $T(\xi) = T(|\xi|e_1)$ so that $T$ is a function of $|\xi|$ only, i.e. there exists $\tau : \mathbf{R}_+ \to \mathbf{R}$ such that $T(\xi) = \tau(|\xi|)$.

*Case* $m = 1$. In that case $T : \mathbf{R}^N \to \mathbf{R}^N$ satisfies

$$T(Q\xi) = QT(\xi) \text{ for each } Q \in O_N(\mathbf{R}) .$$

For $\xi = 0$, specializing the identity above to $Q = -I$, one has $T(0) = -T(0) = 0$.

For $\xi \neq 0$, let $Q$ run through the group $O_N(\mathbf{R})_\xi$ of orthogonal matrices leaving $\xi$ invariant. This group is isomorphic to the set of orthogonal linear transformations on $(\mathbf{R}\xi)^\perp$. Thus, given $\zeta_1 \neq \zeta_2 \in (\mathbf{R}\xi)^\perp$ such that $|\zeta_1| = |\zeta_2|$, there exists $Q \in O_N(\mathbf{R})_\xi$ such that $Q\zeta_1 = \zeta_2$, i.e. the subgroup $O_N(\mathbf{R})_\xi$ acts transitively on the spheres of $(\mathbf{R}\xi)^\perp$ centered at 0. Since

$$QT(\xi) = T(\xi) \text{ for each } Q \in O_N(\mathbf{R})_\xi ,$$

setting $e_\xi = \xi/|\xi|$, one has

$$Q\left(T(\xi) - (e_\xi \cdot T(\xi))e_\xi\right) = T(\xi) - (e_\xi \cdot T(\xi))e_\xi \text{ for each } Q \in O_N(\mathbf{R})_\xi$$

and since $T(\xi) - (e_\xi \cdot T(\xi))e_\xi \perp \xi$ we conclude[6] that

$$T(\xi) - (e_\xi \cdot T(\xi))e_\xi = 0 .$$

In other words, $T(\xi) = t(\xi)\xi$ for all $\xi \neq 0$, with $t(Q\xi) = t(Q\xi)$ for all $\xi \in \mathbf{R}^N$ and $Q \in O_N(\mathbf{R})$. One concludes with the result for the case $m = 0$.

---

[5]Consider the endomorphism of $(\mathbf{R}^N)^{\otimes 2}$ defined by

$$u \otimes v \mapsto (u \otimes v)^\sigma = v \otimes u .$$

An element $T$ of $(\mathbf{R}^N)^{\otimes 2}$ is said to be symmetric if and only if $T^\sigma = T$.

[6]Let $G$ be a subgroup of $O_N(\mathbf{R})$ and $V$ be a linear subspace of $\mathbf{R}^N$. Assume that $G$ leaves $V$ invariant, i.e. $gV \subset V$ for each $g \in G$, and that $G$ acts transitively on the spheres of $V$ centered at 0, i.e. if for each $v_1, v_2 \in V$ such that $|v_1| = |v_2|$, there exists $g \in G$ satisfying $gv_1 = v_2$. Then, the only vector $v \in V$ such that $gv = v$ for each $g \in G$ is $v = 0$. Indeed, if $v \neq 0$, one has $|v| = |-v|$ and therefore there exists $g \in G$ such that $gv = -v \neq v$.

*Case* $m = 2$. First we use the canonical identification $(\mathbf{R}^N)^{\otimes 2} \simeq M_N(\mathbf{R})$ defined by the formula $(v \otimes w)\xi := (w \cdot \xi)v$ for each $v, w, \xi \in \mathbf{R}^N$. In this way, the tensor $Q \cdot (v \otimes w) = (Qv) \otimes (Qw)$ is identified with $Q(v \otimes w)Q^T$.

With this identification $T : \mathbf{R}^N \to (\mathbf{R}^N)^{\otimes 2}$ satisfies

$$T(\xi) = T(\xi)^T \text{ and } T(Q\xi) = Q \cdot T(\xi) = QT(\xi)Q^T \text{ for each } Q \in O_N(\mathbf{R}) .$$

The case $\xi = 0$ is obvious: the symmetric matrix with real entries $T(0)$ satisfies $T(0) = QT(0)Q^T$ for all $Q \in O_N(\mathbf{R})$. Since $T(0)$ is diagonalizable and possesses an orthonormal basis of eigenvectors, $T(0)$ must be diagonal (take $Q$ to be the matrix whose columns form an orthonormal basis of eigenvectors of $T(0)$). If $T(0)$ is not of the form $\lambda I$, let $u$ and $v$ to be unitary eigenvectors of $T(0)$ associated to different eigenvalues, taking $Q$ to be the rotation of an angle $\pm\frac{\pi}{4}$ in the plane $\mathbf{R}u \oplus \mathbf{R}v$ leads to a contradiction, since $QT(0)Q^T$ is not diagonal.

Let $\xi \neq 0$, and consider the vector field $S$ defined by $S(\xi) := T(\xi) \cdot \xi$ for each $\xi \in \mathbf{R}^N$. Since

$$S(Q\xi) = QT(\xi)Q^T Q\xi = Q(T(\xi) \cdot \xi) = QS(\xi) ,$$

the result already established in the case $m = 1$ implies that $S$ is of the form

$$S(\xi) = \alpha(|\xi|)\xi , \quad \xi \in \mathbf{R}^N .$$

Since $T(\xi)$ is identified with a symmetric matrix with real entries and $\xi$ is an eigenvector of $T(\xi)$, the space $(\mathbf{R}\xi)^\perp$ is stable under $T(\xi)$, and can be decomposed as an orthogonal direct sum of eigenspaces of $T(\xi)$. On the other hand, since

$$QT(\xi) = T(\xi)Q \quad \text{ for each } Q \in O_N(\mathbf{R})_\xi ,$$

each eigenspace of $T(\xi)$ is stable under $Q$ for each $Q \in O_N(\mathbf{R})_\xi$. Since $O_N(\mathbf{R})_\xi$ acts transitively on spheres of $(\mathbf{R}\xi)^\perp$ centered at 0, this [7] implies that $(\mathbf{R}\xi)^\perp$ is itself an eigenspace of $T(\xi)$. In other words, for each $\xi \in \mathbf{R}^N \setminus \{0\}$, the tensor $T(\xi)$ is of the form

$$T(\xi) = a(\xi)e_\xi \otimes e_\xi + b(\xi)(I - e_\xi \otimes e_\xi) ,$$

with

$$a(\xi) = a(Q\xi) \quad \text{ and } b(\xi) = b(Q\xi) \quad \text{ for all } Q \in O_N(\mathbf{R}) .$$

[7]Let $G$ be a subgroup of $O_N(\mathbf{R})$ and let $V$ be a linear subspace of $\mathbf{R}^N$. Assume that $G$ acts transitively on spheres of $V$ centered at 0. Then the only linear subspaces $W$ of $V$ such that $gW \subset W$ for each $g \in G$ are $\{0\}$ and $V$. Indeed, if $W$ is a linear subspace of $V$ different from either $\{0\}$ or $V$, let $w \in W$ and $z \in V \setminus W$ satisfy $|w| = |z| \neq 0$. By the transitivity assumption above, there exists $g \in G$ such that $gw = z$ and therefore $gW$ is not included in $W$.

Therefore, appealing to the result already proved in the case $m = 0$, one finds that $T$ is of the form

$$T(\xi) = \alpha(|\xi|)e_\xi \otimes e_\xi + \beta(|\xi|)(I - e_\xi \otimes e_\xi)\,.$$

With $\lambda_1 = \beta$ and $\lambda_2(r) = (\alpha(r) - \beta(r))/r^2$, this concludes the proof.

## Isotropic Tensors and Rotation Invariant Averages of Monomials

We first recall an almost trivial result.

**Lemma 9.** *Let $\chi \equiv \chi(|v|)$ be a measurable radial function defined a.e. on $\mathbf{R}^N$ and such that*

$$\int_{\mathbf{R}^N} |\chi(|v|)||v|^2 dv < \infty\,.$$

*Then, for all $i, j = 1, \dots, N$, one has*

$$\int_{\mathbf{R}^N} \chi(|v|)v_i v_j dv = \frac{1}{N}\delta_{ij} \int_{\mathbf{R}^N} \chi(|v|)|v|^2 dv\,.$$

*Proof.* Let $\chi \equiv \chi(|v|)$ be a measurable radial function defined a.e. on $\mathbf{R}^N$ and such that

$$\int_{\mathbf{R}^N} |\chi(|v|)||v|^2 dv < \infty\,.$$

Set

$$T_{i,j} := \int_{\mathbf{R}^N} \chi(|v|)v_i v_j dv\,, \quad i, j = 1, \dots, N\,.$$

Consider the vector field $T$ defined on $\mathbf{R}^N$ by the formula

$$T(\xi) := \int_{\mathbf{R}^N} \chi(|v|)(v \cdot \xi)v\, dv\,,$$

or equivalently

$$T(\xi)_i := \sum_{j=1}^{N} T_{ij}\xi_j\,.$$

Obviously, for each $R \in O_N(\mathbf{R})$, one has

$$T(R\xi) = \int_{\mathbf{R}^N} \chi(|v|)(v \cdot R\xi)v\,dv = \int_{\mathbf{R}^N} \chi(|v|)(R^T v \cdot \xi)v\,dv$$
$$= \int_{\mathbf{R}^N} \chi(|w|)(w \cdot \xi)Rw\,dw = RT(\xi)\,,$$

where the third equality follows from the substitution $w = R^T v$ in the integral. By Lemma 8, $T$ is of the form

$$T(\xi) = \tau(|\xi|)\xi\,,$$

and since $T$ is obviously linear in $\xi$, the function $\tau$ is a constant, so that

$$T(\xi) = \tau\xi\,,$$

or equivalently

$$T_{ij} = \tau\delta_{ij}\,.$$

In particular

$$N\tau = \sum_{i=1}^{N} T_{ii} = \int_{\mathbf{R}^N} \chi(|v|)|v|^2 dv\,,$$

which gives the formula for $\tau$.

Of course, one could also have observed that the matrix with entries

$$\int_{\mathbf{R}^N} \chi(|v|)v_i v_j\,dv$$

for $i, j = 1, \ldots, N$ is real and symmetric, and commutes with every orthogonal matrix. As already explained in the proof of Lemma 8 (case $m = 2$ and $\xi = 0$), such a matrix is proportional to the identity matrix.

Equivalently, one can also notice that

$$i \neq j \quad \Rightarrow \quad \int_{\mathbf{R}^N} \chi(|v|)v_i v_j\,dv = 0$$

since the integrand is an odd function of $v_i$. On the other hand

$$\chi(|v|)v_i^2 dv = \chi(|v|)v_j^2 dv = \frac{1}{N}\chi(|v|)|v|^2 dv\,,$$

where the first equality follows from the substitution

$$(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_N) \mapsto (v_1, \ldots, v_j, \ldots, v_i, \ldots, v_N)$$

which obviously is a linear isometry of $\mathbf{R}^N$ and therefore leaves $\chi(|v|)$ and the Lebesgue measure invariant.

At variance with these elementary arguments, the (slightly) more complicated proof given above is easily generalized to the case of rotation invariant averages of quartic monomials discussed below.

**Lemma 10.** *Let $\chi \equiv \chi(|v|)$ be a measurable radial function defined a.e. on $\mathbf{R}^N$ and such that*

$$\int_{\mathbf{R}^N} |\chi(|v|)||v|^4 dv < \infty .$$

*Set*

$$T_{ijkl} := \int_{\mathbf{R}^N} \chi(|v|) v_i v_j v_k v_l dv , \quad i, j, k, l = 1, \ldots, N .$$

*Then $T_{ijkl}$ is of the form*

$$T_{ijkl} := t_0(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) ,$$

*where*

$$t_0 = \tfrac{1}{N(N+2)} \int_{\mathbf{R}^N} \chi(|v|)|v|^4 dv .$$

*Proof.* Consider the map $T$ defined by

$$T : \mathbf{R}^N \ni \xi \mapsto \int_{\mathbf{R}^N} \chi(|v|)(\xi \cdot v)^2 v \otimes v \, dv \in (\mathbf{R}^N)^{\otimes 2} .$$

Obviously $T(\xi)$ is a symmetric tensor (as an integral linear combination of symmetric tensors $v \otimes v$) and

$$T(\xi) = \sum_{i,j,k,l} T_{ijkl} \xi_k \xi_l e_i \otimes e_j$$

where $e_i$ is the $i$th vector of the canonical basis of $\mathbf{R}^N$, or equivalently

$$T(\xi)_{ij} = \sum_{k,l} T_{ijkl} \xi_k \xi_l .$$

Moreover, for each $R \in O_N(\mathbf{R})$, one has

$$\begin{aligned} T(R\xi) &= \int_{\mathbf{R}^N} \chi(|v|)(R\xi \cdot v)^2 v \otimes v\, dv \\ &= \int_{\mathbf{R}^N} \chi(|v|)(\xi \cdot R^T v)^2 v \otimes v\, dv \\ &= \int_{\mathbf{R}^N} \chi(|w|)(\xi \cdot w)^2 (Rw) \otimes (Rw) dw = RT(\xi)R^T = R \cdot T(\xi)\,, \end{aligned}$$

where the third equality follows from the substitution $w = R^T v$ in the integral.

In other words, $T$ is an isotropic symmetric tensor field of order 2. By Lemma 8, this tensor field is of the form

$$T(\xi) = \tau_0(|\xi|)I + \tau_1(|\xi|)\xi \otimes \xi\,.$$

Besides, $T$ is quadratic in $\xi$, which implies that $\tau_0(|\xi|) = t_0|\xi|^2$ for some constant $t_0 \in \mathbf{R}$, while $\tau_1(|\xi|) = t_1$ is a constant. Finally

$$T(\xi) = t_0|\xi|^2 I + t_1 \xi \otimes \xi\,.$$

In particular, $T$ is of class $C^\infty$ on $\mathbf{R}^N$, and one has

$$2T_{ijpq} = \frac{\partial^2}{\partial \xi_p \partial \xi_q} T(\xi)_{ij} = 2t_0 \delta_{pq}\delta_{ij} + t_1(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})\,.$$

Since $T_{ijpq} = T_{ipjq}$, one has $t_1 = 2t_0$.

Finally

$$\int_{\mathbf{R}^N} \chi(|v|)|v|^4 dv = \sum_{i,k=1}^{N} T_{ikik} = t_0 \sum_{i,k=1}^{N} (\delta_{ik}\delta_{ik} + \delta_{ii}\delta_{kk} + \delta_{ik}\delta_{ik}) = t_0 N(N+2)\,,$$

which concludes the proof.

## Appendix 2: Invariance Properties of the Linearized Collision Integral and Applications

For all $\rho, \theta > 0$ and $u \in \mathbf{R}^3$, we designate by $\mathscr{L}_{\rho,u,\theta}$ the linearization at $\mathscr{M}_{(\rho,u,\theta)}$ of the Boltzmann collision integral, i.e.

$$\mathscr{L}_{\rho,u,\theta}\phi(v)$$

$$:= \iint_{\mathbf{R}^3 \times \mathbf{S}^2} (\phi(v) + \phi(v_*) - \phi(v') - \phi(v'_*))|(v - v_*) \cdot \omega| \mathscr{M}_{(\rho,u,\theta)}(v_*) dv_* d\omega\,.$$

First we examine the translation and scale invariance of the linearized collision operator.

**Lemma 11.** *For all $u \in \mathbf{R}^3$ and $\lambda > 0$ denote $\tau_u$ and $\sigma_\lambda$ the translation and scaling transformations defined by*

$$\tau_u z := z + u\,, \qquad \text{and } \sigma_\lambda z := \lambda z\,.$$

*Then, for each $\phi \in \mathrm{Dom}(\mathscr{L}_{\rho,u,\theta})$, the function $\phi \circ \tau_u \circ \sigma_{\sqrt{\theta}}$ belongs to $\mathrm{Dom}(\mathscr{L}_{1,0,1})$ and one has*

$$(\mathscr{L}_{\rho,u,\theta}\phi) \circ \tau_u \circ \sigma_{\sqrt{\theta}} = \rho\sqrt{\theta}\mathscr{L}_{1,0,1}(\phi \circ \tau_u \circ \sigma_{\sqrt{\theta}})\,.$$

*Proof.* Since $\mathscr{M}_{(\rho,u,\theta)} = \rho\mathscr{M}_{(1,u,\theta)}$, one has

$$\mathscr{L}_{\rho,u,\theta} = \rho\mathscr{L}_{1,u,\theta}\,.$$

Next, observe (by direct inspection on the formulas (1)) that

$$\begin{cases} v'(v+u, v_*+u, \omega) = v'(v, v_*, \omega) + u\,, \\ v'_*(v+u, v_*+u, \omega) = v'_*(v, v_*, \omega) + u\,. \end{cases}$$

Since the Lebesgue measure is invariant by translation

$$\begin{aligned}
&(\mathscr{L}_{1,u,\theta}\phi)(v+u) \\
&= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(v+u) + \phi(w_*) - \phi(v'(v+u, w_*, \omega)) - \phi(v'_*(v+u, w_*, \omega))) \\
&\qquad |(v+u-w_*)\cdot\omega|\mathscr{M}_{(1,u,\theta)}(w_*)dw_*d\omega \\
&= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(v+u) + \phi(w_*) - \phi(v'(v+u, w_*, \omega)) - \phi(v'_*(v+u, w_*, \omega))) \\
&\qquad |(v+u-w_*)\cdot\omega|\mathscr{M}_{(1,0,\theta)}(w_*-u)dw_*d\omega \\
&= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(v+u) + \phi(v_*+u) - \phi(v'(v+u, v_*+u, \omega)) - \phi(v'_*(v+u, v_*+u, \omega))) \\
&\qquad |(v-v_*)\cdot\omega|\mathscr{M}_{(1,0,\theta)}(v_*)dv_*d\omega \\
&= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(v+u) + \phi(v_*+u) - \phi(v'(v, v_*, \omega)+u) - \phi(v'_*(v, v_*, \omega)+u)) \\
&\qquad |(v-v_*)\cdot\omega|\mathscr{M}_{(1,0,\theta)}(v_*)dv_*d\omega \\
&= \mathscr{L}_{1,0,\theta}(\phi\circ\tau_u)(v)
\end{aligned}$$

so that

$$(\mathscr{L}_{1,u,\theta}\phi) \circ \tau_u = \mathscr{L}_{1,0,\theta}(\phi \circ \tau_u)\,.$$

Finally, observing that the map $(v, v_*) \mapsto (v'(v, v_*, \omega), v'_*(v, v_*, \omega))$ is homogeneous of degree 1 for each $\omega \in \mathbf{S}^2$ (see formulas (1)), one has

$$\begin{aligned}
&\mathscr{L}_{1,0,\theta}\phi(\sqrt{\theta}v)\\
&= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(\sqrt{\theta}v) + \phi(w_*) - \phi(v'(\sqrt{\theta}v, w_*, \omega)) - \phi(v'_*(\sqrt{\theta}v, w_*, \omega)))\\
&\qquad |(\sqrt{\theta}v - w_*)\cdot\omega|\mathscr{M}_{(1,0,\theta)}(w_*)dw_*d\omega\\
&= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(\sqrt{\theta}v) + \phi(w_*) - \phi(v'(\sqrt{\theta}v, w_*, \omega)) - \phi(v'_*(\sqrt{\theta}v, w_*, \omega)))\\
&\qquad |(\sqrt{\theta}v - w_*)\cdot\omega|\mathscr{M}_{(1,0,1)}(w_*/\sqrt{\theta})\theta^{-3/2}dw_*d\omega\\
&= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(\sqrt{\theta}v) + \phi(\sqrt{\theta}v_*) - \phi(v'(\sqrt{\theta}v, \sqrt{\theta}v_*, \omega)) - \phi(v'_*(\sqrt{\theta}v, \sqrt{\theta}v_*, \omega)))\\
&\qquad |(\sqrt{\theta}v - \sqrt{\theta}v_*)\cdot\omega|\mathscr{M}_{(1,0,1)}(v_*)dv_*d\omega\\
&= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(\sqrt{\theta}v) + \phi(\sqrt{\theta}v_*) - \phi(\sqrt{\theta}v'(v, v_*, \omega)) - \phi(\sqrt{\theta}v'_*(v, v_*, \omega)))\\
&\qquad \sqrt{\theta}|(v - v_*)\cdot\omega|\mathscr{M}_{(1,0,1)}(v_*)dv_*d\omega\\
&= \sqrt{\theta}\mathscr{L}_{1,0,1}(\phi\circ\sigma_{\sqrt{\theta}})(v)\,,
\end{aligned}$$

so that

$$(\mathscr{L}_{1,0,\theta}\phi)\circ\sigma_{\sqrt{\theta}} = \sqrt{\theta}\mathscr{L}_{1,0,1}(\phi\circ\sigma_{\sqrt{\theta}})\,.$$

The previous lemma shows that we can restrict our attention to $\mathscr{L}_{1,0,1}$, henceforth denoted by $\mathscr{L}$ for simplicity, as in the main body of this text. Then we discuss the invariance of $\mathscr{L}$ under orthogonal transformations.

**Lemma 12.** *For each $R \in O_3(\mathbf{R})$ and each $\phi \in \mathrm{Dom}(\mathscr{L})$, the function $\phi\circ R$ also belongs to* $\mathrm{Dom}(\mathscr{L})$ *and one has*

$$(\mathscr{L}\phi)\circ R = \mathscr{L}(\phi\circ R)\,.$$

*Proof.* Let $R \in O_3(\mathbf{R})$ and $\phi \equiv \phi(v)$ be an element of $\mathrm{Dom}(\mathscr{L})$. Then, elementary substitutions show that

$$\begin{aligned}
\mathscr{L}\phi(Rv) = \iint_{\mathbf{R}^3\times\mathbf{S}^2} &(\phi(Rv) + \phi(w_*) - \phi(v'(Rv, w_*, u)) - \phi(v'_*(Rv, w_*, u)))\\
&|(Rv - w_*)\cdot u|M(w_*)dw_*du
\end{aligned}$$

$$= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(Rv) + \phi(Rv_*) - \phi(v'(Rv, Rv_*, u)) - \phi(v'_*(Rv, Rv_*, u)))$$
$$|(Rv - Rv_*)\cdot u| M(Rv_*) dv_* du$$
$$= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(Rv) + \phi(Rv_*) - \phi(v'(Rv, Rv_*, R\omega)) - \phi(v'_*(Rv, Rv_*, R\omega)))$$
$$|(Rv - Rv_*)\cdot R\omega| M(v_*) dv_* d\omega$$
$$= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(Rv) + \phi(Rv_*) - \phi(v'(Rv, Rv_*, R\omega)) - \phi(v'_*(Rv, Rv_*, R\omega)))$$
$$|(v - v_*)\cdot \omega| M(v_*) dv_* d\omega\,.$$

Formulas (1) show that

$$\begin{cases} v'(Rv, Rv_*, R\omega) = Rv'(v, v_*, \omega)\,, \\ v'_*(Rv, Rv_*, R\omega) = Rv'_*(v, v_*, \omega)\,. \end{cases}$$

Therefore, the computation above implies that

$$(\mathscr{L}\phi)\circ R = \mathscr{L}(\phi\circ R)\,.$$

Next we define the functions $\alpha$ and $\beta$ used in the computation of the viscosity and heat diffusion in the compressible Navier-Stokes system—see Lecture 1, especially formulas (6).

**Lemma 13.** *Let*

$$A(v) = v^{\otimes 2} - \tfrac{1}{3}|v|^2\,, \qquad B(v) = \tfrac{1}{2}(|v|^2 - 5)v\,, \qquad v \in \mathbf{R}^3\,.$$

*Then, for each $i, j, k = 1, 2, 3$, one has $A_{ij}$ and $B_k \in \operatorname{Ran}\mathscr{L}$.*

*Proof.* First we check that

$$A_{ij} \perp \operatorname{Ker}\mathscr{L}\,, \qquad \text{and } B_k \perp \operatorname{Ker}\mathscr{L}\,.$$

for each $i, j, k = 1, 2, 3$.

The orthogonality relations

$$A_{ij} \perp v_k\,, \quad B_k \perp 1\,, \quad \text{and } B_k \perp |v|^2\,, \quad \text{for all } i, j, k = 1, 2, 3$$

are obvious, since the corresponding inner products are integrals of odd summable functions on $\mathbf{R}^3$. That

$$A_{ij} \perp 1 \text{ and } A_{ij} \perp |v|^2\,, \quad \text{for all } i, j, k = 1, 2, 3$$

follows from Lemma 9. Indeed, for each measurable radial function $\phi \equiv \phi(|v|)$ such that

$$\int_{\mathbf{R}^3} |\phi(|v|)||v|^2 M(v)dv < \infty ,$$

one has

$$\int_{\mathbf{R}^3} \phi(|v|)A_{ij}(v)M(v)dv = c\delta_{ij}$$

by Lemma 9, and

$$c = \tfrac{1}{3}\int_{\mathbf{R}^3} \phi(|v|)\,\mathrm{trace}(A(v))M(v)dv = 0 .$$

Finally

$$\int_{\mathbf{R}^3} v_i B_j(v)M(v)dv = \int_{\mathbf{R}^3} v_i v_j(|v|^2 - 5)M(v)dv = c'\delta_{ij}$$

again by Lemma 9 and a straightforward computation shows that

$$c' = \tfrac{1}{3}\int_{\mathbf{R}^3} (|v|^4 - 5|v|^2)M(v)dv = 0 .$$

Since $\mathscr{L}$ is a self-adjoint Fredholm operator on $L^2(\mathbf{R}^3, M\,dv)$ with null space

$$\mathrm{Ker}\,\mathscr{L} = \mathrm{span}(\{1, v_1, v_2, v_3, |v|^2\})$$

by Hilbert's theorem (Theorem 16), the orthogonality properties above imply that

$$A_{ij} \text{ and } B_k \in \mathrm{Ran}\,\mathscr{L} .$$

**Lemma 14.** *Let $\hat{A}$ be the unique symmetric tensor field of order 2 on $\mathbf{R}^3$ such that $\hat{A}_{ij} \in \mathrm{Dom}\,\mathscr{L} \cap (\mathrm{Ker}\,\mathscr{L})^\perp$ for all $1 \le i, j \le 3$ and*

$$\mathscr{L}\hat{A}_{ij} = A_{ij} , \quad 1 \le i, j \le 3 .$$

*Then, there exists a radial measurable function $\alpha \equiv \alpha(|v|)$ defined on $\mathbf{R}^3$ such that*

$$\hat{A}(v) = \alpha(|v|)A(v) , \quad \text{for a.e. } v \in \mathbf{R}^3 .$$

*Likewise, let $\hat{B}$ be the unique vector field on $\mathbf{R}^3$ such that, for each $i = 1, 2, 3$, one has $\hat{B}_i \in \mathrm{Dom}\,\mathscr{L} \cap (\mathrm{Ker}\,\mathscr{L})^\perp$ and*

$$\mathscr{L}\hat{B}_i = B_i , \quad 1 \le i \le 3 .$$

*Then, there exists a radial measurable function $\beta \equiv \beta(|v|)$ defined on $\mathbf{R}^3$ such that*

$$\hat{B}(v) = \beta(|v|)B(v)\,, \quad \textit{for a.e. } v \in \mathbf{R}^3\,.$$

*Proof.* Applying the identity in Lemma 12 to each component of $\hat{A} \in \operatorname{Dom}\mathscr{L} \cap (\operatorname{Ker}\mathscr{L})^{\perp}$ such that

$$\mathscr{L}\hat{A} = A \text{ componentwise}$$

shows that

$$\mathscr{L}(\hat{A} \circ R) = (\mathscr{L}\hat{A}) \circ R = A \circ R = R \cdot A = RAR^T = R(\mathscr{L}\hat{A})R^T = \mathscr{L}(R\hat{A}R^T)\,.$$

Since $\hat{A} \circ R$ and $R\hat{A}R^T$ are both orthogonal to $\operatorname{Ker}\mathscr{L}$ componentwise, we deduce from Fredholm's alternative that

$$\hat{A} \circ R = R\hat{A}R^T \quad \text{for all } R \in O_3(\mathbf{R})\,.$$

Likewise

$$\hat{A} = \hat{A}^T\,;$$

indeed $\hat{A}$ and $\hat{A}^T \perp \operatorname{Ker}\mathscr{L}$ componentwise and $\mathscr{L}(\hat{A} - \hat{A}^T) = A - A^T = 0$, so that $\hat{A} - \hat{A}^T \in \operatorname{Ker}\mathscr{L} \cap (\operatorname{Ker}\mathscr{L})^T$.

By Lemma 8, the tensor field $\hat{A}$ is therefore of the form

$$\hat{A}(v) = \tau_0(|v|)I + \tau_1(|v|)v \otimes v\,.$$

Besides

$$\mathscr{L}(\operatorname{trace}\hat{A}) = \operatorname{trace}(\mathscr{L}\hat{A}) = \operatorname{trace} A = 0 \quad \text{and } \operatorname{trace}\hat{A} \perp \operatorname{Ker}\mathscr{L}\,.$$

Therefore

$$\operatorname{trace}\hat{A} = 3\tau_0(|v|) + |v|^2\tau_1(|v|) = 0\,,$$

which leads to the announced formula for $\hat{A}$.

The case of the integral equation involving the vector field $B$ is treated in the same way. One finds that $\hat{B} \circ R = RB$ for each $R \in O_3(\mathbf{R})$, so that $\hat{B}$ is of the form $\hat{B}(v) = \tau(|v|)v$; the radial function $\beta$ is defined for all $r \neq \sqrt{5}$ by the formula

$$\beta(r) = \tau(r)/(r^2 - 5)\,.$$

Finally we prove formulas and (7) and Lemma 5.

**Lemma 15.** *Let* $u \in \mathbf{R}^3$ *and* $\theta > 0$, *and define*

$$A_{u,\theta}(v) := A\left(\frac{v-u}{\sqrt{\theta}}\right), \quad B_{u,\theta}(v) := B\left(\frac{v-u}{\sqrt{\theta}}\right).$$

*There exist a unique tensor field* $\hat{A}_{u,\theta}$ *and a unique vector field* $\hat{B}_{u,\theta}$, *both belonging to* $\operatorname{Dom}\mathscr{L}_{1,u,\theta} \cap (\operatorname{Ker}\mathscr{L}_{1,u,\theta})^{\perp}$ *componentwise and such that*

$$\mathscr{L}_{1,u,\theta}\hat{A}_{u,\theta} = A_{u,\theta}, \quad \mathscr{L}_{1,u,\theta}\hat{B}_{u,\theta} = B_{u,\theta}.$$

*Moreover*

$$\begin{cases} \hat{A}_{u,\theta}(v) = \dfrac{1}{\sqrt{\theta}}\alpha\left(\dfrac{|v-u|}{\sqrt{\theta}}\right) A\left(\dfrac{v-u}{\sqrt{\theta}}\right), \\ \hat{B}_{u,\theta}(v) = \dfrac{1}{\sqrt{\theta}}\beta\left(\dfrac{|v-u|}{\sqrt{\theta}}\right) B\left(\dfrac{v-u}{\sqrt{\theta}}\right). \end{cases}$$

*Proof.* Define

$$\hat{A}_{u,\theta}(v) = \frac{1}{\sqrt{\theta}}\hat{A}\left(\frac{v-u}{\sqrt{\theta}}\right),$$

so that

$$\hat{A}_{u,\theta} \circ \tau_u \circ \sigma_{\sqrt{\theta}} = \frac{1}{\sqrt{\theta}}\hat{A}.$$

Using Lemmas 14 and 11 shows that, if

$$A = \mathscr{L}_{1,0,\theta}(\hat{A}) = \sqrt{\theta}\mathscr{L}_{1,0,\theta}(\hat{A}_{u,\theta} \circ \tau_u \circ \sigma_{\sqrt{\theta}}) = (\mathscr{L}_{1,u,\theta}\hat{A}_{u,\theta}) \circ \tau_u \circ \sigma_{\sqrt{\theta}}.$$

Equivalently

$$\mathscr{L}_{1,u,\theta}\hat{A}_{u,\theta} = A_{u,\theta},$$

since

$$A_{u,\theta} \circ \tau_u \circ \sigma_{\sqrt{\theta}} = A.$$

That $\hat{A}_{u,\theta}$ $\operatorname{Dom}\mathscr{L}_{1,u,\theta} \cap (\operatorname{Ker}\mathscr{L}_{1,u,\theta})^{\perp}$ componentwise is obvious since the tensor field $\hat{A}$ satisfies $\hat{A} \in \operatorname{Dom}\mathscr{L} \cap (\operatorname{Ker}\mathscr{L})^{\perp}$ componentwise.

The case of the vector field $B_{u,\theta}$ is treated in the same manner.

In other words,

$$\begin{cases} \hat{A}_{u,\theta}(v) = \tilde{\alpha}\left(\theta, \frac{|v-u|}{\sqrt{\theta}}\right) A\left(\frac{v-u}{\sqrt{\theta}}\right) \\ \hat{B}_{u,\theta}(v) = \tilde{\beta}\left(\theta, \frac{|v-u|}{\sqrt{\theta}}\right) B\left(\frac{v-u}{\sqrt{\theta}}\right), \end{cases}$$

with

$$\tilde{\alpha}\,(\theta, r) = \frac{1}{\sqrt{\theta}}\alpha(r)\,, \quad \text{and } \tilde{\beta}\,(\theta, r) = \frac{1}{\sqrt{\theta}}\beta(r)\,.$$

These last formulas and formulas (6) obviously imply formulas (7).

*Proof of Lemma 5.* By Lemma 10, for each radial measurable function $\chi \equiv \chi(|v|)$ such that

$$\int_{\mathbf{R}^3} |\chi(|v|)||v|^4 dv < \infty\,,$$

one has

$$\int_{\mathbf{R}^3} \chi(|v|)A_{ij}(v)A_{kl}(v)dv = t_0(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - (2t_1 - t_2)\delta_{ij}\delta_{kl}\,,$$

where

$$t_1 = \langle \chi \tfrac{1}{3}|v|^2 \rangle\,, \quad \text{and } t_2 = \langle \chi \tfrac{1}{9}|v|^4 \rangle\,.$$

In particular

$$\begin{aligned} \sum_{i=1}^{3} \int_{\mathbf{R}^3} \chi(|v|)A_{ii}(v)A_{kl}(v)dv &= \int_{\mathbf{R}^3} \chi(|v|)\,\mathrm{trace}(A(v))A_{kl}(v)dv \\ &= t_0(3\delta_{kl} + \delta_{kl} + \delta_{kl}) - 3(2t_1 - t_2)\delta_{kl} = 0\,, \end{aligned}$$

so that

$$(2t_1 - t_2) = \tfrac{5}{3}t_0\,.$$

Therefore

$$\int_{\mathbf{R}^3} \chi(|v|)A_{ij}(v)A_{kl}(v)dv = t_0(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - \tfrac{2}{3}\delta_{ij}\delta_{kl})\,.$$

Thus

$$\sum_{i,k=1}^{3} \int_{\mathbf{R}^3} \chi(|v|) A_{ik}(v)^2 dv = \sum_{i,k=1}^{3} t_0(\delta_{ik}\delta_{ik} + \delta_{ii}\delta_{kk} - \tfrac{2}{3}\delta_{ik}\delta_{ik})$$
$$= t_0(3 + 3 \cdot 3 - \tfrac{2}{3} \cdot 3) = 10\, t_0 \,.$$

In particular, with $\chi(|v|) = M(v)$, one has

$$\langle A_{ij} A_{kl} \rangle = \tfrac{1}{15} \langle |v|^4 \rangle (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - \tfrac{2}{3}\delta_{ij}\delta_{kl})$$
$$= (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - \tfrac{2}{3}\delta_{ij}\delta_{kl}) \,,$$

since

$$\sum_{i,k=1}^{3} A_{ik} A_{ik} = \tfrac{2}{3} |v|^4 \,,$$

so that

$$\sum_{i,k=1}^{3} \langle A_{ik} A_{ik} \rangle = \langle \tfrac{2}{3} |v|^4 \rangle = 10\, t_0$$

and hence

$$t_0 = \tfrac{1}{15} \langle |v|^4 \rangle = 1 \,.$$

On the other hand, one has $\hat{A}(v) = \alpha(|v|) A(v)$ by Lemma 14. With $\chi(|v|) = \alpha(|v|) M(v)$, the same argument as before shows that

$$\langle \hat{A}_{ij} A_{kl} \rangle = \tfrac{1}{15} \langle \alpha(|v|) |v|^4 \rangle (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - \tfrac{2}{3}\delta_{ij}\delta_{kl}) \,,$$

which is the sought formula with

$$\mu := \tfrac{1}{15} \langle \alpha(|v|) |v|^4 \rangle \,.$$

Finally

$$\langle \hat{A} : A \rangle = 10\mu = \langle \hat{A} : \mathscr{L} \hat{A} \rangle > 0 \,,$$

since $\mathscr{L}$ is a nonnegative operator and $\hat{A} \perp \operatorname{Ker} \mathscr{L}$ componentwise. This completes the proof.

# References


1. Agoshkov, V.I.: Spaces of functions with differential difference characteristics and smoothness of the solutions of the transport equations. Dokl. Akad. Nauk SSSR **276**, 1289–1293 (1984)
2. Arkeryd, L., Nouri, A.: The stationary Boltzmann equation in $\mathbf{R}^n$ with given indata. Ann. Sc. Norm. Sup. Pisa Cl. Sci. (5) **1**, 359–385 (2002)
3. Arnold, V.I.: Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. Ann. Inst. Fourier **16**, 316–361 (1966)
4. Arnold, V.I., Khesin, B.A.: Topological Methods in Hydrodynamics. Springer, New York (1998)
5. Bardos, C., Golse, F.: Différents aspects de la notion d'entropie au niveau de l'équation de Boltzmann et de Navier-Stokes. C. R. Math. Acad. Sci. Paris **299**, 291–294 (1984)
6. Bardos, C., Ukai, S.: The classical incompressible Navier-Stokes limit of the Boltzmann equation. Math. Models Methods Appl. Sci. **1**, 235–257 (1991)
7. Bardos, C., Santos, R., Sentis, R.: Diffusion approximation and computation of the critical size. Trans. Am. Math. Soc. **284**, 617–649 (1984)
8. Bardos, C., Caflisch, R.E., Nicolaenko, B.: The Milne and Kramers problems for the Boltzmann equation of a hard sphere gas. Commun. Pure Appl. Math. **39**, 323–352 (1986)
9. Bardos, C., Golse, F., Levermore, C.D.: Sur les limites asymptotiques de la théorie cinétique conduisant à la dynamique des fluides incompressibles. C. R. Acad. Sci. **309**, 727–732 (1989)
10. Bardos, C., Golse, F., Levermore, C.D.: Fluid dynamic limits of the Boltzmann equation I. J. Stat. Phys. **63**, 323–344 (1991)
11. Bardos, C., Golse, F., Levermore, C.D.: Fluid dynamic limits of kinetic equations II: convergence proofs for the Boltzmann equation. Commun. Pure Appl. Math **46**, 667–753 (1993)
12. Bardos, C., Golse, F., Levermore, C.D.: The acoustic limit for the Boltzmann equation. Arch. Ration. Mech. Anal. **153**, 177–204 (2000)
13. Bardos, C., Golse, F., Sone, Y.: Half-space problems for the Boltzmann equation: a survey. J. Stat. Phys. **124**, 275–300 (2006)
14. Bardos, C., Levermore, C.D., Ukai, S., Yang, T.: Kinetic equations: fluid dynamical limits and viscous heating. Bull. Inst. Math. Acad. Sin. (New Ser.) **3**, 1–49 (2008)
15. Bardos, C., Golse, F., Paillard, L.: The incompressible Euler limit of the Boltzmann equation with accommodation boundary condition. Commun. Math. Sci. **10**, 159–190 (2012)
16. Bhatnagar, P.L., Gross, E.P., Krook, M.: A model for collision processes in gases. I. small amplitude processes in charged and neutral one-component systems. Phys. Rev. **94**, 511–525 (1954)
17. Bobylev, A.V.: Quasistationary hydrodynamics for the Boltzmann equation. J. Stat. Phys. **80**, 1063–1085 (2006)
18. Bobylev, A.V.: Instabilities in the Chapman-Enskog expansion and hyperbolic Burnett equations. J. Stat. Phys. **124**, 371–399 (2006)
19. Bobylev, A.V.: Generalized Burnett hydrodynamics. J. Stat. Phys. **132**, 569–580 (2008)
20. Bouchut, F., Golse, F., Pulvirenti, M.: Stationary quasivariational inequalities with gradient constraint and nonhomogeneous boundary conditions. In: Desvillettes, L., Perthame, B. (eds.) Kinetic Equations and Asymptotic Theory (Editions scientifiques et médicales). Elsevier, Paris (2000)
21. Bourgain, J., Golse, F., Wennberg, B.: On the distribution of free path lengths for the periodic Lorentz gas. Commun. Math. Phys. **190**, 491–508 (1998)
22. Brenier, Y.: Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations. Commun. Pure Appl. Math. **52**, 411–452 (1999)
23. Brenier, Y., DeLellis, C., Székélyhidi, L.: Weak-strong uniqueness for measure-valued solutions. Commun. Math. Phys. **305**, 351–361 (2011)

24. Caflisch, R.: The fluid dynamic limit of the nonlinear Boltzmann equation. Commun. Pure Appl. Math. **33**, 651–666 (1980)
25. Caglioti, E., Golse, F.: The Boltzmann-Grad limit of the periodic Lorentz gas in two space dimensions. C. R. Math. Acad. Sci. Paris **346**, 477–482 (2008)
26. Caglioti, E., Golse, F.: On the Boltzmann-Grad limit for the two dimensional periodic Lorentz gas. J. Stat. Phys. **141**, 264–317 (2010)
27. Cercignani, C.: Bifurcation problems in fluid mechanics. Meccanica J. Ital. Assoc. Theor. Appl. Mech. **5**, 7–16 (1970)
28. Cercignani, C.: Theory and Applications of the Boltzmann Equation. Scottish Academic Press, Edinburgh/London (1975)
29. Chen, G.Q., Levermore, C.D., Liu, T.P.: Hyperbolic conservation laws with stiff relaxation terms and entropy. Commun. Pure Appl. Math. **47**, 787–830 (1994)
30. Constantin, P., Foias, C.: Navier-Stokes Equations. University of Chicago Press, Chicago (1988)
31. Coron, F., Golse, F., Sulem, C.: A classification of well-posed kinetic layer problems. Commun. Pure Appl. Math. **41**, 409–435 (1988)
32. DeLellis, C., Szekelyhidi, L.: The Euler equations as a differential inclusion. Ann. Math. (2) **170**, 1417–1436 (2009)
33. DeMasi, A., Esposito, R., Lebowitz, J.: Incompressible Navier-Stokes and Euler limits of the Boltzmann equation. Commun. Pure Appl. Math. **42**, 1189–1214 (1990)
34. DiPerna, R., Lions, P.-L.: On the Cauchy problem for the Boltzmann equation: global existence and weak stability results. Ann. Math. **130**, 321–366 (1990)
35. DiPerna, R.J., Majda, A.: Oscillations and concentrations in weak solutions of the incompressible fluid equations. Commun. Math. Phys. **108**, 667–689 (1987)
36. DiPerna, R., Lions, P.-L., Meyer, Y.: $L^p$ regularity of velocity averages. Ann. Inst. H. Poincaré Anal. Non Linéaire **8**, 271–288 (1991)
37. Gallavotti, G.: Divergences and approach to equilibrium in the Lorentz and the wind-tree-models. Phys. Rev. **185**, 308–322 (1969)
38. Glimm, J.: Solutions in the large for nonlinear hyperbolic systems of equations. Commun. Pure Appl. Math. **18**, 697–715 (1965)
39. Golse, F.: The Boltzmann equation and its hydrodynamic limits. In: Dafermos, C.M., Feireisl, E. (eds.) Handbook of Differential Equations: Evolutionary Equations, vol. 2, pp. 161–301. Elsevier B.V., Amsterdam/Boston/Paris (2005)
40. Golse, F.: On the periodic Lorentz gas in the Boltzmann-Grad scaling. Ann. Fac. Sci. Toulouse **17**, 735–749 (2008)
41. Golse, F.: Analysis of the boundary layer equation in the kinetic theory of gases. Bull. Inst. Math. Acad. Sin. (New Series) **3**, 211–242 (2008)
42. Golse, F.: From the kinetic theory of gases to continuum mechanics. In: Harold Grad Lecture, 27th International Symposium on Rarefied Gas Dynamics, 2010, Pacific Grove. AIP Conference Proceedings, vol. 1333, pp. 15–27 (2011)
43. Golse, F.: From the Boltzmann equation to hydrodynamic equations in thin layers. Boll. Unione Mat. Ital. (9) **4**, 163–186 (2011)
44. Golse, F.: From the Boltzmann equation to the Euler equations in the presence of boundaries. Comput. Math. Appl. **65**, 815–830 (2013)
45. Golse, F., Levermore, C.D.: The Stokes-Fourier and acoustic limits for the Boltzmann equation. Commun. Pure Appl. Math. **55**, 336–393 (2002)
46. Golse, F., Levermore, C.D.: Hydrodynamic limits of kinetic models. In: Topics in Kinetic Theory. Fields Institute Communications, vol. 48, pp. 1–75. American Mathematical Society, Providence (2005)
47. Golse, F., Saint-Raymond, L.: Velocity averaging in $L^1$ for the transport equation. C. R. Acad. Sci. **334**, 557–562 (2002)
48. Golse, F., Saint-Raymond, L.: The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels. Invent. Math. **155**(1), 81–161 (2004)

49. Golse, F., Saint-Raymond, L.: Hydrodynamic limits for the Boltzmann equation. Riv. Mat. Univ. Parma (7) **4* ***, 1–144 (2005)
50. Golse, F., Saint-Raymond, L.: The incompressible Navier-Stokes limit of the Boltzmann equation for hard cutoff potentials. J. Math. Pures Appl. **91**, 508–552 (2009)
51. Golse, F., Perthame, B., Sentis, R.: Un résultat de compacité pour les équations de transport et application au calcul de la limite de la valeur propre principale de l'opérateur de transport. C. R. Acad. Sci. **301**, 341–344 (1985)
52. Golse, F., Lions, P.-L., Perthame, B., Sentis, R.: Regularity of the moments of the solution of a transport equation. J. Funct. Anal. **76**, 110–125 (1988)
53. Golse, F., Perthame, B., Sulem, C.: On a boundary layer problem for the nonlinear Boltzmann equation. Arch. Ration. Mech. Anal. **103**, 81–96 (1988)
54. Grad, H.: On the kinetic theory of rarefied gases. Commun. Pure Appl. Math. **2**, 331–407 (1949)
55. Grad, H.: Principles of the kinetic theory of gases. In: Flügge, S. (ed.) Handbook der Physik Band XII, pp. 205–294. Springer, Berlin (1958)
56. Grad, H.: Asymptotic theory of the Boltzmann equation II. In: Laurmann, J.A. (ed.) Rarefied Gas Dynamics, vol. I, pp. 26–59. Academic, New York (1963)
57. Guiraud, J.-P.: Problème aux limites intérieur pour l'équation de Boltzmann linéaire. J. Mécanique **9**, 443–490 (1970)
58. Guiraud, J.-P.: Problème aux limites intérieur pour l'équation de Boltzmann en régime stationnaire, faiblement non linéaire. J. Mécanique **11**, 183–231 (1972)
59. Hilbert, D.: Begründung der kinetischen Gastheorie. Math. Ann. **72**, 562–577 (1912)
60. Hilbert, D.: Mathematical problems. In: International Congress of Mathematicians, Paris 1900. (Translated and reprinted in Bull. Am. Math. Soc. **37**, 407–436 (2000))
61. Jabin, P.-E., Perthame, B.: Regularity in kinetic formulations via averaging lemmas, a tribute to J. L. Lions. ESAIM Control Optim. Calc. Var. **8**, 761–774 (2002)
62. Jin, S., Slemrod, M.: Regularization of the Burnett equations via relaxation. J. Stat. Phys. **103**, 1009–1033 (2001)
63. Kogan, M.N., Galkin, V.S., Fridlender, O.G.: Stresses produced in gases by temperature and concentration inhomogeneities. New types of free convection. Sov. Phys. Usp. **19**, 420–428 (1976)
64. Lachowicz, M.: On the initial layer and the existence theorem for the nonlinear Boltzmann equation. Math. Methods Appl. Sci. **9**, 342–366 (1987)
65. Lanford, O.E.: The evolution of large classical system. In: Moser, J. (ed.) Dynamical Systems, Theory and Applications. Lecture Notes in Physics, vol. 38, pp. 1–111. Springer, Heidelberg (1975)
66. Larsen, E.W., Keller, J.B.: Asymptotics solutions of neutron transport problems for small mean free paths. J. Math. Phys. **15**, 75–81 (1974)
67. Leray, J.: Essai sur le mouvement d'un liquide visqueux emplissant l'espace. Acta Math. **63**, 193–248 (1934)
68. Levermore, C.D., Masmoudi, N.: From the Boltzmann equation to an incompressible Navier-Stokes-Fourier system. Arch. Ration. Mech. Anal. **196**, 753–809 (2010)
69. Lifhsitz, E., Pitayevski, L.: Physical Kinetics. Pergamon Press, Oxford/New York (1981)
70. Lions, P.-L.: Conditions at infinity for Boltzmann's equation. Commun Partial Differ. Equ. **19**, 335–367 (1994)
71. Lions, P.-L.: Compactness in Boltzmann's equation via Fourier integral operators and applications. II. J. Math. Kyoto Univ. **34**, 429–461 (1994)
72. Lions, P.-L.: Mathematical Topics in Fluid Mechanics, Vol. 1: Incompressible Models. The Clarendon Press/Oxford University Press, New York (1996)
73. Lions, P.-L., Masmoudi, N.: From Boltzmann equation to the Navier-Stokes and Euler equations I. Arch. Ration. Mech. Anal. **158**, 173–193 (2001)
74. Lions, P.-L., Perthame, B., Tadmor, E.: A kinetic formulation of multidimensional scalar conservation laws and related equations. J. Am. Math. Soc. **7**, 169–191 (1994)

75. Liu, T.-P.: Solutions in the large for the equations of nonisentropic gas dynamics. Indiana Univ. Math. J. **26**, 147–177 (1977)
76. Marklof, J., Strömbergsson, A.: The Boltzmann-Grad limit of the periodic Lorentz gas. Ann. Math. **174**, 225–298 (2011)
77. Masmoudi, N., Saint-Raymond, L.: From the Boltzmann equation to the Stokes-Fourier system in a bounded domain. Commun. Pure Appl. Math. **56**, 1263–1293 (2003)
78. Mischler, S.: Kinetic equations with Maxwell boundary conditions. Ann. Sci. Ecole Norm. Sup. **43**, 719–760 (2010)
79. Nirenberg, L.: An abstract form of the nonlinear Cauchy-Kowalewski theorem. J. Differ. Geom. **6**, 561–576 (1972)
80. Nishida, T.: A note on a theorem of Nirenberg. J. Differ. Geom. **12**, 629–633 (1977)
81. Nishida, T.: Fluid dynamical limit of the nonlinear Boltzmann equation to the level of the compressible Euler equation. Commun. Math. Phys. **61**, 119–148 (1978)
82. Papanicolaou, G.C.: Asymptotic analysis of transport processes. Bull. Am. Math. Soc. **81**, 330–392 (1975)
83. Perthame, B.: Kinetic Formulation of Conservation Laws. Oxford University Press, Oxford (2002)
84. Perthame, B., Pulvirenti, M.: Weighted $L^\infty$ bounds and uniqueness for the Boltzmann BGK model. Arch. Ration. Mech. Anal. **125**, 289–295 (1993)
85. Saint-Raymond, L.: Du modèle BGK de l'équation de Boltzmann aux équations d'Euler des fluides incompressibles. Bull. Sci. Math. **126**, 493–506 (2002)
86. Saint-Raymond, L.: Discrete time Navier-Stokes limit for the BGK Boltzmann equation. Commun. Partial Differ. Equ. **27**, 149–184 (2002)
87. Saint-Raymond, L.: Convergence of solutions to the Boltzmann equation in the incompressible Euler limit. Arch. Ration. Mech. Anal. **166**, 47–80 (2003)
88. Saint-Raymond, L.: Hydrodynamic Limits of the Boltzmann Equation. Lecturer Notes in Mathematics, vol. 1971. Springer, Berlin/Heidelberg (2009)
89. Scheffer, V.: Hausdorff measure and the Navier-Stokes equations. Commun. Math. Phys. **55**, 97–112 (1977)
90. Shnirelman, A.: On the nonuniqueness of weak solutions of the Euler equation. Commun. Pure Appl. Math. **50**, 1261–1286 (1997)
91. Sideris, T.: Formation of Singularities in 3D compressible fluids. Commun. Math. Phys. **101**, 475–485 (1985)
92. Sone, Y.: Continuum gas dynamics in the light of kinetic theory and new features of rarefied gas flows. In: Shen, C. (ed.) Harold Grad Lecture, Rarefied Gas Dynamics, pp. 3–24. Peking University Press, Beijing (1997)
93. Sone, Y.: Molecular Gas Dynamics. Theory, Techniques, and Applications. Birkhäuser, Boston (2002)
94. Sone, Y.: Kinetic Theory and Fluid Dynamics. Birkhäuser, Boston (2007)
95. Sone, Y., Aoki, K., Takata, S., Sugimoto, H., Bobylev, A.V.: Inappropriateness of the heat-conduction equation for description of a temperature field of a stationary gas in the continuum limit: examination by asymptotic analysis and numerical computation of the Boltzmann equation. Phys. Fluids **8**, 628–638 (1996)
96. Ukai, S., Yang, T., Yu, S.-H.: Nonlinear boundary layers of the Boltzmann equation. I. Existence. Commun. Math. Phys. **236**, 373–393 (2003)
97. Villani, C.: Limites hydrodynamiques de l'équation de Boltzmann. In: Séminaire Bourbaki, Exp. No. 893, pp. 364–405 (2000–2001)
98. Wiedemann, E.: Existence of weak solutions for the incompressible Euler equations. Ann. Inst. H. Poincaré Anal. Non Linéaire **28**, 727–730 (2011)
99. Yau, H.-T.: Relative entropy and hydrodynamics of Ginzburg-Landau models. Lett. Math. Phys. **22**, 63–80 (1991)

# Part II
# Short Papers

# Stationary Quasivariational Inequalities with Gradient Constraint and Nonhomogeneous Boundary Conditions

**Assis Azevedo, Fernando Miranda, and Lisa Santos**

## 1 Introduction and Main Results

If we want to solve the well known problem of finding $u \in H_0^1(\Omega)$ such that

$$\min\{-\Delta u - f, u - \psi\} = 0 \qquad \text{a.e. in } \Omega,$$

for a given $\psi$, the easiest approach is to solve the variational inequality: to find $u \in K_\psi = \{v \in H_0^1(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}$ such that

$$\int_\Omega \nabla u \cdot \nabla(v - u) \geq \int_\Omega f(v - u), \qquad \forall v \in K_\psi. \tag{1}$$

Existence of solution for stationary variational inequalities like the considered above is immediate (see [6]). Quasivariational inequalities are similar, but implicit, problems where the convex set depends on the solution. For instance, we consider the problem (1), with $K_\psi$ substituted by $K_{F(u)}$, for a given function $F \in \mathscr{C}(\mathbb{R})$. The proof of existence of solution is no more a trivial problem and different approaches can be used, such as a fixed point argument or approximation of the quasivariational inequality by a family of penalized equations, for which existence is known, using a priori estimates to pass to the limit.

Here we are interested in variational and quasivariational inequalities with gradient constraint, whose convex sets are of the following type:

$$\mathbb{K}_\varphi = \{v \in W^{1,p}(\Omega) : |\nabla v| \leq \varphi, \text{ a.e. in } \Omega\}, \tag{2}$$

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or

$$\mathscr{K}_\varphi = \{v \in W^{1,p}(\Omega) : |\nabla v| \le \varphi, \text{ a.e. in } \Omega,\ v_{|\partial\Omega} = g\}, \tag{3}$$

for $\varphi \ge 0$ in the variational case and $\varphi = F(u)$, in the quasivariational case, where $F \in \mathscr{C}(\mathbb{R})$ and $g \in \mathscr{C}(\partial\Omega)$.

The first model of this type was the elastoplastic torsion problem, a stationary variational inequality with gradient constraint 1 and homogeneous Dirichlet boundary condition ([18], [3] or [4]). Sand piles and river networks ([13] or [15]) or electromagnetic problems ([14], [17], [2], [10] or [11]) can be modeled by variational or quasivariational inequalities with gradient or curl constraint. As far as the authors know, the first work in quasivariational inequalities with gradient constraint and nonhomogeneous boundary condition is [1]. This work generalizes the existence results for quasivariational inequalities presented in that paper, improving the growth condition imposed on $F$ (details will be given later). We also present another situation where no growth condition is imposed on $F$, assuming that the operator considered is $\boldsymbol{a}(x, \boldsymbol{u}) = a(x)|\boldsymbol{u}|^{p-2}\boldsymbol{u}$ and assuming a little more on the regularity of the data. We notice that, assuming nonhomegeneous conditions on the boundary introduces additional difficulties when seeking for solutions of quasivariational inequalities. The proof of existence of solution may be done either using a fixed point theorem or by approximating the quasivariational inequality by a family of equations. In both cases, given a function in a certain convex set (depending on the constraint of the gradient and on the boundary condition), we need to find out a function in another convex set and estimate their distance. This procedure, not easy even when null boundary conditions are considered in both convex sets, becomes harder when the boundary conditions change, situation scarcely considered in the literature.

In this paper, we consider $\Omega$ a bounded open subset of $\mathbb{R}^N$ with smooth boundary. Given $1 < p < \infty$, let $\boldsymbol{a} : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a Carathéodory function satisfying the structural conditions (4a), (4b) and (4c) or (4c')

$$\boldsymbol{a}(x, \boldsymbol{u}) \cdot \boldsymbol{u} \ge a_*|\boldsymbol{u}|^p, \tag{4a}$$

$$|\boldsymbol{a}(x, \boldsymbol{u})| \le a^*|\boldsymbol{u}|^{p-1}, \tag{4b}$$

$$\big(\boldsymbol{a}(x, \boldsymbol{u}) - \boldsymbol{a}(x, \boldsymbol{v})\big) \cdot (\boldsymbol{u} - \boldsymbol{v}\big) > 0, \text{ if } \boldsymbol{u} \ne \boldsymbol{v}, \tag{4c}$$

$$\big(\boldsymbol{a}(x, \boldsymbol{u}) - \boldsymbol{a}(x, \boldsymbol{v})\big) \cdot (\boldsymbol{u} - \boldsymbol{v}\big) \ge \begin{cases} a_*|\boldsymbol{u} - \boldsymbol{v}|^p & \text{if } p \ge 2, \\ a_*\big(|\boldsymbol{u}| + |\boldsymbol{v}|\big)^{p-2}|\boldsymbol{u} - \boldsymbol{v}|^2 & \text{if } p < 2, \end{cases} \tag{4c'}$$

for given constants $0 < a_* < a^*$, for all $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

Let $q$ be the critical Sobolev exponent of $p$, if $p \ne N$, i.e.,

$$q = \frac{Np}{N-p} \ \text{ if } 1 < p < N, \quad q = \infty \ \text{ if } p > N,$$

and $q > 1$, if $p = N$. Observe that, given $v \in W^{1,p}(\Omega)$, we have the following inequality

$$\|v\|_{L^q(\Omega)} \leq C_q \|v\|_{W^{1,p}(\Omega)}, \tag{5}$$

being $C_q > 0$.

Let $r$ be the critical Sobolev exponent of $p$ for the trace embedding, if $p \neq N$, i.e.,

$$r = \frac{(N-1)p}{N-p} \ \text{ if } 1 < p < N, \quad r = \infty \ \text{ if } p > N,$$

and $r > 1$, if $p = N$. Then, given $v \in W^{1,p}(\Omega)$, there exists $C_r > 0$ such that

$$\|v\|_{L^r(\partial\Omega)} \leq C_r \|v\|_{W^{1,p}(\Omega)}. \tag{6}$$

Given

$$F \in \mathscr{C}(\mathbb{R};\mathbb{R}^+), \ f \in L^{q'}(\Omega), \ g \in L^{r'}(\partial\Omega), \ c \in L^\infty(\Omega), \ c \geq c_*, \tag{7}$$

where $c_*$ is a nonnegative constant, consider the following quasivariational inequality with Neumann type boundary condition: to find $u \in \mathbb{K}_{F(u)}$ such that

$$\int_\Omega \boldsymbol{a}(x,\nabla u)\cdot\nabla(v-u) + \int_\Omega c\,|u|^{p-2}u(v-u)$$
$$\geq \int_\Omega f(v-u) + \int_{\partial\Omega} g(v-u), \qquad \forall v \in \mathbb{K}_{F(u)}, \tag{8}$$

where $\mathbb{K}_{F(u)}$ is defined in (2).

The following two theorems give sufficient conditions for existence of solution of the above quasivariational inequality.

**Theorem 1.** *Assume* (4a)*,* (4b)*,* (4c') *and* (7)*, with* $c_* > 0$*. If* $p \leq N$ *suppose, in addition, that there exist positive constants* $c_0$ *and* $c_1$ *such that*

$$F(s) \leq c_0 + c_1|s|^\alpha, \qquad \forall s \in \mathbb{R},$$

*being* $\alpha \geq 0$ *if* $p = N$ *and* $0 \leq \alpha < \frac{N}{N-p}$ *if* $p < N$.

*Then the quasivariational inequality* (8) *has a solution.*

*Remark 1.* We point out that the condition $0 \leq \alpha < \frac{p}{N-p}$ when $p < N$ assumed in [1] is here improved to $0 \leq \alpha < \frac{N}{N-p}$.

The following theorem states existence of solution for problem (8) with homogeneous Neumann boundary condition, imposing no growth condition on $F$ but assuming the strict positivity of $F$, the boundedness of $f$ and a restriction on the operator $\boldsymbol{a}$.

**Theorem 2.** *Assume that* $\boldsymbol{a}(x,\boldsymbol{u}) = a(x)|\boldsymbol{u}|^{p-2}\boldsymbol{u}$ *with* $0 < a_* \le a \le a^*$. *Assume, in addition, that* $f \in L^\infty(\Omega)$, $g \equiv 0$, $c \in L^\infty(\Omega)$, *with* $c \ge c_*$, *and* $F \in \mathscr{C}(\mathbb{R};\mathbb{R}^+)$, *with* $F \ge F_*$, *where* $c_*$ *and* $F_*$ *are positive constants.*

*Then the quasivariational inequality* (8) *has a solution.*

Consider the quasivariational inequality with Dirichlet type boundary condition: to find $u \in \mathscr{K}_{F(u)}$ such that

$$\int_\Omega \boldsymbol{a}(x,\nabla u)\cdot\nabla(v-u) + \int_\Omega c\,|u|^{p-2}u(v-u) \ge \int_\Omega f(v-u), \qquad \forall v \in \mathscr{K}_{F(u)}, \tag{9}$$

where $\mathscr{K}_{F(u)}$ is defined in (3).

We present two theorems which give sufficient conditions for the existence of solution of the above quasivariational inequality.

**Theorem 3.** *Consider the assumptions of Theorem 1, with* $c_* \ge 0$ *and* $F \ge F_* > 0$, *where* $c_*$ *and* $F_*$ *are constants. Assume, in addition, that there exists* $k \in [0,1)$ *such that*

$$|g(x) - g(y)| \le kF_*\bar{d}(x,y) \qquad \textit{for } x, y \in \partial\Omega, \tag{10}$$

*where* $\bar{d}$ *is the geodesic distance in* $\Omega$.

*Then the quasivariational inequality* (9) *has a solution.*

We observe that the above theorem generalizes a result of [7], where Dirichlet homogeneous boundary condition was considered as well as a more restrictive growth assumption on $F$, for $1 < p \le N$.

**Theorem 4.** *Assume that* $\boldsymbol{a}(x,\boldsymbol{u}) = a(x)|\boldsymbol{u}|^{p-2}\boldsymbol{u}$ *with* $0 < a_* \le a \le a^*$. *Assume in addition, that* (10) *is verified for some* $k < \frac{a_*}{a^*}$, $f \in L^\infty(\Omega)$, $c \in L^\infty(\Omega)$, *with* $c \ge c_*$, $F \in \mathscr{C}(\mathbb{R};\mathbb{R}^+)$, *with* $F \ge F_*$, *where* $c_*, F_*$ *are constants,* $c_* \ge 0$ *and* $F_* > 0$.

*Then the quasivariational inequality* (9) *has a solution.*

## 2 The Case with Neumann Boundary Condition

In this section we consider the quasivariational inequality with Neumann boundary condition. The proof of Theorem 1 uses a fixed point theorem and the proof of Theorem 2 is done by approximating the quasivariational inequality by a family of penalized and regularized equations.

Given $\varphi \in L^\infty(\Omega)$, $\varphi \geq 0$, we consider the variational inequality: to find $u \in \mathbb{K}_\varphi$ such that

$$\int_\Omega \boldsymbol{a}(x, \nabla u) \cdot \nabla(v-u) + \int_\Omega c\,|u|^{p-2}u(v-u) \geq \int_\Omega f(v-u) + \int_{\partial\Omega} g(v-u), \quad \forall v \in \mathbb{K}_\varphi, \quad (11)$$

where $\mathbb{K}_\varphi$ is defined in (2). In this section we assume (4a), (4b), (4c) and (7) with $c_* > 0$. Under these assumptions, this problem has a unique solution (see [8, Theorem 8.2]).

**Proposition 1.** *Let $u$ be the solution of problem* (11). *Then*

$$\|u\|_{W^{1,p}(\Omega)} \leq M\big(\|f\|_{L^{q'}(\Omega)} + \|g\|_{L^{r'}(\partial\Omega)}\big)^{\frac{1}{p-1}}.$$

*where $M = \left(\frac{\max\{C_q, C_r\}}{\min\{a_*, c_*\}}\right)^{\frac{1}{p-1}}$, for $C_q$ and $C_r$ defined in* (5) *and in* (6).

*Proof.* Considering $v = 0$ in the variational inequality (11) we obtain,

$$\begin{aligned}\int_\Omega \boldsymbol{a}(x, \nabla u) \cdot \nabla u + \int_\Omega c\,|u|^p &\leq \int_\Omega f u + \int_{\partial\Omega} gu \\ &\leq \|f\|_{L^{q'}(\Omega)}\|u\|_{L^q(\Omega)} + \|g\|_{L^{r'}(\partial\Omega)}\|u\|_{L^r(\partial\Omega)} \\ &\leq C\big(\|f\|_{L^{q'}(\Omega)} + \|g\|_{L^{r'}(\partial\Omega)}\big)\|u\|_{W^{1,p}(\Omega)},\end{aligned}$$

where $C = \max\{C_q, C_r\}$. But, as

$$\begin{aligned}\min\{a_*, c_*\}\|u\|^p_{W^{1,p}(\Omega)} &\leq a_*\|\nabla u\|^p_{L^p(\Omega)} + c_*\|u\|^p_{L^p(\Omega)} \\ &\leq \int_\Omega \boldsymbol{a}(x, \nabla u) \cdot \nabla(u) + \int_\Omega c\,|u|^p,\end{aligned}$$

the conclusion follows. □

We present now a continuous dependence result on the gradient constraints that will be necessary to apply later a fixed point theorem. A more general result can be found in [1], where the dependence on $f$ and $g$ is also considered.

**Proposition 2.** *For $\varphi, \psi \in L^\infty(\Omega)$ with a positive lower bound $\eta$ and* $\boldsymbol{a}$ *verifying* (4a), (4b) *and* (4c'), *the solutions $u_\varphi$ and $u_\psi$ of problem* (11) *satisfy*

$$\|u_\varphi - u_\psi\|^{\max\{p,2\}}_{W^{1,p}(\Omega)} \leq C\,\|\varphi - \psi\|_{L^\infty(\Omega)},$$

*where $C = C(\eta)$ is a positive constant.*

*Proof.* Letting $A(u,v)=\int_\Omega \boldsymbol{a}(x,\nabla u)\cdot\nabla v+\int_\Omega c\,|u|^{p-2}uv$, then, for $\lambda\in\mathbb{R}$,

$$\begin{aligned}A(u,u-v)&-A(v,u-v)\\&=A(u,u-\lambda v)+A(v,v-\lambda u)+(\lambda-1)\left[A(u,v)+A(v,u)\right].\end{aligned}\tag{12}$$

Recall that $\eta$ is a positive lower bound of $\varphi$ and $\psi$ and set $\lambda=\frac{\eta}{\eta+\|\varphi-\psi\|_\infty}$. Then, as $\lambda u_\psi\in K_\varphi$ and $\lambda u_\varphi\in K_\psi$, using $\lambda u_\varphi$ as test function in (11) with convex set $\mathbb{K}_\psi$ and $\lambda u_\psi$ as test function in (11) with convex set $\mathbb{K}_\varphi$ we have,

$$\begin{aligned}A(u_\varphi,u_\varphi-\lambda u_\psi)+A(u_\psi,u_\psi-\lambda u_\varphi)&\le(1-\lambda)\Big(\int_\Omega f(u_\varphi+u_\psi)+\int_{\partial\Omega}g(u_\varphi+u_\psi)\Big)\\&\le(1-\lambda)C\big(\|f\|_{L^{q'}(\Omega)}+\|g\|_{L^{r'}(\partial\Omega)}\big)\big(\|u_\varphi\|_{W^{1,p}(\Omega)}+\|u_\psi\|_{W^{1,p}(\Omega)}\big)\\&\le\frac{D}{\eta}\|\varphi-\psi\|_{L^\infty(\Omega)},\end{aligned}$$

where $C=\max\{C_q,C_r\}$ and $D=D(\|f\|_{L^{q'}(\Omega)},\|g\|_{L^{r'}(\partial\Omega)})$ is a positive constant. The last inequality is true by Proposition 1 and because

$$1-\lambda=\frac{\|\varphi-\psi\|_{L^\infty(\Omega)}}{\eta+\|\varphi-\psi\|_{L^\infty(\Omega)}}\le\frac{\|\varphi-\psi\|_{L^\infty(\Omega)}}{\eta}.$$

On the other hand, recalling the constant $M$ defined in Proposition 1,

$$\begin{aligned}\big|A(u_\varphi,u_\psi)\big|&\le a^*\int_\Omega|\nabla u_\varphi|^{p-1}|\nabla u_\psi|+\|c\|_{L^\infty(\Omega)}\int_\Omega|u_\varphi|^{p-1}|u_\psi|\\&\le a^*\|\nabla u_\varphi\|^{p-1}_{L^p(\Omega)}\|\nabla u_\psi\|_{L^p(\Omega)}+\|c\|_{L^\infty(\Omega)}\|u_\varphi\|^{p-1}_{L^p(\Omega)}\|u_\psi\|_{L^p(\Omega)}\\&\le(a^*+\|c\|_{L^\infty(\Omega)})\,\|u_\varphi\|^{p-1}_{W^{1,p}(\Omega)}\|u_\psi\|_{W^{1,p}(\Omega)}\\&\le(a^*+\|c\|_{L^\infty(\Omega)})\,M^p\big(\|f\|_{L^{q'}(\Omega)}+\|g\|_{L^{r'}(\partial\Omega)}\big)^{p'}\end{aligned}$$

and, analogously,

$$\big|A(u_\psi,u_\varphi)\big|\le(a^*+\|c\|_{L^\infty(\Omega)})\,M^p\big(\|f\|_{L^{q'}(\Omega)}+\|g\|_{L^{r'}(\partial\Omega)}\big)^{p'}.$$

So, using (12), there exists $C=C(\|f\|_{L^{q'}(\Omega)},\|g\|_{L^{r'}(\partial\Omega)},\eta)>0$ such that

$$A(u_\varphi,u_\varphi-u_\psi)-A(u_\psi,u_\varphi-u_\psi)\le C\|\varphi-\psi\|_{L^\infty(\Omega)}.$$

On the other hand, by (4c')

$$A(u_\varphi, u_\varphi - u_\psi) - A(u_\psi, u_\varphi - u_\psi) \geq \min\{a_*, c_*\}\|u_\varphi - u_\psi\|^p_{W^{1,p}(\Omega)} \quad \text{if } p \geq 2.$$

Using the reverse Hölder inequality in the case $p < 2$, we get

$$\begin{aligned} A(u_\varphi, u_\varphi - u_\psi) &- A(u_\psi, u_\varphi - u_\psi) \\ &\geq a_* \| |\nabla u_\varphi| + |\nabla u_\psi| \|^{p-2}_{L^p(\Omega)} \|\nabla u_\varphi - \nabla u_\psi\|^2_{L^p(\Omega)} \\ &\qquad + c_* \| |u_\varphi| + |u_\psi| \|^{p-2}_{L^p(\Omega)} \|u_\varphi - u_\psi\|^2_{L^p(\Omega)} \end{aligned}$$

and then, by Proposition 1, the conclusion follows also in this case. □

The following proposition will be used in the proof of Theorem 1.

**Proposition 3.** *Let $N \in \mathbb{N}$, $p > 1$, $s > N$, $\frac{N}{N-1} < \alpha$ and $\alpha < \frac{N}{N-p}$ if $p < N$. Consider the sequence $(s_n)_n$ defined by*

$$s_1 = s \quad \textit{and} \quad s_{n+1} = \frac{\alpha N s_n}{N + \alpha s_n}.$$

*Then there exists $n \in \mathbb{N}$ such that $1 < s_n \leq p$.*

*Proof.* Using the inequality $\frac{N}{N-1} < \alpha$ it is easy to prove, by induction, that $s_n > 1$ for all $n \in \mathbb{N}$. On the other hand $(s_n)_n$ is a decreasing sequence, because $s_2 < s_1$ and, for $n > 2$,

$$s_{n+1} < s_n \Leftrightarrow \frac{\alpha s_n N}{N + \alpha s_n} < \frac{\alpha s_{n-1} N}{N + \alpha s_{n-1}} \Leftrightarrow \frac{s_n}{N + \alpha s_n} < \frac{s_{n-1}}{N + \alpha s_{n-1}} \Leftrightarrow s_n < s_{n-1}.$$

So $(s_n)_n$ is convergent. Using the equality $s_{n+1} = \frac{\alpha s_n N}{N + \alpha s_n}$, we see that the limit is $N(1 - \frac{1}{\alpha})$. To conclude we just need to observe that $N(1 - \frac{1}{\alpha}) < p$. This is true because if $p < N$, $\alpha < \frac{N}{N-p}$. □

We are now able to prove our first result.

*Proof of Theorem 1.* Consider a sequence $(p_n)_n$ such that $p_1 = p$ and, for $i \geq 1$, $p_i$ is a critical Sobolev exponent of $p_{i-1}$. Let $s$ be the first element of this sequence greater than $N$. Applying repeatedly the Sobolev type inequality (5) one has

$$\exists\, C > 0\ \forall u \in W^{1,s}(\Omega) \quad \|u\|_{W^{1,s}(\Omega)} \leq C \left(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^s(\Omega)}\right). \tag{13}$$

Observe that, if $\varphi \in \mathscr{C}(\bar{\Omega})$ and $u \in \mathscr{K}_{F(\varphi)}$ then $u \in W^{1,s}(\Omega)$, as $\nabla u \in L^\infty(\Omega)$. In particular, the operator $T : \mathscr{C}(\bar{\Omega}) \longrightarrow W^{1,s}(\Omega)$ such that $T(\varphi) = u_\varphi$, where $u_\varphi$ is the solution of problem (11) with $\mathscr{K}_{F(\varphi)}$ replacing $\mathscr{K}_\varphi$, is well-defined.

To prove that $T$ is continuous, consider $\varphi \in \mathscr{C}(\bar{\Omega})$ and let $\delta > 0$ be such that $\|F(\psi)\|_{\mathscr{C}(\bar{\Omega})} \leq \|F(\varphi)\|_{\mathscr{C}(\bar{\Omega})} + 1$ if $\|\varphi - \psi\|_{\mathscr{C}(\bar{\Omega})} \leq \delta$. For those $\psi$ we have,

$$
\begin{aligned}
|\nabla u_\varphi - \nabla u_\psi|^s &= |\nabla u_\varphi - \nabla u_\psi|^{s-p} |\nabla u_\varphi - \nabla u_\psi|^p \\
&\leq \big(|\nabla u_\varphi| + |\nabla u_\psi|\big)^{s-p} |\nabla u_\varphi - \nabla u_\psi|^p \leq (F(\varphi) + F(\psi))^{s-p} |\nabla u_\varphi - \nabla u_\psi|^p \\
&\leq \big(2\|F(\varphi)\|_{\mathscr{C}(\bar{\Omega})} + 1\big)^{s-p} |\nabla u_\varphi - \nabla u_\psi|^p
\end{aligned}
$$

and then, using (13),

$$
\begin{aligned}
&\|u_\varphi - u_\psi\|_{W^{1,s}(\Omega)} \\
&\qquad \leq C\Big(\|u_\varphi - u_\psi\|_{L^p(\Omega)} + \big(2\|F \circ \varphi\|_{\mathscr{C}(\bar{\Omega})} + 1\big)^{\frac{s-p}{s}} \|\nabla(u_\varphi - u_\psi)\|^{\frac{p}{s}}_{L^p(\Omega)}\Big).
\end{aligned}
$$

Noticing that $F(\varphi)$ and $F(\psi)$ has a positive lower bound, as $\varphi, \psi \in \mathscr{C}(\bar{\Omega})$ and $F \in \mathscr{C}(\mathbb{R}; \mathbb{R}^+)$, this last inequality together with the Proposition 2, proves that $T$ is continuous.

In order to apply a fixed point theorem we consider

$$
S = i \circ T : \mathscr{C}(\bar{\Omega}) \longrightarrow \mathscr{C}(\bar{\Omega}),
$$

where $i$ is the compact inclusion of $W^{1,s}(\Omega)$ in $\mathscr{C}(\bar{\Omega})$. If $p > N$ then $s = p$ and Proposition 1 shows that $T$ is bounded and so, as $s > N$, the image of $S$ is compact and the conclusion follows from the Schauder fixed point theorem.

If $p \leq N$ we use the Leray-Schauder fixed point theorem. As $i$ is compact we only need to prove the boundedness in $W^{1,s}(\Omega)$ of the set

$$
\mathscr{A} = \big\{\varphi \in \mathscr{C}(\bar{\Omega}) : \varphi = \lambda S(\varphi) \text{ for some } \lambda \in [0, 1]\big\}.
$$

Notice that we can suppose that $\alpha > \frac{N}{N-1}$. Consider the sequence defined in Proposition 3 starting with $s$ and let $n$ be such that $1 < s_n \leq p$.

If $\varphi \in \mathscr{A}$ we have

$$
|\nabla u_\varphi| \leq F(\varphi) \leq c_0 + c_1|\varphi|^\alpha = c_0 + c_1\lambda^\alpha |u_\varphi|^\alpha
$$

and then, for $i < n$, there exist $A, D > 0$, such that

$$
\begin{aligned}
\|u_\varphi\|_{W^{1,s_{i-1}}(\Omega)} &\leq A\big(\|u_\varphi\|_{L^{\alpha s_{i-1}}(\Omega)} + \|\nabla u_\varphi\|_{L^{s_{i-1}}(\Omega)}\big), \quad \text{as } \alpha > 1 \\
&\leq A\big(\|u_\varphi\|_{L^{\alpha s_{i-1}}(\Omega)} + c_0|\Omega|^{\frac{1}{s_i - 1}} + c_1\lambda^\alpha \|u_\varphi\|^\alpha_{L^{\alpha s_{i-1}}(\Omega)}\big) \\
&\leq A\big(D\|u_\varphi\|_{W^{1,s_i}(\Omega)} + c_0|\Omega|^{\frac{1}{s_i - 1}} + c_1\lambda^\alpha D\|u_\varphi\|^\alpha_{W^{1,s_i}(\Omega)}\big) \\
&\qquad \text{as } \alpha s_{i-1} \text{ is the critical Sobolev exponent of } s_i.
\end{aligned}
$$

By consequence, using Proposition 1 and since $s_n \leq p$, we obtain the boundedness of $\mathscr{A}$ in $W^{1,s}(\Omega)$. So $T$ has a fixed point and this fixed point solves the quasivariational inequality. □

The proof of Theorem 2 will be done using a family of approximating problems, obtained by regularizing and penalizing the quasivariational inequality.

Given $0 < \varepsilon < 1$, consider the family of quasilinear elliptic problems

$$-\nabla \cdot \Big(k_\varepsilon\big(|\nabla u^\varepsilon|^p - F_\varepsilon^p(u^\varepsilon)\big)\, a_\varepsilon(x)\big(|\nabla u^\varepsilon|^2 + \varepsilon\big)^{\frac{p-2}{2}} \nabla u^\varepsilon + \varepsilon \nabla u^\varepsilon\Big) + c\,|u^\varepsilon|^{p-2}u^\varepsilon = f^\varepsilon \quad \text{in } \Omega, \tag{14a}$$

$$\Big(k_\varepsilon\big(|\nabla u^\varepsilon|^p - F_\varepsilon^p(u^\varepsilon)\big)\, a_\varepsilon(x)\big(|\nabla u^\varepsilon|^2 + \varepsilon\big)^{\frac{p-2}{2}} \nabla u^\varepsilon + \varepsilon \nabla u^\varepsilon\Big)\cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{14b}$$

where $a_\varepsilon$, $f^\varepsilon$ and $F_\varepsilon$ are approximations by convolution of $a$, $f$ and $F$, and $k_\varepsilon$ is a smooth nondecreasing function such that

$$k_\varepsilon(s) = \begin{cases} 1 & \text{if } s \leq 0, \\ e^{\frac{s}{\varepsilon}} & \text{if } \varepsilon \leq s. \end{cases} \tag{15}$$

This problem has a unique solution $u^\varepsilon \in \mathscr{C}^{2,\alpha}(\Omega) \cap \mathscr{C}(\bar{\Omega})$, being the proof a simple adaptation of [5, Theorem 5.19] for the case with Neumann homogeneous boundary condition.

Before proving Theorem 2 we need the following auxiliary result.

**Proposition 4.** *Let $u^\varepsilon$ be a solution of problem (14). Then there exist positive constants $C_1$, $C_2$ and $D_q$, independent of $\varepsilon$, such that*

$$\|u^\varepsilon\|_{L^\infty(\Omega)} \leq C_1, \tag{16}$$

$$\|k_\varepsilon(|\nabla u^\varepsilon|^p - F^p(u^\varepsilon))\|_{L^1(\Omega)} \leq C_2, \tag{17}$$

$$\forall\, 1 < q < \infty \quad \|\nabla u_\varepsilon\|_{L^q(\Omega)} \leq D_q. \tag{18}$$

*Proof.* Denote, for simplicity, $w = |\nabla u^\varepsilon|^p - F_\varepsilon^p(u^\varepsilon)$. Consider $\gamma \in \mathbb{R}^+$, to be chosen later. Multiplying equation (14a) by $(u^\varepsilon - \gamma)^+$ and integrating over $\Omega$, we get

$$\int_\Omega k_\varepsilon(w)\, a_\varepsilon(x)\big(|\nabla u^\varepsilon|^2 + \varepsilon\big)^{\frac{p-2}{2}} \nabla u^\varepsilon \cdot \nabla(u^\varepsilon - \gamma)^+ + \varepsilon \int_\Omega \nabla u^\varepsilon \cdot \nabla(u^\varepsilon - \gamma)^+ + \int_\Omega c\,|u^\varepsilon|^{p-2}u^\varepsilon(u^\varepsilon - \gamma)^+ = \int_\Omega f^\varepsilon(u^\varepsilon - \gamma)^+$$

and so

$$\int_\Omega k_\varepsilon(w)\, a_\varepsilon(x)\big(|\nabla(u^\varepsilon-\gamma)^+|^2+\varepsilon\big)^{\frac{p-2}{2}}|\nabla(u^\varepsilon-\gamma)^+|^2$$
$$+\,\varepsilon\int_\Omega|\nabla(u^\varepsilon-\gamma)^+|^2+\int_\Omega c\,|u^\varepsilon|^{p-1}(u^\varepsilon-\gamma)^+\le\|f^\varepsilon\|_{L^\infty(\Omega)}\int_\Omega(u^\varepsilon-\gamma)^+.$$

Observing that the two first terms of the above inequality are nonegative and choosing $\gamma>\Big(\frac{\|f\|_{L^\infty(\Omega)}}{c_*}\Big)^{\frac{1}{p-1}}$, we get

$$(c_*\gamma^{p-1}-\|f^\varepsilon\|_{L^\infty(\Omega)})\int_\Omega(u^\varepsilon-\gamma)^+\le 0,$$

and so $(u^\varepsilon-\gamma)^+\equiv 0$. Proceeding similarly, we obtain $(u^\varepsilon+\gamma)^-\equiv 0$, concluding (16).

Multiply now Eq. (14a) by $u^\varepsilon$ and integrate in $\Omega$. Then

$$\int_\Omega k_\varepsilon(w)\, a_\varepsilon(x)\big(|\nabla u^\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}}|\nabla u^\varepsilon|^2+\varepsilon\int_\Omega|\nabla u^\varepsilon|^2+\int_\Omega c\,|u^\varepsilon|^p=\int_\Omega f^\varepsilon u^\varepsilon. \quad (19)$$

Observe that

$$\int_\Omega k_\varepsilon(w)\, a_\varepsilon(x)\big(|\nabla u^\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}}|\nabla u^\varepsilon|^2=\int_\Omega k_\varepsilon(w)\, a_\varepsilon(x)\big(|\nabla u^\varepsilon|^2+\varepsilon\big)^{\frac{p}{2}}$$
$$-\,\varepsilon\int_\Omega k_\varepsilon(w)\, a_\varepsilon(x)\big(|\nabla u^\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}} \quad (20)$$

and it can be easily seen that

$$\varepsilon\int_\Omega k_\varepsilon(w)\, a_\varepsilon(x)(|\nabla u^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\le\alpha_\varepsilon\int_\Omega k_\varepsilon(w)|\nabla u^\varepsilon|^p+\beta_\varepsilon\int_\Omega k_\varepsilon(w), \quad (21)$$

where $\alpha_\varepsilon\underset{\varepsilon\to0}{\longrightarrow}0$ and $\beta_\varepsilon\underset{\varepsilon\to0}{\longrightarrow}0$.

Noticing that $k_\varepsilon(w)F_*^p\le k_\varepsilon(w)|\nabla u^\varepsilon|^p+F_\varepsilon^p(u^\varepsilon)$ since $k_\varepsilon(w)=1$ if $w\le 0$ and, when $w>0$, we have $|\nabla u^\varepsilon|\ge F_\varepsilon(u^\varepsilon)\ge F_*$, we obtain

$$\int_\Omega k_\varepsilon(w)\le\frac{1}{F_*^p}\Big(\int_\Omega k_\varepsilon(w)|\nabla u^\varepsilon|^p+\int_\Omega F_\varepsilon^p(u^\varepsilon)\Big). \quad (22)$$

As $F$ is continuous and $(u^\varepsilon)_\varepsilon$ is uniformly bounded in $L^\infty(\Omega)$, $\big(F_\varepsilon^p(u^\varepsilon)\big)_\varepsilon$ is also uniformly bounded in $L^\infty(\Omega)$. Using (20) and (21) we obtain, from (19), that

$$\left(a_* - a^*\Big(\alpha_\varepsilon - \frac{\beta_\varepsilon}{F_*^p}\Big)\right)\int_\Omega k_\varepsilon(w)\,|\nabla u^\varepsilon|^p \le \frac{\beta_\varepsilon}{F_*^p}\|F_\varepsilon^p(u^\varepsilon)\|_{L^\infty(\Omega)} + \|f_\varepsilon\|_{L^1(\Omega)}\|u^\varepsilon\|_{L^\infty(\Omega)},$$

The right hand side of the above inequality is bounded by a positive constant $C$ independent of $\varepsilon$. Choosing $\varepsilon$ sufficiently small such that $\alpha_\varepsilon - \frac{\beta_\varepsilon}{F_*^p} \le \frac{a_*}{2a^*}$ we get

$$\int_\Omega k_\varepsilon(w)|\nabla u^\varepsilon|^p \le \frac{2}{a_*}C$$

and, using this inequality and (22), we immediately obtain (17).

Denote $A_\varepsilon = \{x \in \Omega : |\nabla u^\varepsilon(x)|^p > F_\varepsilon^p(u^\varepsilon(x)) + \varepsilon\}$. Observe that, for $q > p$,

$$\int_\Omega |\nabla u^\varepsilon|^q = \int_{\Omega\setminus A_\varepsilon} |\nabla u^\varepsilon|^q + \int_{A_\varepsilon} |\nabla u^\varepsilon|^q$$
$$\le |\Omega|\|F_\varepsilon^p + \varepsilon\|_{L^\infty(-M,M)}^{\frac{q}{p}} + 2^{\frac{q}{p}-1}\Big(\int_{A_\varepsilon}\big(|\nabla u^\varepsilon|^p - F_\varepsilon^p(u^\varepsilon)\big)^{\frac{q}{p}} + \int_{A_\varepsilon} F_\varepsilon^q(u^\varepsilon)\Big)$$

and to conclude the boundedness of $\|\nabla u^\varepsilon\|_{L^q(\Omega)}$, we only need to control the second term of the right hand side of the above inequality. As, for all $j \in \mathbb{N}$ and $s > 0$ we have $e^s \ge s^j/j!$, then, for $\frac{q}{p} \in \mathbb{N}$, we get, by the definition of $k_\varepsilon$,

$$\int_{A_\varepsilon} w^{\frac{q}{p}} \le \varepsilon^{\frac{q}{p}}\big(\tfrac{q}{p}\big)!\int_{A_\varepsilon} k_\varepsilon(w)$$

and, by (17), the conclusion follows, first for $q$ such that $\frac{q}{p} \in \mathbb{N}$ and after for any $1 < q < \infty$. □

*Proof of Theorem 2.* Let $u^\varepsilon$ be the solution of problem (14). From (16) to (18), we get that there exists $u \in W^{1,q}(\Omega)$ such that, at least for a subsequence,

$$\nabla u^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \nabla u \text{ weakly in } L^q(\Omega), \text{ for any } 1 < q < \infty,$$
$$u^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u \text{ in } \mathscr{C}(\bar\Omega).$$

Let us prove that $u \in \mathbb{K}_{F(u)}$. Set

$$B_\varepsilon = \big\{x \in \Omega : |\nabla u^\varepsilon(x)|^p - F_\varepsilon^p(u^\varepsilon(x)) \ge \sqrt{\varepsilon}\big\}.$$

Then, as $k_\varepsilon$ is nondecreasing, and using (17)

$$|B_\varepsilon| = \int_{B_\varepsilon} 1 \le \int_{B_\varepsilon} \frac{k_\varepsilon\big(|\nabla u^\varepsilon(x)|^p - F_\varepsilon^p(u^\varepsilon(x))\big)}{k_\varepsilon(\sqrt{\varepsilon})} \le Ce^{-\frac{1}{\sqrt{\varepsilon}}}. \tag{23}$$

Let $\omega$ be any measurable subset of $\Omega$. As

$$|\nabla u^\varepsilon|^p - F_\varepsilon^p(u^\varepsilon) - \sqrt{\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} |\nabla u|^p - F^p(u) \text{ weakly in } L^1(\Omega),$$

then

$$\begin{aligned}\int_\omega (|\nabla u|^p - F^p(u)) &= \lim_{\varepsilon\to 0}\int_\omega (|\nabla u^\varepsilon|^p - F_\varepsilon^p(u^\varepsilon) - \sqrt{\varepsilon})\\ &\le \lim_{\varepsilon\to 0}\int_{\omega\cap B_\varepsilon} |\nabla u^\varepsilon|^p\\ &= \lim_{\varepsilon\to 0} |\omega\cap B_\varepsilon|^{\frac{1}{2}} \|\nabla u^\varepsilon\|^p_{L^{2p}(\Omega)} = 0, \quad \text{using (23) and (18)},\end{aligned}$$

concluding that $|\nabla u| \le F(u)$ a.e. in $\Omega$.

Let us now see that $u$ solves the quasivariational inequality (8). Given $v \in \mathbb{K}_{F(u)}$ we define $\gamma_\varepsilon = \|F(u) - F_\varepsilon(u^\varepsilon)\|_{\mathscr{C}(\bar\Omega)}$ and $v^\varepsilon = \frac{F_*}{F_*+\gamma_\varepsilon} v$. Observe that $v^\varepsilon \in \mathbb{K}_{F_\varepsilon(u^\varepsilon)}$ and $v^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} v$ in $W^{1,p}(\Omega)$. Besides, denoting again $w = |\nabla u^\varepsilon|^p - F_\varepsilon^p(u^\varepsilon)$,

$$\begin{aligned}&(k_\varepsilon(w)-1)a_\varepsilon(x)\big(|\nabla u^\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}}\nabla u^\varepsilon\cdot\nabla(v^\varepsilon-u^\varepsilon)\\ &\qquad\le (k_\varepsilon(w)-1)a_\varepsilon(x)\big(|\nabla u^\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}}|\nabla u^\varepsilon|\,\big(|\nabla v^\varepsilon| - |\nabla u^\varepsilon|\big)\le 0, \quad (24)\end{aligned}$$

as, when $k_\varepsilon(w) > 1$ then $|\nabla u^\varepsilon| \ge F_\varepsilon(u^\varepsilon) \ge |\nabla v^\varepsilon|$.

So, multiplying equation (14a) by $v^\varepsilon - u^\varepsilon$, and using (24), we obtain

$$\begin{aligned}&\int_\Omega a_\varepsilon(x)\big(|\nabla u^\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}}\nabla u^\varepsilon\cdot\nabla(v^\varepsilon-u^\varepsilon) + \varepsilon\int_\Omega \nabla u^\varepsilon\cdot\nabla(v^\varepsilon-u^\varepsilon)\\ &\qquad + \int c|u^\varepsilon|^{p-2}u^\varepsilon(v^\varepsilon-u^\varepsilon) \ge \int_\Omega f^\varepsilon(v^\varepsilon-u^\varepsilon).\end{aligned}$$

Using the strict monotonicity of the $p$-laplacian operator, we get

$$\begin{aligned}&\int_\Omega a_\varepsilon(x)\big(|\nabla v^\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}}\nabla v^\varepsilon\cdot\nabla(v^\varepsilon-u^\varepsilon) + \varepsilon\int_\Omega \nabla u^\varepsilon\cdot\nabla(v^\varepsilon-u^\varepsilon)\\ &\qquad + \int c|u^\varepsilon|^{p-2}u^\varepsilon(v^\varepsilon-u^\varepsilon) \ge \int_\Omega f^\varepsilon(v^\varepsilon-u^\varepsilon)\end{aligned}$$

and, letting $\varepsilon \to 0$ and, as the term $\int_\Omega \nabla u^\varepsilon\cdot\nabla(v^\varepsilon-u^\varepsilon)$ is bounded, we have

$$\int_\Omega a(x)|\nabla v|^{p-2}\nabla v\cdot\nabla(v-u) + \int c|u|^{p-2}u(v-u) \ge \int_\Omega f(v-u),$$

which implies, by applying a kind of Minty's Lemma, that

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u\cdot\nabla(v-u)+\int c|u|^{p-2}u(v-u)\geq\int_\Omega f(v-u)$$

which concludes the proof of the theorem. □

## 3 The Case with Dirichlet Boundary Condition

In this section we consider the quasivariational case with nonhomogeneous Dirichlet boundary condition, correspondent to the convex sets defined in (3), with $\varphi$ substituted by $F(u)$. As it was already referred, one main concern is to avoid the emptiness of these sets. So we introduce the assumption (10), based on a compatibility condition between the boundary condition $g$, the minimum of the gradient constraint function $F$ and the geometry of the domain.

Consider the variational inequality: to find $u\in\mathscr{K}_\varphi$ such that

$$\int_\Omega \boldsymbol{a}(x,\nabla u)\cdot\nabla(v-u)+\int_\Omega c\,|u|^{p-2}u(v-u)\geq\int_\Omega f(v-u),\qquad\forall v\in\mathscr{K}_\varphi,\tag{25}$$

where $\mathscr{K}_\varphi$ is defined in (3).

*Proof of Theorem 3.* The proof follows the steps of the proof of Theorem 1. The main difference consists in proving the continuity of the operator $T:\mathscr{C}(\bar\Omega)\longrightarrow W^{1,p}(\Omega)$, where $T(\varphi)$ is the solution of problem (25), with $F(\varphi)$ in the place of $\varphi$. We will sketch the proof of the Mosco convergence of $\mathscr{K}_{F(\varphi_n)}$ to $\mathscr{K}_{F(\varphi)}$, where $(\varphi_n)_n$ converges to $\varphi$ in $\mathscr{C}(\bar\Omega)$, from which we immediately deduce the continuity of $T$ (see [12] or [16, Theorem 4.1]). So, we only need to prove the following two conditions:

$$\forall v\in\mathscr{K}_{F(\varphi)}\ \forall n\in\mathbb{N}\ \exists v_n\in\mathscr{K}_{F(\varphi_n)}:\quad v_n\underset{n}{\longrightarrow}v\ \text{ in } W^{1,p}(\Omega),\tag{26a}$$

$$\text{if, for all } n\in\mathbb{N}, v_n\in\mathscr{K}_{F(\varphi_n)} \text{ and } v_n\underset{n}{\rightharpoonup}v\ \text{ in } W^{1,p}(\Omega),\ \text{then } v\in\mathscr{K}_{F(\varphi)}.\tag{26b}$$

Using the assumption (10) we may extend the function $g$ to $\bar\Omega$ (still calling it by $g$) satisfying the condition $|\nabla g|=k\,F_*$ (see [9]).

To prove (26a) consider, for given $v\in\mathscr{K}_{F(\varphi)}$ and, for $n\in\mathbb{N}$,

$$G_n=\min\{F(\varphi_n),\ F(\varphi)\}$$

and $v_n = b_n v + (1 - b_n)g$, where

$$b_n = \min_{x\in\bar{\Omega}} \frac{G_n(x) - kF_*}{F(\varphi(x)) - kF_*}.$$

So, $0 < b_n \le 1$ and $\left(\frac{G_n - kF_*}{F(\varphi) - kF_*}\right)_n$ converges to 1 in $\mathscr{C}(\bar{\Omega})$ and, as $\bar{\Omega}$ is compact, $b_n \underset{n}{\longrightarrow} 1$. Note that $v_n \in \mathscr{K}_{F(\varphi_n)}$ as $v_{n|\partial\Omega} = g$ and

$$|\nabla v_n(x)| \le b_n F(\varphi(x)) + (1 - b_n)kF_* \le G_n(x),$$

because $b_n \le \frac{G_n(x) - kF_*}{F(\varphi(x)) - kF_*}$. We have

$$\int_\Omega |\nabla(v_n - v)|^p = (1 - b_n)^p \int_\Omega |\nabla(g - v)|^p \underset{n}{\longrightarrow} 0.$$

To prove (26b), let $(v_n)_n$ be a sequence in $\mathscr{K}_{F(\varphi_n)}$, converging weakly in $W^{1,p}(\Omega)$ to $v$. As $v_{n|\partial\Omega} = g$ then $v_{|\partial\Omega} = g$. Given any measurable set $\omega \subset \Omega$,

$$\int_\omega |\nabla v| \le \liminf_n \int_\omega |\nabla v_n| \le \liminf_n \int_\omega F(\varphi_n) = \int_\omega F(\varphi),$$

so $|\nabla v| \le F(\varphi)$ a.e. in $\Omega$, which means $v \in \mathscr{K}_{F(\varphi)}$. This concludes the proof of the continuity of $T$.

We present now an a priori estimate for the $W^{1,p}(\Omega)$ norm of $u_\varphi = T(\varphi)$, independent of $\varphi$.

Choosing $g$ as test function in (25) and recalling that $f \in L^{q'}(\Omega)$ we have

$$\begin{aligned}
&\int_\Omega \boldsymbol{a}(x, \nabla u_\varphi) \cdot \nabla u_\varphi + \int_\Omega c|u_\varphi|^p \\
&\qquad \le \int_\Omega \boldsymbol{a}(x, \nabla u_\varphi) \cdot \nabla g + \int_\Omega f u_\varphi - \int_\Omega fg + \int_\Omega c|u_\varphi|^{p-2}u_\varphi g \\
&\qquad \le \int_\Omega |\nabla u_\varphi|^{p-1} a^* kF_* + \int_\Omega |u_\varphi|^{p-1} \|cg\|_{L^\infty(\Omega)} \\
&\qquad\qquad + \|f\|_{L^{q'}(\Omega)} \|u_\varphi\|_{L^q(\Omega)} + \|f\|_{L^{q'}(\Omega)} \|g\|_{L^q(\Omega)}.
\end{aligned}$$

As $\|u_\varphi\|_{L^q(\Omega)} \le C_q \|u_\varphi\|_{W^{1,p}(\Omega)}$ we have, for $\delta > 0$,

$$\begin{aligned}
a_* \|\nabla u_\varphi\|^p_{L^p(\Omega)} &\le \frac{\delta^{p'}}{p'} \|u_\varphi\|^p_{W^{1,p}(\Omega)} + \frac{|\Omega|}{\delta^p p} \left((a^* kF_*)^p + \|cg\|^p_{L^\infty(\Omega)}\right) \\
&\quad + \frac{\delta^p}{p} \|u_\varphi\|^p_{W^{1,p}(\Omega)} + \frac{C_q^{p'}}{\delta^{p'} p'} \|f\|^{p'}_{L^{q'}(\Omega)} + \|f\|_{L^{q'}(\Omega)} \|g\|_{L^q(\Omega)}.
\end{aligned}$$

Applying the Poincaré inequality to $u_\varphi$, we have

$$\|u_\varphi\|^p_{W^{1,p}(\Omega)} \le c_p\big(\|\nabla u_\varphi\|^p_{L^p(\Omega)} + \|g\|^p_{L^p(\partial\Omega)}\big).$$

Choosing $\delta$ such that $\big(\frac{\delta p'}{p'} + \frac{\delta p}{p}\big)c_p < a_*$, we conclude that there exists a positive constant $C = C(\|f\|_{L^{q'}(\Omega)}, \|g\|_{L^\infty(\Omega)})$ such that $\|\nabla u_\varphi\|^p_{L^p(\Omega)} \le C$. Applying again the Poincaré inequality, there exists another positive constant $C$ such that

$$\|u_\varphi\|^p_{W^{1,p}(\Omega)} \le C.$$

As we proved the continuity of the operator $T$ and the above estimate, the conclusion follows as in the proof of Theorem 1. □

The proof of Theorem 4 will be done using, as in the proof of Theorem 2, a family of approximating problems, obtained by regularizing and penalizing the quasivariational inequality. For $0 < \varepsilon < 1$, consider the approximating family of problems:

$$-\nabla\cdot\Big(k_\varepsilon\big(|\nabla u^\varepsilon|^p - F^p_\varepsilon(u^\varepsilon)\big)\,a_\varepsilon(x)\big(|\nabla u^\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}}\nabla u^\varepsilon + \varepsilon\nabla u^\varepsilon\Big) + c\,|u^\varepsilon|^{p-2}u^\varepsilon = f^\varepsilon \quad \text{in } \Omega, \tag{27a}$$

$$u^\varepsilon_{|\partial\Omega} = g^\varepsilon, \tag{27b}$$

where $a_\varepsilon$, $g^\varepsilon$, $f^\varepsilon$ and $F_\varepsilon$ are approximations by convolution of $a$, $g$, $f$ and $F$, and $k_\varepsilon : \mathbb{R} \to \mathbb{R}$ is a smooth nondecreasing function as in (15).

This problem has a unique solution $u^\varepsilon \in \mathscr{C}^{2,\alpha}(\Omega) \cap \mathscr{C}(\bar\Omega)$ (see [5, Theorem 5.19]).

Consider an extension $g^\varepsilon$ to $\bar\Omega$, still denoted by $g^\varepsilon$, such that $|\nabla g^\varepsilon| = kF_*$ in $\Omega$. Notice that such a function exists because (10) is verified and $g^\varepsilon \in W^{1,\infty}(\Omega)$.

**Proposition 5.** *Under the assumptions of Theorem 4 there exist positive constants $C$ and $D_q$, independent of $\varepsilon$, such that*

$$\|u^\varepsilon\|_{L^\infty(\Omega)} \le C, \tag{28}$$

$$\|k_\varepsilon(|\nabla u^\varepsilon|^p - F^p_\varepsilon(u^\varepsilon))\|_{L^1(\Omega)} \le C, \tag{29}$$

$$\forall\, 1 < q < \infty \quad \|\nabla u_\varepsilon\|_{L^q(\Omega)} \le D_q. \tag{30}$$

*Proof.* By the strong maximum principle for quasilinear elliptic equations, the $L^\infty(\Omega)$-norm of $u^\varepsilon$ depends only on $\|f\|_{L^\infty(\Omega)}$ and $\|g\|_{L^\infty(\partial\Omega)}$.

Denote once again $|\nabla u^\varepsilon|^p - F^p_\varepsilon(u^\varepsilon)$ by $w$. Multiplying by $u^\varepsilon - g^\varepsilon$ the Eq. (27a) and integrating over $\Omega$ we obtain

$$\int_\Omega k_\varepsilon(w)a_\varepsilon(x)(|\nabla u^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u^\varepsilon|^2+\varepsilon\int_\Omega|\nabla u^\varepsilon|^2+\int_\Omega c|u^\varepsilon|^p$$
$$=\int_\Omega k_\varepsilon(w)a_\varepsilon(x)(|\nabla u^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u^\varepsilon\cdot\nabla g^\varepsilon+\varepsilon\int_\Omega\nabla u^\varepsilon\cdot\nabla g^\varepsilon$$
$$+\int_\Omega c|u^\varepsilon|^{p-2}u^\varepsilon g^\varepsilon+\int_\Omega f^\varepsilon(u^\varepsilon-g^\varepsilon).$$

We can rewrite the above equality as

$$\int_\Omega k_\varepsilon(w)a_\varepsilon(x)(|\nabla u^\varepsilon|^2+\varepsilon)^{\frac{p}{2}}+\varepsilon\int_\Omega|\nabla u^\varepsilon|^2+\int_\Omega c|u^\varepsilon|^p$$
$$=\varepsilon\int_\Omega k_\varepsilon(w)a_\varepsilon(x)(|\nabla u^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}+\int_\Omega k_\varepsilon(w)a_\varepsilon(x)(|\nabla u^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u^\varepsilon\cdot\nabla g^\varepsilon$$
$$+\varepsilon\int_\Omega\nabla u^\varepsilon\cdot\nabla g^\varepsilon+\int_\Omega c|u^\varepsilon|^{p-2}u^\varepsilon g^\varepsilon+\int_\Omega f^\varepsilon(u^\varepsilon-g^\varepsilon)$$

and then, using (28) and (30), there exists a constant $C_1>0$, independent of $\varepsilon$, such that

$$a_*\int_\Omega k_\varepsilon(w)(|\nabla u^\varepsilon|^2+\varepsilon)^{\frac{p}{2}}\le a^*\varepsilon\int_\Omega k_\varepsilon(w)(|\nabla u^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}$$
$$+a^*kF_*\int_\Omega k_\varepsilon(w)(|\nabla u^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u^\varepsilon|+C_1. \quad (31)$$

Observe that

$$a^*kF_*\int_\Omega k_\varepsilon(w)(|\nabla u^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u^\varepsilon|$$
$$\le a^*kF_*\int_\Omega k_\varepsilon(w)(|\nabla u^\varepsilon|^2+\varepsilon)^{\frac{p-1}{2}}\le a^*kF_*\int_\Omega k_\varepsilon(w)\Big(\frac{|\nabla u^\varepsilon|^2+\varepsilon)^{\frac{p}{2}}}{p'\delta^{p'}}+\frac{\delta^p}{p}\Big),$$

for any $\delta>0$. Choosing $\delta=F_*^{\frac{1}{p'}}$, we obtain

$$a^*kF_*\int_\Omega k_\varepsilon(w)(|\nabla u^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u^\varepsilon|$$
$$\le\frac{a^*k}{p'}\int_\Omega k_\varepsilon(w)(|\nabla u^\varepsilon|^2+\varepsilon)^{\frac{p}{2}}+\frac{a^*kF_*^p}{p}\int_\Omega k_\varepsilon(w). \quad (32)$$

Inequalities (31) and (21) allow us to obtain

$$\Big(a_* - \frac{a^*k}{p'} - \alpha_\varepsilon\Big)\int_\Omega k_\varepsilon(w)|\nabla u^\varepsilon|^p \le \Big(\frac{a^*kF_*^p}{p} + \beta_\varepsilon\Big)\int_\Omega k_\varepsilon(w) + C_1.$$

Recalling (22), we obtain

$$\big(F_*^p(a_* - a^*k) - (F_*^p\alpha_\varepsilon + \beta_\varepsilon)\big)\int_\Omega k_\varepsilon(w) \le \Big(a_* - \frac{a^*k}{p'} - \alpha_\varepsilon\Big)\int_\Omega F_\varepsilon^p(u^\varepsilon) + C_1.$$

Observing, as in the proof of Proposition 4, that $\big(F_\varepsilon^p(u^\varepsilon)\big)_\varepsilon$ is uniformly bounded in $L^\infty(\Omega)$ and $F_*^p\alpha_\varepsilon + \beta_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0$, there exists $D > 0$ such that

$$F_*^p\frac{a_* - a^*k}{2}\int_\Omega k_\varepsilon(w) \le D$$

and the conclusion (29) follows.

The proof of (30) is similar to the case of Neumann boundary condition. □

*Remark 2.* If $a_* = a^*$, as in the $p$-laplacian case, the only restriction on $k$ is $0 < k < 1$.

*Proof of Theorem 4.* After the previous proposition, the proof of this theorem is similar to the proof of Theorem 2. Here the constant $c_*$ may be zero. However, after obtaining the uniform control of $\|\nabla u^\varepsilon\|_{L^q(\Omega)}$, the Poincaré inequality implies the uniform boundedness of $\|u^\varepsilon\|_{W^{1,q}(\Omega)}$. The verification that $u_{|\partial\Omega} = g$ is a consequence of the equality $u^\varepsilon_{|\partial\Omega} = g^\varepsilon$ and convergence of $(u^\varepsilon)_\varepsilon$ to $u$ in $\mathscr{C}(\bar{\Omega})$. □

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## References

1. Azevedo, A., Miranda, F., Santos, L.: Variational and quasivariational inequalities with first order constraints. J. Math. Anal. Appl. **397**, 738–756 (2013)
2. Barrett, J.W., Prigozhin, L.: A quasi-variational inequality problem in superconductivity. Math. Models Methods Appl. Sci. **20**(5), 679–706 (2010)
3. Brezis, H., Caffarelli, L.A., Friedman, A.: Reinforcement problems for elliptic equations and variational inequalities. Ann. Mat. Pura Appl. (4) **123**, 219–246 (1980)
4. Caffarelli, L.A., Friedman, A.: Reinforcement problems in elastoplasticity. Rocky Mt. J. Math. **10**(1), 155–184 (1980)
5. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Classics in Mathematics. Springer, Berlin (2001)

6. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Volume 88 of Pure and Applied Mathematics. Academic, New York (1980)
7. Kunze, M., Rodrigues, J.-F.: An elliptic quasi-variational inequality with gradient constraints and some of its applications. Math. Methods Appl. Sci. **23**(10), 897–908 (2000)
8. Lions, J.-L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, Paris (1969)
9. Lions, P.-L.: Generalized solutions of Hamilton-Jacobi equations. Volume 69 of Research Notes in Mathematics. Pitman, Boston (1982)
10. Miranda, F., Rodrigues, J.F., Santos, L.: A class of stationary nonlinear Maxwell systems. Math. Models Methods Appl. Sci. **19**(10), 1883–1905 (2009)
11. Miranda, F., Rodrigues, J.F., Santos, L.: On a p-curl system arising in electromagnetism. Discret. Contin. Dyn. Syst. Ser. S **5**(3), 605–629 (2012)
12. Mosco, U.: Convergence of convex sets and of solutions of variational inequalities. Adv. Math. **3**, 510–585 (1969)
13. Prigozhin, L.: Sandpiles and river networks: extended systems with nonlocal interactions. Phys. Rev. E (3) **49**(2), 1161–1167 (1994)
14. Prigozhin, L.: On the Bean critical-state model in superconductivity. Eur. J. Appl. Math. **7**(3), 237–247 (1996)
15. Prigozhin, L.: Variational model of sandpile growth. Eur. J. Appl. Math. **7**(3), 225–235 (1996)
16. Rodrigues, J.F.: Obstacle Problems in Mathematical Physics. North-Holland, Amsterdam (1987)
17. Rodrigues, J.-F., Santos, L.: A parabolic quasi-variational inequality arising in a superconductivity model. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29**(1), 153–169 (2000)
18. Ting, T.W.: Elastic-plastic torsion problem II. Arch. Ration. Mech. Anal. **25**, 342–366 (1967)

# Shocks and Antishocks in the ASEP Conditioned on a Low Current

**Vladimir Belitsky and Gunter M. Schütz**

## 1 Introduction

The Asymmetric Simple Exclusion Process (ASEP) [14, 20] on a one-dimensional lattice of $L$ sites can be informally defined as follows.

1. Each particle attempts to jump independently of the other particles after an exponentially distributed random time with parameter $1/(c_r + c_\ell)$ with probability $c_r/(c_r + c_\ell)$ to the next site on the right (clockwise) and $c_\ell/(c_r + c_\ell)$ to the next site on the left (counterclockwise)
2. The hopping attempt is rejected if the site to which the particle tries to move is occupied

We introduce the hopping asymmetry $q^2 := c_r/c_\ell$ and assume without loss of generality a hopping bias to the right, i.e., $q > 1$. For convenience we take $L = 2M$ even. We consider two kinds of boundary conditions, the periodic system with $N$ particles on the torus $\mathbb{T}_L := \mathbb{Z}/L\mathbb{Z}$ (with sites $k$ modulo $L$ in the principal domain $-M+1, -M+2, \ldots, M$), and the open system where at the left boundary site $-M+1$ particles are injected with rate $\alpha = c_r\rho_1$ and extracted with rate $\gamma = c_\ell(1-\rho_1)$ and at the right boundary site $M$ particles are injected with rate $\delta = c_\ell\rho_2$ and extracted with rate $\beta = c_r(1-\rho_2)$. Injection attempts onto occupied sites are rejected. The parameters $\rho_{1,2} \in [0, 1]$ can be interpreted as particle reservoir densities. Because of the hopping bias and the boundary conditions the dynamics is not reversible and there is a non-zero stationary particle current $j^*$ that depends on

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the conserved particle density $\rho = N/L$ in the periodic case and on the boundary densities $\rho_{1,2}$ in the open case.

In the last decade, following the seminal papers [4, 9], the large deviation theory for the ASEP has been developed in considerable detail. Here we focus on large deviations of the time-integrated current $J(t)$ for finite $t$. This random number is given by $J(t) = J^+(t) - J^-(t)$ where $J^{\pm}(t) \in \mathbb{Z}$ is the total number of jumps of all particles to the right/left up to time $t$, starting from some initial distribution of the particles. In a finite system one has asymptotically $J(t) \sim j^* L t$ as time $t$ tends to infinity, expressing the fact that $J(t)$ is extensive in system size $L$ and time $t$. Therefore, for studying large times, we define also the global space-time average $j(t) = J(t)/(Lt)$. The probability to observe for a long time interval $t$ an untypical mean $j \neq j^*$ is exponentially small in $L$ and $t$. This is expressed in the large deviation property $\mathrm{Prob}\,[\,J(t) = J\,] \propto \exp(-f(j)Lt)$ [8] where $f(j)$ is the rate function. It is then natural to introduce a generalized fugacity $y = \mathrm{e}^s$ with generalized chemical potential $s$ and to study the generating function $Y_s(t) := \sum_{J\in\mathbb{Z}} y^J \mathrm{Prob}\,[\,J(t) = J\,]$. The cumulant function $g(s) = \lim_{t\to\infty} \ln Y_s(t)/(Lt)$ is the Legendre transform of the rate function, i.e., $g(s) = \max_j [js - f(j)]$. The intensive variable $s$ is thus conjugate to the mean current density $j$.

The large deviation theory developed on the basis [4, 9] provides information about the large scale spatio-temporal structure of the process conditioned on realizing a prolonged untypical behaviour of the current. Bodineau and Derrida have studied in some detail the weakly asymmetric exclusion process (WASEP) where $c_r - c_\ell = \nu/L$ is small [5, 6]. For periodic boundary conditions [5] the optimal macroscopic density profile $\rho(x,t)$ that realizes a large deviation with strictly positive $s$ (i.e., for any current $j$ conditioned on $j > j^*$) is time-independent and flat and hence equal to the typical profile $\rho(x,t) = N/L =: \rho$. However, for a current conditioned to be *below* the typical value (corresponding to $s < 0$) there is a dynamical phase transition: The flat profile becomes unstable below some critical value $s_c$ and a travelling wave of the form $\rho(x - vt)$ develops. In the limit $\nu \to \infty$ (expected to correspond to finite asymmetry in the ASEP) the optimal profile in this regime is predicted to have the form of a step function with two constant values $\rho_1$ and $\rho_2$. This is a profile consisting of a shock discontinuity where the density jumps from $\rho_1$ to $\rho_2 > \rho_1$ at a position $x_1(t)$ and an antishock where the density jumps from $\rho_2$ to $\rho_1$ at position $x_2(t)$.

Subsequently an antishock was found also for the WASEP with open boundaries in the maximal current phase in the limit $\nu \to \infty$ [6], and more recently for conditioning on atypically low activities $K(t) = J^+(t) + J^-(t)$ in the symmetric simple exclusion process [13]. Notice that the macroscopic large deviation theory provides only information about the density profile on Euler scale. Moreover, it cannot predict the macroscopic position of the antishock. For unconditioned dynamics antishocks are unstable and dissolve into a rarefaction wave [18]. The fact that under conditioning on a sufficiently low current an antishock in the WASEP is stable therefore raises a number of interesting questions.

Here we address the existence of an antishock in the ASEP, its position and its microscopic structure. The techniques used are an adaptation of our earlier work

[1] on the microscopic structure of shocks under unconditioned dynamics. This approach, which is essentially algebraic, has a probabilistic interpretation as self-duality [7, 10, 12, 19]. However, it constrains our approach to a specific family of initial measures with densities $\rho_{1,2}$ which are determined by the hopping asymmetry, see conditions S, S′ below. As in [16, 17, 21] we construct the generator of the conditioned dynamics for non-zero $s$. Fixing some $s < 0$ corresponds to studying realizations of the process where the current fluctuates around some non-typical mean $j < j^*$. We shall refer to this approach as grandcanonical conditioning, as opposed to a canonical condition where the current $J(t)$ would be conditioned to have some fixed value $J$. Unlike [16, 17, 21] we focus on finite times, choosing specific shock or antishock measures as initial distributions of the conditioned dynamics.

## 2 Quantum Hamiltonian Formalism

Here we review some useful tools for treating the conditioned time evolution of the ASEP. We refer the reader to [2] for more details.

### 2.1 Master Equation and Conditioned Dynamics

For interacting particle systems with state space $\mathbb{V}$ the Markov generator $L$ acting on cylinder functions $f(\eta)$ of a particle configuration $\eta \in \mathbb{V}$ is usually defined through the relation

$$Lf(\eta) = \sum_{\eta'} w_{\eta',\eta}[f(\eta') - f(\eta)] \tag{1}$$

where $w_{\eta',\eta}$ is the transition rate from a configuration $\eta$ to $\eta'$. For a given probability measure $\mu(t)$ on then has

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle f \rangle_\mu = \langle Lf \rangle_\mu \tag{2}$$

where $\langle \cdot \rangle_\mu$ denotes expectation w.r.t. $\mu(t)$. By taking $f$ to be the indicator function $\mathbf{1}_\eta$ on a fixed configuration $\eta$ we can construct from this relation the forward evolution equation, called *master equation*,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu(\eta;t) = \sum_{\substack{\eta' \in \mathbb{V} \\ \eta' \neq \eta}} \left[ w_{\eta,\eta'}\mu(\eta';t) - w_{\eta',\eta}\mu(\eta;t) \right] \tag{3}$$

for the time evolution of the probability $\mu(\eta; t)$ of finding the configuration $\eta$ at time $t$. A stationary distribution, i.e., an invariant measure satisfying $\mathrm{d}\mu/\mathrm{d}t = 0$ is denoted by $\mu^*$ and a stationary probability of a configuration $\eta$ is denoted $\mu^*(\eta)$.

For particle systems it is convenient to write the master equation (3) in a matrix form often called quantum Hamiltonian formalism [15, 20]. One assigns to each configuration $\eta$ a column vector $|\,\eta\,\rangle$ which together with the transposed vectors $\langle\,\eta\,|$ form an orthogonal basis of a complex vector space with inner product $\langle\,\eta\,|\,\eta'\,\rangle = \delta_{\eta,\eta'}$. Here $\delta_{\eta,\eta'}$ is the Kronecker symbol which is equal to 1 if the two arguments are equal and zero otherwise. With this convention a measure can be written as a probability vector

$$|\,\mu(t)\,\rangle = \sum_{\eta\in\mathbb{V}} \mu(\eta; t)\,|\,\eta\,\rangle. \tag{4}$$

whose components are the probabilities $\mu(\eta; t) = \langle\,\eta\,|\,\mu(t)\,\rangle$. The notation for vectors is an elegant tool borrowed from quantum mechanics.

Next we define a matrix $H$ whose off-diagonal matrix elements $H_{\eta,\eta'}$ are the (negative) transition rates $w_{\eta,\eta'}$ and the diagonal entries $H_{\eta,\eta}$ are the sum of all outgoing transition rates $w_{\eta',\eta}$ from configuration $\eta$. The master equation (3) then takes the form of a Schrödinger equation in imaginary time,

$$\frac{d}{dt}|\,\mu(t)\,\rangle = -H\,|\,\mu(t)\,\rangle \tag{5}$$

with the formal solution

$$|\,\mu(t)\,\rangle = \mathrm{e}^{-Ht}\,|\,\mu(0)\,\rangle \tag{6}$$

reflecting the semi-group property.

For the ASEP $\eta = (\eta(-M+1), \eta(-M+2), \ldots, \eta(M))$ and $\mathbb{V} = \{0,1\}^L$ and we choose a tensor basis with basis vectors $|\,\eta\,\rangle = |\,\eta(-M+1)\,\rangle \otimes \cdots \otimes |\,\eta(M)\,\rangle$ where $|\,\eta(k)\,\rangle$ is a two-dimensional column vector whose upper component is 1 (0) and lower component is 0 (1) if $\eta(k) = 0$ ($\eta(k) = 1$). The generator of the ASEP can then be constructed from the usual Pauli matrices $\sigma_k^{x,y,z}$ acting locally on site $k$ of the lattice by introducing the particle creation and annihilation operators $\sigma_k^{\pm} = (\sigma_k^x \pm i\sigma_k^y)$ and the projectors $\hat{n}_k = (1-\sigma_k^z)/2$ on particles and $\hat{v}_k = 1-n_k$ on vacancies on site $k$. With these definitions we find for periodic boundary conditions

$$H_{per} := -\sum_{k=-M+1}^{M} e_k \tag{7}$$

with hopping matrices ($k$ modulo $L$)

$$e_k = c_r(\sigma_k^+\sigma_{k+1}^- - \hat{n}_k\hat{v}_{k+1}) + c_\ell(\sigma_k^-\sigma_{k+1}^+ - \hat{v}_k\hat{n}_{k+1}) \tag{8}$$

and for open boundary conditions

$$H_{open} := -\left[ b_{-M+1} + \sum_{k=-M+1}^{M-1} e_k + b'_M \right] \tag{9}$$

with the boundary matrices

$$\begin{aligned} b_{-M+1} &= \alpha(\sigma^-_{-M+1} - \hat{v}_{-M+1}) + \gamma(\sigma^+_{-M+1} - \hat{n}_{-M+1}), \\ b'_M &= \delta(\sigma^-_M - \hat{v}_M) + \beta(\sigma^+_M - \hat{n}_M). \end{aligned} \tag{10}$$

The generator $H$ defined above describes the unconditioned hopping dynamics. Following Refs. [8, 11] the grandcanonically conditioned dynamics is generated by a matrix $H(s)$ which is obtained from $H = H(0)$ by multiplying the offdiagonal matrices that correspond to jumps to the right (either in the bulk or from bulk to the reservoirs) by a factor $\mathrm{e}^s$ and correspondingly for jumps to the left by $\mathrm{e}^{-s}$. For open boundary conditions this yields

$$H_{open}(s) := -\left[ b_{-M+1}(s) + \sum_{k=-M+1}^{M-1} e_k(s) + b'_M(s) \right] \tag{11}$$

with

$$e_k(s) = c_r(\mathrm{e}^s \sigma^+_k \sigma^-_{k+1} - \hat{n}_k \hat{v}_{k+1}) + c_\ell(\mathrm{e}^{-s} \sigma^-_k \sigma^+_{k+1} - \hat{v}_k \hat{n}_{k+1}) \tag{12}$$

and

$$b_{-M+1}(s) = c_r \rho_1(\mathrm{e}^s \sigma^-_{-M+1} - \hat{v}_{-M+1}) + c_\ell(1-\rho_1)(\mathrm{e}^{-s} \sigma^+_{-M+1} - \hat{n}_{-M+1}), \tag{13}$$

$$b'_M(s) = c_\ell \rho_2(\mathrm{e}^{-s} \sigma^-_M - \hat{v}_M) + c_r(1-\rho_2)(\mathrm{e}^s \sigma^+_M - \hat{n}_M). \tag{14}$$

where we have used (10). For periodic boundary conditions one has

$$H_{per}(s) := -\sum_{k=-M+1}^{M} e_k(s). \tag{15}$$

Since periodic and open boundary conditions are presented below in different subsections we shall drop the subscript on $H$.

With these conventions the (unnormalized) conditioned time evolution is given by

$$|\, \tilde{\mu}_s(t) \,\rangle := \mathrm{e}^{-H(s)t} |\, \mu(0) \,\rangle. \tag{16}$$

and

$$Y_s(t) = \langle\, s \,| e^{-H(s)t} |\, \mu(0) \,\rangle \tag{17}$$

for the normalization. Here $\langle\, s \,| := (1, 1, \ldots, 1) = \sum_{\eta \in \{0,1\}^L} \langle\, \eta \,|$ is called the summation vector. The normalized conditional probability vector

$$|\, \mu_s(t) \,\rangle := |\, \tilde{\mu}_s(t) \,\rangle / Y_s(t) \tag{18}$$

describes the approach to the long-time large deviation regime from a given initial distribution and hence provides information about the microscopic space-time structure of the long-time large deviation regime. In order to avoid heavy notation we shall drop the subscript $s$.

## 2.2 *Shock and Antishock Measures*

We shall consider the evolution of two distinct types of measures which are product measures with space-dependent densities $\rho(k)$ which we express in terms of fugacities defined by

$$z := \frac{\rho}{1-\rho}. \tag{19}$$

Type I shock/antishock measures for a finite system of $L$ sites are defined as follows:

**Definition 1.** A shock or antishock measure $\mu_m^I$ of type I and $-M \leq m \leq M$ is a Bernoulli product measure with fugacities

$$z(k) = \begin{cases} z_1 \text{ at the set of sites } -M < k \leq m \\ z_2 \text{ at the set of sites } m < k \leq M \end{cases} \tag{20}$$

For $c_r > c_\ell$ and $z_2 > z_1$ such a measure is called a shock measure and for $c_r > c_\ell$ and $z_1 > z_2$ it is called an antishock measure for the ASEP.

We call site $m$ the *microscopic position of the shock* (or antishock) in the shock/antishock measure of type I.

In vector representation we have

$$|\, \mu_m^I \,\rangle = \frac{1}{A_m} \begin{pmatrix} 1 \\ z_1 \end{pmatrix}^{\otimes(m+M)} \otimes \begin{pmatrix} 1 \\ z_2 \end{pmatrix}^{\otimes(M-m)} \tag{21}$$

with $A_m = (1+z_1)^{M+m}(1+z_2)^{M-m}$. Tensor products with exponent 0 are defined to be absent. For studying periodic boundary conditions we need the (unnormalized) restriction of $\mu_m^I$ to the sector with $N$ particles. This is the measure

$$|\,\mu_m^{I,N}\,\rangle = P_N|\,\mu_m^I\,\rangle \tag{22}$$

where the projector on configurations with $N$ particles is defined by

$$P_N|\,\eta\,\rangle = \begin{cases} |\,\eta\,\rangle \text{ if } \sum_{k\in\mathbb{T}_L}\eta(k) = N \\ 0 \quad \text{otherwise} \end{cases}. \tag{23}$$

The density profile seen from the shock (antishock) position is the lattice analogue of a step function with densities $\rho_{1,2}$. For $m = \pm M$ the step vanishes and the step function measures reduce to the usual Bernoulli product measures which we denote by

$$|\,\mu_z\,\rangle = \frac{1}{(1+z)^L}\begin{pmatrix}1\\z\end{pmatrix}^{\otimes L} = \begin{pmatrix}1-\rho\\ \rho\end{pmatrix}^{\otimes L} \tag{24}$$

where $z$ is the fugacity (19). In particular,

$$|\,\mu^I_{M,M}\,\rangle \equiv |\,\mu^I_M\,\rangle = |\,\mu_{z_1}\,\rangle, \quad |\,\mu^I_{-M,M}\,\rangle \equiv |\,\mu^I_{-M}\,\rangle = |\,\mu_{z_2}\,\rangle. \tag{25}$$

Notice that $\mu^I_M \neq \mu^I_{-M}$.

We note the transformation property

**Lemma 1.** *Let* $|\,\mu_z\,\rangle$ *be the vector representation of the Bernoulli product measure with fugacity* $z$ *for* $L$ *sites and* $\hat{N}_m = \sum_{k=m+1}^{M}\hat{n}_k$ *be the partial number operator. Then* $\forall z_1, z_2 \in (0,\infty)$ *and* $-M \le m \le M$

$$|\,\mu^I_m\,\rangle = \left(\frac{1+z_1}{1+z_2}\right)^{M-m}\left(\frac{z_2}{z_1}\right)^{\hat{N}_m}|\,\mu_{z_1}\,\rangle, \tag{26}$$

*and for fixed particle number* $N$

$$|\,\mu^{I,N}_{-M}\,\rangle = \left(\frac{1+z_1}{1+z_2}\right)^{L}\left(\frac{z_2}{z_1}\right)^{N}|\,\mu^N_{z_1}\,\rangle. \tag{27}$$

*Proof.* With the matrix representation of the number operator $\hat{n}$ for a single site

$$y^{\hat{n}} = \mathbb{1} + (y-1)\hat{n} = \begin{pmatrix}1 & 0\\ 0 & y\end{pmatrix},$$

the tensor property $y^{\hat{N}_m} = \mathbb{1}^{\otimes M+m}\otimes(y^{\hat{n}})^{\otimes M-m}$ and the vector representation (21) the proof of the first equality becomes elementary multilinear algebra. The second equality follows from $\hat{N}_{-M} = \hat{N}$ and the fact that the projection on $N$ sites can be interchanged with the number operator and that the projected vector is an eigenstate with eigenvalue $N$ of the number operator $\hat{N}$. □

The densities $\rho_{1,2}$ are, in principle, free parameters, both from the interval [0, 1]. We shall consider two distinct submanifolds,

**Condition S:**

$$\frac{z_2}{z_1} = q^2 \tag{28}$$

and

**Condition S′:**

$$\frac{z_1}{z_2} = q^2 \tag{29}$$

Condition S implies $\rho_2 > \rho_1$, i.e., the corresponding measure is a shock measure, while Condition S′ implies $\rho_2 < \rho_1$, i.e., the corresponding measure is an antishock measure. Notice that when we work with condition S we can write Lemma 1 in the form

$$|\,\mu_m^I\,\rangle = \left(\frac{1+z_1}{1+q^2 z_1}\right)^{M-m} q^{2\hat{N}_m}|\,\mu_{z_1}\,\rangle \tag{30}$$

and

$$|\,\mu_{-M}^{I,N}\,\rangle = \left(\frac{1+z_1}{1+q^2 z_1}\right)^{L} q^{2N}|\,\mu_{z_1}^N\,\rangle. \tag{31}$$

A second one-parameter family of shock measures is defined on the torus $\mathbb{T}_L$ [2] as follows.

**Definition 2.** A shock measure $\mu_m^{II}$ of type II on $\mathbb{T}_L$ with shock at position $m \in \{-M+1, -M+2, \ldots, \leq M-1\}$ is a Bernoulli product measure with space-dependent fugacities

$$z(k) = \left\{ \begin{array}{ll} z_1 q^{2\frac{m-M-k}{L}} & \text{for } -M < k \leq m \\ z_1 q^{2\frac{m+M-k}{L}} & \text{for } m < k \leq M \end{array} \right\}. \tag{32}$$

Furthermore, $\mu_{-M}^{II} \equiv \mu_M^{II}$ is a Bernoulli product measure with space-dependent fugacities

$$z(k) = z_1 q^{\frac{-2k}{L}} \text{ for } -M < k \leq M. \tag{33}$$

The unnormalized restrictions of these measures on the sectors with $N$ particles are denoted $\mu_m^{II,N}$.

As the following lemma shows, there is a similarity transformation that relates shock measures of type I and type II.

**Lemma 2.** *Assume condition S and let*

$$U := q^{-\frac{2}{L}\sum_{k\in\mathbb{T}_L} k\hat{n}_k}. \tag{34}$$

*Then for* $-M \le m \le M$ *we have*

$$|\,\mu_m^{II}\,\rangle = \frac{1}{Y_m} q^{\frac{2(m-M)}{L}\hat{N}} U\,|\,\mu_m^{I}\,\rangle \tag{35}$$

*with*

$$Y_m = \prod_{k=-M+1}^{m} \left( \frac{1+z_1 q^{2\frac{m-M-k}{L}}}{1+z_1} \right) \prod_{k=m+1}^{M} \left( \frac{1+z_1 q^{2\frac{m+M-k}{L}}}{1+q^2 z_1} \right). \tag{36}$$

*Moreover, for fixed particle number N*

$$|\,\mu_m^{II,N}\,\rangle = \frac{1}{Y_m} q^{\frac{2(m-M)}{L}N} U\,|\,\mu_m^{I,N}\,\rangle. \tag{37}$$

The proof is analogous to the proof of Lemma 1. The central fact is that $P_N$, $\hat{N}$ and $U$ are all diagonal and hence commute. The density profile seen from the shock position is the lattice analogue of a hyperbolic tangent with a transition region of width $\sim L/(\ln q)$. Notice that $z_1$ parametrizes the total mass which is conserved in the case of periodic boundary conditions.

## 3 Main Results

We are now in a position to state and prove our main results on the time evolution of the ASEP under conditioned dynamics, viz. for type II shocks for periodic boundary conditions [2] where we work with condition S and for type I antishocks in the case of open boundary conditions [3] where we shall work with condition S′.

### 3.1 Periodic Boundary Conditions

Our objective is to characterize the time evolution of shocks of type II. To this end we define modified jump rates

$$\tilde{c}_r := c_r q^{-\frac{2N}{L}}, \quad \tilde{c}_\ell := c_\ell q^{\frac{2N}{L}}. \tag{38}$$

**Theorem 1.** *Consider the ASEP on* $\mathbb{T}_L$ *with N particles and hopping rates* $c_\ell$ *and* $c_r$ *to the left and to the right respectively. Let the initial distribution at time* $t = 0$

*be the type II shock measure* $\mu_m^{II,N}$ *and let* $\mu_m^{II,N}(t)$ *denote the grandcanonically conditioned measure of the ASEP at time* $t$ *under global conditioning with weight* $\mathrm{e}^{-sL} = q^2$*. Then for* $m \in \mathbb{T}_L$ *and any* $t \geq 0$

$$\mu_m^{II,N}(t) = \sum_{l=-M+1}^{M} p_t(l \mid m) \mu_l^{II,N} \tag{39}$$

*where* $p_t(l \mid m)$ *is the probability that a particle that performs a continuous-time simple random walk on* $\mathbb{T}_L$ *with hopping rates* $\tilde{c}_r$ *to the right and* $\tilde{c}_\ell$ *to the left, respectively, is at site* $l$ *at time* $t$*, starting from site* $m$*.*

We remark that the random walk transition probability can be written in the form

$$p_t(l \mid m) = e^{-(\tilde{c}_r+\tilde{c}_\ell)t} \sum_{p=-\infty}^{\infty} \left(\frac{\tilde{c}_r}{\tilde{c}_\ell}\right)^{(l-m+pL)/2} I_{m-l+pL}(2\sqrt{\tilde{c}_r\tilde{c}_\ell}\, t) \tag{40}$$

with the modified Bessel function $I_n(\cdot)$. The interpretation of (39) is that at any time $t \geq 0$ the shock position performs a biased random walk and that the distribution of this process seen from the position of the shock retains its product structure.

*Proof.* In this subsection we define $\tilde{H} = H(-2\ln q/L)$. In order to prove (39) we observe that with (30) and with Lemma 2 we can write the initial measure

$$|\, \mu_m^{II,N} \,\rangle = V_m |\, \mu_{z_1}^N \,\rangle \tag{41}$$

with

$$V_m = \frac{1}{W} q^{\frac{2}{L}\left(\sum_{k=-M+1}^{m}(-M+m-k)\hat{n}_k + \sum_{k=m+1}^{M}(M+m-k)\hat{n}_k\right)} \tag{42}$$

where the normalization factor

$$W = (1+z_1)^L \prod_{k=-M+1}^{M} \left(1 + z_1 q^{-\frac{2k}{L}}\right) \tag{43}$$

is independent of $m$. Recalling (5) this leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} |\, \tilde{\mu}_m^{II,N}(t) \,\rangle = -V_m \tilde{H}^{(m)} |\, \tilde{\mu}_{z_1}^N(t) \,\rangle \tag{44}$$

for the unnormalized measure with the transformed generator $\tilde{H}^{(m)} = V_m^{-1} \tilde{H} V_m$.

Straightforward computation yields

$$\mathrm{e}^{-a_k \hat{n}_k} \sigma_l^{\pm} \mathrm{e}^{a_k \hat{n}_k} = \begin{cases} \sigma_l^{\pm} & k \neq l \\ \mathrm{e}^{\pm a_l} \sigma_l^{\pm} & k = l \end{cases} . \tag{45}$$

and therefore, with Condition S,

$$\tilde{H}^{(m)} = - \sum_{i=-M+1}^{M}{}' \left[ c_r(\sigma_i^+\sigma_{i+1}^- - \hat{n}_i\hat{v}_{i+1}) + c_\ell(\sigma_i^-\sigma_{i+1}^+ - \hat{v}_i\hat{n}_{i+1}) \right] \tag{46}$$

$$- \left[ c_\ell\sigma_m^+\sigma_{m+1}^- - c_r\hat{n}_m\hat{v}_{m+1} + c_r\sigma_m^-\sigma_{m+1}^+ - c_\ell\hat{v}_m\hat{n}_{m+1} \right] \tag{47}$$

where the prime at the summation indicates the absence of the term with $i = m$. We add $-(c_r - c_\ell)(\hat{n}_{i+1} - \hat{n}_i)$ to each term in (46) and $-(c_r - c_\ell)(\hat{n}_{m+1} - \hat{n}_m)$ to (47) to compensate. Thus we arrive at

$$\tilde{H}^{(m)} = - \sum_{i=-M+1}^{M}{}' \left[ c_r(\sigma_i^+\sigma_{i+1}^- - \hat{v}_i\hat{n}_{i+1}) + c_\ell(\sigma_i^-\sigma_{i+1}^+ - \hat{n}_i\hat{v}_{i+1}) \right] \tag{48}$$

$$- \left[ c_\ell(\sigma_m^+\sigma_{m+1}^- - \hat{n}_m\hat{v}_{m+1}) + c_r(\sigma_m^-\sigma_{m+1}^+ - \hat{v}_m\hat{n}_{m+1}) \right] \tag{49}$$

which we write in the form $\tilde{H}^{(m)} = \tilde{H}_b^{(m)} + B^{(m)}$ with bulk term $\tilde{H}_b^{(m)} = \sum_i' \tilde{h}_i^b$ given by (48) transformation term $B^{(m)}$ given by (49).

Using the properties of the particle creation and annihilation operators it follows that $h_i^b | \mu_{z_1}^N \rangle = 0$. On the other hand, $B^{(m)} | \mu_{z_1}^N \rangle = -(c_r - c_\ell)(\hat{n}_{m+1} - \hat{n}_m) | \mu_{z_1}^N \rangle$ which implies $V_m B^{(m)} | \mu_{z_1}^N \rangle = -(c_r - c_\ell)(\hat{n}_{m+1} - \hat{n}_m) V_m | \mu_{z_1}^N \rangle$ since $V_m$ and the number operators $\hat{n}_i$ are both diagonal and hence commute. With the projector property $\hat{n}_k^2 = \hat{n}_k$ resulting from the exclusion principle one has $q^{-2N/L} V_{m+1} = (1 + (q^{-2} - 1)\hat{n}_{m+1}) V_m$ and $q^{2N/L} V_{m-1} = (1 + (q^2 - 1)\hat{n}_m) V_m$. Therefore $c_r q^{-2N/L} V_{m+1} + c_\ell q^{2N/L} V_{m-1} - (c_r + c_\ell) V_m = -(c_r - c_\ell)(\hat{n}_{m+1} - \hat{n}_m) V_m$. Putting these results together yields the evolution equation for the unnormalized conditioned measure

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{\mu}_m^{II,N}(t) = c_r q^{-\frac{2N}{L}} \tilde{\mu}_{m+1}^{II,N}(t) + c_l q^{\frac{2N}{L}} \tilde{\mu}_{m-1}^{II,N}(t) - (c_r + c_\ell) \tilde{\mu}_m^{II,N}(t) \tag{50}$$

for any $m \in \mathbb{T}_L$ and any $t \geq 0$.

This relation brings us into a position to compute the normalization $R_m(t) = \langle s | \mathrm{e}^{-\tilde{H}t} | \mu_m^{II,N} \rangle$ and from this the normalized measure $\mu_k^{II,N}(t) = \tilde{\mu}_k^{II,N}(t)/R(t)$. Using (50) we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} R_m(t) &= -\langle s | \mathrm{e}^{\tilde{H}t} \tilde{H} | \mu_m^{II,N} \rangle \\ &= c_r q^{-2N/L} R_{m+1}(t) + c_\ell q^{2N/L} R_{m-1}(t) - (c_r + c_\ell) R_m(t) \qquad (51) \\ &= \left[ c_r q^{-2N/L} + c_\ell q^{2N/L} - (c_r + c_\ell) \right] R_m(t) \end{aligned}$$

where the last line follows from translation invariance. Integrating yields

$$\mu_k^{II,N}(t) = \exp\left[ (-c_r q^{-2N/L} - c_\ell q^{2N/L} + c_r + c_\ell)t \right] \tilde{\mu}_k^{II,N}(t)/R(0) \tag{52}$$

which implies the system of linear ODE's

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_k^{II,N}(t) = \tilde{c}_r \mu_{k+1}^{II,N}(t) + \tilde{c}_\ell \mu_{k-1}^{II,N}(t) - (\tilde{c}_r + \tilde{c}_\ell)\, \mu_k^{II,N}(t). \tag{53}$$

In order to show that (39) satisfies this system of ODE's with the initial condition $\mu_m^{II}(0) = \mu_m^{II}$ we note that the random walk transition probability of the theorem satisfies the forward evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t} p_t(l\,|\,m) = \tilde{c}_r\, p_t(l-1\,|\,m) + \tilde{c}_\ell\, p_t(l+1\,|\,m) - (\tilde{c}_r + \tilde{c}_\ell)\, p_t(l\,|\,m) \tag{54}$$

with initial condition $p_0(l\,|\,m) = \delta_{l,m}$. The theorem thus follows from translation invariance $p_t(l+r\,|\,m+r) = p_t(l\,|\,m)\ \forall r \in Z$ and periodicity $p_t(l+pL\,|\,m) = p_t(l\,|\,m)\ \forall p \in Z$ of the transition probability. □

## 3.2 Open Boundary Conditions

In contrast to the periodic case we consider for open boundary conditions antishocks satisfying condition S′ (29) [3]. We define

$$\delta_1 = (c_r - c_\ell)\frac{\rho_1(1-\rho_1)}{\rho_1 - \rho_2} \tag{55}$$

and

$$\delta_2 = (c_r - c_\ell)\frac{\rho_2(1-\rho_2)}{\rho_1 - \rho_2} \tag{56}$$

which, as the following theorems shows, are the hopping rates of the antishock.

**Theorem 2.** *Let the initial distribution of the ASEP with open boundary conditions defined in (10) be given by the type I antishock measure $\mu_m^I$ satisfying condition S′ and let $\mu_m^I(t)$ denote the grandcanonically conditioned measure at time $t$ under conditioning* $\mathrm{e}^{-s} = q^2$. *Then we have* $\forall\, t \geq 0$

$$\mu_m^I(t) = \sum_{k=-M+1}^{M} p_t(k\,|\,m)\mu_k^I \tag{57}$$

*where $p_t(k\,|\,m)$ is the transition probability of a simple continuous-time random walk with hopping rates $\delta_1$ to the left and $\delta_2$ to the right and reflecting boundary conditions at sites $n = -M, M$.*

The interpretation of this result for the antishock is analogous to interpretation of (1) for the shock. The antishock performs a random walk and the internal structure of the measure seen from the position of the antishock is invariant. It is remarkable that the antishock hopping rates are similar to those of a shock satisfying condition S under unconditioned dynamics [1].

*Proof.* In this subsection we define $-\tilde{H} = -H(-2\ln q) = \tilde{b}_{-M+1} + \sum_{k=-M+1}^{M-1} \tilde{e}_k + \tilde{b}'_M$ and $\tilde{Y}(t) = Y_{-2\ln q}(t)$. Using $\langle s | \sigma_k^- = \langle s | \hat{v}_k$ and $\langle s | \sigma_k^+ = \langle s | \hat{n}_k$ one has $\langle s | \tilde{e}_k = (c_r - c_\ell)(\hat{n}_{k+1} - \hat{n}_k)$ and, summing up all terms, obtains $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{Y}(t) = -\langle s | \tilde{H} \mathrm{e}^{-\tilde{H}t} | \mu(0) \rangle = -\epsilon_0 \tilde{Y}(t)$ with

$$\epsilon_0 = -(c_r - c_\ell)(\rho_2 - \rho_1) \tag{58}$$

for any initial distribution $\mu(0)$. Since $\mu(0)$ is normalized to one it follows that

$$\tilde{Y}(t) = \mathrm{e}^{-\epsilon_0 t} \tag{59}$$

We conclude that the normalized grandcanonically conditioned time evolution is generated by

$$\hat{H} = \tilde{H} - \epsilon_0 \tag{60}$$

Next we consider the time derivative $-\tilde{H} | \mu_n^I \rangle$, using arguments completely analogous to those of [1]. One finds $\tilde{b}_{-M+1} | \mu_n^I \rangle = 0$ for $n \neq -M$, $\tilde{b}'_M | \mu_n^- \rangle = 0$ for $n \neq M$ and $\tilde{e}_k | \mu_n^- \rangle = 0$ for $n \neq k$, and, for $n \neq \pm M$,

$$-\tilde{H} | \mu_n^- \rangle = \tilde{e}_n | \mu_n^- \rangle = \delta_1 | \mu_{n-1}^- \rangle + \delta_2 | \mu_{n+1}^- \rangle - (c_r + c_\ell) | \mu_n^- \rangle \tag{61}$$

with

$$\delta_1 = c_\ell \frac{\rho_1}{\rho_2}, \quad \delta_2 = c_r \frac{\rho_2}{\rho_1}. \tag{62}$$

Conditions S′ (29) then leads to the expressions (55), (56). Similarly one gets

$$-\tilde{H} | \mu_{-M}^I \rangle = \tilde{b}_{-M+1} | \mu_{-M}^I \rangle = \delta_2 | \mu_{-M+1}^I \rangle - f_1 | \mu_{-M}^I \rangle \tag{63}$$

and

$$-\tilde{H} | \mu_M^I \rangle = \tilde{b}'_M | \mu_M^I \rangle = \delta_1 | \mu_{M-1}^I \rangle - f_2 | \mu_M^I \rangle \tag{64}$$

with

$$f_1 = c_r \rho_1 + c_\ell (1 - \rho_1) = c_r + c_\ell - \delta_1, \quad f_2 = c_r (1 - \rho_2) + c_\ell \rho_2 = c_r + c_\ell - \delta_2. \tag{65}$$

On the other hand, condition S′ gives after some straightforward algebra

$$\delta_1 + \delta_2 - c_r - c_\ell = -\epsilon_0. \tag{66}$$

Therefore, for $-M \leq n \leq M$, one has for the unnormalized time evolution $-\tilde{H}|\, \mu_n^I \,\rangle = \delta_1(1-\delta_{n,-M})\left(|\, \mu_{n-1}^I \,\rangle - |\, \mu_n^I \,\rangle\right) + \delta_2(1-\delta_{n,M})\left(|\, \mu_{n+1}^I \,\rangle - |\, \mu_n^I \,\rangle\right) - \epsilon_0|\, \mu_n^I \,\rangle$. The last diagonal term is compensated by the normalization (60) and we finally obtain for the normalized grandcanonically evolution of antishock measures satisfying (29)

$$-\hat{H}|\, \mu_n^I \,\rangle = \delta_1(1-\delta_{n,-M})\left(|\, \mu_{n-1}^I \,\rangle - |\, \mu_n^I \,\rangle\right) + \delta_2(1-\delta_{n,M})\left(|\, \mu_{n+1}^I \,\rangle - |\, \mu_n^I \,\rangle\right). \tag{67}$$

This is the forward evolution equation of a biased random walk with hopping rate $\delta_1$ to the left and $\delta_2$ to the right and reflecting boundary conditions at sites $n = -M, M$. Formal integration of this evolution equation leads to the statement (57) of the theorem. □

The stationary solution of the random walk problem is a geometric distribution of antishock positions. This leads us to the corollary

**Corollary 1.** *Let* $|\, \mu^{I*} \,\rangle = \lim_{t\to\infty} \tilde{\mu}_m^I(t)$ *be the (unique) stationary distribution of the conditioned dynamics. Then*

$$|\, \mu^{I*} \,\rangle = \frac{1}{Z} \sum_{k=-M}^{M} u^k |\, \mu_k^I \,\rangle \tag{68}$$

*with parameter* $u = \delta_2/\delta_1$ *and normalization* $Z = (1-u^{L+1})/(1-u)$ *for* $u \neq 1$ *and* $Z = L+1$ *for* $u = 1$.

Notice that for $u < 1$ the position of the antishock is concentrated at the right boundary, leaving the bulk at density $\rho_1$. For $u > 1$ the position of the antishock is concentrated at the left boundary, leaving the bulk at density $\rho_2$. At $u = 1$, where the random walk is unbiased, the position of the antishock is uniformly distributed.

## 4 Relation to Macroscopic Large Deviation Theory

With regard to the macroscopic large-deviation theory of [5, 6] we remark that our results provide exact information for any *finite* $t \geq 0$ and for *finite* lattice distances. Notice however, that our results require the large deviation parameter $s$ to take a special value for a given (but arbitrary) asymmetry $q$, while the phenomena observed in [5, 6] are valid for any $s$, but require weak asymmetry.

A less technical difference between our work and that of [5, 6] is the setting in which the conditioning is considered. We look at the final distribution of the conditioned process after a finite time interval, not at the process in the middle of a large interval. Nevertheless there is a striking similarity, viz., the biased random walk of the shock position together with the invariance of the density profile seen from the shock position on the one hand, and the existence of macroscopic travelling wave solutions on the other hand. It would be interesting to investigate the precise relation between our hyperbolic tangent solution and the macroscopic shock-antishock profiles obtained from the large-deviation theory of [5].

We have also shown that under dynamics where the time-integrated current of the ASEP with open boundaries is conditioned to be atypically low one can have stable antishocks. Taking as initial distribution an antishock which is a Bernoulli product measure with segments of different densities $\rho_{1,2}$, the position of the antishock performs a biased random walk with rates similar to that of a shock. The microscopic structure of this antishock seen from the position of the antishock remains unchanged in time. Measures of type I satisfying condition S′ are a microscopic realization of an antishock in a macroscopic step function density profile with density $\rho_1$ ($\rho_2$) in the interval $[-1/2, x)$ ($[x, 1/2)$) of rescaled coordinates $m \to x$ under suitable rescaling of space. On macroscopic scale the position of the antishock is shown to move with a speed determined by mass conservation and to fluctuate around its mean position diffusively with a diffusion coefficient $D = (\delta_1 + \delta_2)/2$.

For zero speed this property sheds light on the fact that the large deviation theory cannot predict the position of the antishock. For the long-time regime our result provides a direct microscopic proof that each antishock position is equally likely. For non-zero speed its macroscopic position at late times is either at the left boundary (for negative speed) or at the right boundary (for positive speed). The fluctuations of the microscopically sharp antishock near the boundary correspond to a microscopic boundary layer of width $\ln(c_r/c_\ell)$, and macroscopically to a jump discontinuity from $\rho_1$ to $\rho_2$. On the other hand, macroscopic large deviation theory generically predicts three discontinuities, one at each boundary and one for the antishock. Our result suggests that there could be special curves in parameter space where there is only one discontinuity, for all phases.

For general values of $s$ our approach does not allow us to obtain microscopic results. However, the arguments of [1] suggest that a similar approach should be possible for a positive-integer family of values $s_p$ and a corresponding family of microscopic shock measures $\mu_{k_1,\ldots,k_p}$ with $p$ shocks that perform interacting random walks. This might lead to a density profile different from the hyperbolic tangent considered here. This is an open problem which is beyond the scope of the present work.

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## References

1. Belitsky, V., Schütz, G.M.: Diffusion and coalescence of shocks in the partially asymmetric exclusion process. Electron. J. Probab. **7**(Paper No. 11), 1–21 (2002)

2. Belitsky, V., Schütz, G.M.: Microscopic structure of shocks and antishocks in the ASEP conditioned on low current. J. Stat. Phys. **152**, 93–111 (2013)
3. Belitsky, V., Schütz, G.M.: Antishocks in the ASEP with open boundaries conditioned on low current. J. Phys. A Math. Theor. **46**, 295004 (2013)
4. Bertini, L., De Sole, A., Gabrielli, D., Jona Lasinio, G., Landim, C.: Macroscopic fluctuation theory for stationary non-equilibrium states. J. Stat. Phys. **107**, 635–675 (2002)
5. Bodineau, T., Derrida, B.: Distribution of current in non-equilibrium diffusive systems and phase transitions. Phys. Rev. E **72**, 066110 (2005)
6. Bodineau, T., Derrida, B.: Current large deviations for asymmetric exclusion processes with open boundaries. J. Stat. Phys. **123**(2), 277–300 (2006)
7. Borodin, A., Corwin, I., Sasamoto, T.: From duality to determinants for Q-TASEP and ASEP. arXiv:1207.5035 [math.PR]. http://arxiv.org/pdf/1207.5035v1 (2012). Cited 20 Jul 2012
8. Derrida, B., Lebowitz, J.L.: Exact large deviation function in the asymmetric exclusion process. Phys. Rev. Lett. **80**(2), 209–213 (1998)
9. Derrida, B., Lebowitz, J.L., Speer, E.R.: Large deviation of the density profile in the steady state of the open symmetric simple exclusion process. J. Stat. Phys. **107**, 599–634 (2002)
10. Giardinà, C., Kurchan, J., Redig, F., Vafayi, K.: Duality and hidden symmetries in interacting particle systems. J. Stat. Phys. **135**, 25–55 (2009)
11. Harris, R.J., Schütz, G.M.: Fluctuation theorems for stochastic dynamics. J. Stat. Mech. P07020 (2007)
12. Imamura, T., Sasamoto, T.: Current moments of 1D ASEP by duality. J. Stat. Phys. **142**(5), 919–930 (2011)
13. Lecomte, V., Garrahan, J.P., Van Wijland, F.: Inactive dynamical phase of a symmetric exclusion process on a ring. J. Phys. A Math. Theor. **45**, 175001 (2012)
14. Liggett, T.M.: Stochastic Interacting Systems: Contact, Voter and Exclusion Processes. Springer, Berlin (1999)
15. Lloyd, P., Sudbury, A., Donnelly, P.: Quantum operators in classical probability theory: I. "Quantum spin" techniques and the exclusion model of diffusion. Stoch. Process. Appl. **61**(2), 205–221 (1996)
16. Popkov, V., Schütz, G.M.: Transition probabilities and dynamic structure factor in the ASEP conditioned on strong flux. J. Stat. Phys. **142**(3), 627–639 (2011)
17. Popkov, V., Simon, D., Schütz, G.M.: ASEP on a ring conditioned on enhanced flux. J. Stat. Mech. P10007 (2010)
18. Rezakhanlou, F.: Hydrodynamic limit for attractive particle systems on $\mathbb{Z}^d$. Commun. Math. Phys. **140** 417–448 (1991)
19. Schütz, G.M.: Duality relations for asymmetric exclusion process. J. Stat. Phys. **86**(5/6), 1265–1287 (1997)
20. Schütz, G.M.: Exactly solvable models for many-body systems far from equilibrium. In: Domb, C., Lebowitz, J. (eds.) Phase Transitions and Critical Phenomena, vol. 19, pp. 1–251. Academic, London (2001)
21. Simon, D.: Construction of a coordinate Bethe ansatz for the asymmetric simple exclusion process with open boundaries. J. Stat. Mech. P07017 (2009)

# Superdiffusion of Energy in Hamiltonian Systems Perturbed by a Conservative Noise

**Cédric Bernardin**

## 1 Introduction

Transport properties of one-dimensional Hamiltonian systems consisting of coupled oscillators on a lattice have been the subject of many theoretical and numerical studies, see the review papers [7, 8, 12]. Despite many efforts, our knowledge of the fundamental mechanisms necessary and/or sufficient to have a normal diffusion remains very limited.

Consider a 1-dimensional chain of oscillators indexed by $x \in \mathbb{Z}$, whose formal Hamiltonian is given by

$$\mathscr{H} = \sum_{x \in \mathbb{Z}} \left[ \frac{p_x^2}{2} + V(r_x) \right],$$

where $r_x = q_{x+1} - q_x$ is the "deformation" of the lattice, $q_x$ being the displacement of the atom $x$ from its equilibrium position and $p_x$ its momentum. The interaction potential $V$ is a smooth positive function growing at infinity fast enough. The energy $e_x$ of atom $x \in \mathbb{Z}$ is defined by

$$e_x = \frac{p_x^2}{2} + V(r_x).$$

Our goal is to understand the macroscopic energy diffusion properties for the corresponding Hamiltonian dynamics

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$$\frac{dr_x}{dt} = p_{x+1} - p_x, \quad \frac{dp_x}{dt} = V'(r_x) - V'(r_{x-1}), \quad x \in \mathbb{Z}.$$

Under suitable conditions on $V$, the infinite dynamics is well defined for a large class of initial conditions.

Apart from the total energy $\sum_x e_x$, observe that the total momentum $\sum_x p_x$ and the total deformation $\sum_x r_x$ of the lattice are formally conserved. This is a consequence of the following microscopic continuity equations:

$$\frac{de_x}{dt} + \nabla[j^e_{x-1,x}] = 0, \quad j^e_{x,x+1} = -p_{x+1}V'(r_x), \tag{1}$$

$$\frac{dp_x}{dt} + \nabla[-V'(r_{x-1})] = 0, \tag{2}$$

$$\frac{dr_x}{dt} + \nabla[-p_x] = 0. \tag{3}$$

The function $j^e_{x,x+1}$ is the current of energy going from $x$ to $x+1$. The main open problem [11, 17] concerning the foundation of statistical mechanics based on classical mechanics is precisely to show that the three quantities above are the only quantities which are conserved by the dynamics. In some sense, it means that the dynamics, evolving on the manifold defined by fixing the total energy, the total momentum and the total deformation, is ergodic. Of course, the last sentence does not make sense since we are in infinite volume and $\sum_x e_x, \sum_x p_x$ and $\sum_x r_x$ are typically infinite. Nevertheless, an alternative meaningful definition will be proposed and discussed in Sect. 2.

Numerical simulations provide a strong evidence of the fact that one dimensional chains of anharmonic oscillators conserving momentum are[1] superdiffusive. It shall be noticed that there is no explanation of this, apart from heuristic considerations, and that some models which do not conserve momentum can also display anomalous diffusion of energy (see [10]).

An interesting area of current research consists in studying this problem for hybrid models where a stochastic perturbation is superposed to the deterministic evolution. Even if the problem is considerably simplified, several open challenging questions can be addressed for these systems. The first benefit of the introduction of stochasticity in the models is to guarantee the ergodicity that we are not able to show for purely deterministic systems. The added noise must be carefully chosen in order not to destroy the conservation laws we are interested in. In particular, the noise shall conserve energy. But we will consider a noise conserving also some of the other quantities conserved by the underlying Hamiltonian dynamics, e.g. the momentum, the deformation or any linear combination of them.

[1] See however the coupled-rotor model which displays normal behavior (see [12], Sect. 6.4). This is probably due to the fact that the position space is compact.

The paper is organized as follows. In Sect. 2 we discuss the problem of the ergodicity of the infinite dynamics mentioned above and the possible stochastic perturbations we can add to the deterministic dynamics to obtain ergodic dynamics. In Sect. 3 we review some results obtained in the context of harmonic chains perturbed by a conservative noise and we discuss the case of anharmonic chains in the last section.

## 2 Ergodicity

Let us first generalize the models introduced above [6]. Let $U$ and $V$ be smooth positive potentials growing at infinity fast enough and let $\mathscr{H} := \mathscr{H}_{U,V}$ be the Hamiltonian

$$\mathscr{H}_{U,V} = \sum_{x\in\mathbb{Z}} [U(p_x) + V(r_x)] \,.$$

The corresponding Hamiltonian dynamics satisfy

$$\frac{dr_x}{dt} = U'(p_{x+1}) - U'(p_x), \quad \frac{dp_x}{dt} = V'(r_x) - V'(r_{x-1}), \quad x \in \mathbb{Z}. \tag{4}$$

The energy of particle $x$ is defined by $e_x = U(p_x) + V(r_x)$. The three formal quantities $\sum_x e_x$, $\sum_x r_x$ and $\sum_x p_x$ are conserved by the dynamics. The fundamental question we address in this section is: are they the only ones? In finite volume, i.e. replacing the lattice $\mathbb{Z}$ by a finite box $\Lambda$, this would correspond to the usual notion of ergodicity for Hamiltonian flows with a finite number of degrees of freedom. But since we consider the dynamics in infinite volume the notion of conserved quantity has to be properly defined. The way we follow to attack the problem is to detect the existence of a non-trivial conserved quantity through the existence of a non-trivial invariant state for the infinite dynamics.

Let $\Omega = (\mathbb{R}\times\mathbb{R})^{\mathbb{Z}}$ be the phase space of the dynamics and let us denote a typical configuration by $\omega = (r,p) \in \Omega$. For simplicity we assume that for any $(\beta,\lambda,\lambda') \in (0,+\infty)\times\mathbb{R}\times\mathbb{R}$, the partition function

$$Z(\beta,\lambda,\lambda') = \int_{\mathbb{R}\times\mathbb{R}} e^{-\beta[U(a)+V(b)]-\lambda b-\lambda' a} da\, db$$

is finite. Let $\mu_{\beta,\lambda,\lambda'}$ be the product Gibbs measures on $\Omega$ defined by

$$d\mu_{\beta,\lambda,\lambda'}(\omega) = \prod_{x\in\mathbb{Z}} \frac{1}{Z(\beta,\lambda,\lambda')} \exp\left[-\beta[U(p_x)+V(r_x)] - \lambda r_x - \lambda' p_x\right] dr_x dp_x .$$

We assume that (4) is well defined for a subset $\Omega_{\beta,\lambda,\lambda'}$ of full measure with respect to $\mu_{\beta,\lambda,\lambda'}$, that the latter is invariant for (4), and that it is possible to define a strongly continuous semigroup in $\mathbb{L}^2(\mu_{\beta,\lambda,\lambda'})$ with formal generator

$$\mathscr{A}_{U,V} = \sum_{x\in\mathbb{Z}} \left[ (U'(p_{x+1}) - U'(p_x))\partial_{r_x} + (V'(r_x) - V'(r_{x-1}))\partial_{p_x} \right].$$

All that can be proved under suitable assumptions on $U$ and $V$ [5,9].

In order to explain what is meant by ergodicity of the infinite volume dynamics we need to introduce some notation. For any topological space $X$ equipped with its Borel $\sigma$-algebra we denote by $\mathscr{P}(X)$ the convex set of probability measures on $X$. The relative entropy $H(\nu|\mu)$ of $\nu \in \mathscr{P}(X)$ with respect to $\mu \in \mathscr{P}(X)$ is defined as

$$H(\nu|\mu) = \sup_{\phi} \left\{ \int \phi \, d\nu - \log \left( \int e^{\phi} \, d\mu \right) \right\}, \tag{5}$$

where the supremum is carried over all bounded measurable functions $\phi$ on $X$.

Let $\theta_x, x \in \mathbb{Z}$, be the shift by $x$: $(\theta_x \omega)_z = \omega_{x+z}$. For any function $g$ on $\Omega$, $\theta_x g$ is the function such that $(\theta_x g)(\omega) = g(\theta_x \omega)$. For any probability measure $\mu \in \mathscr{P}(\Omega)$, $\theta_x \mu \in \mathscr{P}(\Omega)$ is the probability measure such that, for any bounded function $g : \Omega \to \mathbb{R}$, it holds $\int_\Omega g \, d(\theta_x \mu) = \int_\Omega \theta_x g \, d\mu$. If $\theta_x \mu = \mu$ for any $x$ then $\mu$ is said to be translation invariant.

If $\Lambda$ is a finite subset of $\mathbb{Z}$ the marginal of $\mu \in \mathscr{P}(\Omega)$ on $\mathbb{R}^\Lambda$ is denoted by $\mu|_\Lambda$. The relative entropy of $\nu \in \mathscr{P}(\Omega)$ with respect to $\mu \in \mathscr{P}(\Omega)$ in the box $\Lambda$ is defined by $H(\nu|_\Lambda \,|\, \mu|_\Lambda)$ and is denoted by $H_\Lambda(\nu|\mu)$. We say that a translation invariant probability measure $\nu \in \mathscr{P}(\Omega)$ has finite entropy density (with respect to $\mu$) if there exists a finite positive constant $C$ such that for any finite $\Lambda \subset \mathbb{Z}$, $H_\Lambda(\nu|\mu) \le C|\Lambda|$. In fact, if this condition is satisfied, then the limit

$$\overline{H}(\nu|\mu) = \lim_{|\Lambda|\to\infty} \frac{H_\Lambda(\nu|\mu)}{|\Lambda|}$$

exists and is finite (see [9]). It is called the entropy density of $\nu$ with respect to $\mu$.

We are now in position to define ergodicity.

**Definition 1.** We say that the infinite volume dynamics with infinitesimal generator $\mathscr{A}_{U,V}$ is *ergodic* if the following claim is true: If $\nu \in \mathscr{P}(\Omega)$ is a probability measure invariant by translation, invariant by the dynamics generated by $\mathscr{A}_{U,V}$ and with finite entropy density with respect to $\mu_{1,0,0}$, then $\nu$ is a mixture of the $\mu_{\beta,\lambda,\lambda'}, \beta > 0, \lambda, \lambda' \in \mathbb{R}$.

In the harmonic case ($U(z) = V(z) = z^2/2$) and for the Toda lattice ($U(z) = z^2/2$, $V(z) = e^{-z} + z - 1$), the infinite system is completely integrable and an infinite number of conserved quantities can be explicitly written. It follows that they are not ergodic in the sense above. Nevertheless we expect that for a very large class of potentials, the Hamiltonian dynamics are ergodic and that these two cases are exceptional.

In order that the infinite dynamics enjoy good ergodic properties, we superpose to the deterministic evolution a stochastic noise.

Given a sequence $u = (u_y)_{y\in\mathbb{Z}} \in \mathbb{R}^\mathbb{Z}$ and a site $x \in \mathbb{Z}$, we denote by $u^x$ (resp. $u^{x,x+1}$) the sequence defined by $(u^x)_y = u_y$ if $y \neq x$ and $(u^x)_x = -u_x$ (resp.

$(u^{x,x+1})_y = u_y$ if $y \neq x, x+1$, $(u^{x,x+1})_x = u_{x+1}$ and $(u^{x,x+1})_{x+1} = u_x$). We consider the following noises (jump processes) whose generators are defined by their actions on functions $f : \Omega \to \mathbb{R}$ according to:

1. $(\mathscr{S}_{flip}^{p} f)(r, p) = \sum_x [f(r, p^x) - f(r, p)]$.
2. $(\mathscr{S}_{flip}^{r} f)(r, p) = \sum_x [f(r^x, p) - f(r, p)]$.
3. $(\mathscr{S}_{ex}^{p} f)(r, p) = \sum_x \left[f(r, p^{x,x+1}) - f(r, p)\right]$.
4. $(\mathscr{S}_{ex}^{r} f)(r, p) = \sum_x \left[f(r^{x,x+1}, p) - f(r, p)\right]$.

If $U$ is even then the noise $\mathscr{S}_{flip}^{p}$ conserves the energy, the deformation but not the momentum; if $U$ is odd the noise has little interest for us since the energy conservation is destroyed. Similarly, if $V$ is even the noise $\mathscr{S}_{flip}^{r}$ conserves the energy and the momentum but not the deformation. The noises $\mathscr{S}_{ex}^{p}$ and $\mathscr{S}_{ex}^{r}$ conserve the energy, the deformation and the momentum.

Let now $\gamma > 0$ and denote by $\mathscr{L}$ the generator of the infinite Hamiltonian dynamics generated by $\mathscr{A}_{U,V}$ perturbed by one of the previous noise $\mathscr{S}$ with intensity $\gamma$, i.e. $\mathscr{L} = \mathscr{A}_{U,V} + \gamma \mathscr{S}$.

**Theorem 1 ([5, 6, 9]).** *The dynamics generated by $\mathscr{L}$ is ergodic in the sense that if $\nu \in \mathscr{P}(\Omega)$ is a probability measure invariant by translation, invariant by the dynamics generated by $\mathscr{L}$ and with finite entropy density with respect to $\mu_{1,0,0}$, then it holds:*

1. *If $U$ even and $\mathscr{S} = \mathscr{S}_{flip}^{p}$ then $\nu$ is a mixture of the $\mu_{\beta,\lambda,0}$;*
2. *If $V$ is even and $\mathscr{S} = \mathscr{S}_{flip}^{r}$ then $\nu$ is a mixture of the $\mu_{\beta,0,\lambda'}$.*
3. *If $\mathscr{S} = \mathscr{S}_{ex}^{p}$ or $\mathscr{S} = \mathscr{S}_{ex}^{r}$ then $\nu$ is a mixture of the $\mu_{\beta,\lambda,\lambda'}$.*

The main motivation to establish such a theorem is that by using Yau's relative entropy method [19] in the spirit of Olla-Varadhan-Yau [14], it is possible to show that if the infinite volume dynamics is ergodic then the propagation of local equilibrium holds in the hyperbolic time scale, before the appearance of the shocks. As a consequence, the dynamics has a set of compressible Euler equations as hydrodynamic limits [5, 6]. Observe that this is true also for the deterministic dynamics so that the rigorous derivation of the Euler equations from the first principles of the mechanics in the smooth regime is "reduced" to prove that the dynamics generated by $\mathscr{A}_{U,V}$ is ergodic.

## 3 Harmonic Chains

### *3.1 Role of the Conservation of Momentum and Deformation*

We consider here the specific (harmonic) case $V(z) = U(z) = z^2/2$. The dynamics is then linear and can be solved analytically using Fourier transform. Let us introduce a new macroscopic variable $\eta \in \mathbb{R}^{\mathbb{Z}}$ defined from $(p, r) \in \Omega$ by setting

$$\eta_{2x} = r_x, \quad \eta_{2x+1} = p_{x+1}, \quad x \in \mathbb{Z}. \tag{6}$$

Then, the Hamiltonian dynamics can be rewritten in the form

$$\frac{d\eta_x}{dt} = V'(\eta_{x+1}) - V'(\eta_{x-1}), \quad x \in \mathbb{Z}. \tag{7}$$

We introduce the $k$th mode $\hat{\eta}(k,\cdot)$ for $k \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, the one-dimensional torus of length 1:

$$\hat{\eta}(t,k) = \sum_{x \in \mathbb{Z}} \eta_x(t)\, e^{2i\pi k x}.$$

Then, the equations of motion are equivalent in the sense of distributions to the following decoupled system of first order differential equations:

$$\frac{d\hat{\eta}}{dt}(t,k) = i\omega(k)\, \hat{\eta}(t,k),$$

where the dispersion relation $\omega(k)$ reads

$$\omega(k) = -2\sin(2\pi k),$$

and the group velocity $v_g$ is

$$v_g(k) = \omega'(k) = -4\pi\cos(2\pi k).$$

By inverting the Fourier transform, the solution can be written as

$$\eta_x(t) = \int_{\mathbb{T}} \hat{\eta}(t,k)\, e^{-2i\pi k x}\, dk.$$

If the initial configuration $\eta(0)$ is in $\ell_2$ the well defined energy of the $k$th mode

$$E_k(t) = \frac{1}{4\pi} |\hat{\eta}(t,k)|^2 = E_k(0)$$

is conserved by the time evolution, and the total energy current $\tilde{J}^e = \sum_{x \in \mathbb{Z}} j^e_{x,x+1}$ takes the simple form

$$\tilde{J}^e = \int_{\mathbb{T}} v_g(k) E_k\, dk.$$

We interpret the waves $\hat{\eta}(k,t)$ as fictitious particles (phonons in solid state physics). In the absence of nonlinearities, they travel the chain without scattering. The diffusion of energy is then said to be ballistic. If the potential is non-quadratic, it may be expected that the nonlinearities produce a scattering responsible for the diffusion of the energy. Nevertheless, the conservation of the deformation and of the momentum implies that $\sum_x (r_x + p_x)$ is conserved

$$\hat{\eta}(t,0) = \hat{\eta}(0,0). \tag{8}$$

The identity (8) is valid even if $U \neq V$ and $U, V$ are not quadratic. It means that the 0th mode is not scattered at all and crosses the chain ballistically. In fact, the modes with small wave number $k$ do not experience a strong scattering and they therefore contribute to the observed anomalous diffusion of energy.

It is usually explained that momentum conservation plays a major role in the anomalous diffusion of energy but it is clear that the deformation conservation plays exactly the same role as momentum and that it is the conservation of their sum which is the real ingredient producing anomalous diffusion of energy (see Theorems 2 and 4).

### 3.2 Green-Kubo Formula

The signature of an anomalous diffusion of energy can be seen at the level of the Green-Kubo formula. When transport of energy is normal, meaning that the macroscopic equations such as the Fourier's law or heat equation hold, the transport coefficient appearing in these equations can be expressed by the famous Green-Kubo formula. In order to define the latter we need to introduce some notations. Since the discussion about the Green-Kubo formula is not restricted to the harmonic case we go back to a generic anharmonic model in the rest of the Subsection.

Recall that the probability measures $\mu_{\beta,\lambda,\lambda'}$ form a family of invariant probability measures for the infinite dynamics generated by $\mathscr{A}_{U,V}$. The following thermodynamic relations (which are valid since we assumed that the partition function $Z$ is well defined on $(0,+\infty)\times\mathbb{R}\times\mathbb{R}$) relate the chemical potentials $\beta, \lambda, \lambda'$ to the mean energy $e$, the mean deformation $u$, the mean momentum $\pi$ under $\mu_{\beta,\lambda,\lambda'}$:

$$e(\beta,\lambda,\lambda') = \mu_{\beta,\lambda,\lambda'}(U(p_x) + V(r_x)) = -\partial_\beta\Big(\log Z(\beta,\lambda,\lambda')\Big), \tag{9}$$

$$u(\beta,\lambda,\lambda') = \mu_{\beta,\lambda,\lambda'}(r_x) = -\partial_\lambda\Big(\log Z(\beta,\lambda,\lambda')\Big), \tag{10}$$

$$\pi(\beta,\lambda,\lambda') = \mu_{\beta,\lambda,\lambda'}(p_x) = -\partial_{\lambda'}\Big(\log Z(\beta,\lambda,\lambda')\Big). \tag{11}$$

These relations can be inverted by a Legendre transform to express $\beta$, $\lambda$ and $\lambda'$ as a function of $e$, $u$ and $\pi$. Define the thermodynamic entropy $S : (0,+\infty)\times\mathbb{R}\times\mathbb{R} \to [0,+\infty]$ as

$$S(e,u,\pi) = \inf_{\lambda,\lambda' \in \mathbb{R}^2, \beta>0} \Big\{ \beta e + \lambda u + \lambda' \pi + \log Z(\beta,\lambda,\lambda') \Big\}.$$

Let $\mathscr{U}$ be the convex domain of $(0,+\infty)\times\mathbb{R}\times\mathbb{R}$ where $S(e,u,\pi) < +\infty$ and $\mathring{\mathscr{U}}$ its interior. Then, for any $(e,u,\pi) := (e(\beta,\lambda,\lambda'), u(\beta,\lambda,\lambda'), \pi(\beta,\lambda,\lambda')) \in \mathring{\mathscr{U}}$, the parameters $\beta,\lambda,\lambda'$ can be obtained as

$$\beta = (\partial_e S)(e,u,\pi), \qquad \lambda = (\partial_u S)(e,u,\pi), \qquad \lambda' = (\partial_\pi S)(e,u,\pi) \tag{12}$$

These thermodynamic relations allow us to parameterize the Gibbs states by the average values of the conserved quantities $(e,u,\pi)$ rather than by the chemical potentials $(\beta,\lambda,\lambda')$. Thus, we denote by $\nu_{e,u,\pi}$ the Gibbs measure $\mu_{\beta,\lambda,\lambda'}$ where $(e,u,\pi)$ are related to $(\beta,\lambda,\lambda')$ by (12). Let $J^e := J^e(e,u,\pi) = \nu_{e,u,\pi}(j^e_{x,x+1})$ be the average of the energy current $j^e_{x,x+1} = -U'(p_x)V'(r_x)$ and define the normalized energy current $\hat{j}^e_{x,x+1}$ by

$$\hat{j}^e_{x,x+1} = j^e_{x,x+1} - J^e - (\partial_e J^e)(e_x - e) - (\partial_u J^e)(r_x - u) - (\partial_\pi J^e)(p_x - \pi).$$

The normalized energy current is the part of the centered energy current which is orthogonal in $\mathbb{L}^2(\nu_{e,u,\pi})$ to the space spanned by the conserved quantities.

Up to multiplicative thermodynamic parameters (see [15] for details) that we neglect to simplify the notations, the Green-Kubo formula[2] is nothing but

$$\kappa(e,u,\pi) := \int_0^\infty \sum_{x\in\mathbb{Z}} \mathbb{E}_{\nu_{e,u,\pi}} \Big[ \hat{j}^e_{x,x+1}(\omega(t))\ \hat{j}^e_{0,1}(\omega(0)) \Big]\, dt$$

where $\mathbb{E}_{\nu_{e,u,\pi}}$ denotes the expectation corresponding to the law of the infinite volume dynamics $(\omega(t))_{t\geq 0}$ generated by $\mathscr{A}_{U,V}$ with initial condition $\omega(0)$ distributed according to the equilibrium Gibbs measure $\nu_{e,u,\pi}$. The definition of $\kappa(e,u,\pi)$ is formal but the way we adopt to give it a mathematically well posed definition is to introduce a small parameter $z>0$ and define $\kappa(e,u,\pi)$ as

$$\kappa(e,u,\pi) = \limsup_{z\to 0} \ll \hat{j}^e_{0,1}\,,\ (z-\mathscr{A}_{U,V})^{-1}\hat{j}^e_{0,1} \gg_{e,u,\pi} \tag{13}$$

where the inner-product $\ll \cdot,\cdot \gg_{e,u,\pi}$ is defined for local square integrable functions $f,g : \Omega \to \mathbb{R}$ by

[2]The transport coefficient is in fact a matrix whose size is the number of conserved quantities. Since we are interested in the energy diffusion, we only consider the entry corresponding to the energy-energy flux.

$$\ll f\,,\,g \gg_{e,u,\pi} \;=\; \sum_{x\in\mathbb{Z}}\left[\left(\int f\theta_x g\, d\nu_{e,u,\pi}\right)-\left(\int f d\nu_{e,u,\pi}\right)\left(\int g d\nu_{e,u,\pi}\right)\right].$$

Since $(z-\mathscr{A}_{U,V})^{-1}\hat{j}^{e}_{0,1}$ is not a local function, the term on the RHS of (13) has to be interpreted in the Hilbert space obtained by the completion of the space of local bounded functions with respect to the inner product $\ll \cdot,\cdot \gg_{e,u,\pi}$.

The superdiffusion (resp. normal diffusion) of energy corresponds to an infinite (resp. finite) value for $\kappa(e,u,\pi)$. In order to study the superdiffusion, it is of interest to estimate the time decay of the autocorrelation of the normalized current

$$C(t) := C_{e,u,\pi}(t) = \sum_{x\in\mathbb{Z}} \mathbb{E}_{\nu_{e,u,\pi}}\left[\hat{j}^{e}_{x,x+1}(\omega(t))\; \hat{j}^{e}_{0,1}(\omega(0))\right].$$

It is in general easier to estimate the behavior of the Laplace transform $L(z) = \int_0^\infty e^{-zt}C(t)dt$ as $z\to 0$. Roughly, if $L(z)\sim z^{-\delta}$ for some $\delta\geq 0$ then $C(t)\sim t^{\delta-1}$ as $t\to+\infty$. Observe also that

$$L(z) = \ll \hat{j}^{e}_{0,1}\,,\,(z-\mathscr{A}_{U,V})^{-1}\hat{j}_{0,1} \gg_{e,u,\pi}\,.$$

### *3.3 Harmonic Chain Perturbed by a Conservative Stochastic Noise*

We consider now the particular case $U(z) = V(z) = z^2/2$ and study the Green-Kubo formula for the perturbed dynamics generated by $\mathscr{L} = \mathscr{A}_{U,V} + \gamma\mathscr{S}$ where $\mathscr{S}$ is one of the noises introduced in Sect. 2. Since, depending of the form of the noise, the momentum conservation law (resp. deformation conservation law) can be suppressed, the corresponding Green-Kubo formula shall be modified by setting $\pi = 0$ and $\partial_\pi J^e = 0$ (resp. $u=0$ and $\partial_u J^e = 0$).

We have the following theorem which shows that if momentum conservation law or deformation conservation law is destroyed by the noise then a normal behavior occurs.

**Theorem 2 ([4]).** *Let $U$ and $V$ be quadratic potentials.*

1. *Consider the system generated by $\mathscr{L} = \mathscr{A}_{U,V}+\gamma\mathscr{S}^{p}_{flip}$, $\gamma>0$. Then the following limit*

$$\lim_{z\to 0} \ll \hat{j}^{e}_{0,1}\,,\,(z-\mathscr{L})^{-1}\hat{j}^{e}_{0,1} \gg_{e,u,0}$$

   *exists, is finite and strictly positive and can be explicitly computed.*
2. *Consider the system generated by $\mathscr{L} = \mathscr{A}_{U,V}+\gamma\mathscr{S}^{r}_{flip}$, $\gamma>0$. Then the following limit*

$$\lim_{z \to 0} \ll \hat{j}^e_{0,1} \, , \, (z - \mathscr{L})^{-1} \hat{j}^e_{0,1} \gg_{e,0,\pi}$$

*exists, is finite and strictly positive and can be explicitly computed.*

It shall be noticed that the second statement is a direct consequence of the first one since the process of the second item is equal to the first one by the transformation

$$r_x \to p_x, \quad p_x \to r_{x-1}.$$

However, the interest of the second statement is to show that *even if momentum is conserved*, a normal diffusion of energy occurs. This is because the deformation is no longer conserved.

The following theorem shows that if the noise added conserves momentum *and* deformation then the situation is very different since an anomalous diffusion of energy is observed.

**Theorem 3 ([1,2]).** *Let $U$ and $V$ be quadratic potentials.*

1. *Consider the system generated by $\mathscr{L} = \mathscr{A}_{U,V} + \gamma \mathscr{S}^p_{ex}$, $\gamma > 0$. Then the following limit*

$$\lim_{z \to 0} z^{1/2} \ll \hat{j}^e_{0,1} \, , \, (z - \mathscr{L})^{-1} \hat{j}^e_{0,1} \gg_{e,u,\pi}$$

   *exists, is finite and strictly positive and can be explicitly computed.*
2. *Consider the system generated by $\mathscr{L} = \mathscr{A}_{U,V} + \gamma \mathscr{S}^r_{ex}$, $\gamma > 0$. Then the following limit*

$$\lim_{z \to 0} z^{1/2} \ll \hat{j}^e_{0,1} \, , \, (z - \mathscr{L})^{-1} \hat{j}^e_{0,1} \gg_{e,u,\pi}$$

   *exists, is finite and strictly positive and can be explicitly computed.*

*In particular, in each of the previous case the Green-Kubo formula yields an infinite conductivity.*

## 4 Anharmonic Chains

We consider now the anharmonic case. For deterministic chains generated by $\mathscr{A}_{U,V}$ we expect usually a superdiffusive behavior of the energy. If a noise $\mathscr{S}$ is superposed to the dynamics, we expect that transport is normal for $\mathscr{S} = \mathscr{S}^p_{flip}$ and $\mathscr{S} = \mathscr{S}^r_{flip}$ and superdiffusive if $\mathscr{S} = \mathscr{S}^p_{ex}$ or $\mathscr{S} = \mathscr{S}^r_{ex}$.

The following theorem generalizes Theorem 2 to the anharmonic case showing that a noise destroying momentum conservation law or deformation conservation law produces normal transport. This shows that, also in the anharmonic case, momentum conservation alone is not responsible of anomalous diffusion of energy but that deformation conservation law plays a similar role.

**Theorem 4 ([4]).** *Let $U$ and $V$ be smooth potentials such that there exists a constant $c > 0$ such that*

$$c \leq U'' \leq c^{-1}, \quad c \leq V'' \leq c^{-1}.$$

1. *Assume $U$ even and consider the system generated by $\mathscr{L} = \mathscr{A}_{U,V} + \gamma \mathscr{S}_{flip}^{p}$, $\gamma > 0$. Then the following limit*

$$\lim_{z \to 0} \ll \hat{j}_{0,1}^{e}\, , \, (z - \mathscr{L})^{-1} \hat{j}_{0,1}^{e} \gg_{e,u,0}$$

   *exists and is finite.*
2. *Assume $V$ even and consider the system generated by $\mathscr{L} = \mathscr{A}_{U,V} + \gamma \mathscr{S}_{flip}^{r}$, $\gamma > 0$. Then the following limit*

$$\lim_{z \to 0} \ll \hat{j}_{0,1}^{e}\, , \, (z - \mathscr{L})^{-1} \hat{j}_{0,1}^{e} \gg_{e,0,\pi}$$

   *exists and is finite.*

*Proof.* The second statement is a direct consequence of the first one by the symmetry argument evoked for Theorem 2. The upper bounds on $U''$ and $V''$ are here to assure the existence of the infinite volume dynamics.

For simplicity assume that $u = 0$ and $\beta := \beta(e, u, 0) = 1$. The first statement has been proved in [4] in the particular case $U(z) = z^2/2$. The generalization to a non quadratic smooth even potential $U$ is straightforward. In [4], since $U(z) = z^2/2$, we used Hermite polynomials which are orthogonal w.r.t. the Gaussian measure $d\mu(z) = (2\pi)^{-1/2} \exp\{-z^2/2\} dz$. In the present case, the only difference is that we have to replace the Hermite basis by any orthogonal polynomial basis $\{P_n\}_{n \geq 0}$ with respect to the probability measure $\mathscr{N}^{-1} \exp(-U(z)) dz$ (with $\mathscr{N}$ a normalization constant) which satisfies $P_n$ odd if $n$ odd and even otherwise. Then the proof is exactly the same.

It would be now of interest to show that if we perturb the dynamics generated by $\mathscr{A}_{U,V}$ by $\mathscr{S}_{ex}^{p}$ or by $\mathscr{S}_{ex}^{r}$ then anomalous diffusion of energy occurs.[3] This is an open question and as far as we know the only result going in this direction has been obtained in [3].

The model considered in [3] is the dynamics generated by $\mathscr{A}_{U,V}$ with $U = V$ taking the particular form $V(z) = e^{-z} + z - 1$, perturbed by a noise $\mathscr{S}$ which conserves energy and $\sum_{x \in \mathbb{Z}} (r_x + p_x)$. More exactly, let us rewrite the Hamiltonian dynamics (4) by using the variable $\eta := (\eta_x)_{x \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ defined by (6). Then we get

[3] However, if $U$ or $V$ is bounded, like for the rotors model, we expect that diffusion is normal.

the equations of motion given by (7). With these new variables, the total energy is $2\sum_x V(\eta_x)$, the total deformation is $\sum_x \eta_{2x}$ and the total momentum is $\sum_x \eta_{2x+1}$. The noise $\mathscr{S}$ superposed to the dynamics acts on local functions $f : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ according to

$$(\mathscr{S} f)(\eta) = \sum_{x\in\mathbb{Z}} \left[ f(\eta^{x,x+1}) - f(\eta) \right].$$

Observe that the noise conserves the energy, destroys the momentum and the deformation conservation laws but conserves $\sum_x \eta_x = \sum_x (p_x + r_x)$, which as explained above is the quantity (that we call the "volume" to follow the terminology used in [3]) responsible of the anomalous diffusion of energy. Since we have now only two conserved quantities (the energy and the volume), the Gibbs states of the perturbed dynamics are given by $\{\mu_{\beta,\lambda,\lambda}\}_{\beta>0,\lambda}$ or equivalently by $\{\nu_{e,\pi,\pi}\,;\, e > 0, \pi\}$. The normalized energy current is given by

$$\hat{j}^{e}_{x,x+1}(\eta) = -2V'(\eta_x)V'(\eta_{x+1}) + 2\tau^2 + 2\partial_e(\tau^2)\,(2V(\eta_x) - e) + 2\partial_\pi(\tau^2)\,(\eta_x - \pi)$$

with $\tau := \tau(e,\pi) = \int V'(\eta_x) d\nu_{e,\pi,\pi}$.

**Theorem 5 ([3]).** *Let $(e,\pi) \in (0,+\infty) \times \mathbb{R}$ such that $\nu_{e,\pi,\pi}$ is well defined. Consider the dynamics with generator $\mathscr{L} = \mathscr{A}_{exp} + \gamma\mathscr{S}$, $\gamma > 0$, where*

$$\mathscr{A}_{exp} = \sum_x (V'(\eta_{x+1}) - V'(\eta_{x-1}))\partial_{\eta_x}, \tag{14}$$

*and $V(z) = e^{-z} + z - 1$. Then there exists a constant $c > 0$ such that for any $z > 0$*

$$c z^{-1/4} \le \ll \hat{j}^e_{0,1}, (z - \mathscr{L})^{-1} \hat{j}^e_{0,1} \gg_{e,\pi,\pi} \le c^{-1} z^{-1/2}.$$

*It follows that the Green-Kubo formula of the energy transport coefficient yields an infinite value.*

We expect that the system above belongs to the KPZ class so that $\ll \hat{j}^e_{0,1}, (z - \mathscr{L})^{-1} \hat{j}^e_{0,1} \gg_{e,\pi,\pi}$ should diverge like $z^{-1/3}$. In the present state of the art no robust technique is available to show such result apart from the non-rigorous (but powerful) mode-coupling theory [13, 16, 18]. A second open problem is to generalize the previous theorem to other interaction potentials $V$. Numerical simulations have been reported in [6].

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## References


1. Basile, G., Bernardin, C., Olla, S.: Momentum conserving model with anomalous thermal conductivity in low dimensional systems. Phys. Rev. Lett. **96**, 204303 (2006)
2. Basile, G., Bernardin, C., Olla, S.: Thermal conductivity for a momentum conserving model. Commun. Math. Phys. **287**(1), 67–98 (2009)
3. Bernardin, C., Gonąlves, P.: Anomalous fluctuations for a perturbed Hamiltonian system with exponential interactions, to appear in Commun. Math. Phys. **325**(1), 291–332 (2013)
4. Bernardin, C., Olla, S.: Transport properties of a chain of anharmonic oscillators with random flip of velocities. J. Stat. Phys. **145**, (2011)
5. Bernardin, C., Olla, S.: Non-equilibrium macroscopic dynamics of chains of anharmonic oscillators, in preparation, available at http://perso.ens-lyon.fr/cedric.bernardin/Files/springs.pdf
6. Bernardin, C., Stoltz, G.: Anomalous diffusion for a class of systems with two conserved quantities. Nonlinearity **25**(4), 1099–1133 (2012)
7. Bonetto, F., Lebowitz, J.L., Rey-Bellet, L.: Fourier's law: a challenge to theorists. In: Fokas, A. et al. (eds.) Mathematical Physics 2000, pp. 128–150. Imperial College Press, London (2000)
8. Dhar, A.: Heat transport in low-dimensional systems. Adv. Phys. **57**, 457 (2008)
9. Fritz, J., Funaki, T., Lebowitz, J.L.: Stationary states of random Hamiltonian systems. Probab. Theory Relat. Fields **99**, 211–236 (1994)
10. Gerschenfeld, A., Derrida, B., Lebowitz, J.L.: Anomalous Fourier's law and long range correlations in a 1D non-momentum conserving mechanical model. J. Stat. Phys. **141**, 757–766 (2010)
11. Lebowitz, J.L., Penrose, O.: Modern ergodic theory. Phys. Today **26**, 155–175 (1973)
12. Lepri, S., Livi, R., Politi, A.: Thermal conduction in classical low-dimensional lattices. Phys. Rep. **377**, 1–80 (2003)
13. Mendl, C.B., Spohn, H.: Dynamic correlators of FPU chains and nonlinear fluctuating hydrodynamics (2013), arXiv:1305.1209
14. Olla, S., Varadhan, S.R.S., Yau, H.T.: Hydrodynamic limit for a Hamiltonian system with weak noise. Commun. Math. Phys. **155**, 523–560 (1993)
15. Spohn, H.: Large Scale Dynamics of Interacting Particles. Springer, Berlin/New York (1991)
16. Spohn, H.: Nonlinear fluctuating hydrodynamics for anharmonic chains (2013), arXiv:1305.6412
17. Szász, D.: Boltzmann's ergodic hypothesis, a conjecture for centuries? In: Hard Ball Systems and the Lorentz Gas. Encyclopaedia Mathematical Science. vol. 101, 421–448. Springer, Berlin/London (2000)
18. van Beijeren, H.: Exact results for anomalous transport in one dimensional Hamiltonian systems. Phys. Rev. Lett. **28**, 180601 (2012)
19. Yau, H.T.: Relative entropy and hydrodynamics of Ginzburg-Landau models. Lett. Math. Phys. **22**(1), 63–80 (1991)

# Equilibrium Fluctuations of Additive Functionals of Zero-Range Models

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## 1 Introduction and Model Assumptions

We consider zero-range processes which follow a collection of random walks interacting on $\mathbb{Z}^d$ in the following way: When there are $k$ particles at a location $x$, one of them displaces by $y$ with rate $[g(k)/k]p(y)$. Here, $g : \{0, 1, 2, \dots\} \to \mathbb{R}_+$ is a prescribed function such that $g(0) = 0$ and $g(k) > 0$ for $k \geq 1$, and $p$ is a jump probability on $\mathbb{Z}^d$. Another way to think of the process is that each location $x \in \mathbb{Z}^d$ has a clock which rings at rate $g(k)$ where $k$ is the particle number at $x$. Once the clock rings at $x$, a particle selected at random displaces by $y$ with probability $p(y)$. This well-studied model has been used in the modeling of traffic, queues, granular media, fluids, etc. [2], and also includes the case of independent random walks when $g(k) \equiv k$.

In this note, we study the equilibrium fluctuations of additive functionals in a class of zero-range processes, namely those which are 'asymmetric', 'attractive', and for which a 'spectral gap' estimate holds. When the model is 'symmetric', the fluctuation behaviors are found in [15] and [12]. Also, part of the motivation of this

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note is to complement the much detailed work on fluctuations in simple exclusion processes (cf. Chap. 5 of [6]), as much less is known for zero-range systems. The arguments make use of a combination of techniques in the literature. We now define more carefully the model and related terms.

## 1.1 Jump Rates and Construction

We will assume that the function $g$ allows motion: $g(0) = 0$ and $g(k) > 0$ for $k \geq 1$ and the following two conditions.

(LIP) There is a constant $K$ such that $|g(k+1) - g(k)| < K$ for $k \geq 0$
(INC) $g$ is increasing: $g(k+1) \geq g(k)$ for $k \geq 0$.

Condition (LIP) is usually assumed in order to construct the process on an infinite lattice. However, although also often assumed, (INC) is a more technical condition which makes available a certain 'basic coupling' that we will use later.

Assume also that $p$ is such that the symmetrization $s(x) = (p(x) + p(-x))/2$ is irreducible and

(FR) $p$ is finite-range: There is an $R$ such that $p(z) = 0$ for $|z| > R$.

Here, $|z| = \max\{|z_i| : i = 1, \ldots, d\}$. The assumption (FR) might presumably be relaxed in favor of a $p$ with rapidly diminishing tail behavior, although we do not consider this case or when $p$ might have heavy tails where certainly the results would differ.

Under these conditions, more restrictive than necessary, the zero-range system $\eta_t = \{\eta_t(x) : x \in \mathbb{Z}^d\}$ can be constructed as a Markov process on the state space $\Omega := \mathbb{N}_0^{\mathbb{Z}^d}$ with generator $L$ whose action on local functions is given by

$$Lf(\eta) = \sum_{x,y \in \mathbb{Z}^d} g(\eta(x))p(y)\big[f(\eta^{x,x+y}) - f(\eta)\big].$$

Here, $\eta_t(x)$ is the occupation number at $x$ at time $t$, $\eta^{x,x+y}$ is the state obtained from $\eta$ by decreasing and increasing the occupation numbers at $x$ and $y$ by one respectively, and a local function is one which depends only on a finite number of coordinates $\{\eta(z)\}$. See [1] for the construction and weakening of the assumptions.

We will say that the system is symmetric, mean-zero asymmetric or asymmetric with drift if $p$ satisfies $p(z) = p(-z)$, $\sum_{z \in \mathbb{Z}^d} zp(z) = 0$ but $p$ is not symmetric, and $\sum_{z \in \mathbb{Z}^d} zp(z) \neq 0$ respectively.

## 1.2 Invariant Measures

Part of the appeal of zero-range processes is that they possess a family of invariant measures which are fairly explicit product measures. For $\alpha \geq 0$, define

$$\mathcal{Z}(\alpha) := \sum_{k\geq 0} \frac{\alpha^k}{g(k)!}$$

where $g(k)! = g(1)\cdots g(k)$ for $k \geq 1$ and $g(0)! = 1$. Let $\alpha^*$ be the radius of convergence of this power series and notice that $\mathcal{Z}$ increases on $[0, \alpha^*)$. Fix $0 \leq \alpha < \alpha^*$ and let $\bar{\nu}_\alpha$ be the product measure on $\mathbb{N}^{\mathbb{Z}}$ whose marginal at the site $x$ is given by

$$\bar{\nu}_\alpha\{\eta : \eta(x) = k\} = \begin{cases} \dfrac{1}{\mathcal{Z}(\alpha)} \dfrac{\alpha^k}{g(k)!} & \text{when } k \geq 1 \\ \dfrac{1}{\mathcal{Z}(\alpha)} & \text{when } k = 0. \end{cases}$$

We may reparametrize these measures in terms of the 'density'. Let $\rho(\alpha) := E_{\bar{\nu}_\alpha}[\eta(0)] = \alpha\mathcal{Z}'(\alpha)/\mathcal{Z}(\alpha)$. By computing the derivative, we obtain that $\rho(\alpha)$ is strictly increasing on $[0, \alpha^*)$. Then, let $\alpha(\cdot)$ denote its inverse. Now, we define

$$\nu_\rho(\cdot) := \bar{\nu}_{\alpha(\rho)}(\cdot),$$

so that $\{\nu_\rho : 0 \leq \rho < \rho^*\}$ is a family of invariant measures parameterized by the density. Here, $\rho^* = \lim_{\alpha\uparrow\alpha^*} \rho(\alpha)$, which may be finite or infinite depending on whether $\lim_{\alpha\to\alpha^*} \mathcal{Z}(\alpha)$ converges or diverges. In this notation, $\alpha(\rho) = E_{\nu_\rho}[g(\eta(0)]$ is a 'fugacity' parameter.

One may check that the measures $\{\nu_\rho : 0 \leq \rho < \rho^*\}$ are invariant for the zero-range process [1]. Moreover, we remark, by the construction in [14], which extends the construction in [1] to an $L^2(\nu_\rho)$ process, we have that $L$ is a Markov $L^2(\nu_\rho)$ generator whose core can be taken as the space of all local $L^2(\nu_\rho)$ functions. Also, one may compute that the adjoint $L^*$ is the zero-range process with jump probability $p^*(z) = p(-z)$ for $z \in \mathbb{Z}^d$. The operator

$$S = (L + L^*)/2$$

may be seen as the generator for the symmetrized process with jump law $s$. In particular, when $p$ is symmetric, the process is reversible with respect to $\nu_\rho$ with generator $L = L^* = S = (L + L^*)/2$.

It is also known that the family $\{\nu_\rho : \rho < \rho^*\}$ are all extremal invariant measures, and hence when the process is started from one of them, the system will be ergodic with respect to time shifts. Let us now fix one of these invariant measures $\nu_\rho$ throughout the article.

### 1.3 Spectral Gap

To state results, we will need to detail the spectral gap properties of the system. For $\ell \geq 1$, let $\Lambda_\ell = \{x \in \mathbb{Z}^d : |x| \leq \ell\}$. Consider the 'symmetrized' process restricted to $\Lambda_\ell$ with generator

$$S_\ell f(\eta) = \sum_{\substack{|x|\leq \ell, |y|=1 \\ |x+y|\leq \ell}} s(y) g(\eta(x)) \big[f(\eta^{x,x+y}) - f(\eta)\big].$$

Given the number particles in $\Lambda_\ell$, say $\sum_{|x|\leq \ell} \eta(x) = M$, one can verify that the process is a reversible finite-state Markov chain with unique invariant measure $\nu_\rho\big(\cdot \mid \sum_{|x|\leq\ell} \eta(x) = M\big)$ (which does not depend on $\rho$). Hence, since the chain is irreducible, there is a gap in the spectrum of $S_\ell$ between the eigenvalue 0 and the next one which is strictly negative. Let $W(M, \ell)$ be the reciprocal of the absolute value of this 'spectral gap'. Also, $W(M, \ell)$ can be captured as the smallest constant $c$ such that the Poincaré inequality, $E_{\nu_\rho}[f^2] \leq c E_{\nu_\rho}[f(-S_\ell f)]$, holds for all local mean-zero function $f$, $E_{\nu_\rho}[f] = 0$.

We will assume that the following estimate holds:

(G) There is a constant $C = C(\rho)$ such that for all $\ell \geq 1$, we have

$$E_{\nu_\rho}\Big[W^2\Big(\sum_{|x|\leq \ell} \eta(x), \ell\Big)\Big] \leq C(\rho)\ell^4.$$

Such an estimate is a further condition on $g$ and $p$ and holds in a number of cases. Usually, one tries to bound the spectral gap for the corresponding nearest-neighbor process. Given assumption (FR), by comparing the associated Poincaré inequalities, the order of $W(M, \ell)$, asymptotically in $\ell$, with respect to $S_\ell$ and the nearest-neighbor version will be the same.

- If $g$ is not too different from the independent case, that is $g(x) \equiv x$, for which the gap is of order $O(\ell^{-2})$ uniform in $x$, one expects similar behavior as for a single particle. This has been proved for $d \geq 1$ in [7] under assumptions (LIP) and

  (U) There exists $x_0 \geq 1$ and $\varepsilon_0 > 0$ such that $g(x + x_0) - g(x) \geq \varepsilon_0$ for all $x \geq 0$.

- If $g$ is sublinear, that is $g(x) = x^\gamma$ for $0 < \gamma < 1$, then it has been shown that the spectral gap depends on the number of particles $k$, namely the gap for $d \geq 1$ is $O((1 + \beta)^{-\gamma}\ell^{-2})$ where $\beta = k/(2\ell + 1)^d$ [10].
- If $g(x) = \mathbf{1}_{x\geq 1}$, then it has been shown in $d \geq 1$ that the gap is $O((1+\beta)^{-2}\ell^{-2})$ where $\beta = k/(2\ell + 1)^d$ [9]. In $d = 1$, this is true because of the connection between the zero-range and simple exclusion processes for which the gap estimate is well-known [11]: The number of spaces between consecutive particles

in simple exclusion correspond to the number of particles in the zero-range process.

In all these cases, (G) follows readily by straightforward moment calculations.

### *1.4 Attractivity*

A main technical tool we will use is the 'basic coupling' for interacting particle systems. Such a coupling is available for zero-range processes when $g$ is an increasing function.

Namely, consider two zero-range systems $\eta_t^0$ and $\eta_t^1$ started from initial configurations $\eta_0^0 = \eta^0$ and $\eta_0^1 = \eta^1$ such that $\eta^0(x) \leq \eta^1(x)$ for all $x \in \mathbb{Z}^d$. Then, one can couple the two systems so that whenever a '0' particle moves, a corresponding '1' particle makes the same jump. That is, a particle at $x$ in the '0' and '1' systems displaces by $y$ with rate $g(\eta^0(x))p(y)$, and also with rate $[g(\eta^1(x))-g(\eta^0(x))]p(y)$ one of the particles at $x$ in the '1' system displaces by $y$. In particular, one can write $\eta_t^1 = \eta_t^0 + \xi_t$. Here, $\xi_t$ counts the 'second-class' or 'discrepancy' particles: $\xi_t(x)$ is the number of second-class particles at $x$ at time $t$. See [8] for more details.

## 2 Results

By an additive functional, we mean the time integral of a local function $f$ with respect to the zero-range process:

$$A_f(t) = \int_0^t f(\eta_s)ds.$$

Since $\nu_\rho$ is extremal, as alluded to earlier, the ergodic theorem captures the law of large numbers behavior

$$\lim_{t\to\infty} \frac{1}{t} A_f(t) \;=\; E_{\nu_\rho}[f].$$

In this context, the results of this note are on the second-order terms, the fluctuations of $A_f(t)$ about its mean. Let $\bar{f} = f - E_{\nu_\rho}[f]$ and

$$\sigma_t^2(\rho, f) \;=\; E_{\nu_\rho}\Big[A_{\bar{f}}(t)^2\Big]$$

be the variance at time $t$. One can compute $\sigma_t^2(\rho, f)$, using stationarity, as follows:

$$\sigma_t^2(\rho, f) = 2\int_0^t (t-s)E_{\nu_\rho}[\bar{f}(\eta_s)\bar{f}(\eta_0)]ds$$
$$= 2\int_0^t (t-s)E_{\nu_\rho}[\bar{f}(\eta_0)(P_s\bar{f})(\eta_0)]ds$$

where $P_t$ is the semigroup of the process.

One of the main questions is to understand the order of the variance $\sigma_t^2(\rho, f)$ as $t \uparrow \infty$. Perhaps, surprisingly, this order may or may not be diffusive, that is of order $t$, depending on the function $f$. When the limit exists, we denote

$$\sigma^2(\rho, f) \;:=\; \lim_{t\uparrow\infty} \frac{1}{t}\sigma_t^2(\rho, f).$$

To explore this point, consider the occupation function $h(\eta) = 1(\eta(0) \geq 1)$ which indicates when the origin is occupied. Then, $A_{\bar{h}}(t)$ is the centered occupation time of the origin up to time $t$. When the jump probability $p$ is symmetric, particles tend to stay put more and in $d \leq 2$, when $p$ is recurrent, $A_h(t)$ is quite volatile and the variance $\sigma_t^2(\rho, h)$ is super-diffusive. However, in the transient case, $d \geq 3$, the behavior is more regular and $A_h(t)$ has a diffusive variance.

On the other hand, if a function $b$ is somewhat 'smooth', say the difference $b(\eta) = \eta(0) - \eta(1)$ which casts $A_{\bar{b}}(t)$ as the difference of two additive functionals, then one might suspect the variance to be less volatile than under $h$. This is indeed the case, and in all dimensions $d \geq 1$, $\sigma_t^2(\rho, b)$ is diffusive.

This phenomenon is summarized by the following result which is Theorem 1.2 of [15]. We say a local function $f$ is admissible if

$$\limsup_{t\uparrow\infty} \frac{1}{t}\sigma_t^2(\rho, f) \;<\; \infty$$

and not admissible otherwise.

**Proposition 1 (Theorem 1.2 [15]).** *Suppose assumptions (LIP), (FR), (G) hold, and in addition suppose $p$ is symmetric so that the zero-range process is reversible. Let $f \in L^4(\nu_\rho)$ be a local function supported on coordinates in $\Lambda_\ell$. Then, starting from $\nu_\rho$, $f$ is admissible if and only if*

$$E_{\nu_\rho}[f] = E_{\nu_\rho}\Big[f(\eta)\sum_{x\in\Lambda_\ell}\eta(x)\Big] = E_{\nu_\rho}\Big[f(\eta)\Big(\sum_{x\in\Lambda_\ell}\eta(x)\Big)^2\Big] = 0 \text{ in dimension } d=1$$

$$E_{\nu_\rho}[f] = E_{\nu_\rho}\Big[f(\eta)\sum_{x\in\Lambda_\ell}\eta(x)\Big] = 0 \text{ in dimension } d=2$$

$$E_{\nu_\rho}[f] = 0 \text{ in dimension } d \geq 3.$$

Motivated by the proposition, we will say a mean-zero function local $f$ supported on coordinates in $\Lambda_\ell$ has degree $n \geq 0$ if

$$E_{\nu_\rho}\big[f(\eta)\big(\sum_{x\in\Lambda_\ell}\eta(x)\big)^n\big] \neq 0$$

but

$$E_{\nu_\rho}\big[f(\eta)\big(\sum_{x\in\Lambda_\ell}\eta(x)\big)^r\big] = 0 \quad \text{when} \quad r < n.$$

Let $\tilde{f}(y) = E_{\nu_y}[f]$. Since

$$E_{\nu_y}[f] \;=\; \frac{1}{E_{\nu_\rho}[e^{\lambda(y)\eta(0)}]^{|\Lambda_\ell|}} E_{\nu_\rho}\Big[f(\eta)e^{\lambda(y)\sum_{x\in\Lambda_\ell}\eta(x)}\Big]$$

$f$ is of degree $n$ exactly when

$$d^n/dy^n\,\tilde{f}(\rho) \neq 0 \quad \text{but} \quad d^r/dy^r\,\tilde{f}(\rho) = 0 \quad \text{when} \quad r < n.$$

We remark that when $p$ is symmetric, the limiting variance can be computed from monotone convergence since

$$E_{\nu_\rho}[\bar{f}P_s\bar{f}] = E_{\nu_\rho}[(P_{s/2}\bar{f})^2] \geq 0$$

then

$$\begin{aligned}\sigma^2(\rho, f) &= \lim_{t\uparrow\infty} 2\int_0^t (1 - s/t)E_{\nu_\rho}\big[\big(P_{s/2}\bar{f}\big)^2\big]ds \\ &= 2\int_0^\infty E_{\nu_\rho}[\bar{f}(\eta_0)\bar{f}(\eta_s)]ds \;:=\; \sigma^2(\rho, f).\end{aligned}$$

One can see from the formula that $\sigma^2(\rho, f) > 0$ in the symmetric case.

To relate the limiting variance to so-called $H_{-1,\lambda,L}$ resolvent norms of $f$, define

$$\begin{aligned}\|f\|^2_{-1,\lambda,L} &:= E_{\nu_\rho}\big[\bar{f}, (\lambda - L)^{-1}\bar{f}\big] \\ &= \int_0^\infty e^{-\lambda s}E_{\nu_\rho}\big[\bar{f}P_s\bar{f}\big]ds.\end{aligned}$$

Also, when the limit exists, define the $H_{-1,L}$ norm of $f$ by

$$\|f\|_{-1,L} \;:=\; \lim_{\lambda\downarrow 0}\|f\|_{-1,\lambda,L}.$$

In the symmetric case, when $L = S$, we will call

$$\|f\|_{-1,\lambda} := \|f\|_{-1,\lambda,S}$$

and

$$\|f\|_{-1} := \|f\|_{-1,S}.$$

Then, for the general process,

$$\begin{aligned}\|f\|^2_{-1,\lambda,L} &= \lambda^2 \int_0^\infty e^{-\lambda t} \int_0^t \int_0^s E_{\nu_\rho}[\bar{f} P_u \bar{f}] du\, ds\, dt \\ &= \frac{\lambda^2}{2} \int_0^\infty e^{-\lambda t} \sigma_t^2(\rho, f) dt. \end{aligned} \tag{1}$$

In the symmetric case, as

$$E_{\nu_\rho}\big[\bar{f} P_s \bar{f}\big] = E_{\nu_\rho}\big[\big(P_{s/2}\bar{f}\big)^2\big] \geq 0,$$

we observe the limiting variance is well-defined (although possibly infinite):

$$\begin{aligned}\sigma^2(\rho, f) &= 2\int_0^\infty E_{\nu_\rho}[\bar{f}(\eta_0)\bar{f}(\eta_s)] ds \\ &= 2 \lim_{\lambda \downarrow 0} \|f\|^2_{-1,\lambda} \;=\; 2\|f\|^2_{-1}.\end{aligned}$$

We remark, in the $H_{-1}$ notation, the admissibility conditions for a function $f$ in Proposition 1 are equivalent to $\|f\|_{-1} < \infty$. Although we will not need it, but to complete the discussion, we note $\|f\|_{-1}$ is often represented (cf. Chaps. 2 and 5 of [6]) as

$$\|f\|^2_{-1} \;=\; \sup_{\phi \text{ local}} \left\{ \frac{E_{\nu_\rho}[\bar{f}\phi]}{D_{\nu_\rho}(\phi)^{1/2}} \right\}$$

where the Dirichlet form

$$D_{\nu_\rho}(\phi) = E_{\nu_\rho}[\phi(-S\phi)] = \frac{1}{4} \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} E_{\nu_\rho}\big[g(\eta(x))\big(\phi(\eta^{x,x+y}) - \phi(\eta)\big)^2\big] s(y).$$

On the other hand, when $p$ is asymmetric, the limiting variance $\sigma^2(\rho, f)$ can be shown to exist for certain functions. We say that $f$ is coordinatewise increasing if $\eta(x) \leq \zeta(x)$ for all $x \in \mathbb{Z}^d$ then $f(\eta) \leq f(\zeta)$. For such a function,

$$P_s f(\eta) = E[f(\eta(s)) | \eta(0) = \eta]$$

is itself a coordinatewise increasing function: When $\xi \leq \zeta$ coordinatewise, let $\beta(x) = \zeta(x) - \xi(x)$ for $x \in \mathbb{Z}^d$. Let $\eta_t^0$ and $\eta_t^1$ be processes starting in $\xi$ and $\zeta$ respectively. Then, by the basic coupling, $\eta_t^1 = \eta_t^0 + \beta_t$ where $\beta_t$ follows second-class particles. In particular, as $\bar{f}$ is increasing, with respect to the coupling measure $\hat{P}$,

$$P_s \bar{f}(\zeta) - P_s \bar{f}(\xi) = \hat{E}[\bar{f}(\eta_s^1) - \bar{f}(\eta_s^0 + \beta_s)] \geq 0.$$

Then, for nontrivial coordinatewise increasing functions, $E_{\nu_\rho}[\bar{f} P_s \bar{f}] > 0$ as $\nu_\rho$, being a product measure, is a FKG measure. Therefore, the limit

$$\sigma^2(\rho, f) = 2 \int_0^t E_{\nu_\rho}[\bar{f} P_s \bar{f}] ds > 0$$

for such functions.

For the general process, when $f$ is admissible, it is natural to ask if a functional central limit theorem holds for the diffusively scaled additive functional. When $p$ is symmetric, by the Kipnis-Varadhan CLT for reversible Markov processes, this is indeed the case [5, 15] and the limit in the uniform topology is a Brownian motion with diffusion coefficient $\sigma^2(\rho, f)$. Moreover, for nearest-neighbor systems and a class of functions $f$ in $d \leq 2$ such that $\tilde{f}'(\rho) \neq 0$, which are not admissible, the super-diffusive orders of $\sigma_t^2(\rho, f)$ and the limit laws of $A_{\bar{f}}(t)$ scaled by the standard deviation have also been found [12]. To give an example, in dimension $d = 1$ when $E_{\nu_\rho}[f] = 0$ and $E_{\nu_\rho}[f \sum_{x \in \Lambda_\ell} \eta(x)] \neq 0$, the limit law is a fractional Brownian motion with Hurst parameter 3/4. See [12] for the full statements.

The purpose of this note is to understand the fluctuation behaviors under mean-zero asymmetric and asymmetric with drift zero-range processes. When $p$ is mean-zero, we will show that the generator $L$ satisfies a 'sector inequality'. As a consequence, by the method in [16], the variance behaviors in terms of orders are the same as if the process were symmetric. When $f$ is admissible, the limit

$$\sigma^2(\rho, f) = \lim_{t \uparrow \infty} \frac{1}{t} E_{\nu_\rho}\left[A_{\bar{f}}(t)^2\right]$$

converges, and the diffusively scaled additive functional still tends to a Brownian motion.

**Theorem 1.** *Suppose g and p satisfy assumptions (LIP), (FR), (G), and in addition suppose p is mean-zero. Let $f$ be a local function supported on coordinates in $\Lambda_\ell$. Then, $f$ is admissible if and only if the conditions in Proposition 1 are met.*

*In the case, $f$ is admissible, the limiting variance $\sigma^2(\rho, f)$ converges and we have, in the uniform topology,*

$$\lim_{\lambda \uparrow \infty} \frac{1}{\sqrt{\lambda}} A_{\bar{f}}(\lambda t) \Rightarrow \sigma(\rho, f) B_t$$

*where $B_t$ is the standard Brownian motion on $\mathbb{R}$.*

*Remark 1.* In the mean-zero case, for inadmissible $f$, it remains open to derive the limit laws under the appropriate scalings.

When the system is asymmetric with drift, one might have the intuition that the admissibility of a function should follow what happens in the 'transient' regime in the symmetric case. With the additional assumption of attractivity, this is indeed the case.

**Theorem 2.** *Suppose assumptions (LIP), (INC) (FR), (G) hold, and in addition suppose $p$ is asymmetric with drift. Let $f$ be a local function supported on coordinates in $\Lambda_\ell$. Then, $f$ is admissible if and only if $E_{\nu_\rho}[f] = 0$.*

*When, $f$ is an increasing function, the limiting variance exists and is finite, $\sigma^2(\rho, f) < \infty$ and in the uniform topology*

$$\lim_{\lambda\uparrow\infty} \frac{1}{\sqrt{\lambda}} A_{\bar{f}}(\lambda t) \;\Rightarrow\; \sigma(\rho, f) B_t.$$

*Remark 2.* When $f$ is the difference of two increasing functions, $\sigma^2(\rho, f)$ exists and the last statement of the theorem holds (see [13]). However, for more general $f$, in the asymmetric with drift case, this is an open question.

## 3 Proof of Theorem 1: Mean-Zero Dynamics

The main step is the following sector inequality, whose proof is deferred to the end of the section. Recall the definition of the Dirichlet form $D_{\nu_\rho}(\phi)$.

**Proposition 2.** *Under the assumptions of Theorem 1, there is a constant $C = C(\rho, p, d)$ such that for local functions $\phi, \psi : \Omega \to \mathbb{R}$ we have*

$$|E_{\nu_\rho}[\phi L \psi]| \;\le\; C D_{\nu_\rho}(\phi)^{1/2} D_{\nu_\rho}(\psi)^{1/2}.$$

When the process is symmetric, since $S$ is a nonpositive operator, a sector inequality trivially holds by Schwarz inequality and the constant $C = 1$.

*Proof of Theorem 1.* The main argument follows the argument in [13] for mean-zero simple exclusion processes, which compares $H_{-1}$ norms with respect to $L$ and the symmetrized generator $S$.

As in Lemma 4.4 of [13], we have for a constant $C_1 > 0$ that

$$C_1^{-1} \|f\|_{-1,\lambda} \;\le\; \|f\|_{-1,\lambda,L} \;\le\; C_1 \|f\|_{-1,\lambda}.$$

Next, when the sector inequality in Proposition 2 holds, as computed in [16], the limit exists,

$$\sigma^2(\rho, f) = \lim_{t\uparrow\infty} t^{-1}\sigma_t^2(\rho, f) = 2\lim_{\lambda\downarrow 0} \|f\|_{-1,\lambda,L} = 2\|f\|_{-1,L}.$$

Moreover,

$$\|f\|_{-1,L} = \lim_{\lambda\downarrow 0} \|f\|_{-1,\lambda,L} \;\le\; C_1 \lim_{\lambda\downarrow 0} \|f\|_{-1,\lambda} \;=\; C_1\|f\|_{-1}.$$

Then, given that $f$ satisfies the admissibility conditions in Theorem 1, we have

$$\|f\|_{-1,L} \le C_1\|f\|_{-1} < \infty.$$

Conversely, suppose $f$ does not satisfy the admissibility conditions in Theorem 1, and $\sup_{t>0} t^{-1}\sigma_t^2(\rho, f) \;\le\; C_2$. Then, (1) is bounded by $C_2\int_0^\infty e^{-u}u\,du$ uniformly in $\lambda$. Hence, $\|f\|_{-1,\lambda}$ is uniformly bounded in $\lambda$. But, a contradiction arises as then

$$\lim_{\lambda\downarrow 0} \|f\|_{-1,\lambda} = \|f\|_{-1} < \infty,$$

which means $f$ is admissible. Therefore,

$$\limsup_{t\uparrow\infty} t^{-1}\sigma_t^2(\rho, f) = \infty.$$

Finally, the functional CLT follows exactly the same proof given in [16] for mean-zero simple exclusion processes. □

*Proof of Proposition 2.* Since $p$ is mean-zero and finite-range, it decomposes into a finite number of 'irreducible cycles' by Lemma 5.3 of [16]. That is,

$$p = \sum_{i=1}^{r} \alpha_i \pi_i$$

where $\pi_i$ places weight $1/k$ on $k$ points $a_1, \ldots, a_k$ such that $a_1 + \cdots + a_k = 0$ and $y_0 = 0$,

$$\{y_\ell = \sum_{j=1}^{\ell} a_j : 1 \le \ell \le k\}$$

have no double points. We call the $B_i = \{0, y_1, \ldots, y_k\}$ as the $i$th cycle. For example $a_1 = -1$, $a_2 = 2$ and $a_3 = -1$ corresponding to $y_0 = 0$, $y_1 = -1$, $y_2 = 1$ and $y_3 = 0$ is an irreducible cycle.

Let $A_B$ be the zero-range process on the cycle $\{0, y_1, \ldots, y_k\}$ with jump probability $\pi_B$ where $\pi_B(a_i) = 1/k$ for $1 \le i \le k$. Then,

$$L_B = \sum_{x \in \mathbb{Z}^d} A_{B+x}$$

and

$$L = \sum_{i=1}^{r} \alpha_i L_{B_i}$$

(cf. [16], Lemma 5.4).

It is enough to show the sector inequality with respect to $L_B$ for a specific cycle $B$. Indeed, if such a sector inequality holds, by a Schwarz inequality, we can write

$$\begin{aligned} E_{\nu_\rho}\Big[\phi \sum_{i=1}^{r} \alpha_i L_{B_i} \psi\Big] &= \sum_{i=1}^{r} \alpha_i E_{\nu_\rho}\Big[\phi L_{B_i} \psi\Big] \\ &\leq \sum_{i=1}^{r} \alpha_i C_i D^i_{\nu_\rho}(\phi)^{1/2} D^i_{\nu_\rho}(\psi)^{1/2} \\ &\leq C \sum_{i=1}^{r} \alpha_i \Big(\frac{\epsilon}{2} D^i_{\nu_\rho}(\phi) + \frac{1}{2\epsilon} D^i_{\nu_\rho}(\psi)\Big) \\ &= C \Big(\frac{\epsilon}{2} \sum_{i=1}^{r} \alpha_i D^i_{\nu_\rho}(\phi) + \frac{1}{2\epsilon} \sum_{i=1}^{r} \alpha_i D^i_{\nu_\rho}(\psi)\Big) \end{aligned}$$

where $D^i_{\nu_\rho}$ is the Dirichlet form with respect to $S_{B_i}$, the symmetrization of $L_{B_i}$, $C = \max_{i=1,\cdots,r}\{C_i\}$ and $\epsilon > 0$. Taking the infimum over $\epsilon > 0$, allows to bound the left-hand side by

$$C D_{\nu_\rho}(\phi)^{1/2} D_{\nu_\rho}(\psi)^{1/2}.$$

Moreover, it will be enough to show the sector inequality with respect to $A_B$. Indeed, if so, by the same Schwarz inequality as above, we can write

$$E_{\nu_\rho}\Big[\phi \sum_{x \in \mathbb{Z}^d} A_{B+x} \psi\Big] \leq C_B \Big(\frac{\epsilon}{2} \sum_{x \in \mathbb{Z}^d} D^x_{\nu_\rho}(\phi) + \frac{1}{2\epsilon} \sum_{x \in \mathbb{Z}^d} D^x_{\nu_\rho}(\psi)\Big)$$

where $D^x_{\nu_\rho}$ is the Dirichlet form with respect to symmetrization of $A_{B+x}$. Since no bond $(z, w)$ is double counted,

$$\sum_{x \in \mathbb{Z}^d} D^x_{\nu_\rho}(\phi) = D^B_{\nu_\rho}(\phi)$$

where $D^B_{\nu_\rho}$ is the Dirichlet form with respect to $L_B$.

Following the scheme in [16], we now write, with $y_{k+1} = 0$, that

$$E_{\nu_\rho}\Big[\phi A_B \psi\Big] = \frac{1}{k}\sum_{i=0}^{k} E_{\nu_\rho}\Big[\phi(\eta)\cdot g(\eta(y_i))\big(\psi(\eta^{y_i,y_{i+1}}) - \psi(\eta)\big)\Big]$$
$$= \alpha(\rho)\sum_{i=0}^{k} E_{\nu_\rho}\Big[\phi(\eta+\delta(y_i))\big(\psi(\eta+\delta(y_{i+1})) - \psi(\eta+\delta(y_i))\big)\Big].$$

Here, $\delta(a)$ is the configuration which puts exactly one particle at location $a$. We have used the identity

$$E_{\nu_\rho}[g(\eta(a))h(\eta)] = \alpha(\rho)E_{\nu_\rho}[h(\eta+\delta(a))]$$

where $\alpha(\rho)$ is the fugacity mentioned in the introduction.

Now, since the sum

$$\sum_{i=0}^{k}\psi(\eta+\delta(y_i)) - \psi(\eta+\delta(y_{i+1})) = 0,$$

the right-hand side equals

$$\alpha(\rho)\sum_{i=0}^{k} E_{\nu_\rho}\Big[\Big(\phi(\eta+\delta(y_i)) - \phi(\eta+\delta(0))\Big)\Big(\psi(\eta+\delta(y_{i+1})) - \psi(\eta+\delta(y_i))\Big)\Big].$$

Note that

$$\alpha(\rho)E_{\nu_\rho}\Big[\Big(\phi(\eta+\delta(y_i)) - \phi(\eta+\delta(0))\Big)^2\Big]$$
$$\leq\; k\sum_{i=1}^{k} E_{\nu_\rho}\Big[g(\eta(y_i))\Big(\phi(\eta^{y_i,y_{i+1}}) - \phi(\eta)\Big)^2\Big].$$

Then, the sector inequality for $A_B$ follows from Schwarz inequality with a constant depending on the length of the cycle $k$. □

## 4 Proof of Theorem 2: Asymmetric with Drift Dynamics

We will make use of the following results to prove Theorem 2.

**Proposition 3.** *Suppose that assumption (LIP) holds and $f$ is a local function which is mean-zero, $E_{\nu_\rho}[f] = 0$. Then,*

$$\sigma_t^2(\rho, f) \;\le\; 2t \| f \|_{-1}^2 .$$

A proof of the proposition can be found in Appendix 1.6 of [4].

**Proposition 4.** *Under the assumptions of Theorem 2, we have that*

$$f_1(\eta) = g(\eta(x)) - \alpha(\rho)$$

*and*

$$f_2(\eta) = \Big(g(\eta(x)) - \alpha(\rho)\Big)\Big(g(\eta(y)) - \alpha(\rho)\Big) \quad \textit{for} \quad x \neq y,$$

*are admissible functions.*

The proof of Proposition 4 is deferred to the end of the section.

*Proof of Theorem 2.* We consider cases depending on the degree of the function $f$ and dimension $d$. When $f$ is admissible for the symmetrized dynamics, that is when $\| f \|_{-1} < \infty$, by Proposition 3, $\sigma_t^2(\rho, f) = O(t)$, and hence $f$ is admissible for the asymmetric model.

We now argue in the exceptional cases when $\| f \|_{-1} = \infty$ that $f$ is however still admissible for the asymmetric with drift model. It will be helpful to note that $\tilde{f}_1'(\rho) = \alpha'(\rho)$ and $\tilde{f}_2''(\rho) = 2\alpha'(\rho)$.

**Case 1.** In $d = 2$, if $f$ is a mean-zero degree $n = 1$ function, let

$$h = f - \frac{\tilde{f}'(\rho)}{\alpha'(\rho)} f_1 .$$

Then, as $\tilde{h}(\rho) = \tilde{h}'(\rho) = 0$, $h$ is a degree $n \ge 2$ function. Hence, $\|h\|_{-1} < \infty$ and $h$ is admissible by Proposition 3 for the asymmetric with drift model. But, since $f_1$ is admissible by Proposition 4, we have

$$\sigma_t^2(\rho, f) \le 2\sigma_t^2(\rho, h) + 2\sigma_t^2(f_1, \rho) = O(t)$$

and therefore $f$ is admissible also.

**Case 2.** In $d = 1$, if $f$ is a mean-zero degree $n = 2$ function, consider

$$h = f - \frac{\tilde{f}''(\rho)}{2(\alpha'(\rho))^2} f_2 .$$

Since $\tilde{h}(\rho) = \tilde{h}'(\rho) = \tilde{h}''(\rho) = 0$, $h$ is at degree $n \ge 3$ function and hence admissible. Since $f_2$ is also admissible (Proposition 4), $f$ is admissible by the reasoning at the end of Case 1.

On the other hand, if $f$ is a mean-zero degree $n = 1$ function, consider

$$k = f - \frac{\tilde{f}'(\rho)}{\alpha'(\rho)} f_1.$$

Again, as $\tilde{k}(\rho) = \tilde{k}'(\rho) = 0$, $k$ is a degree $n \geq 2$ function. By the conclusion just above, if $k$ is a degree $n = 2$ function, it is admissible. If $k$ is a degree $n \geq 3$ function, it is already admissible. Since $f_1$ is also admissible (Proposition 4), we conclude then that $f$ is admissible.

Finally, when $f$ is an increasing coordinatewise function, the same argument as given for Theorem 1.1 of [13], making use of a Newman-Wright CLT yields the weak convergence in the theorem. □

*Proof of Proposition 4.* We follow a technique given in [3]. We prove that $f_2$ is admissible. The argument for admissibility of $f_1$ is simpler and omitted.

Recall that $\bar{h} = h - E_{\nu_\rho}[h]$. Since $f_2$ is increasing, $E_{\nu_\rho}[f_2 P_s f_2] \geq 0$ and the variance of the additive functional is bounded

$$\sigma_t^2(\rho, f_2) \;\leq\; 2t \int_0^t E_{\nu_\rho}[\bar{g}(\eta_s(x)\bar{g}(\eta_s(y) \cdot \bar{g}(\eta_0(x))\bar{g}(\eta_0(y))]ds.$$

The integrand, using stationarity and the basic coupling, can be written as

$$\begin{aligned}
&E_{\nu_\rho}[\bar{g}(\eta_s(x)\bar{g}(\eta_s(y)) \cdot \bar{g}(\eta_0(x))\bar{g}(\eta_0(y))] \qquad (2)\\
&= \alpha^2(\rho)\Big\{E_{\nu_\rho}[P_s f_2(\eta + \delta(x) + \delta(y))] - E_{\nu_\rho}[P_s f_2(\eta)]\Big\}\\
&= \alpha^2(\rho)\hat{E}\Big[g(\eta_s(x) + \xi_s(x) + \chi_s(x))g(\eta_s(y) + \xi_s(y) + \chi_s(y))\\
&\quad -g(\eta_s(x))g(\eta_s(y))\Big]
\end{aligned}$$

where $\xi_s$ and $\chi_s$ are the processes following second-class particles starting at $x$ and $y$ respectively.

Note by (LIP) and explicit computation,

$$\begin{aligned}
E_{\nu_\rho}[g^2(\eta(x) + 2)] &= 2E_{\nu_\rho}\Big[\big(g(\eta(x) + 2) - g(\eta(x))\big)^2\Big] + 2E_{\nu_\rho}[g^2(\eta(x))]\\
&\leq 2K^2 + 2\alpha(\rho)E_{\nu_\rho}[g(\eta(x) + 1)]\\
&\leq 2K^2 + 2\alpha(\rho)[K + \alpha(\rho)] \;<\; \infty.
\end{aligned}$$

Then, by adding and subtracting terms and Schwarz inequality, one can bound the integrand (2) by $C \sum_{z=x,y}[\hat{P}(\chi_s(z) = 1) + \hat{P}(\xi_s(z) = 1)]$. For instance,

$$\Big(\hat{E}\,[g(\eta_s(z) + \xi_s(z) + \chi_s(z)) - g(\eta_s(z) + \xi_s(z))]^2\Big)^{1/2} \;\leq\; K\hat{P}(\chi_s(z) = 1).$$

To finish the proof, we now bound the integral

$$\int_0^\infty \hat{P}(\chi_s(x) = 1)ds \ < \ \infty$$

as the integrals of $\hat{P}(\chi_s(y) = 1)$ and $\hat{P}(\xi_s(z) = 1)$ for $z = x, y$ are similar.

Construct, following Kipnis's paper, the motion of the second-class particles. We follow the two particles individually $(\xi_s, \chi_s)$. From the basic coupling, the rate of the jumps from a site $x$ is given by $g(\eta_s(x) + \xi_s(x) + \chi_s(x)) - g(\eta_s(x))$. We will suppose, if both particles are at $x$, then one of them is chosen at random to make the jump.

Let $X_0 = y$ and $\{X_i : i \geq 1\}$ be the position of a random walk on $\mathbb{Z}^d$ according to the transient jump probability $p$. Let also $\{T_i : i \geq 1\}$ be independent random variables with exponential distribution with mean 1. Then, we define jump times $\{\tau_i : i \geq 1\}$ and process values as follows:

$$\tau_1 = \inf\Big\{u > 0 : \int_0^u \frac{\chi_s(X_0)}{\xi_s(X_0) + \chi_s(X_0)} \cdot\Big(g(\eta_s(X_0) + \xi_s(X_0) + \chi_s(X_0)) - g(\eta_s(X_0) + \xi_s(X_0))\Big)ds \geq T_1\Big\}.$$

Set $\chi_s(x) = \mathbf{1}_{X_0(x)}$ for $0 \leq s < \tau_1$. Also, for $r \geq 1$,

$$\tau_{r+1} = \inf\Big\{u > \tau_{r-1}^u : \int_{\tau_{r-1}}^u \frac{\chi_s(X_r)}{\xi_s(X_r) + \chi_s(X_r)} \cdot\Big(g(\eta_s(X_r) + \xi_s(X_r) + \chi_s(X_r)) - g(\eta_s(X_r) + \xi_s(X_r))\Big)ds \geq T_{r+1}\Big\}$$

and $\chi_s(x) = \mathbf{1}_{X_r(x)}$ for $\tau_r \leq s < \tau_{r+1}$.

The dynamics for $\xi_s$ is similarly defined. Note that with respect to $\nu_\rho$, since $g$ is increasing,

$$\begin{aligned}
&\int_0^\infty \frac{\chi_s(X_r)}{\xi_s(X_r) + \chi_s(X_r)} \left(g(\eta_s(X_r) + \xi_s(X_r) + \chi_s(X_r)) - g(\eta_s(X_r) + \xi_s(X_r))\right) ds \\
&\geq \frac{1}{2} \int_0^\infty \min\Big\{g(\eta_s(X_r) + 2) - g(\eta_s(X_r) + 1), g(\eta_s(X_r) + 1) - g(\eta_s(X_r))\Big\}ds \\
&\geq \frac{1}{2} \inf_{x\in\mathbb{Z}^d} \int_0^\infty \min\Big\{g(\eta_s(x) + 2) - g(\eta_s(x) + 1), g(\eta_s(x) + 1) - g(\eta_s(x))\Big\}ds.
\end{aligned}$$

By the ergodic theorem, for each $x \in \mathbb{Z}^d$ in the countable space $\mathbb{Z}^d$,

$$\int_0^\infty \min\{g(\eta_s(x) + 2) - g(\eta_s(x) + 1), g(\eta_s(x) + 1) - g(\eta_s(x))\}ds = \infty \ \text{ a.s.}$$

Therefore, all the times $\tau_r$ are finite a.s.

Then,

$$\int_0^\infty \chi_s(x)ds = \sum_{j=0}^\infty T_j 1_x(X_j).$$

Take expectation on both sides to obtain

$$\int_0^\infty \hat{P}(\chi_s(x) = 1)ds = \sum_{j=0}^\infty P(X_j = x) < \infty$$

since $\{X_j\}$ is transient. □

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## References

1. Andjel, E.D.: Invariant measures for the zero range process. Ann. Probab. **10**, 525–547 (1982)
2. Evans, M.R., Hanney, T.: Nonequilibrium statistical mechanics of the zero-range process and related models. J. Phys. A: Math. Gen. **38**, 195–240 (2005)
3. Kipnis, C.: Central limit theorems for infinite series of queues and applications to simple exclusion. Ann. Probab. **14**, 397–408, (1986)
4. Kipnis, C., Landim, C.: Scaling Limits of Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften, vol. 320. Springer, Berlin (1999)
5. Kipnis, C., Varadhan, S.R.S.: Central limit theorem for additive functionals of reversible Markov processes. Commun. Math. Phys. **104**, 1–19 (1986)
6. Komorowski, T., Landim, C., Olla, S.: Fluctuations in Markov Processes. Springer, New York (2012)
7. Landim, C., Sethuraman, S., Varadhan, S.: Spectral gap for zero-range dynamics. Ann. Probab. **24**, 1871–1902 (1996)
8. Liggett, T.: Interacting Particle Systems. Springer, New York (1985)
9. Morris, B.: Spectral gap for the zero-range process with constant rate. Ann. Probab. **34**, 1645–1664 (2006)
10. Nagahata, Y.: Spectral gap for zero-range processes with jump rate $g(x) = x^\gamma$. Stoch. Process. Appl. **120**, 949–958 (2010)

11. Quastel, J.: Diffusion of color in the simple exclusion process. Commun. Pure Appl. Math. **45**, 623–679 (1992)
12. Quastel, J., Jankowski, H., Sheriff, J.: Central limit theorem for zero-range processes. special issue dedicated to Daniel W. Stroock and Srinivasa S. R. Varadhan on the occasion of their 60th birthday. Methods Appl. Anal. **9**(3), 393–406 (2002)
13. Sethuraman, S.: Central limit theorems for additive functionals of the simple exclusion process. Ann. Probab. **28**, 277–302 (2000); Correction (2006) **34**, 427–428
14. Sethuraman, S.: On extremal measures for conservative particle systems. Ann. Inst. Henri Poincare Prob. et Statistiques **37**, 139–154 (2001)
15. Sethuraman, S., Xu, L.: A central limit theorem for reversible exclusion and zero-range particle systems. Ann. Probab. **24**, 1842–1870 (1996)
16. Varadhan, S.R.S.: Self-diffusion of a tagged particle in equilibrium for asymmetric mean zero random walk with simple exclusion. Ann. Inst. Henri Poincare Prob. et Statistiques. **31**, 273–285 (1995)

# A Survey on Bogoliubov Generating Functionals for Interacting Particle Systems in the Continuum

**Dmitri L. Finkelshtein and Maria João Oliveira**

## 1 Bogoliubov Generating Functionals

Let $\Gamma := \Gamma_{\mathbb{R}^d}$ be the configuration space over $\mathbb{R}^d$, $d \in \mathbb{N}$,

$$\Gamma := \left\{\gamma \subset \mathbb{R}^d : |\gamma \cap \Lambda| < \infty \text{ for every compact } \Lambda \subset \mathbb{R}^d\right\},$$

where $|\cdot|$ denotes the cardinality of a set. As usual we identify each $\gamma \in \Gamma$ with the non-negative Radon measure $\sum_{x\in\gamma} \delta_x$ on the Borel $\sigma$-algebra $\mathscr{B}(\mathbb{R}^d)$, where $\delta_x$ is the Dirac measure with mass at $x$, $\sum_{x\in\emptyset} \delta_x := 0$. This allows to endow $\Gamma$ with the vague topology, that is, the weakest topology on $\Gamma$ with respect to which all mappings

$$\Gamma \ni \gamma \longmapsto \langle f, \gamma\rangle := \int_{\mathbb{R}^d} d\gamma(x)\, f(x) = \sum_{x\in\gamma} f(x)$$

are continuous for all continuous functions $f$ on $\mathbb{R}^d$ with compact support. In the sequel we denote the corresponding Borel $\sigma$-algebra on $\Gamma$ by $\mathscr{B}(\Gamma)$.

**Definition 1.** Let $\mu$ be a probability measure on $(\Gamma, \mathscr{B}(\Gamma))$. The Bogoliubov generating functional (shortly GF) $B_\mu$ corresponding to $\mu$ is a functional defined at each $\mathscr{B}(\mathbb{R}^d)$-measurable function $\theta$ by

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$$B_\mu(\theta) := \int_\Gamma d\mu(\gamma) \prod_{x\in\gamma}(1+\theta(x)), \tag{1}$$

provided the right-hand side exists for $|\theta|$, i.e., $B_\mu(|\theta|) < \infty$.

Observe that for each $\theta > -1$ such that the right-hand side of (1) exists, one may equivalently rewrite (1) as

$$B_\mu(\theta) := \int_\Gamma d\mu(\gamma)\, e^{\langle \ln(1+\theta),\gamma\rangle},$$

showing that $B_\mu$ is a modified Laplace transform.

From Definition 1, it is clear that the existence of $B_\mu(\theta)$ for $\theta \neq 0$[1] depends on the underlying probability measure $\mu$. However, it follows also from Definition 1 that if the GF $B_\mu$ corresponding to a probability measure $\mu$ exists, then the domain of $B_\mu$ depends on $\mu$. Conversely, the domain of $B_\mu$ reflects special properties over the measure $\mu$ [21]. For instance, if $\mu$ has finite local exponential moments, i.e., for all $\alpha > 0$ and all bounded Borel sets $\Lambda \subseteq \mathbb{R}^d$,

$$\int_\Gamma d\mu(\gamma)\, e^{\alpha|\gamma\cap\Lambda|} < \infty,$$

then $B_\mu$ is well-defined, for instance, on all bounded functions $\theta$ with compact support. The converse is also true and it follows from the fact that, for each $\alpha > 0$ and for each $\Lambda$ described as before, the latter integral is equal to $B_\mu((e^\alpha - 1)\mathbb{1}_\Lambda)$, where $\mathbb{1}_\Lambda$ is the indicator function of $\Lambda$. In this situation, to a such measure $\mu$ one may associate the so-called correlation measure $\rho_\mu$.

In order to introduce the notion of correlation measure, for any $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ let

$$\Gamma^{(n)} := \{\gamma \in \Gamma : |\gamma| = n\},\ n \in \mathbb{N}, \quad \Gamma^{(0)} := \{\emptyset\}.$$

Clearly, each $\Gamma^{(n)}$, $n \in \mathbb{N}$, can be identified with the symmetrization of the set $\{(x_1,\dots,x_n) \in (\mathbb{R}^d)^n : x_i \neq x_j \text{ if } i \neq j\}$ under the permutation group over $\{1,\dots,n\}$, which induces a natural (metrizable) topology on $\Gamma^{(n)}$ and the corresponding Borel $\sigma$-algebra $\mathscr{B}(\Gamma^{(n)})$. Moreover, for the Lebesgue product measure $(dx)^{\otimes n}$ fixed on $(\mathbb{R}^d)^n$, this identification yields a measure $m^{(n)}$ on $(\Gamma^{(n)}, \mathscr{B}(\Gamma^{(n)}))$. This leads to the space of finite configurations

$$\Gamma_0 := \bigsqcup_{n=0}^{\infty} \Gamma^{(n)}$$

[1] Of course, for any probability measure $\mu$ on $(\Gamma, \mathscr{B}(\Gamma))$ one has $B_\mu(0) = 1$.

endowed with the topology of disjoint union of topological spaces and the corresponding Borel $\sigma$-algebra $\mathscr{B}(\Gamma_0)$, and to the so-called Lebesgue-Poisson measure on $(\Gamma_0, \mathscr{B}(\Gamma_0))$,

$$\lambda := \lambda_{dx} := \sum_{n=0}^{\infty} \frac{1}{n!} m^{(n)}, \quad m^{(0)}(\{\emptyset\}) := 1. \tag{2}$$

Given a probability measure $\mu$ on $(\Gamma, \mathscr{B}(\Gamma))$ with finite local exponential moments, the correlation measure $\rho_\mu$ corresponding to $\mu$ is a measure on $(\Gamma_0, \mathscr{B}(\Gamma_0))$ defined for all complex-valued exponentially bounded $\mathscr{B}(\Gamma_0)$-measurable functions $G$ with local support[2] by

$$\int_{\Gamma_0} d\rho_\mu(\eta)\, G(\eta) = \int_{\Gamma} d\mu(\gamma) \sum_{\substack{\eta \subset \gamma \\ |\eta| < \infty}} G(\eta). \tag{3}$$

As a consequence, for every bounded $\mathscr{B}(\mathbb{R}^d)$-measurable function $\theta$ with compact support and $G = e_\lambda(\theta)$,

$$e_\lambda(\theta, \eta) := \prod_{x \in \eta} \theta\,(x)\,,\ \eta \in \Gamma_0 \setminus \{\emptyset\}, \quad e_\lambda(\theta, \emptyset) := 1,$$

definition (3) leads to

$$B_\mu(\theta) = \int_{\Gamma} d\mu(\gamma) \prod_{x \in \gamma} (1 + \theta(x)) = \int_{\Gamma} d\mu(\gamma) \sum_{\substack{\eta \subset \gamma \\ |\eta| < \infty}} e_\lambda(\theta, \eta) = \int_{\Gamma_0} d\rho_\mu(\eta)\, e_\lambda(\theta, \eta),$$

yielding a description of the GF $B_\mu$ in terms of either the correlation measure $\rho_\mu$ or the so-called correlation function $k_\mu := \frac{d\rho_\mu}{d\lambda}$ corresponding to $\mu$, if $\rho_\mu$ is absolutely continuous with respect to the Lebesgue-Poisson measure $\lambda$:

$$B_\mu(\theta) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta, \eta) k_\mu(\eta). \tag{4}$$

Throughout this work we will consider GF defined on the whole $L^1 := L^1(\mathbb{R}^d, dx)$ space of complex-valued functions. Furthermore, we will assume that the GF are entire. For a comprehensive presentation of the general theory of holomorphic functionals on Banach spaces see e.g. [1,5]. We recall that a functional $A : L^1 \to \mathbb{C}$ is entire on $L^1$ whenever $A$ is locally bounded and for all $\theta_0, \theta \in L^1$

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[2]That is, $G \upharpoonright_{\Gamma_0 \setminus \Gamma_\Lambda} \equiv 0$, $\Gamma_\Lambda := \{\eta \in \Gamma : \eta \subset \Lambda\}$, for some bounded Borel set $\Lambda \subseteq \mathbb{R}^d$ and there are $C_1, C_2 > 0$ such that $|G(\eta)| \leq C_1 e^{C_2 |\eta|}$ for all $\eta \in \Gamma_0$.

the mapping $\mathbb{C} \ni z \mapsto A(\theta_0 + z\theta) \in \mathbb{C}$ is entire. Thus, at each $\theta_0 \in L^1$, every entire functional $A$ on $L^1$ has a representation in terms of its Taylor expansion,

$$A(\theta_0 + z\theta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} d^n A(\theta_0; \theta, \dots, \theta), \quad z \in \mathbb{C}, \theta \in L^1.$$

**Theorem 1.** *Let $A$ be an entire functional on $L^1$. Then each differential $d^n A(\theta_0; \cdot), n \in \mathbb{N}, \theta_0 \in L^1$ is defined by a symmetric kernel*

$$\delta^n A(\theta_0; \cdot) \in L^\infty(\mathbb{R}^{dn}) := L^\infty\left((\mathbb{R}^d)^n, (dx)^{\otimes n}\right)$$

*called the variational derivative of $n$-th order of $A$ at the point $\theta_0$. More precisely,*

$$\begin{aligned} d^n A(\theta_0; \theta_1, \dots, \theta_n) &:= \frac{\partial^n}{\partial z_1 \dots \partial z_n} A\left(\theta_0 + \sum_{i=1}^n z_i \theta_i\right)\Bigg|_{z_1 = \dots = z_n = 0} \\ &=: \int_{(\mathbb{R}^d)^n} dx_1 \dots dx_n \, \delta^n A(\theta_0; x_1, \dots, x_n) \prod_{i=1}^n \theta_i(x_i) \end{aligned}$$

*for all $\theta_1, \dots, \theta_n \in L^1$. Moreover, the operator norm of the bounded $n$-linear functional $d^n A(\theta_0; \cdot)$ is equal to $\|\delta^n A(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^{dn})}$ and for all $r > 0$ one has*

$$\|\delta A(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{r} \sup_{\|\theta'\|_{L^1} \le r} |A(\theta_0 + \theta')| \tag{5}$$

*and, for $n \ge 2$,*

$$\|\delta^n A(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^{dn})} \le n! \left(\frac{e}{r}\right)^n \sup_{\|\theta'\|_{L^1} \le r} |A(\theta_0 + \theta')|. \tag{6}$$

*Remark 1.* 1. According to Theorem 1, the Taylor expansion of an entire functional $A$ at a point $\theta_0 \in L^1$ may be written in the form

$$\begin{aligned} A(\theta_0 + \theta) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} dx_1 \dots dx_n \, \delta^n A(\theta_0; x_1, \dots, x_n) \prod_{i=1}^n \theta(x_i) \\ &= \int_{\Gamma_0} d\lambda(\eta) \, \delta^n A(\theta_0; \eta) e_\lambda(\theta, \eta), \end{aligned}$$

where $\lambda$ is the Lebesgue-Poisson measure defined in (2).

2. Concerning Theorem 1, we observe that the analogous result does not hold neither for other $L^p$-spaces, nor Banach spaces of continuous functions, or

Sobolev spaces. For a detailed explanation see the proof of Theorem 1 and Remark 7 in [21].

The first part of Theorem 1 stated for GF and their variational derivatives at $\theta_0 = 0$ yields the next result. In particular, it shows that the assumption of entireness on $L^1$ is a natural environment, namely, to recover the notion of correlation function.

**Proposition 1.** *Let $B_\mu$ be an entire GF on $L^1$. Then the measure $\rho_\mu$ is absolutely continuous with respect to the Lebesgue-Poisson measure $\lambda$ and the Radon-Nykodim derivative $k_\mu = \frac{d\rho_\mu}{d\lambda}$ is given by*

$$k_\mu(\eta) = \delta^{|\eta|} B_\mu(0;\eta) \quad \textit{for } \lambda\textit{-a.a. } \eta \in \Gamma_0.$$

*Remark 2.* Proposition 1 shows that the correlation functions $k_\mu^{(n)} := k_\mu \upharpoonright_{\Gamma^{(n)}}$ are the Taylor coefficients of the GF $B_\mu$. In other words, $B_\mu$ is the generating functional for the correlation functions $k_\mu^{(n)}$. This was also the reason why N. N. Bogoliubov [4] introduced these functionals. Furthermore, GF are also related to the general infinite dimensional analysis on configuration spaces, cf., e.g. [19]. Namely, through the unitary isomorphism $S_\lambda$ defined in [19] between the space $L^2(\Gamma_0, \lambda)$ of complex-valued functions and the Bargmann-Segal space one finds $B_\mu = S_\lambda(k_\mu)$.

Concerning the second part of Theorem 1, namely, estimates (5) and (6), we note that $A$ being entire does not ensure that for every $r > 0$ the supremum appearing on the right-hand side of (5), (6) is always finite. This will hold if, in addition, the entire functional $A$ is of bounded type, that is,

$$\forall\, r > 0, \quad \sup_{\|\theta\|_{L^1} \le r} |A(\theta_0 + \theta)| < \infty, \;\; \forall\, \theta_0 \in L^1.$$

Hence, as a consequence of Proposition 1, it follows from (5) and (6) that the correlation function $k_\mu$ of an entire GF of bounded type on $L^1$ fulfills the so-called generalized Ruelle bound, that is, for any $0 \le \varepsilon \le 1$ and any $r > 0$ there is some constant $C \ge 0$ depending on $r$ such that

$$k_\mu(\eta) \le C\, (|\eta|!)^{1-\varepsilon} \left(\frac{e}{r}\right)^{|\eta|}, \quad \lambda\text{–a.a. } \eta \in \Gamma_0. \tag{7}$$

In our case, $\varepsilon = 0$. We observe that if (7) holds for $\varepsilon = 1$ and for at least one $r > 0$, then condition (7) is the classical Ruelle bound. In terms of GF, the latter means that $|B_\mu(\theta)| \le C \exp\left(\frac{e}{r}\|\theta\|_{L^1}\right)$, as it can be easily checked using representation (4) and the following equality [20],

$$\int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(f,\eta) = \exp\left(\int_{\mathbb{R}^d} dx\, f(x)\right), \quad f \in L^1.$$

This special case motivates the definition of the family of Banach spaces $\mathcal{E}_\alpha$, $\alpha > 0$, of all entire functionals $B$ on $L^1$ such that

$$\|B\|_\alpha := \sup_{\theta \in L^1} \left( |B(\theta)| \, e^{-\frac{1}{\alpha}\|\theta\|_{L^1}} \right) < \infty, \tag{8}$$

cf. [21, Proposition 23], which plays an essential role in the study of stochastic dynamics of infinite particle systems (Sect. 2).

For more details and proofs and for further results concerning GF see [21] and the references therein.

## 2 Stochastic Dynamic Equations

The stochastic evolution of an infinite particle system might be described by a Markov process on $\Gamma$, which is determined heuristically by a Markov generator $L$ defined on a suitable space of functions on $\Gamma$. If such a Markov process exists, then it provides a solution to the (backward) Kolmogorov equation

$$\frac{d}{dt} F_t = L F_t, \qquad F_t|_{t=0} = F_0. \tag{9}$$

However, the construction of the Markov process seems to be often a difficult question and at the moment it has been successfully accomplished only for very restrictive classes of generators, see [16] and [24].

Besides this technical difficulty, in applications it turns out that one needs a knowledge on certain characteristics of the stochastic evolution in terms of mean values rather than pointwise, which do not follow neither from the construction of the Markov process nor from the study of (9). These characteristics concern e.g. observables, that is, functions defined on $\Gamma$, for which expected values are given by

$$\langle F, \mu \rangle = \int_\Gamma d\mu(\gamma) \, F(\gamma),$$

where $\mu$ is a probability measure on $\Gamma$, that is, a state of the system. This leads to the time evolution problem on states,

$$\frac{d}{dt} \langle F, \mu_t \rangle = \langle LF, \mu_t \rangle, \quad \mu_t|_{t=0} = \mu_0. \tag{10}$$

Technically, to proceed further, first we shall exploit definition (3), namely, the sum appearing therein, which concerns the so-called $K$-transform introduced by A. Lenard [26]. That is a mapping which maps functions defined on $\Gamma_0$ into functions defined on the space $\Gamma$. More precisely, given a complex-valued bounded

$\mathscr{B}(\Gamma_0)$-measurable function $G$ with bounded support[3] (shortly $G \in B_{\text{bs}}(\Gamma_0)$), the $K$-transform of $G$ is a mapping $KG : \Gamma \to \mathbb{C}$ defined at each $\gamma \in \Gamma$ by

$$(KG)(\gamma) := \sum_{\substack{\eta \subset \gamma \\ |\eta| < \infty}} G(\eta). \tag{11}$$

It has been shown in [18] that the $K$-transform is a linear and invertible mapping. Thus, definition (3) shows, in particular, that for any probability measure $\mu$ on $(\Gamma, \mathscr{B}(\Gamma))$ with finite local exponential moments, one has $B_{\text{bs}}(\Gamma_0) \subseteq L^1(\Gamma_0, \rho_\mu)$. Moreover, on the dense set $B_{\text{bs}}(\Gamma_0)$ in $L^1(\Gamma_0, \rho_\mu)$ the inequality $\|KG\|_{L^1(\mu)} \leq \|G\|_{L^1(\rho_\mu)}$ holds, which allows an extension of the $K$-transform to a bounded operator $K : L^1(\Gamma_0, \rho_\mu) \to L^1(\Gamma, \mu)$ in such a way that equality (3) still holds for any $G \in L^1(\Gamma_0, \rho_\mu)$. For the extended operator the explicit form (11) still holds, now $\mu$-a.e. This means, in particular,

$$\left(Ke_\lambda(f)\right)(\gamma) = \prod_{x \in \gamma} (1 + f(x)), \quad \mu\text{-a.a.}\, \gamma \in \Gamma,$$

for all $\mathscr{B}(\mathbb{R}^d)$-measurable functions $f$ such that $e_\lambda(f) \in L^1(\Gamma_0, \rho_\mu)$, cf. e.g. [18].

In terms of the time evolution description (10) on the states $\mu_t$ of an infinite particle system, these considerations imply that for $F$ being of the type $F = KG$, $G \in B_{\text{bs}}(\Gamma_0)$, (10) may be rewritten in terms of the correlation functions $k_t := k_{\mu_t}$ corresponding to the states $\mu_t$, provided these functions exist (or, more generally, in terms of correlation measures $\rho_t := \rho_{\mu_t}$), yielding

$$\frac{d}{dt} \langle\langle G, k_t \rangle\rangle = \langle\langle \hat{L} G, k_t \rangle\rangle, \quad k_t|_{t=0} = k_{\mu_0}, \tag{12}$$

where $\hat{L} := K^{-1} L K$ and $\langle\langle \cdot, \cdot \rangle\rangle$ is the usual pairing

$$\langle\langle G, k \rangle\rangle := \int_{\Gamma_0} d\lambda(\eta)\, G(\eta) k(\eta). \tag{13}$$

Of course, a stronger version of (12) is

$$\frac{d}{dt} k_t = \hat{L}^* k_t, \quad k_t|_{t=0} = k_{\mu_0}, \tag{14}$$

for $\hat{L}^*$ being the dual operator of $\hat{L}$ in the sense defined in (13).

[3] That is, $G \upharpoonright_{\Gamma_0 \setminus \left(\bigsqcup_{n=0}^{N} \Gamma_\Lambda^{(n)}\right)} \equiv 0$, $\Gamma_\Lambda^{(n)} := \{\eta \in \Gamma : \eta \subset \Lambda\} \cap \Gamma^{(n)}$, for some $N \in \mathbb{N}_0$ and for some bounded Borel set $\Lambda \subseteq \mathbb{R}^d$.

Representation (4) combined with (12), (13) gives us a way to widen the dynamical description towards the GF $B_t := B_{\mu_t}$ corresponding to $\mu_t$ [13, 21], provided these functionals exist. Informally,

$$\frac{\partial}{\partial t} B_t(\theta) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta, \eta) \frac{\partial}{\partial t} k_t(\eta) = \int_{\Gamma_0} d\lambda(\eta)\, (\hat{L} e_\lambda(\theta))(\eta) k_t(\eta). \tag{15}$$

In other words, given the operator $\tilde{L}$ defined at $B(\theta) := \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta, \eta) k(\eta)$, $k : \Gamma_0 \to [0, +\infty)$, by

$$(\tilde{L} B)(\theta) := \int_{\Gamma_0} d\lambda(\eta)\, (\hat{L} e_\lambda(\theta))(\eta) k(\eta), \tag{16}$$

heuristically (15) means that $B_t, t \geq 0$, is a solution to the Cauchy problem

$$\frac{\partial}{\partial t} B_t = \tilde{L} B_t, \quad B_t|_{t=0} = B_{\mu_0}. \tag{17}$$

According to the considerations above, there is a close connection between the Markov evolution (10) and the Cauchy problems (12), (14), and (17). More precisely, given a solution $\mu_t, t \geq 0$, to (10), if additionally the correlation function $k_{\mu_t}$ corresponding to each state $\mu_t$ exists, then $k_t := k_{\mu_t}$ is a solution to (12). Similarly, the informal sequence of equalities (15) shows that if the GF $B_{\mu_t}$ exists for each time $t \geq 0$, then $B_t := B_{\mu_t}$ solves (17). Conversely, given a solution $k_t$ to (12), or to (14), or a solution $B_t, t \geq 0$, to (17), for $k_{\mu_0}$ and $B_{\mu_0}$ being, respectively, the correlation function and the GF corresponding to the initial state $\mu_0$ of the system, an additional analysis is needed in order to check that each $k_t$ (resp., $B_t$) is indeed a correlation function (resp., a GF) corresponding to some measure $\mu_t$. If so, then, by construction, $\mu_t, t \geq 0$, is a solution to (10) and $k_t = k_{\mu_t}$ (resp., $B_t = B_{\mu_t}$). For more details concerning the aforementioned analysis see e.g. [10] for the case of correlation functions, and [21, 25] for the GF case.

*Remark 3.* Although correlation functions appear in this work as a side remark, we note that the study of the properties of correlation functions of a dynamics is a classical problem in mathematical physics. In order to analyze the existence of solutions to (12), (14), and the properties of such solutions, some approaches have been proposed. One of them is based on semigroup techniques, which for birth-and-death dynamics has been accomplished in e.g. [7, 10, 12, 22, 23] and summarized in a recent article [11]. Another approach is based on the so-called Ovsyannikov technique and it has been successfully applied in the analysis of birth-and-death as well as hopping particle systems (on a finite time interval), see e.g. [2, 3, 6].

In most concrete applications, to find a solution to (17) on a Banach space seems to be often a difficult question. However, this problem may be simplified within the framework of scales of Banach spaces. We recall that a scale of Banach spaces is a one-parameter family of Banach spaces $\{\mathbb{B}_s : 0 < s \leq s_0\}$ such that $\mathbb{B}_{s''} \subseteq \mathbb{B}_{s'}$,

$\|\cdot\|_{s'} \leq \|\cdot\|_{s''}$ for any pair $s', s''$ such that $0 < s' < s'' \leq s_0$, where $\|\cdot\|_s$ denotes the norm in $\mathbb{B}_s$. As an example, it is clear from definition (8) that for each $\alpha_0 > 0$ the family $\{\mathscr{E}_\alpha : 0 < \alpha \leq \alpha_0\}$ is a scale of Banach spaces.

Within this framework, one has the following existence and uniqueness result (see e.g. [27]). For concreteness, in subsections below we will analyze two examples of applications.

**Theorem 2.** *On a scale of Banach spaces $\{\mathbb{B}_s : 0 < s \leq s_0\}$ consider the initial value problem*

$$\frac{du(t)}{dt} = Au(t), \quad u(0) = u_0 \in \mathbb{B}_{s_0} \tag{18}$$

*where, for each $s \in (0, s_0)$ fixed and for each pair $s', s''$ such that $s \leq s' < s'' \leq s_0$, $A : \mathbb{B}_{s''} \to \mathbb{B}_{s'}$ is a linear mapping so that there is an $M > 0$ such that for all $u \in \mathbb{B}_{s''}$*

$$\|Au\|_{s'} \leq \frac{M}{s'' - s'} \|u\|_{s''}.$$

*Here $M$ is independent of $s', s''$ and $u$, however it might depend continuously on $s, s_0$.*

*Then, for each $s \in (0, s_0)$, there is a constant $\delta > 0$ (which depends on $M$) such that there is a unique function $u : [0, \delta(s_0 - s)) \to \mathbb{B}_s$ which is continuously differentiable on $(0, \delta(s_0 - s))$ in $\mathbb{B}_s$, $Au \in \mathbb{B}_s$, and solves (18) in the time-interval $0 \leq t < \delta(s_0 - s)$.*

## *2.1 The Glauber Dynamics*

The Glauber dynamics is an example of a birth-and-death model where, in this special case, particles appear and disappear according to a death rate identically equal to 1 and to a birth rate depending on the interaction between particles. More precisely, let $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a pair potential, that is, a $\mathscr{B}(\mathbb{R}^d)$-measurable function such that $\phi(-x) = \phi(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$, which we will assume to be non-negative and integrable. Given a configuration $\gamma \in \Gamma$, the birth rate of a new particle at a site $x \in \mathbb{R}^d \setminus \gamma$ is given by $\exp(-E(x, \gamma))$, where $E(x, \gamma)$ is a relative energy of interaction between a particle located at $x$ and the configuration $\gamma$ defined by

$$E(x, \gamma) := \sum_{y \in \gamma} \phi(x - y) \in [0, +\infty]. \tag{19}$$

Informally, in terms of Markov generators, this means that the behavior of such an infinite particle system is described by

$$(L_G F)(\gamma) := \sum_{x\in\gamma} (F(\gamma \setminus \{x\}) - F(\gamma)) + z \int_{\mathbb{R}^d} dx\, e^{-E(x,\gamma)} (F(\gamma \cup \{x\}) - F(\gamma)), \tag{20}$$

where $z > 0$ is an activity parameter (for more details see e.g. [13, 23]). Thus, according to Sect. 2, the operator $\tilde{L}_G$ defined in (16) is given cf. [13] by

$$(\tilde{L}_G B)(\theta) = -\int_{\mathbb{R}^d} dx\, \theta(x) \Big( \delta B(\theta; x) - zB\Big(\theta e^{-\phi(x-\cdot)} + e^{-\phi(x-\cdot)} - 1\Big)\Big). \tag{21}$$

The Glauber dynamics is an example where semigroups theory can be apply to study the time evolution in terms of correlation functions, see e.g. [10, 12, 23]. However, within the context of GF, semigroup techniques seem do not work (see e.g. [17]). This is partially due to the fact that given the natural class of Banach spaces $\mathscr{E}_\alpha$, the operator $\tilde{L}_G$ maps elements of a Banach space $\mathscr{E}_\alpha$, $\alpha > 0$, on elements of larger Banach spaces $\mathscr{E}_{\alpha'}$, $0 < \alpha' < \alpha$ [14]:

$$\|\tilde{L}_G B\|_{\alpha'} \le \frac{\alpha'}{\alpha - \alpha'} \left(1 + z\alpha e^{\frac{\|\phi\|_{L^1}}{\alpha} - 1}\right) \|B\|_\alpha, \quad B \in \mathscr{E}_\alpha.$$

However, this estimate of norms and an application of Theorem 2 lead to the following existence and uniqueness result.

**Proposition 2 ([14, Theorem 3.1]).** *Given an $\alpha_0 > 0$, let $B_0 \in \mathscr{E}_{\alpha_0}$. For each $\alpha \in (0, \alpha_0)$ there is a $T > 0$ (which depends on $\alpha, \alpha_0$) such that there is a unique solution $B_t$, $t \in [0, T)$, to the initial value problem $\frac{\partial B_t}{\partial t} = \tilde{L}_G B_t$, (21), $B_{t|t=0} = B_0$ in the space $\mathscr{E}_\alpha$.*

*Remark 4.* 1. Concerning the initial conditions considered in Proposition 2, observe that, in particular, $B_0$ can be an entire GF $B_{\mu_0}$ on $L^1$ such that, for some constants $\alpha_0, C > 0$, $|B_{\mu_0}(\theta)| \le C \exp(\frac{\|\theta\|_{L^1}}{\alpha_0})$ for all $\theta \in L^1$. As we have mentioned before, in such a situation an additional analysis is required in order to guarantee that for each time $t \in [0, T)$ the solution $B_t$ given by Proposition 2 is a GF. If so, then clearly each $B_t$ is the GF corresponding to the state of the particle system at the time $t$. For more details see [14, Remark 3.6].

2. If the initial condition $B_0$ is an entire GF on $L^1$ such that the corresponding correlation function $k_0$ (given by Proposition 1) fulfills the Ruelle bound $k_0(\eta) \le z^{|\eta|}$, $\eta \in \Gamma_0$, where $z$ is the activity parameter appearing in definition (20), then the local solution given by Proposition 2 might be extend to a global one, that is, to a solution defined on the whole time interval $[0, +\infty)$. For more details and the proof see [14, Corollary 3.7].

## 2.2 The Kawasaki Dynamics

The Kawasaki dynamics is an example of a hopping particle model where, in this case, particles randomly hop over the space $\mathbb{R}^d$ according to a rate depending on the interaction between particles. More precisely, let $a : \mathbb{R}^d \to [0, +\infty)$ be an even and integrable function and let $\phi : \mathbb{R}^d \to [0, +\infty]$ be a pair potential, which we will assume to be integrable. A particle located at a site $x$ in a given configuration $\gamma \in \Gamma$ hops to a site $y$ according to a rate given by $a(x-y)\exp(-E(y,\gamma))$, where $E(y,\gamma)$ is a relative energy of interaction between the site $y$ and the configuration $\gamma$ defined similarly to (19). Informally, the behavior of such an infinite particle system is described by

$$(L_K F)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, a(x-y)e^{-E(y,\gamma)} \left(F(\gamma\setminus\{x\}\cup\{y\}) - F(\gamma)\right), \tag{22}$$

meaning in terms of the operator $\tilde{L}_K$ defined in (16) that

$$(\tilde{L}_K B)(\theta)$$
$$= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y)e^{-\phi(x-y)}(\theta(y)-\theta(x))\delta B(\theta e^{-\phi(y-\cdot)} + e^{-\phi(y-\cdot)} - 1; x), \tag{23}$$

cf. [13]. In this case the following estimate of norms holds

$$\|\tilde{L}_K B\|_{\alpha'} \le 2e^{\frac{\|\phi\|_{L^1}}{\alpha}} \frac{\alpha'}{\alpha-\alpha'} \|a\|_{L^1} \|B\|_\alpha, \quad B \in \mathcal{E}_\alpha, \alpha' < \alpha,$$

which, by an application of Theorem 2, yields the following statement.

**Proposition 3 ([15, Theorem 3.1]).** *Given an $\alpha_0 > 0$, let $B_0 \in \mathcal{E}_{\alpha_0}$. For each $\alpha \in (0, \alpha_0)$ there is a $T > 0$ (which depends on $\alpha, \alpha_0$) such that there is a unique solution $B_t$, $t \in [0, T)$, to the initial value problem $\frac{\partial}{\partial t} B_t = \tilde{L}_K B_t$, (23), $B_{t|t=0} = B_0$ in the space $\mathcal{E}_\alpha$.*

# 3 Vlasov Scaling

We proceed to investigate the Vlasov-type scaling proposed in [8] for generic continuous particle systems and accomplished in [9] and [2] for the Glauber and the Kawasaki dynamics, respectively, now in terms of GF. As explained in these references, we start with a rescaling of an initial correlation function $k_0$, denoted, respectively, by $k_{G,0}^{(\varepsilon)}$, $k_{K,0}^{(\varepsilon)}$, $\varepsilon > 0$, which has a singularity with respect to $\varepsilon$ of the type $k_{G,0}^{(\varepsilon)}(\eta), k_{K,0}^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|} r_0(\eta)$, $\eta \in \Gamma_0$, being $r_0$ a function independent of $\varepsilon$. The aim is to construct a scaling for the operator $L_G$ (resp., $L_K$) defined in (20) (resp., (22)), $L_{G,\varepsilon}$ (resp., $L_{K,\varepsilon}$), $\varepsilon > 0$, in such a way that the following two conditions are fulfilled. The first one is that under the scaling $L \mapsto L_{\#,\varepsilon}$, $\# = G, K$, the solution $k_{\#,t}^{(\varepsilon)}$, $t \geq 0$, to

$$\frac{\partial}{\partial t} k_{\#,t}^{(\varepsilon)} = \hat{L}_{\#,\varepsilon}^* k_{\#,t}^{(\varepsilon)}, \quad k_{\#,t}^{(\varepsilon)}{}_{|t=0} = k_{\#,0}^{(\varepsilon)}$$

preserves the order of the singularity with respect to $\varepsilon$, that is, $k_{\#,t}^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|} r_{\#,t}(\eta)$, $\eta \in \Gamma_0$. The second condition is that the dynamics $r_0 \mapsto r_{\#,t}$ preserves the Lebesgue-Poisson exponents, that is, if $r_0$ is of the form $r_0 = e_\lambda(\rho_0)$, then each $r_{\#,t}$, $t > 0$, is of the same type, i.e., $r_{\#,t} = e_\lambda(\rho_{\#,t})$, where $\rho_{\#,t}$ is a solution to a non-linear equation (called a Vlasov-type equation). As shown in [8, Example 8], [9], in the case of the Glauber dynamics this equation is given by

$$\frac{\partial}{\partial t} \rho_{G,t}(x) = -\rho_{G,t}(x) + z e^{-(\rho_{G,t} * \phi)(x)}, \quad x \in \mathbb{R}^d, \tag{24}$$

where $*$ denotes the usual convolution of functions. Existence of classical solutions $0 \leq \rho_{G,t} \in L^\infty$ to (24) has been discussed in [6,9]. For the Kawasaki dynamics, the corresponding Vlasov-type equation is given by

$$\frac{\partial}{\partial t} \rho_{K,t}(x) = (\rho_{K,t} * a)(x) e^{-(\rho_{K,t} * \phi)(x)} - \rho_{K,t}(x)(a * e^{-(\rho_{K,t} * \phi)})(x), \quad x \in \mathbb{R}^d, \tag{25}$$

cf. [8, Example 12], [2]. In this case, existence of classical solutions $0 \leq \rho_{K,t} \in L^\infty$ to (25) has been discussed in [2].

Therefore, it is natural to consider the same scalings, but in terms of GF.

## 3.1 The Glauber Dynamics

The previous scheme was accomplished in [9] through the scale transformations $z \mapsto \varepsilon^{-1} z$ and $\phi \mapsto \varepsilon \phi$ of the operator $L_G$, that is,

$$(L_{G,\varepsilon} F)(\gamma) := \sum_{x \in \gamma} (F(\gamma \setminus \{x\}) - F(\gamma)) + \frac{z}{\varepsilon} \int_{\mathbb{R}^d} dx\, e^{-\varepsilon E(x,\gamma)} (F(\gamma \cup \{x\}) - F(\gamma)).$$

To proceed towards GF, let us consider $k_{G,t}^{(\varepsilon)}$ defined as before and $k_{G,t,\mathrm{ren}}^{(\varepsilon)}(\eta) := \varepsilon^{|\eta|} k_{G,t}^{(\varepsilon)}(\eta)$. In terms of GF, these yield

$$B^{(\varepsilon)}_{G,t}(\theta) := \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta) k^{(\varepsilon)}_{G,t}(\eta),$$

and

$$B^{(\varepsilon)}_{G,t,\mathrm{ren}}(\theta) := \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta) k^{(\varepsilon)}_{G,t,\mathrm{ren}}(\eta) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\varepsilon\theta,\eta) k^{(\varepsilon)}_{G,t}(\eta) = B^{(\varepsilon)}_{G,t}(\varepsilon\theta),$$

leading, as in (16) and (17), to the initial value problem

$$\frac{\partial}{\partial t} B^{(\varepsilon)}_{G,t,\mathrm{ren}} = \tilde{L}_{G,\varepsilon,\mathrm{ren}} B^{(\varepsilon)}_{G,t,\mathrm{ren}}, \quad B^{(\varepsilon)}_{G,t,\mathrm{ren}\,|t=0} = B^{(\varepsilon)}_{G,0,\mathrm{ren}}, \tag{26}$$

where, for all $\theta \in L^1$,

$$(\tilde{L}_{G,\varepsilon,\mathrm{ren}} B)(\theta) = -\int_{\mathbb{R}^d} dx\, \theta(x) \left( \delta B(\theta,x) - zB\left( \theta e^{-\varepsilon\phi(x-\cdot)} + \frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon} \right)\right),$$

cf. [14]. Concerning this operator, it has been also shown in [14, Proposition 4.2] that if $B \in \mathscr{E}_\alpha$ for some $\alpha > 0$, then, for all $\theta \in L^1$, $(\tilde{L}_{G,\varepsilon,\mathrm{ren}} B)(\theta)$ converges as $\varepsilon$ tends zero to

$$(\tilde{L}_{G,V} B)(\theta) := -\int_{\mathbb{R}^d} dx\, \theta(x) \left(\delta B(\theta;x) - zB\left(\theta - \phi(x-\cdot)\right)\right).$$

Furthermore, fixed $0 < \alpha < \alpha_0$, if $B \in \mathscr{E}_{\alpha''}$ for some $\alpha'' \in (\alpha,\alpha_0]$, then $\left\{\tilde{L}_{G,\varepsilon,\mathrm{ren}} B, \tilde{L}_{G,V} B\right\} \subset \mathscr{E}_{\alpha'}$ for all $\alpha \le \alpha' < \alpha''$, and one has

$$\|\tilde{L}_\# B\|_{\alpha'} \le \frac{\alpha_0}{\alpha''-\alpha'}\left(1 + z\alpha_0 e^{\frac{\|\phi\|_{L^1}}{\alpha}-1}\right)\|B\|_{\alpha''},$$

where $\tilde{L}_\# = \tilde{L}_{G,\varepsilon,\mathrm{ren}}$ or $\tilde{L}_\# = \tilde{L}_{G,V}$. That is, the estimate of norms for $\tilde{L}_{G,\varepsilon,\mathrm{ren}}$, $\varepsilon > 0$, and the limiting mapping $\tilde{L}_{G,V}$ are similar. Therefore, given any $B_{G,0,V}, B^{(\varepsilon)}_{G,0,\mathrm{ren}} \in \mathscr{E}_{\alpha_0}$, $\varepsilon > 0$, it follows from Theorem 2 that for each $\alpha \in (0,\alpha_0)$, there is a constant $\delta > 0$ such that there is a unique solution $B^{(\varepsilon)}_{G,t,\mathrm{ren}} : [0,\delta(\alpha_0-\alpha)) \to \mathscr{E}_\alpha$, $\varepsilon > 0$, to each initial value problem (26) and a unique solution $B_{G,t,V} : [0,\delta(\alpha_0-\alpha)) \to \mathscr{E}_\alpha$ to the initial problem

$$\frac{\partial}{\partial t} B_{G,t,V} = \tilde{L}_{G,V} B_{G,t,V}, \quad B_{G,t,V\,|t=0} = B_{G,0,V}. \tag{27}$$

In other words, independent of the initial value problem under consideration, the solutions obtained are defined on the same time-interval and with values in the same Banach space. Therefore, it is natural to analyze under which conditions the solutions to (26) converge to the solution to (27). This follows from the following general result [14]:

**Theorem 3.** *On a scale of Banach spaces* $\{\mathbb{B}_s : 0 < s \leq s_0\}$ *consider a family of initial value problems*

$$\frac{du_\varepsilon(t)}{dt} = A_\varepsilon u_\varepsilon(t), \quad u_\varepsilon(0) = u_\varepsilon \in \mathbb{B}_{s_0}, \quad \varepsilon \geq 0, \tag{28}$$

*where, for each* $s \in (0, s_0)$ *fixed and for each pair* $s', s''$ *such that* $s \leq s' < s'' \leq s_0$, $A_\varepsilon : \mathbb{B}_{s''} \to \mathbb{B}_{s'}$ *is a linear mapping so that there is an* $M > 0$ *such that for all* $u \in \mathbb{B}_{s''}$

$$\|A_\varepsilon u\|_{s'} \leq \frac{M}{s'' - s'} \|u\|_{s''}.$$

*Here* $M$ *is independent of* $\varepsilon, s', s''$ *and* $u$*, however it might depend continuously on* $s, s_0$*. Assume that there is a* $p \in \mathbb{N}$ *and for each* $\varepsilon > 0$ *there is an* $N_\varepsilon > 0$ *such that for each pair* $s', s''$, $s \leq s' < s'' \leq s_0$*, and all* $u \in \mathbb{B}_{s''}$

$$\|A_\varepsilon u - A_0 u\|_{s'} \leq \sum_{k=1}^{p} \frac{N_\varepsilon}{(s'' - s')^k} \|u\|_{s''}.$$

*In addition, assume that* $\lim_{\varepsilon\to 0} N_\varepsilon = 0$ *and* $\lim_{\varepsilon\to 0} \|u_\varepsilon(0) - u_0(0)\|_{s_0} = 0$.

*Then, for each* $s \in (0, s_0)$*, there is a constant* $\delta > 0$ *(which depends on* $M$*) such that there is a unique solution* $u_\varepsilon : [0, \delta(s_0 - s)) \to \mathbb{B}_s$, $\varepsilon \geq 0$*, to each initial value problem* (28) *and for all* $t \in [0, \delta(s_0 - s))$ *we have*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon(t) - u_0(t)\|_s = 0.$$

We observe that if $0 \leq \phi \in L^1 \cap L^\infty$, then, given $\alpha_0 > \alpha > 0$, for all $B \in \mathcal{E}_{\alpha''}$, $\alpha'' \in (\alpha, \alpha_0]$, one finds [14, Proposition 4.4]

$$\|\tilde{L}_{G,\varepsilon,\mathrm{ren}} B - \tilde{L}_{G,V} B\|_{\alpha'} \leq \varepsilon z \|\phi\|_{L^\infty} \|B\|_{\alpha''} e^{\frac{\|\phi\|_{L^1}}{\alpha}} \left( \frac{\|\phi\|_{L^1}\alpha_0}{\alpha'' - \alpha'} + \frac{4\alpha_0^3}{(\alpha'' - \alpha')^2 e} \right)$$

for all $\alpha'$ such that $\alpha \leq \alpha' < \alpha''$ and all $\varepsilon > 0$. Thus, given the local solutions $B^{(\varepsilon)}_{G,t,\mathrm{ren}}, B_{G,t,V}$, $t \in [0, \delta(\alpha_0 - \alpha))$, in $\mathcal{E}_\alpha$ to the initial value problems (26) and (27), respectively, with $B^{(\varepsilon)}_{G,0,\mathrm{ren}}, B_{G,0,V} \in \mathcal{E}_{\alpha_0}$, if $\lim_{\varepsilon\to 0} \|B^{(\varepsilon)}_{G,0,\mathrm{ren}} - B_{G,0,V}\|_{\alpha_0} = 0$, then, by an application of Theorem 3, $\lim_{\varepsilon\to 0} \|B^{(\varepsilon)}_{G,t,\mathrm{ren}} - B_{G,t,V}\|_\alpha = 0$, for each $t \in [0, \delta(\alpha_0 - \alpha))$. Moreover [14, Theorem 4.5], if $B_{G,0,V}(\theta) = \exp\left(\int_{\mathbb{R}^d} dx\, \rho_0(x)\theta(x)\right)$, $\theta \in L^1$, for some function $0 \leq \rho_0 \in L^\infty$ such that $\|\rho_0\|_{L^\infty} \leq \frac{1}{\alpha_0}$, and if $\max\{\frac{1}{\alpha_0}, z\} < \frac{1}{\alpha}$ then, for each $t \in [0, \delta(\alpha_0 - \alpha))$,

$$B_{G,t,V}(\theta) = \exp\left(\int_{\mathbb{R}^d} dx\, \rho_t(x)\theta(x)\right), \quad \theta \in L^1,$$

where $0 \leq \rho_t \in L^\infty$ is a classical solution to Eq. (24) such that, for each $t \in [0, \delta(\alpha_0 - \alpha))$, $\|\rho_t\|_{L^\infty} \leq \frac{1}{\alpha}$. For more results and proofs see [14].

## 3.2 The Kawasaki Dynamics

In this example one shall consider the scale transformation $\phi \mapsto \varepsilon\phi$ of the operator $L_K$ cf. [2], that is,

$$(L_{K,\varepsilon}F)(\gamma) := \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, a(x-y)e^{-\varepsilon E(y,\gamma)} \left(F(\gamma \setminus \{x\} \cup \{y\}) - F(\gamma)\right).$$

To proceed towards GF we consider $k^{(\varepsilon)}_{K,t}$, $k^{(\varepsilon)}_{K,t,\text{ren}}$ and $B^{(\varepsilon)}_{K,t}$ defined as before, which lead to the Cauchy problem

$$\frac{\partial}{\partial t} B^{(\varepsilon)}_{K,t,\text{ren}} = \tilde{L}_{K,\varepsilon,\text{ren}} B^{(\varepsilon)}_{K,t,\text{ren}}, \quad B^{(\varepsilon)}_{K,t,\text{ren}|t=0} = B^{(\varepsilon)}_{K,0,\text{ren}}, \tag{29}$$

with

$$(\tilde{L}_{K,\varepsilon,\text{ren}}B)(\theta) = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y)e^{-\varepsilon\phi(x-y)}(\theta(y) - \theta(x)) \times \delta B\left(\theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; x\right),$$

for all $\varepsilon > 0$ and all $\theta \in L^1$. Similar arguments show [15] that given a $B \in \mathcal{E}_\alpha$ for some $\alpha > 0$, then, for all $\theta \in L^1$, $(\tilde{L}_{K,\varepsilon,\text{ren}}B)(\theta)$ converges as $\varepsilon$ tends to zero to

$$(\tilde{L}_{K,V}B)(\theta) := \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y)(\theta(y) - \theta(x))\delta B(\theta - \phi(y - \cdot); x).$$

In addition, given $0 < \alpha < \alpha_0$, if $B \in \mathcal{E}_{\alpha''}$ for some $\alpha'' \in (\alpha, \alpha_0]$, then $\left\{\tilde{L}_{K,\varepsilon,\text{ren}}B, \tilde{L}_{K,V}B\right\} \subset \mathcal{E}_{\alpha'}$ for all $\alpha \leq \alpha' < \alpha''$, and the following inequality of norms holds

$$\|\tilde{L}_\# B\|_{\alpha'} \leq 2\|a\|_{L^1} \frac{\alpha_0}{(\alpha'' - \alpha')} e^{\frac{\|\phi\|_{L^1}}{\alpha}} \|B\|_{\alpha''},$$

where $\tilde{L}_\# = \tilde{L}_{K,\varepsilon,\text{ren}}$ or $\tilde{L}_\# = \tilde{L}_{K,V}$. Now, let us assume that $0 \leq \phi \in L^1 \cap L^\infty$ and let $\alpha_0 > \alpha > 0$ be given. Then, for all $B \in \mathcal{E}_{\alpha''}$, $\alpha'' \in (\alpha, \alpha_0]$, the following estimate holds [15, Proposition 4.3]

$$\|\tilde{L}_{K,\varepsilon,\mathrm{ren}}B - \tilde{L}_{K,V}B\|_{\alpha'}$$
$$\leq 2\varepsilon\|a\|_{L^1}\|\phi\|_{L^\infty}\frac{e\alpha_0}{\alpha}\|B\|_{\alpha''}e^{\frac{\|\phi\|_{L^1}}{\alpha}}\left(\left(2e\|\phi\|_{L^1}+\frac{\alpha_0}{e}\right)\frac{1}{\alpha''-\alpha'}+\frac{8\alpha_0^2}{(\alpha''-\alpha')^2}\right)$$

for all $\alpha'$ such that $\alpha \leq \alpha' < \alpha''$ and all $\varepsilon > 0$, meaning that one may apply Theorem 3.

**Proposition 4 ([15, Theorem 4.4]).** *Given an* $0 < \alpha < \alpha_0$*, let* $B^{(\varepsilon)}_{K,t,\mathrm{ren}}, B_{K,t,V}$*,* $t \in [0, T)$*, be the local solutions in* $\mathscr{E}_\alpha$ *to the initial value problems* (29)*,*

$$\frac{\partial}{\partial t}B_{K,t,V} = \tilde{L}_{K,V}B_{K,t,V}, \quad B_{K,t,V\,|t=0} = B_{K,0,V},$$

*with* $B^{(\varepsilon)}_{K,0,\mathrm{ren}}, B_{K,0,V} \in \mathscr{E}_{\alpha_0}$*. If* $0 \leq \phi \in L^1 \cap L^\infty$ *and* $\lim_{\varepsilon\to 0}\|B^{(\varepsilon)}_{K,0,\mathrm{ren}} - B_{K,0,V}\|_{\alpha_0} = 0$*, then, for each* $t \in [0, T)$*,* $\lim_{\varepsilon\to 0}\|B^{(\varepsilon)}_{K,t,\mathrm{ren}} - B_{K,t,V}\|_\alpha = 0$*. Moreover, if* $B_{K,0,V}(\theta) = \exp\left(\int_{\mathbb{R}^d} dx\, \rho_0(x)\theta(x)\right)$*,* $\theta \in L^1$*, for some function* $0 \leq \rho_0 \in L^\infty$ *such that* $\|\rho_0\|_{L^\infty} \leq \frac{1}{\alpha_0}$*, then for each* $t \in [0, T)$*,* $B_{K,t,V}(\theta) = \exp\left(\int_{\mathbb{R}^d} dx\, \rho_t(x)\theta(x)\right)$*,* $\theta \in L^1$*, where* $0 \leq \rho_t \in L^\infty$ *is a classical solution to Eq.* (25)*.*

## References


1. Barroso, J.A.: Introduction to Holomorphy. Volume 106 of Mathematics Studies. North-Holland, Amsterdam (1985)
2. Berns, C., Kondratiev, Yu.G., Kozitsky, Yu., Kutoviy, O.: Kawasaki dynamics in continuum: micro- and mesoscopic descriptions. J. Dyn. Diff. Equat. **25**, 1027–1056 (2013)
3. Berns, C., Kondratiev, Yu.G., Kutoviy, O.: Construction of a state evolution for Kawasaki dynamics in continuum. Anal. Math. Phys. **3**(2), 97–117 (2013)
4. Bogoliubov, N.N.: Problems of a Dynamical Theory in Statistical Physics. Gostekhisdat, Moscow (1946) (in Russian). English translation in de Boer, J., Uhlenbeck, G.E. (eds.) Studies in Statistical Mechanics, vol. 1, pp. 1–118. North-Holland, Amsterdam (1962)
5. Dineen, S.: Complex Analysis in Locally Convex Spaces. Volume 57 of Mathematics Studies. North-Holland, Amsterdam (1981)
6. Finkelshtein, D.L., Kondratiev, Yu.G., Kozitsky, Yu.: Glauber dynamics in continuum: a constructive approach to evolution of states. Discret. Contin. Dyn. Syst. **33**(4), 1431–1450 (2013)
7. Finkelshtein, D.L., Kondratiev, Yu.G., Kutoviy, O.: Individual based model with competition in spatial ecology. SIAM J. Math. Anal. **41**, 297–317 (2009)
8. Finkelshtein, D.L., Kondratiev, Yu.G., Kutoviy, O.: Vlasov scaling for stochastic dynamics of continuous systems. J. Stat. Phys. **141**, 158–178 (2010)
9. Finkelshtein, D.L., Kondratiev, Yu.G., Kutoviy, O.: Vlasov scaling for the Glauber dynamics in continuum. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **14**(4), 537–569 (2011)
10. Finkelshtein, D.L., Kondratiev, Yu.G., Kutoviy, O.: Correlation functions evolution for the Glauber dynamics in continuum. Semigroup Forum **85**, 289–306 (2012)
11. Finkelshtein, D.L., Kondratiev, Yu.G., Kutoviy, O.: Semigroup approach to non-equilibrium birth-and-death stochastic dynamics in continuum. J. Funct. Anal. **262**, 1274–1308 (2012)

12. Finkelshtein, D., Kondratiev, Yu., Kutoviy, O., Zhizhina, E.: An approximative approach to construction of the Glauber dynamics in continuum. Math. Nachr. **285**, 223–235 (2012)
13. Finkelshtein, D.L., Kondratiev, Yu.G., Oliveira, M.J.: Markov evolutions and hierarchical equations in the continuum. I: one-component systems. J. Evol. Equ. **9**(2), 197–233 (2009)
14. Finkelshtein, D.L., Kondratiev, Yu.G., Oliveira, M.J.: Glauber dynamics in the continuum via generating functionals evolution. Complex Anal. Oper. Theory **6**(4), 923–945 (2012)
15. Finkelshtein, D.L., Kondratiev, Yu.G., Oliveira, M.J.: Kawasaki dynamics in the continuum via generating functionals evolution. Methods Funct. Anal. Topol. **18**(1), 55–67 (2012)
16. Garcia, N.L., Kurtz, T.G.: Spatial birth and death processes as solutions of stochastic equations. ALEA Lat. Am. J. Probab. Math. Stat. **1**, 281–303 (2006)
17. Kondratiev, Yu., Kozitsky, Yu., Shoikhet, D.: Dynamical systems on sets of holomorphic functions. In: Complex Analysis and Dynamical Systems IV, Part 1. Contemporary Mathematics, vol. 553. American Mathematical Society, Providence (2011)
18. Kondratiev, Yu.G., Kuna, T.: Harmonic analysis on configuration space I. General theory. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **5**(2), 201–233 (2002)
19. Kondratiev, Yu.G., Kuna, T., Oliveira, M.J.: Analytic aspects of Poissonian white noise analysis. Methods Funct. Anal. Topol. **8**(4), 15–48 (2002)
20. Kondratiev, Yu.G., Kuna, T., Oliveira, M.J.: On the relations between Poissonian white noise analysis and harmonic analysis on configuration spaces. J. Funct. Anal. **213**(1), 1–30 (2004)
21. Kondratiev, Yu.G., Kuna, T., Oliveira, M.J.: Holomorphic Bogoliubov functionals for interacting particle systems in continuum. J. Funct. Anal. **238**(2), 375–404 (2006)
22. Kondratiev, Yu., Kutoviy, O., Minlos, R.: On non-equilibrium stochastic dynamics for interacting particle systems in continuum. J. Funct. Anal. **255**, 200–227 (2008)
23. Kondratiev, Yu., Kutoviy, O., Zhizhina, E.: Nonequilibrium Glauber-type dynamics in continuum. J. Math. Phys. **47**(11), 113501 (2006)
24. Kondratiev, Y., Skorokhod, A.: On contact processes in continuum. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **9**(2), 187–198 (2006)
25. Kuna, T.: Studies in configuration space analysis and applications. PhD thesis, Bonner Mathematische Schriften Nr. 324, University of Bonn (1999)
26. Lenard, A.: States of classical statistical mechanical systems of infinitely many particles II. Arch. Ration. Mech. Anal. **59**, 241–256 (1975)
27. Treves, F.: Ovcyannikov Theorem and Hyperdifferential Operators. Volume 46 of Notas de Matemática. IMPA, Rio de Janeiro (1968)

# Interacting Particle Systems: Hydrodynamic Limit Versus High Density Limit

**Tertuliano Franco**

## 1 Introduction

A central question in Statistical Mechanics is about the passage from discrete systems to the *continuum*. And, consequently, how the intrinsic properties of a discrete systems are inherited by the *continuum*. Aiming for rigorous results in this scope, fruitful mathematical theories have been developed since the last century.

An important class of discrete systems are the so-called *interacting particle systems*. Roughly speaking, an interacting particle system is a discrete system that evolves in time according to random clocks under some interaction among particles. To clarify ideas, two examples of interacting particle systems are presented in the Sect. 2.

A quantity of interest associated to a particle system is the spatial density of particles. Since the system evolves in time, its spatial density of particles evolves as well. Is therefore natural to investigate the possible limits for the *time trajectory* of the spatial density of particles.

The limiting object for the time trajectory of the spatial density of particles is usually described as the solution of some partial differential equation. A standard hypothesis is to suppose that, at initial time, the spatial density of particles converges to a profile $\varphi$ as the mesh of the lattice goes to zero. This profile $\varphi$ will be, as reasonable, the initial condition of the respective partial differential equation.

The nature of the convergence (topology, parameters to be rescaled) is the subject of this paper. For sake of clarity, we take the liberty to divide the main types of convergence in two classes: the hydrodynamic limit and the high density limit. The expression *hydrodynamic limit* is widely used in the literature. On the hand, the *high density limit* nomenclature is less known, being employed in the paper [12].

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The hydrodynamic limit consists of the limit for the time trajectory of the spatial density of particles where the parameters to be rescaled are time and space. The space among particles is lead to zero, while the time the system has to evolve is lead to infinity. If the time is taken as the inverse of the space among particles, this situation is called the *ballistic scaling*. If the time is as taken as square of the inverse of the space among particles, it is called the *diffusive scaling*. The initial configuration of particles is randomly chosen according to a distribution related to the fixed profile $\varphi$.

The high density limit consists of the limit for the time trajectory of the spatial density of particles where the parameters to be rescaled are: time, space *and initial quantity of particles per site*. While space among particles is lead to zero, time and initial quantity of particles per site are lead to infinity. As suggested, the fact the initial quantity of particles increases in a meaningful way originated the nomenclature.

A vast literature has been produced about the hydrodynamic limit, which is nowadays an exciting and active research area. For a reference in the subject, we refer the reader to the classical book [10]. For very important techniques in the area, we cite the *Entropy Method*, the *Relative Entropy Method*, non-gradient techniques, attractiveness techniques, among many others. In its turn, the high density limit approach had important papers about as [1, 3–5, 11, 12]. As a more recent paper on the subject, we cite [6].

The hydrodynamic limit is far more studied and known. But the high density limit has interesting characteristics and a plenty of open problems, some of them discussed here. Our goal in this short survey is to compare main aspects of each approach, exemplify them, and debate for whose models each one is more suitable to (in some sense).

The outline of this paper is: in Sect. 2, two interacting particle systems are presented. In Sect. 3, it is made a comparison of results in each approach for the law of large numbers scenario. In Sect. 4, the same for the central limit theorem scenario, and in Sect. 5, the same for the large deviations principle scenario.

## 2 Two Interacting Particle Systems

In this section we present two dynamics of interacting particle systems. For a classical reference on particle systems, we cite the book [13]. The first example is the symmetric simple exclusion process, the second one is a system of independent random walks with birth-and-death dynamics. Denote by

$$T_N = \mathbb{Z}/(N\mathbb{Z}) = \{0, 1, 2, \ldots, N-1\}$$

the discrete one-dimensional torus with $N$ sites.

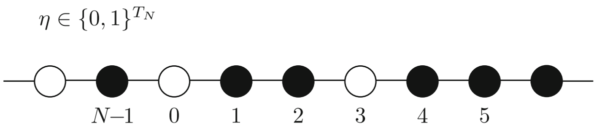


**Fig. 1** A configuration of particles $\eta \in \{0,1\}^{T_N}$. Observe that $\eta(N-1) = 1$, $\eta(0) = 0$, $\eta(1) = 1$, $\eta(2) = 1$, etc. Notice that, since $T_N$ is the discrete torus, $x = 0$ and $x = N$ represent the same site

**The symmetric simple exclusion process** The symmetric simple exclusion process, abbreviated by SSEP, is a quite standard, widely studied model in Probability and Statistical Mechanics. In words, in the SSEP each particle performs an independent continuous-time symmetric random walk except when some particle tries to jump to an already occupied site. When this happens, this jumps is forbidden, and nothing happens. This *exclusion rule* originates the name exclusion process. In several models of physical phenomena, fermions dynamics are constrained by an exclusion rule.

Of course, since a particle can not jump to an already occupied site, the state space in this case is $\{0,1\}^{T_N}$. For $x \in T_N$, we will write down $\eta(x)$ for the number of particles at the site $x$ in the configuration $\eta$ of particles. See Fig. 1 above, where black balls represent particles.

The dynamics is the following: to each *edge* $(x, x+1)$ of the discrete torus $T_N$, it is associated a Poisson point process[1] of parameter $N^2$, all of them independent. At a time arrival of the Poisson process corresponding to the edge $(x, x+1)$, the occupations at $\eta(x)$ and $\eta(x+1)$ are interchanged. In case that both sites $x$ and $x+1$ are occupied, of course, nothing happens.[2] Since the Poisson processes are independent, the probability to observe two marks at same time is zero. Hence, there is no chance to a particle be in "doubt" whether to jump, and the construction is well defined. Figure 2 illustrates ideas.

Given an initial configuration of particles $\eta \in \{0,1\}^{T_N}$, this construction yields the continuous time Markov chain $\{\eta_t\ ;\ t \geq 0\}$, which is the so-called SSEP.

**Independent random walks with birth-and-death dynamics** In this particle system we have a superposition of two standard dynamics. One is given by independent random walks, where each particle has its own random clock (a Poisson process associated to him). When the clock rings, the particle chooses with equal probability one of the neighbor sites and jumps to there. For independent random walks, there is no interaction among particles. The other part of the dynamics is given by birth and death of particles at each site. The birth and death rates at a site $x$ are given by functions $b$ and $d$, respectively, of the number of particles at that site $x$.

[1] A Poisson process can be described as marks in time, being the time between marks i.i.d of exponential distribution. The parameter $N^2$ has to do with the scaling we are going to perform later.

[2] Corroborating the exclusion rule.

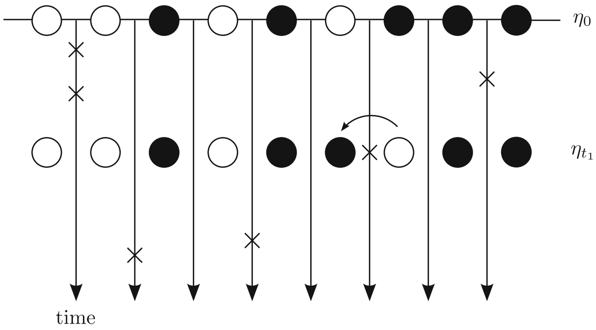


**Fig. 2** At right, an evolution of the initial configuration $\eta_0$ according to the Poisson processes (the marks represent the time arrivals). At time $t_1$, a particle jumps to a neighbor site. Notice that at the three marks in times previous to $t_1$ nothing happened because both sites related to the mark were empty or occupied

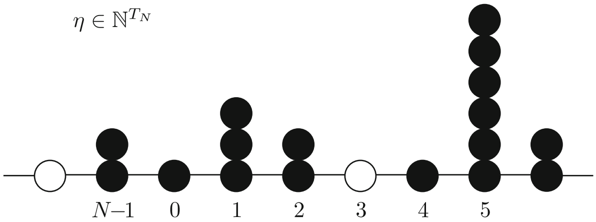


**Fig. 3** A configuration of particles $\eta \in \mathbb{N}^{T_N}$. Observe that $\eta(N-1) = 2$, $\eta(0) = 1$, $\eta(1) = 3$, $\eta(2) = 2$ and $\eta(3) = 0$. Again, since $T_N$ is the discrete torus, $x = 0$ and $x = N$ represent the same site

In each site of $T_N$ we allow a nonnegative integer quantity of indistinguishable particles. A configuration of particles will be denote by $\eta$, which is an element of $\{\mathbb{N} \cup \{0\}\}^{T_N}$. As before, we will write down $\eta(x)$ for the number of particles at the site $x$ in the configuration $\eta$ of particles, see Fig. 3 above.

Next, we construct the system of independent random walks with birth-and-death dynamics. Fix two nonnegative smooth[3] functions $b, d : \mathbb{R}_+ \to \mathbb{R}_+$ such that $d(0) = 0$ and fix $\ell = \ell_N$ a positive parameter. In the next section we will see that this parameter $\ell$ represents the number of particles per site at the initial time.

Consider the following transition rates:

- At rate $N^2\eta(x)$, a particle jumps from $x$ to $x+1$;
- At rate $N^2\eta(x)$, a particle jumps from $x$ to $x-1$;
- At rate $\ell b(\ell^{-1}\eta(x))$, a new particle is created at $x$;
- At rate $\ell d(\ell^{-1}\eta(x))$, a particle is destroyed at $x$.

[3]The smoothness is not necessary here. It will be required only in the later scaling.

The transitions above are assumed for all $x \in \mathbb{T}_N$. Each transition corresponds to an arrival of an independent Poisson process of parameter given by the respective rate. That is, an analogous graphical construction (as the aforementioned for the SSEP) can be made in this case, see [6].

Since there are no assumptions about the growth of $b$, the waiting times of this Markov chain can be summable.[4] In this case we say that the process *explodes* or *blows up*, and we define the state of the process as $\infty$ for times greater or equal than the sum of all the waiting times, that we call $T^N_{\max}$. More precisely, define

$$\tau^N_y := \inf\{t \geq 0 \colon \|\eta(t)\|_\infty \geq y\} \quad \text{and} \quad \tau^N_{\text{blow-up}} := \lim_{y\to\infty} \tau^N_y \,.$$

Then, for $t < \tau^N_{\text{blow-up}}$, we define $\eta(t)$ by means of the rates stated before, and for $t \geq \tau^N_{\text{blow-up}}$, we define $\eta(t) = \infty$. This characterizes a continuous time Markov chain

$$\{\eta_t \ ; \ t \geq 0\}$$

with state space $\mathbb{N}^{T_N} \cup \{\infty\}$.

## 3 Law of Large Numbers Scenario for Each Setting

In order to state the limit for the time trajectory of the spatial density of particles, we need to define first what we mean by a spatial density of particles. We point out that the definition of the spatial density of particles is different for each setting, the hydrodynamic limit or the high density limit. In the first one, given the SSEP $\{\eta_t \ ; \ t \geq 0\}$ described in the Sect. 2, the spatial density of particles, usually called the *empirical measure*, is defined as

$$\pi^N_t \ := \ \frac{1}{N} \sum_{x \in T_N} \eta_t(x)\, \delta_{\frac{x}{N}} \,. \tag{1}$$

As can be easily seen, the empirical measure is:

- A positive measure (since it is a sum of deltas of Dirac);
- A random measure (since it is a function of $\eta_t$, which is random);
- A measure with total mass bounded by one (by the normalization constant $N$);

[4]Meaning that the total quantity of particles has exploded. For more on explosions of Markov chains see [15].

- A measure that gives mass $1/N$ at the point $x/N$ belonging at the continuous one-dimensional torus

$$\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1)$$

if there is a particle at the site $x \in T_N$ (at that time $t$), and gives measure 0 otherwise.

Denote by $\mathscr{M}_+$ the space of positive measures on $\mathbb{T}$ with mass bounded by one and by $\mathscr{D}([0, T], \mathscr{M}_+)$ the set of càdlàg[5] time trajectories taking values on $\mathscr{M}_+$. We notice that the time trajectory of the empirical measure (1) is a random element taking values in $\mathscr{D}([0, T], \mathscr{M}_+)$.

The theorem stated next is what we call the hydrodynamic limit (for the SSEP). For a proof, see [10, Chap. 4]. We notice that the topology assumed in the convergence in distribution ahead is the Skorohod topology on $\mathscr{D}([0, T], \mathscr{M}_+)$. For an exposition on the Skohorod topology, see [2] or [10].

**Theorem 1.** *Fix $\varphi : \mathbb{T} \to [0, 1]$ a smooth function and $T > 0$. Suppose that the initial distribution of particles for the SSEP are chosen in such a way, as $N \to \infty$,*

$$\pi_0^N \longrightarrow \varphi(u)\, du \quad \textit{in probability}.$$

*Then, as $N \to \infty$,*

$$\{\pi_t^N \,;\, 0 \leq t \leq T\} \longrightarrow \{\rho(t, u)\, du \,;\, 0 \leq t \leq T\} \quad \textit{in distribution,}$$

*where $\rho(t, u)$ is the unique solution of the periodic heat equation with initial condition $\varphi$, or else,*

$$\begin{cases} \partial_t \rho(t, u) \;=\; \partial_{uu} \rho(t, u)\,,\; t \geq 0,\; u \in \mathbb{T}\,, \\ \rho(0, u) \;=\; \varphi(u), \qquad\quad u \in \mathbb{T}\,. \end{cases} \tag{2}$$

It is not our intention to resume a huge research area in few words. Nevertheless, let us make some remarks. The hydrodynamic limit has been successfully applied in interacting particles systems as: the zero range model; the symmetric simple exclusion process, the asymmetric simple exclusion process, the Ginzburg-Landau model, the generalized exclusion process, among others.

We point out here the power of the existing methods for proving the hydrodynamic limit of models whose microscopic interactions often lead to non-linear

[5]From the French, right continuous and with side left limits.

partial differential equations. In the above case, it is obtained a linear heat equation. For the zero range process, the partial differential equation would the a non-linear one, with $\partial_{uu}\Phi(\rho)$ replacing $\partial_{uu}\rho$ in (2), where the function $\Phi$ is defined via the microscopic interaction.

Despite the wide applicability, the available techniques for hydrodynamic limit are not suitable for systems with huge birth's rate of new particles. There are some papers on the subject as [14], but in a scheme where the birth's rate of new particles is small in some sense. The reason is simple, in the situation where the particle system explodes in finite time, the expectation of the number of particles is infinity at any positive time. Hence, any method based on expectations is doomed to fail. And most of hydrodynamic techniques are based on expectation techniques.

For $x \in T_N$, let $u_x = x/N \in \mathbb{T}$. We define now $X^N : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}_+$, the spatial density of particles for the high density limit scenario, by

$$X^N(t, u_x) \;=\; \ell^{-1}\eta_t(x)\,.$$

For $u_x < u < u_{x+1}$, we define the spatial density via a linear interpolation, i.e.,

$$X^N(t, u) \;=\; (Nu - x)\; X^N(t, u_{x+1}) + (x + 1 - Nx)\; X^N(t, u_k)\,.$$

If $\eta(t) = \infty$, we say that $X^N(t, \cdot) = \infty$ as well.

Before stating a high density limit theorem, let us say some words about the partial differential equation

$$\begin{cases} \partial_t\rho(t,u) = \partial_{uu}\rho(t,u) + f(\rho(t,u))\,, & t \in [0,T),\ u \in \mathbb{T}\,, \\ \rho(0,u) = \varphi(u) \geq 0\,, & u \in \mathbb{T}\,. \end{cases} \tag{3}$$

Above, $f = b - d$, where $d$ and $b$ are the aforementioned smooth functions that drive the birth and death of particles, and $\varphi$ is smooth and nonnegative. Since there is not restriction about the growth of $f$, the solution of (3) can exhibit a phenomena called *blow-up* or *explosion*. In this case, there is a finite time $T_{\text{blow-up}}$ for which

$$\lim_{t \nearrow T_{\text{blow-up}}} \|\rho(t,\cdot)\|_\infty = \infty\,, \tag{4}$$

and such that $\|\rho(t,\cdot)\|_\infty$ is finite for times smaller than $T_{\text{blow-up}}$. In this case, the solution $\rho$ of (3) is defined only in the time interval $[0, T_{\text{blow-up}})$. If there is no explosion, we would say that $T_{\text{blow-up}} = \infty$.

Remark: a well known condition on the nonlinear term $f$ that assures the existence of solutions with blow-up is being convex, strictly positive in some interval $[a, +\infty)$ and $\int_a^\infty \frac{ds}{f(s)} < \infty$.

**Theorem 2 (F., Groisman '12).** *Assume that*

*(A1)* $\lim_{N\to\infty} \|X^N(0,\cdot) - \varphi(\cdot)\|_\infty = 0$ *almost surely;*
*(A2) for any* $c > 0$, $\sum_{N\geq 0} N^3 e^{-c\,\ell(N)} < \infty$.

*Then, for any* $T \in [0, T_{blow\text{-}up})$,

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \|X^N(t,\cdot) - \rho(t,\cdot)\|_\infty = 0 \quad \textit{almost surely,}$$

*where* $\rho$ *is the solution of* (3) *and* $T_{blow\text{-}up}$ *is given in* (4).

The assumption (A1) allows to interpret $\ell$ as the quantity of particles per site at initial time. Roughly speaking, since $X_0^N$ is $\ell^{-1}\eta_0$, and $X_0^N$ converges in the supremum norm to the function $\varphi$, the initial quantity of particles at a point $u \in \mathbb{T}$ is of order $\ell\,\varphi(u)$.

We remark that $\pi_t^N$ and $X^N(t,u)$ are equivalent in some sense. With due care, the hydrodynamic limit could be stated in terms of piecewise affine functions and, vice versa, the high density limit could be stated in terms of a suitable empirical measure.

The result above is proved in [6] making use of couplings. It is a challenging problem to obtain some analogous results in the hydrodynamic limit setting. For instance, the zero range process has no limit of particles per site. Superposing this dynamics with birth of particles, explosions can occur under suitable choice of rates. The hydrodynamic limit of this model could be studied.

The known techniques for the high density limit are strongly support on three pillars: martingales, Duhamel's Principle and smoothing properties of the heat equation semi-group. Duhamel's Principle is a general idea widely applied in ordinary different equations, partial differential equations, numerical schemes, *etcetera*. For a system whose dynamics has two parts, being one of them linear, the solution (in time) can be expressed as the initial condition evolved by the linear part plus an convolution of the nonlinear part with the semi-group of the linear part.

Keeping this in mind, we can realize why the literature on high density limit is concentrated in dynamics involving independent random walks: in order to apply Duhamel's Principle, it is necessary to have a linear part in the dynamics, and the independent random walks plays this role. It is an open problem to extend the high density limit for others dynamics as the zero range process, for example.

In the paper [14], it was considered the exclusion process superposed with birth dynamics, but with no explosions. It is a challenging problem to prove the hydrodynamic limit for the zero range process with birth of new particles and explosion in finite time.

## 4 Central Limit Theorem Scenario for Each Setting

There are a lot of results on the central limit theorem for both settings, the hydrodynamic limit and the high density limit. In the high density limit setting, we cite [3, 5, 11, 12]. In the hydrodynamic limit setting, we cite [7, 9].

In both settings, the limit is usually described through generalized Ornstein-Uhlenbeck processes, see [8] or [10] on this type of stochastic process.

We cite as an interesting open problem to prove the central limit theorem for the spatial density of particles near the explosion for the model considered in [6].

## 5 Large Deviation Principle Scenario for Each Setting

Recall that $\mathscr{M}_+$ denote the space of positive measures on $\mathbb{T}$ with mass bounded by one and by $\mathscr{D}([0,T],\mathscr{M}_+)$ the set of càdlàg time trajectories with values on $\mathscr{M}_+$.

By a large deviations principle in the hydrodynamic setting, we mean the existence of a lower semicontinuous rate function

$$I\,:\,\mathscr{D}([0,T],\mathscr{M}_+)\to\mathbb{R}_+\cup\{\infty\}$$

such that:

*For each closed set $\mathscr{C}$, and each open set $\mathscr{O}$ of $\mathscr{D}([0,T],\mathscr{M}_+)$,*

$$\limsup_{N\to\infty}\frac{1}{N}\log Q^N[\mathscr{C}]\;\leq\;-\inf_{\pi\in\mathscr{C}}I(\pi)\,,$$

$$\liminf_{N\to\infty}\frac{1}{N}\log Q^N[\mathscr{O}]\;\geq\;-\inf_{\pi\in\mathscr{O}}I(\pi)\,,$$

where $Q^N$ is the probability induced the in the space $\mathscr{D}([0,T],\mathscr{M}_+)$ by the empirical measure. A proof of the large deviation principle for the SSEP can be found in [10, Chap. 10].

For the high density limit, the statement of a LDP is analogous, *mutatis mutandis* with respect to the topology. However, there is no LDP available yet for the high density limit scenario and we list some difficulties for proving it.

An important step in order to obtain a LDP for a model is to obtain a law of large numbers for a class of perturbed process. The Radon-Nykodim between the original process and the perturbed one will further give the cost to observe the limit given by the perturbed one from the point-of-view of the original process. In the proof of a LDP, it is made a careful analysis and precise optimization over the perturbations.

For the SSEP, the perturbations are given by weakly asymmetric exclusion process, where the asymmetry is driven by a smooth function $H$, being the limit for the hydrodynamic limit given by a solution of a heat equation with a non-linear Burgers term added, see [10, page 273].

As commented in the Sect. 3, the known techniques for the high density limit are strongly based on martingales, Duhamel's Principle and smoothing properties of the semi-group corresponding to the linear part of the dynamics. In order to obtain a LDP for the high density limit (let us say, in the case of independent random walks with birth-and-death dynamics) it would be necessary to prove the high density limit for some non-linear situation not attained yet in the literature.

Other obstacle is the presence of two superposed dynamics. To observe a given profile that differs from the expected limit, it would be possible to consider two different perturbations at same time, one about the diffusion and another about the birth-and-death of particles. Performing variational analysis over two competing different perturbations is a complicated situation in LDP.

In resume, LDP is a challenging open problem in the high density limit scenario.

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## References

1. Arnold, L., Theodosopulu, M.: Deterministic limit of the stochastic model of chemical reactions with diffusion. Adv. Appl. Probab. **12**, 367–379 (1980)
2. Billingsley, P.: Convergence of Probability Measures. Wiley Series in Probability and Statistics, 2nd edn. Wiley, New York (1999)
3. Blount, D.: Comparison of stochastic and deterministic models of a linear chemical reaction with diffusion. Ann. Probab. **19**(4), 1440–1462 (1991)
4. Blount, D.: Law of large numbers in the supremum norm for a chemical reaction with diffusion. Ann. Appl. Probab. **2**(1), 131–141 (1992)
5. Blount, D.: Diffusion limits for a nonlinear density dependent space-time population model. Ann. Probab. **24**(2), 639–659 (1996)
6. Franco, T., Groisman, P.: A particle system with explosions: law of large numbers for the density of particles and the blow-up time. J. Stat. Phys. **149**(4), 629–642 (2012)
7. Gonçalves, P.: Central limit theorem for a tagged particle in asymmetric simple exclusion. Stoch. Process Appl **118**, 474–502 (2008)
8. Holley, R., Stroock, D.: Generalized Ornstein-Uhlenbeck processes and infinite particle branching Brownian motions. Publ. Res. Inst. Math. Sci. **14**(3), 741–788 (1978)
9. Jara, M., Landim, C.: Non equilibrium central limit theorem for a tagged particle in symmetric simple exclusion. Ann. Inst. H. Poincaré (B) Probab. Stat. **42**, 567–577 (2006)
10. Kipnis, C., Landim, C.: Scaling Limits of Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 320. Springer, Berlin (1999)
11. Kotelenez, P.: Law of large numbers and central limit theorem for linear chemical reactions with diffusions. Ann. Probab. **14**(1), 173–193 (1986)

12. Kotelenez, P.: High density limit theorems for nonlinear chemical reactions with diffusion. Probab. Theory Relat. Fields **78**, 11–37 (1988)
13. Liggett, T.: Interacting Particle Systems. Springer, New York (1985)
14. Mourragui, M.: Comportement hydrodynamic et entropie relative des processus de sauts, de naissancess et de morts. Ann. Inst. Henri Poincaré **32**(3), 361–385 (1996)
15. Norris, J.R.: Markov Chains. Cambridge University Press, Cambridge/New York (1998)

# Slowed Exclusion Process: Hydrodynamics, Fluctuations and Phase Transitions

**Tertuliano Franco, Patrícia Gonçalves, and Adriana Neumann**

## 1 Introduction

A major question in Statistical Mechanics is how to perform the limit from the discrete to the continuum in such a way that the discretization of the system really gives the correct description of the continuum? This question gave rise to plenty of famous models and results, both in Physics and Mathematics. In the particular context of *particle systems* and *hydrodynamic limits*, the passage of the discrete to the continuum is a consequence of rescaling both time and space. The discrete system consists in a collection of particles with a stochastic dynamics. Depending on the prescribed interaction we are lead to different limits. Therefore the random interaction of the microscopic system is connected to the macroscopic phenomena to be explored.

As the main reference on the subject, we cite the classical book [9], which treats the limit of several particle systems, as the zero range process, the symmetric and asymmetric exclusion process, the generalized $K$-exclusion process, independent

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random walks and some of their scaling limits. We point out some of the possible natures of those scaling limits.

The scaling limit for the time-trajectory of the spatial density of particles is the so-called *hydrodynamic limit* of the system, which is a Law of Large Numbers (L.L.N.) type-theorem. The scaling limit for how the discrete system oscillates around its hydrodynamic limit is usually referred as *fluctuations*, being a Central Limit Theorem (C.L.T.). The study of the rate at which the probability of observing the discrete deviates from the expected limit decreases (roughly, exponentially fast) is the theme of the Large Deviations Principle.

Recently, the scientific community has given attention to particle systems in random and non-homogeneous media, and several approaches have been developed in order to study the problem. In the papers [2, 8, 10], the authors considered random walks in a random environment, as for example the case where the environment is driven by an $\alpha$-subordinator. These works inspired a series of other papers in the context of particle systems, as [1, 6, 7, 11]. The work in [6] was related to the hydrodynamic limit of exclusion processes driven by a general increasing function $W$, not necessarily a toss of an $\alpha$-subordinator. This work, in its hand, inspired the work [3], which dealt with the case $W$ being the distribution function of the Lebesgue measure plus a delta of Dirac measure, being the mass of the delta of Dirac dependent on the scale parameter. The model of [3] can be described as follows. To each site of the discrete torus with $n$ sites, it is allowed to have at most one particle. Each bond has a Poisson clock which is independent of the clocks on other sites. When the Poisson clock of a bond rings, the occupation at the vertices of this bond are interchanged. All the Poisson clocks have parameter one, except one special clock, which has parameter given by $\alpha n^{-\beta}$, with $\alpha > 0$ and $\beta \in [0, \infty]$. This "slower" clock, makes the passage of particles across the corresponding bond more difficult, and for that reason that bond coined the name *slow bond*.

In the scenario of [3], according to the value of $\beta$, three different limits for the time trajectory of the spatial density of particles were obtained. If $\beta \in [0, 1)$ the limit is given by the weak solution of the periodic heat equation, meaning that the slow bond is not slow enough to originate any change in the continuum. If $\beta = 1$, the limit is given by the weak solution of the heat equation with some Robin's boundary conditions representing the Fick's Law of passage of particles. And if $\beta \in (1, \infty]$, the limit is given by the weak solution of the heat equation with Neumann's boundary conditions, meaning that the slow bond in this regime of $\beta$ is slow enough to divide the space in the continuum.

Such dynamical phase transition (based on the strength of a single slow bond) is not limited to the hydrodynamic limit. In the ensuing papers [4, 5], some other dynamical phase transitions were proved. In [4], it was shown that the solutions of the three partial differential equations aforementioned are continuously related to a given boundary's parameter, indicating a dynamical phase transition also at the macroscopic level. In [5], it was proved that the equilibrium fluctuations of the exclusion process with a slow bond evolving on an infinite volume, is also characterized by the same regimes of $\beta$. As before, in each case, namely for

$\beta \in [0, 1)$, $\beta = 1$ or $\beta \in (1, \infty]$, the limit fluctuations of the system are driven by three Ornstein-Uhlenbeck processes. As a consequence of the density fluctuations, we have also obtained the corresponding phase transition for the current of particles through a fixed bond and for a tagged particle.

In these notes we make a synthesis of last results, all of them related to dynamic phase transitions that occur when the strength of a particular slow bond varies. We notice that the theme is not finished at all. There are a lot of particle systems to examine and different limits to prove. As an example, in the cited papers [3–5], the underlying particle systems are only of exclusion constrain and with symmetric dynamics. Therefore, one can exploit other dynamics and obtain other partial differential equations of physical interest. Moreover, even for the symmetric exclusion dynamics with a slow bond, the full scenario for the scaling limits is not closed yet: a Large Deviations Principle is still open. This is subject for future work.

Here follows an outline of these notes. In Sect. 2 we present the exclusion process with a slow bond. Section 3 is devoted to the scaling limits at the level of hydrodynamics. We present the hydrodynamic equations, the hydrodynamic limit and the phase transition for the corresponding partial differential equations. In Sect. 4 we present the scaling limits at the level of fluctuations. We present the Ornstein-Uhlenbeck processes and the fluctuations of the density of particles. We finish in Sect. 5 with a description of the fluctuations of the current of particles and of a tagged particle.

## 2 Exclusion Processes

We are concerned with the study of dynamical phase transitions in particle systems with a single slow bond. Before discussing what we mean by a dynamical phase transition we describe our particle systems. We consider the simple exclusion process (SEP) with a single slow bond. Probabilistic speaking, the SEP is a Markov process that we denote by $\{\eta_t : t \geq 0\}$ and we consider it evolving on the state space $\Omega := \{0, 1\}^{\mathbb{T}_n}$, where $\mathbb{T}_n = \mathbb{Z}/n\mathbb{Z}$ is the one-dimensional discrete torus with $n$ points. A configuration of this Markov process is denoted by $\eta$ and it consists in a vector with $n$ components, each one taking the value 0 or 1. The physical interpretation is that whenever $\eta(x) = 1$ we say that the site $x$ is occupied, otherwise it is empty.

The microscopic dynamics of this process can be informally described as follows. At each bond $\{x, x + 1\}$ of $\mathbb{T}_n$, there is an exponential clock of parameter $a^n_{x,x+1}$. When this clock rings, the value of $\eta$ at the vertices of this bond are interchanged. We choose the parameters of the clocks in all bonds equal to 1, except at the bond $\{-1, 0\}$, in such a way that the passage of particles across this bond is more difficult with respect to other bonds. For $\beta \in [0, \infty]$ and $\alpha > 0$, we consider

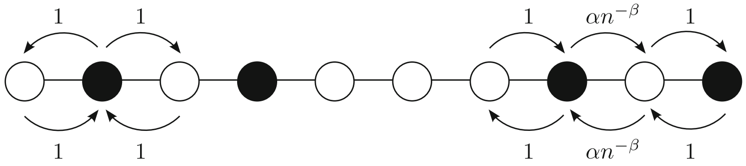


**Fig. 1** SEP with a slow bond with vertices $\{-1, 0\}$, whose jump rates are given by $\alpha n^{-\beta}$. *Black balls* represent occupied sites

$$a^n_{x,x+1} = \begin{cases} \alpha n^{-\beta}, \text{ if } \; x = -1\,, \\ \quad 1, \quad \text{otherwise}\,. \end{cases}$$

This means that particles cross all the bonds at rate 1, except the bond $\{-1, 0\}$, whose dynamics is slowed down as $\alpha n^{-\beta}$, with $\alpha > 0$ and $\beta \in [0, \infty]$, see Fig. 1.

The dynamics described above can be characterized via the infinitesimal generator, which we denote by $\mathcal{L}_n$ and is given on functions $f : \Omega \to \mathbb{R}$ as

$$\mathcal{L}_n f(\eta) = \sum_{x \in \mathbb{T}_n} a^n_{x,x+1} \left[ f(\eta^{x,x+1}) - f(\eta) \right] ,$$

where $\eta^{x,x+1}$ is the configuration obtained from $\eta$ by exchanging the occupation variables $\eta(x)$ and $\eta(x+1)$, namely

$$\eta^{x,x+1}(y) = \begin{cases} \eta(x+1), \text{ if } \; y = x\,, \\ \quad \eta(x), \quad \text{if } \; y = x+1\,, \\ \quad \eta(y), \quad \text{otherwise}\,. \end{cases}$$

Let $\rho \in [0, 1]$ and denote the Bernoulli product measure, defined in $\Omega$ and with parameter $\rho$, by

$$\nu^n_\rho\{\eta \in \Omega : \; \eta(x) = 1, \; \text{ for any } \; x \in A\} = \rho^{\#A},$$

for all set $A \subset \mathbb{T}_n$. Here $\#A$ denotes the cardinality of the set $A$. It is well known that the measures $\nu^n_\rho$ are invariant for the dynamics introduced above. Moreover, these measures are also reversible.

The trajectories of the Markov process $\{\eta_t : t \geq 0\}$ live on the space $\mathcal{D}(\mathbb{R}_+, \Omega)$, that is, the path space of càdlàg trajectories with values in $\Omega$. For a measure $\mu_n$ on $\Omega$, we denote by $\mathbb{P}_{\mu_n}$ the probability measure on $\mathcal{D}(\mathbb{R}_+, \Omega)$ induced by $\mu_n$ and $\{\eta_t : t \geq 0\}$; and we denote by $\mathbb{E}_{\mu_n}$ expectation with respect to $\mathbb{P}_{\mu_n}$.

We notice that we do not index the Markov process, the generator nor the measures, in $\beta$ or $\alpha$ for simplicity of notation.

# 3 Hydrodynamical Phase Transition

The study of the hydrodynamical behavior consists in the analysis of the time evolution of the density of particles. For that purpose we introduce the empirical measure process as follows.

For $t \in [0,T]$, let $\pi_t^n(\eta, du) := \pi^n(\eta_t, du) \in \mathcal{M}$ be defined as

$$\pi^n(\eta_t, du) \;=\; \tfrac{1}{n} \sum_{x \in \mathbb{T}_n} \eta_t(x)\, \delta_{x/n}(du)\,,$$

where $\delta_y$ is the Dirac measure concentrated on $y \in \mathbb{T}$. Above, $\mathbb{T}$ denotes the one-dimensional torus and $\mathcal{M}$ denotes the space of positive measures on $\mathbb{T}$ with total mass bounded by one, endowed with the weak topology.

The hydrodynamic limit can be stated as follows. If we assume a L.L.N. for $\{\pi_0^n\}_{n\in\mathbb{N}}$ to a limit $\rho_0(u)du$ under the initial distribution of the system, then at any time $t > 0$ the L.L.N. holds for $\{\pi_t^n\}_{n\in\mathbb{N}}$ to a limit $\rho(t,u)du$ under the corresponding distribution of the system at time $t$. Moreover, the density $\rho(t,u)$ evolves according to a partial differential equation – the hydrodynamic equation. For this model, depending on the range of the parameter $\beta$, we obtain different hydrodynamic equations for the underlying particle system.

In the next section we describe the hydrodynamic equations we obtained and we precise in which sense $\rho(t,u)$ is a solution to those equations.

## 3.1 Hydrodynamic Equations

We start by describing the hydrodynamic equations that govern the evolution of the density of particles for the models introduced above. Depending on the range of the parameter $\beta$ we obtain hydrodynamic equations which have different behavior. More precisely, we always obtain the heat equation but with different boundary conditions. The first hydrodynamic equation is the heat equation with periodic boundary conditions, namely:

$$\begin{cases} \partial_t \rho(t,u) \;=\; \Delta\rho(t,u)\,, & t \geq 0,\; u \in \mathbb{T}\,,\\ \rho(0,u) \;=\; \rho_0(u), & u \in \mathbb{T}\,. \end{cases} \tag{1}$$

In the hydrodynamic limit scenario, we obtain $\rho(t,u)$ as a weak solution of the corresponding hydrodynamic equation. To make this notion precise, we introduce the following definition:

**Definition 1.** Let $\rho_0 : \mathbb{T} \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times \mathbb{T} \to [0,1]$ is a weak solution of the heat equation with periodic boundary conditions given in (1) if $\rho$ is measurable and, for any $t \in [0,T]$ and any $H \in C^{1,2}([0,T] \times \mathbb{T})$,

$$\int_{\mathbb{T}} \rho(t,u)H(t,u)du - \int_{\mathbb{T}} \rho(0,u)H(0,u)du \\ - \int_0^t \int_{\mathbb{T}} \rho(s,u)\,(\partial_s H(s,u) + \Delta H(s,u))\,du\,ds \;=\; 0\,. \tag{2}$$

Above and in the sequel the space $C^{1,2}([0,T]\times\mathbb{T})$ is the space of real valued functions defined on $[0,T]\times\mathbb{T}$ of class $C^1$ in time and $C^2$ in space.

The second equation we consider is the heat equation with a type of Robin's boundary conditions, that is:

$$\begin{cases} \partial_t \rho(t,u) \;=\; \Delta\rho(t,u)\,, & t \geq 0,\ u \in (0,1)\,,\\ \partial_u \rho(t,0) \;=\; \partial_u\rho(t,1) = \alpha(\rho(t,0)-\rho(t,1))\,, & t \geq 0,\\ \rho(0,u) \;=\; \rho_0(u), & u \in (0,1)\,. \end{cases} \tag{3}$$

To introduce the notion of weak solution of this equation we need to recall the notion of Sobolev's spaces.

**Definition 2.** Let $\mathcal{H}^1$ be the set of all locally summable functions $\zeta : (0,1) \to \mathbb{R}$ such that there exists a function $\partial_u \zeta \in L^2(0,1)$ satisfying

$$\int_{\mathbb{T}} \partial_u G(u)\zeta(u)\;du = -\int_{\mathbb{T}} G(u)\partial_u\zeta(u)\;du\,,$$

for all $G \in C^\infty(0,1)$ with compact support. Let $L^2(0,T;\mathcal{H}^1)$ be the space of all measurable functions $\xi : [0,T] \to \mathcal{H}^1$ such that

$$\|\xi\|^2_{L^2(0,T;\mathcal{H}^1)} \;:=\; \int_0^T \Big(\|\xi\|^2_{L^2[0,1]} + \|\partial_u\xi\|^2_{L^2[0,1]}\Big)\,dt \;<\; \infty\,.$$

Above $\|\cdot\|_{L^2[0,1]}$ denotes the $L^2$-norm in $[0,1]$.

**Definition 3.** Let $\rho_0 : \mathbb{T} \to [0,1]$ be a measurable function. We say that $\rho : [0,T]\times\mathbb{T} \to [0,1]$ is a weak solution of the heat equation with Robin's boundary conditions given in (3) if $\rho \in L^2(0,T;\mathcal{H}^1)$ and for all $t \in [0,T]$ and for all $H \in C^{1,2}([0,T]\times[0,1])$,

$$\begin{aligned} \int_{\mathbb{T}} \rho(t,u)H(t,u)du - \int_{\mathbb{T}} \rho(0,u)H(0,u)du & \\ - \int_0^t \int_{\mathbb{T}} \rho(s,u)\,(\partial_s H(s,u) + \Delta H(s,u))\,du\,ds & \\ - \int_0^t (\rho_s(0)\partial_u H_s(0) - \rho_s(1)\partial_u H_s(1))\,ds & \\ + \int_0^t \alpha(\rho_s(0)-\rho_s(1))(H_s(0)-H_s(1))\,ds = 0\,. & \end{aligned} \tag{4}$$

The last equation we consider is the heat equation with Neumann's boundary conditions given by:

$$\begin{cases} \partial_t \rho(t,u) = \Delta\rho(t,u)\,, & t \geq 0,\ u \in (0,1)\,, \\ \partial_u\rho(t,0) = \partial_u\rho(t,1) = 0\,, & t \geq 0\,, \\ \rho(0,u) = \rho_0(u), & u \in (0,1)\,. \end{cases} \tag{5}$$

**Definition 4.** Let $\rho_0 : \mathbb{T} \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times \mathbb{T} \to [0,1]$ is a weak solution of the heat equation with Neumann's boundary conditions if $\rho \in L^2(0,T;\mathcal{H}^1)$ and for all $t \in [0,T]$ and for all $H \in C^{1,2}([0,T] \times [0,1])$,

$$\begin{aligned} \int_{\mathbb{T}} \rho(t,u)H(t,u)du - \int_{\mathbb{T}} \rho(0,u)H(0,u)du& \\ - \int_0^t \int_{\mathbb{T}} \rho(s,u)\,(\partial_s H(s,u) + \Delta H(s,u))\,du\,ds& \\ - \int_0^t (\rho_s(0)\partial_u H_s(0) - \rho_s(1)\partial_u H_s(1))\,ds = 0\,.& \end{aligned} \tag{6}$$

Our argument to prove the hydrodynamic limit is standard in the theory of stochastic processes and goes through a tightness argument for $\{\pi_t^n\}_{n\in\mathbb{N}}$, which means relatively compactness of $\{\pi_t^n\}_{n\in\mathbb{N}}$. Therefore, there exists a limit point. To have uniqueness of the limit point of $\{\pi_t^n\}_{n\in\mathbb{N}}$ it is sufficient to prove uniqueness of the weak solution of the corresponding hydrodynamic equation. Then, it follows the convergence of the whole sequence $\{\pi_t^n\}_{n\in\mathbb{N}}$ to the unique limit point. For tightness issues we refer the reader to [3] and the uniqueness of the weak solution is stated below.

**Proposition 1.** *Let $\rho_0 : \mathbb{T} \to [0,1]$ be a measurable function. There exists a unique weak solution of the heat equation with periodic boundary conditions given in* (1) *and a unique weak solution of the heat equation with Neumann's boundary conditions given in* (5)*. Moreover, for each $\alpha > 0$, there exists a unique weak solution of the heat equation with Robin's boundary conditions given in* (3)*.*

## *3.2 Hydrodynamic Limit*

Returning to our discussion on the validity of the hydrodynamic limit, we introduce the set of initial measures for which we deduce the result.

**Definition 5.** Let $\rho_0 : \mathbb{T} \to [0,1]$ be a measurable function. A sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ on $\Omega$ is said to be associated to a profile $\rho_0 : \mathbb{T} \to [0,1]$ if, for every $\delta > 0$ and every continuous function $H : \mathbb{T} \to \mathbb{R}$, it holds that

$$\lim_{n\to\infty} \mu_n \Big\{\eta : \Big|\tfrac{1}{n} \sum_{x\in\mathbb{T}_n} H(\tfrac{x}{n})\, \eta(x) - \int_{\mathbb{T}} H(u)\, \rho_0(u) du\Big| > \delta\Big\} \;=\; 0\,. \tag{7}$$

One could ask about the existence of a measure associated to the profile $\rho_0 : \mathbb{T} \to [0,1]$. For instance, we can consider a Bernoulli product measure in $\Omega$ with marginal at $\eta(x)$ given by $\mu_n\{\eta \in \Omega : \eta(x) = 1\} = \rho_0(x/n)$.

For these processes we obtained in [3, 4] that:

**Theorem 1 (L.L.N. for the density of particles).** *Fix* $\beta \in [0,\infty]$ *and* $\rho_0 : \mathbb{T} \to [0,1]$ *a measurable function. Let* $\{\mu_n\}_{n\in\mathbb{N}}$ *be a sequence of probability measures on* $\Omega$ *associated to* $\rho_0$. *Then, for any* $t \in [0,T]$, *for every* $\delta > 0$ *and every continuous function* $H : \mathbb{T} \to \mathbb{R}$*:*

$$\lim_{n\to\infty} \mathbb{P}_{\mu_n} \Big\{\eta_\cdot : \Big|\tfrac{1}{n} \sum_{x\in\mathbb{T}_n} H\left(\tfrac{x}{n}\right)\, \eta_t(x) - \int_{\mathbb{T}} H(u)\rho(t,u) du\Big| > \delta\Big\} \;=\; 0\,,$$

*where:*

- *For* $\beta \in [0,1)$, $\rho(t,\cdot)$ *is the unique weak solution of* (1)*;*
- *For* $\beta = 1$, $\rho(t,\cdot)$ *is the unique weak solution of* (3)*;*
- *For* $\beta \in (1,\infty]$, $\rho(t,\cdot)$ *is the unique weak solution of* (5).

*All equations have the same initial condition* $\rho_0 : \mathbb{T} \to [0,1]$.

### 3.3 Phase Transition for the Hydrodynamic Equations

A puzzling question is whether there is a similar phase transition as described above, but at the macroscopic level. More precisely, does the unique weak solution of the heat equation with Robin's boundary conditions, that we denote by $\rho^\alpha$, converge in any sense to the weak solution of the heat equation with periodic boundary conditions or to the weak solution of the heat equation with Neumann's boundary conditions? In [4] we gave an affirmative answer to this question. We proved that $\rho^\alpha$ converges to the unique weak solution of the heat equation with Neumann's boundary conditions, when $\alpha$ goes to zero and to the unique weak solution of the heat equation with periodic boundary conditions, when $\alpha$ goes to infinity. This is the content of the next theorem.

This result is concerned only with the partial differential equations, having at principle nothing to do with the underlying particle systems. Nevertheless, our approach of proof is based on energy estimates coming from these particle systems.

**Theorem 2 (Phase transition for the heat equation with Robin's boundary conditions).** *For* $\alpha > 0$, *let* $\rho^\alpha : [0,T]\times[0,1] \to [0,1]$ *be the unique weak solution of the heat equation with Robin's boundary conditions:*

$$\begin{cases} \partial_t \rho^\alpha(t,u) = \Delta\rho^\alpha(t,u)\,, & t \geq 0,\ u \in (0,1)\,,\\ \partial_u \rho^\alpha(t,0) = \partial_u\rho^\alpha(t,1) = \alpha(\rho^\alpha(t,0) - \rho^\alpha(t,1))\,, & t \geq 0\,,\\ \rho^\alpha(0,u) = \rho_0(u), & u \in (0,1)\,. \end{cases}$$

*Then,* $\lim_{\alpha\to 0}\rho^\alpha = \rho^0$, *in* $L^2([0,T]\times[0,1])$, *where* $\rho^0 : [0,T]\times[0,1]\to[0,1]$ *is the unique weak solution of the heat equation with Neumann's boundary conditions*

$$\begin{cases} \partial_t \rho^0(t,u) = \Delta\rho^0(t,u)\,, & t \geq 0,\ u \in (0,1)\,,\\ \partial_u \rho^0(t,0) = \partial_u\rho^0(t,1) = 0\,, & t \geq 0\,,\\ \rho^0(0,u) = \rho_0(u)\,, & u \in (0,1) \end{cases}$$

*and* $\lim_{\alpha\to\infty}\rho^\alpha = \rho^\infty$, *in* $L^2([0,T]\times[0,1])$, *where* $\rho^\infty : [0,T]\times[0,1]\to[0,1]$ *is the unique weak solution of the heat equation with periodic boundary conditions*

$$\begin{cases} \partial_t \rho^\infty(t,u) = \Delta\rho^\infty(t,u)\,, & t \geq 0,\ u \in \mathbb{T}\,,\\ \rho^\infty(0,u) = \rho_0(u)\,, & u \in \mathbb{T}\,. \end{cases}$$

## 4 Equilibrium Fluctuations

Above we obtained a L.L.N. for the empirical measure considering the process starting from a measure which is associated to a profile $\rho_0 : \mathbb{T}\to[0,1]$. The natural question that follows is: what are the fluctuations around this "mean" profile? Do we have a C.L.T. for the density of particles? Under what set of initial measures? In the next lines we answer this question for a particular set of initial distributions, namely for the invariant measures $\nu^n_\rho$. In case of non-invariant measures the problem is still open.

In this case we consider the process evolving on $\mathbb{Z}$, being its state space $\{0,1\}^{\mathbb{Z}}$. To define properly our results, we fix $\rho \in [0,1]$, and we introduce the density fluctuation field as follows. For $t \in [0,T]$, let

$$\mathcal{Y}^n_t(\eta,du) = \sqrt{n}\pi^n_{tn^2}(\eta,du) - E_{\nu^n_\rho}[\sqrt{n}\pi^n_{tn^2}(\eta,du)],$$

where $x$ runs through $\mathbb{Z}$ in the definition of $\pi^n_t(\eta,du)$ and $E_{\nu^n_\rho}$ denotes expectation with respect to $\nu^n_\rho$. Then, for any function $H:\mathbb{R}\to\mathbb{R}$ we have that

$$\int_{\mathbb{R}} H(u)\mathcal{Y}^n_t(\eta,du) = \frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} H\left(\frac{x}{n}\right)[\eta_{tn^2}(x)-\rho].$$

By computing the characteristic function of $\mathcal{Y}^n_0$, we obtain that $\{\mathcal{Y}^n_0\}_{n\in\mathbb{N}}$ converges as $n$ goes to $\infty$ to a mean zero gaussian process $\mathcal{Y}_0$. More precisely, for any $H$, $\mathcal{Y}_0(H)$ is a gaussian random variable with mean zero and variance given by

$$\rho(1-\rho)\int_{\mathbb{R}}(H(x))^2dx.$$

Next, we are going to characterize the stochastic partial differential equations governing the evolution of the limit points of $\{\mathcal{Y}_t^n\}_{n\in\mathbb{N}}$.

## 4.1 Ornstein-Uhlenbeck Processes

In order to properly write down the stochastic partial differential equations that we deal with, we need to introduce different sets of test functions and two type of operators defined on these spaces.

**Definition 6.** Define $\mathcal{S}(\mathbb{R}\backslash\{0\})$ as the space of functions $H \in C^\infty(\mathbb{R}\backslash\{0\})$, that are continuous from the right at $x = 0$, for which

$$\|H\|_{k,\ell} \;:=\; \sup_{x\in\mathbb{R}\backslash\{0\}} |(1+|x|^\ell)\, H^{(k)}(x)| \;<\; \infty\,,$$

for all integers $k, \ell \geq 0$, and $H^{(k)}(0^-) = H^{(k)}(0^+)$, for all $k$ integer, $k \geq 1$.

- For $\beta \in [0,1)$, let $\mathcal{S}_\beta(\mathbb{R})$ be the subset of $\mathcal{S}(\mathbb{R}\backslash\{0\})$ composed of functions $H$ satisfying $H(0^-) = H(0^+)$.
- For $\beta = 1$, let $\mathcal{S}_\beta(\mathbb{R})$ as the subset of $\mathcal{S}(\mathbb{R}\backslash\{0\})$ composed of functions $H$ satisfying $H^{(1)}(0^+) = H^{(1)}(0^-) = \alpha(H(0^+) - H(0^-))$.
- For $\beta \in (1,+\infty]$, let $\mathcal{S}_\beta(\mathbb{R})$ be the subset of $\mathcal{S}(\mathbb{R}\backslash\{0\})$ composed of functions $H$ satisfying $H^{(1)}(0^+) = H^{(1)}(0^-) = 0$.

Above and in the sequel, $H^{(k)}(\cdot)$ represents the $k$-th derivative of the function $H$ and $H(0^+)$ (resp. $H(0^-)$) denotes the limit of $H$ from the right (resp. left) of 0.

**Definition 7.** For $\beta \in [0,\infty]$, we define the operators $\Delta_\beta, \nabla_\beta : \mathcal{S}_\beta(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ by

$$\nabla_\beta H(u) \;=\; \begin{cases} H^{(1)}(u), & \text{if } \; u \neq 0\,, \\ H^{(1)}(0^+), & \text{if } \; u = 0\,, \end{cases}$$

and

$$\Delta_\beta H(u) \;=\; \begin{cases} H^{(2)}(u), & \text{if } \; u \neq 0\,, \\ H^{(2)}(0^+), & \text{if } \; u = 0\,, \end{cases}$$

which are essentially the usual derivative and the usual second derivative, but defined in the domains $\mathcal{S}_\beta(\mathbb{R})$. We have the following uniqueness result which is a key point in our approach.

Denote by $T_t^\beta$ the semigroup corresponding to the partial differential equations (1), (3) or (5), if $\beta \in [0,1)$, if $\beta = 1$ or if $\beta \in (1,\infty]$, respectively.

**Proposition 2.** *For each $\beta \in [0, \infty]$ and $\alpha > 0$, there exists an unique random element $\mathcal{Y}_\cdot$ taking values in the space $C([0,T], \mathcal{S}'_\beta(\mathbb{R}))$ such that:*

*(i) For every function $H \in \mathcal{S}_\beta(\mathbb{R})$, $\mathcal{M}_t(H)$ and $\mathcal{N}_t(H)$ given by*

$$\begin{aligned} \mathcal{M}_t(H) &= \mathcal{Y}_t(H) - \mathcal{Y}_0(H) - \int_0^t \mathcal{Y}_s(\Delta_\beta H)ds\,, \\ \mathcal{N}_t(H) &= (\mathcal{M}_t(H))^2 - 2\chi(\rho)\, t\, \|\nabla_\beta H\|^2_{2,\beta} \end{aligned} \tag{8}$$

*are $\mathcal{F}_t$-martingales, where $\mathcal{F}_t := \sigma(\mathcal{Y}_s(H); s \leq t, H \in \mathcal{S}_\beta(\mathbb{R}))$, for $t \in [0,T]$.*

*(ii) $\mathcal{Y}_0$ is a mean zero gaussian field with covariance given on $G, H \in \mathcal{S}_\beta(\mathbb{R})$ as*

$$\mathbb{E}\left[\mathcal{Y}_0(G)\mathcal{Y}_0(H)\right] = \chi(\rho)\int_{\mathbb{R}} G(u)H(u)du\,. \tag{9}$$

*Moreover, for each $H \in \mathcal{S}_\beta(\mathbb{R})$, the stochastic process $\{\mathcal{Y}_t(H)\,;\, t \geq 0\}$ is gaussian, being the distribution of $\mathcal{Y}_t(H)$ conditionally to $\mathcal{F}_s$, for $s < t$, gaussian of mean $\mathcal{Y}_s(T^\beta_{t-s}H)$ and variance $\int_0^{t-s} \|\nabla_\beta T^\beta_r H\|^2_{2,\beta}\, dr$.*

Above and in the sequel $\mathcal{S}'_\beta(\mathbb{R})$ denotes the space of bounded linear functionals $f : \mathcal{S}_\beta(\mathbb{R}) \to \mathbb{R}$ and $\mathcal{D}([0,T], \mathcal{S}'_\beta(\mathbb{R}))$ (resp. $C([0,T], \mathcal{S}'_\beta(\mathbb{R}))$) is the space of càdlàg (resp. continuous) $\mathcal{S}'_\beta(\mathbb{R})$ valued functions endowed with the Skohorod topology. Also $\|H\|^2_{2,\beta} = \|H\|^2_2 + (H(0))^2\mathbf{1}_{\{\beta=1\}}$, where $\|\cdot\|_2$ denotes the $L^2$-norm in $\mathbb{R}$. We call to $\mathcal{Y}_\cdot$ the generalized Ornstein-Uhlenbeck process of characteristic operators $\Delta_\beta$ and $\nabla_\beta$ and it is the formal solution of the following equation

$$d\mathcal{Y}_t = \Delta_\beta \mathcal{Y}_t dt + \sqrt{2\chi(\rho)}\nabla_\beta d\mathcal{W}_t\,,$$

where $\mathcal{W}_t$ is a space-time white noise of unit variance.

## 4.2 Central Limit Theorem

We are in position to state the equilibrium fluctuations for the density of particles. Notice that our initial distribution is $\nu^n_\rho$, an invariant measure.

**Theorem 3 (C.L.T. for the density of particles).** *The sequence of processes $\{\mathcal{Y}^n_t\}_{n\in\mathbb{N}}$ converges in distribution, as $n$ goes to $\infty$, with respect to the Skorohod topology of $\mathcal{D}([0,T], \mathcal{S}'_\beta(\mathbb{R}))$ to a gaussian process $\mathcal{Y}_t$ in $C([0,T], \mathcal{S}'_\beta(\mathbb{R}))$, which is the formal solution of the Ornstein-Uhlenbeck equation given by*

$$d\mathcal{Y}_t = \Delta_\beta \mathcal{Y}_t dt + \sqrt{2\chi(\rho)}\nabla_\beta d\mathcal{W}_t\,. \tag{10}$$

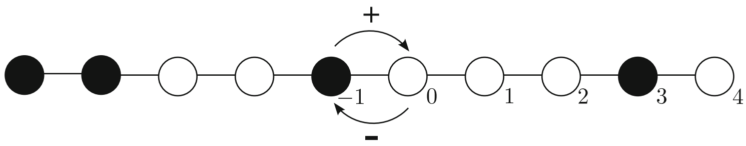


**Fig. 2** Current at the bond $\{-1,0\}$ of the SEP with a slow bond. Every time a particle jumps from $-1$ to 0 (0 to $-1$) the current increases (decreases) by one

# 5 Current and Tagged Particle Fluctuations

In this section we are still restricted to the invariant state $\nu_\rho^n$ and for that purpose we fix a density $\rho$ from now on up to the rest of these notes.

## 5.1 *The Current*

Now, we introduce the notion of current of particles through a fixed bond $\{x, x+1\}$. For a bond $e_x := \{x, x+1\}$, denote by $J^n_{e_x}(t)$ the current of particles over the bond $e_x$, that is $J^n_{e_x}(t)$ counts the total number of jumps from the site $x$ to the site $x+1$ minus the total number of jumps from the site $x+1$ to the site $x$ in the time interval $[0, tn^2]$, see the figure below. More generally, to each macroscopic point $u \in \mathbb{R}$ we can define the current through its associated microscopic bond of vertices $\{\lfloor un \rfloor - 1, \lfloor un \rfloor\}$, as $J^n_u(t) := J^n_{e_{\lfloor un \rfloor -1}}(t)$. Here $\lfloor un \rfloor$ denotes the biggest integer smaller or equal to $un$. As a consequence of the C.L.T. for the density of particles, namely of Theorem 3, it is simple to derive the C.L.T. for the current of particles which we enounce as follows (Fig. 2).

**Theorem 4 (C.L.T. for the current of particles).** *Under $\mathbb{P}_{\nu_\rho^n}$, for every $t \geq 0$ and every $u \in \mathbb{R}$,*

$$\frac{J^n_u(t)}{\sqrt{n}} \xrightarrow[n\to\infty]{} J_u(t)$$

*in the sense of finite-dimensional distributions, where $J_u(t)$ is a gaussian process with mean zero and variance given by*

- *For $\beta \in [0,1)$, $\mathbb{E}_{\nu_\rho^n}[(J_u(t))^2] = 2\chi(\rho)\sqrt{\frac{t}{\pi}}$, that is $J_u(t)$ is a fractional Brownian Motion of Hurst exponent $1/4$;*
- *For $\beta = 1$, $\mathbb{E}_{\nu_\rho^n}[(J_u(t))^2] = 2\chi(\rho)\left(\sqrt{\frac{t}{\pi}} + \frac{\Phi_{2t}(2u+4\alpha t)\, e^{4\alpha u + 4\alpha^2 t} - \Phi_{2t}(2u)}{2\alpha}\right)$;*
- *For $\beta \in (1, +\infty]$, $\mathbb{E}_{\nu_\rho^n}[(J_u(t))^2] = 2\chi(\rho)\left(\sqrt{\frac{t}{\pi}}\left[1 - e^{-u^2/t}\right] + 2u\,\Phi_{2t}(2u)\right)$,*

*where*

$$\Phi_{2t}(x) := \int_x^{+\infty} \frac{e^{-u^2/4t}}{\sqrt{4\pi t}} du\,.$$

It worth to remark the variance at $u = 0$, corresponding to the current of particles through the slow bond $e_{-1}$. If $\beta \in [0, 1)$, the variance corresponds to the one of a fractional Brownian Motion of Hurst exponent $1/4$. If $\beta \in (1, \infty]$, the variance equals to zero as expected. This is a consequence of having Neumann's boundary conditions at $x = 0$ which turns it into an isolated boundary. And for $\beta = 1$, we obtain a family of gaussian processes indexed in $\alpha$ interpolating the two aforementioned processes.

**Corollary 1.** *For $\beta = 1$, denote the limit, as $n \to \infty$, of $J^n_u(t)/\sqrt{n}$ by $J^\alpha_u(t)$.*
*Then for every $t \geq 0$ and every $u \in \mathbb{R}$,*

$$J^\alpha_u(t) \xrightarrow[\alpha\to+\infty]{} J^\infty_u(t),$$

*where $J^\infty_u(t)$ is the fractional Brownian Motion with Hurst exponent $1/4$ and*

$$J^\alpha_u(t) \xrightarrow[\alpha\to 0]{} J^0_u(t)\,,$$

*where $J^0_u(t)$ is the mean zero gaussian process with variance given by* $\mathbb{E}_{\nu^n_\rho}[(J_u(t))^2] = 2\chi(\rho)\left(\sqrt{\frac{t}{\pi}}\left[1 - e^{-u^2/t}\right] + 2u\,\Phi_{2t}(2u)\right)$.
*The convergence is in the sense of finite dimensional distributions.*

## 5.2 Tagged Particle Fluctuations

Our last goal is to present the asymptotic behavior of a tagged particle in the system. The dynamic of this tagged particle is no longer Markovian, since its behavior is influenced by the presence of other particles in the system. Nevertheless, we can relate the position of the tagged particle with the current and the density of particles, and from the previous results we obtain information about the behavior of this particle.

Suppose to start the system from a configuration with a particle at the site $\lfloor un \rfloor$ and in all other sites suppose that the configuration is distributed according to $\nu^n_\rho$. In other words, this means that we consider the Markov process $\{\eta_t : t \geq 0\}$ starting from the measure $\nu^n_\rho$ conditioned to have a particle at the site $\lfloor un \rfloor$, that we denote by $\nu^{n,u}_\rho$. That is, $\nu^{u,n}_\rho(\cdot) := \nu^n_\rho(\,\cdot\,|\eta_{tn^2}(\lfloor un \rfloor) = 1)$ (Fig. 3).

We notice that the previous results were obtained considering the process starting from $\nu^n_\rho$. In order to be able to use them, we couple the process starting from $\nu^{n,u}_\rho$ and starting from $\nu^n_\rho$, in such a way that both processes differ at most by one site at

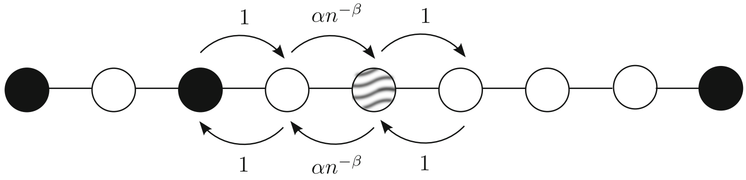


**Fig. 3** The tagged particle of the SEP with a slow bond. At initial time, the tagged particle is at the site 0

any given time. This allow us to derive the same statements of Theorems 3 and 4 for the starting measure $\nu_\rho^{n,u}$.

Now, let $X_u^n(t)$ be the position at the time $tn^2$ of a tagged particle initial at the site $\lfloor un \rfloor$. Since our study is restricted to the one dimensional setting, particles do preserve their order, and it is simple to check that

$$\{X_u^n(t) \geq k\} = \Big\{J_u^n(t) \geq \sum_{x=\lfloor un \rfloor}^{\lfloor un \rfloor + k-1} \eta_{tn^2}(x)\Big\}.$$

We explain briefly how to get the previous equality. Suppose for simplicity that $u = 0$, so that we start the system with the tagged particle at the origin. If this particle is, at time $tn^2$, at the right hand side of $n$, then all the particles that jumped from $-1$ to 0 and did not jump backwards, are somewhere at the sites $\{0, 1, \ldots, X_u^n(t)\}$. It follows that the current through the bond $\{-1, 0\}$ has to be greater or equal than the density of particles in $\{0, \ldots, n\}$. Reasoning similarly, we get the equality between those events.

Finally, last relation together with Theorem 4, implies the following result.

**Theorem 5 (C.L.T. for a tagged particle).** *Under $\mathbb{P}_{\nu_\rho^u}$, for all $\beta \in [0, \infty]$, every $u \in \mathbb{R}$ and $t \geq 0$*

$$\frac{X_u^n(t)}{\sqrt{n}} \xrightarrow[n \to +\infty]{} X_u(t)$$

*in the sense of finite-dimensional distributions, where $X_u(t) = J_u(t)/\rho$ in law and $J_u(t)$ is the same as in Theorem 4. In particular, the variance of the process $X_u(t)$ is given by*

- *For $\beta \in [0, 1)$, $\mathbb{E}_{\nu_\rho^n}[(X_u(t))^2] = 2\dfrac{\chi(\rho)}{\rho^2}\sqrt{\frac{t}{\pi}}$, that is $X_u(t)$ is a fractional Brownian Motion of Hurst exponent $1/4$;*
- *For $\beta = 1$, $\mathbb{E}_{\nu_\rho^n}[(X_u(t))^2] = 2\dfrac{\chi(\rho)}{\rho^2}\left(\sqrt{\frac{t}{\pi}} + \frac{\Phi_{2t}(2u+4\alpha t)\, e^{4\alpha u + 4\alpha^2 t}}{2\alpha}\right)$;*
- *For $\beta \in (1, +\infty]$, $\mathbb{E}_{\nu_\rho^n}[(X_u(t))^2] = 2\dfrac{\chi(\rho)}{\rho^2}\left(\sqrt{\frac{t}{\pi}}\left[1 - e^{-u^2/t}\right] + 2u\,\Phi_{2t}(2u)\right)$.*

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## References

1. Faggionato, A., Jara, M., Landim, C.: Hydrodynamic behavior of one dimensional subdiffusive exclusion processes with random conductances. Probab. Theory Relat. Fields **144**(3–4), 633–667 (2008)
2. Fontes, L.R.G., Isopi, M., Newman, C.M.: Random walks with strongly inhomogeneous rates and singular diffusions: convergence, localization and aging in one dimension. Ann. Probab. **30**(2), 579–604 (2002)
3. Franco, T., Gonçalves, P., Neumann, A.: Hydrodynamical behavior of symmetric exclusion with slow bonds, Ann. Inst. Henri Poincaré Probab. Stat. **49**(2), 402–427 (2013)
4. Franco, T., Gonçalves, P., Neumann, A.: Phase transition of a heat equation with Robin's boundary conditions and exclusion process. arXiv:1210.3662 and accepted for publication in the Transactions of the American Mathematical Society (2013)
5. Franco, T., Gonçalves, P., Neumann, A.: Phase transition in equilibrium fluctuations of symmetric slowed exclusion. Stoch. Process. Appl **123**(12), 4156–4185 (2013)
6. Franco, T., Landim, C.: Hydrodynamic limit of gradient exclusion processes with conductances. Arch. Ration. Mech. Anal. **195**(2), 409–439 (2010)
7. Jara, M.: Hydrodynamic limit of particle systems in inhomogeneous media. Online. ArXiv http://arxiv.org/abs/0908.4120 (2009)
8. Kawazu, K., Kesten, H.: On birth and death processes in symmetric random environment. J. Stat. Phys. **37**, 561–576 (1984)
9. Kipnis, C., Landim, C.: Scaling Limits of Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 320. Springer, Berlin (1999)
10. Nagy, K.: Symmetric random walk in random environment. Period. Math. Ung. **45**, 101–120 (2002)
11. Valentim, F.: Hydrodynamic limit of a d-dimensional exclusion process with conductances. Ann. Inst. Henri Poincaré Probab. Stat. **48**(1), 188–211 (2012)

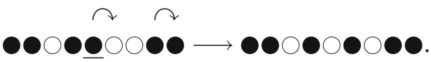
# Exclusion and Zero-Range in the Rarefaction Fan

**Patrícia Gonçalves**

## 1 Introduction

In these notes we review some asymptotic results on two classical interacting particle systems: the totally asymmetric simple exclusion process and the totally asymmetric constant-rate zero-range process, in the presence of particles with different priorities. These processes are taken on $\mathbb{Z}$ and at each site $x \in \mathbb{Z}$ we place a random clock $T_x$, which is distributed according to an exponential law with parameter 1. The collection of clocks $\{T_x\}_{x\in\mathbb{Z}}$ forms a sequence of independent and identically distributed random variables. Initially we randomly distribute particles along the lattice and each time a clock rings, if there is at least one particle at the corresponding site, then one of them jumps to one of its nearest-neighbors. If there is no particle at that site, then nothing happens and the clocks restart.

We consider two types of jumps in these notes. The first type of jump is realized under an exclusion rule, therefore the particle system coined the name simple exclusion process. In this process a jump from a site $x$ to $x + 1$ occurs at rate 1, but the jump is performed if and only if the destination site is empty. In the figure below we represent particles by ● and holes by ○, therefore the jump on the left hand side can occur but not the jump on the right hand side.



In the sequel we denote this Markov process by $\{\eta_t : t \geq 0\}$ and it has state space $\Omega_{EP} := \{0, 1\}^{\mathbb{Z}}$. For this model there is at most one particle per site, so its

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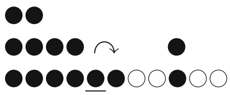
configurations, denoted by $\eta$, consist in vectors whose components are either 0 or 1. Physically the interpretation $\eta(x) = 1$ means that the site $x$ is occupied.

The second type of jump that we consider is described as follows. There is no restriction on the number of particles at each site and if the clock at the site $x$ rings and if there is at least one particle at that site, then one of them jumps to $x + 1$ at rate 1. In this case, the jump occurs independently from the number of particles at the departure and destination sites.

A possible jump is



In the sequel we denote this Markov process by $\{\xi_t : t \geq 0\}$ and it has state space $\Omega_{ZR} := \mathbb{N}_0^{\mathbb{Z}}$. The configurations of this model are denoted by $\xi$ and consist in vectors whose components contain one number of $\mathbb{N}_0$. Physically, the interpretation $\xi(x) = k$, for $k \in \mathbb{N}_0$ means that the site $x$ is occupied with $k$ particles.

We will add to these particle systems a "special" particle, which is seen by the remaining particles as a hole and it is seen by the holes as a particle, therefore this particle is called a *second class particle*. We will first present the Law of Large Numbers (LLN) for this particle starting both processes from initial conditions in the rarefaction fan. Then, we will consider both processes in the presence of a second class particle and a third class particle at its right site. The first and second class particles see the third class particle as a hole, but the third class particle does not distinguish the second class particle from the first class particles. We will prove, by a symmetry argument, that for the exclusion, the probability of the second class particle swapping order with the third class particle is equal to $2/3$. As a consequence, by coupling the exclusion with the zero-range, the probability of the second class particle being at the right hand side or at the same site of the third class particle, in the zero-range, equals $2/3$.

The outline of these notes is described as follows. In Sect. 2, we define the processes, their invariant measures and a set of measures which are not invariant but lead in the hydrodynamics to the rarefaction fan of the associated hydrodynamic equation. In Sect. 3, we describe the hydrodynamics for these processes and in Sect. 4, we state a LLN for a second class particle in a rarefaction setting. In Sect. 5 we present a coupling between both processes and in Sect. 6 we discuss crossing probabilities for second and third class particles.

## 2 The Models

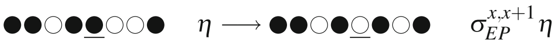
Let $\{\eta_t; t \geq 0\}$ be the one-dimensional totally asymmetric simple exclusion process (TASEP), a continuous time Markov process with state space $\Omega_{EP}$ whose infinitesimal generator is defined on local functions $f : \Omega_{EP} \to \mathbb{R}$ as

$$\mathcal{L}_{EP} f(\eta) = \sum_{x \in \mathbb{Z}} \eta(x)(1 - \eta(x+1))[f(\sigma_{EP}^{x,x+1}\eta) - f(\eta)].$$

Above

$$(\sigma_{EP}^{x,x+1}\eta)(x) = \eta(x+1), \quad (\sigma_{EP}^{x,x+1}\eta)(x+1) = \eta(x)$$

and on other sites $\sigma_{EP}^{x,x+1}\eta$ coincides with $\eta$. As an example see the figure below in which the particle underlined is at the site $x$.



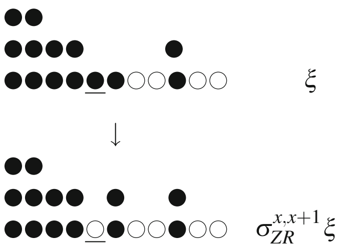
Now, let $\{\xi_t; t \geq 0\}$ be the one-dimensional constant-rate totally asymmetric zero-range process (TAZRP), a continuous time Markov process with state space $\Omega_{ZR}$ whose infinitesimal generator is defined on local functions $f : \Omega_{ZR} \to \mathbb{R}$ as

$$\mathcal{L}_{ZR} f(\xi) = \sum_{x \in \mathbb{Z}} \mathbf{1}\{\xi(x) \geq 1\}[f(\sigma_{ZR}^{x,x+1}\xi) - f(\xi)],$$

where

$$(\sigma_{ZR}^{x,x+1}\xi)(x) = \xi(x) - 1, \quad (\sigma_{ZR}^{x,x+1}\xi)(x+1) = \xi(x+1) + 1$$

and on other sites $\sigma_{ZR}^{x,x+1}\xi$ coincides with $\xi$. As an example see the figure below in which the particle underlined is at the site $x$.



Fore more details on the construction of these models we refer to [2, 10].

Now, we describe briefly the invariant measures for these processes. We start with the TASEP. It is well known that the Bernoulli product measure of parameter $\alpha \in [0, 1]$, that we denote by $\nu_\alpha$, is invariant for the TASEP. This measure is defined on $\Omega_{EP}$, is translation invariant and parameterized by the density $\alpha$, namely: $E_{\nu_\alpha}[\eta(x)] = \alpha$ for any $x \in \mathbb{Z}$. For $x \in \mathbb{Z}$, $k \in \{0, 1\}$ and $\alpha \in [0, 1]$, its marginal is given by

$$\nu_\alpha(\eta : \eta(x) = k) = \alpha^k (1 - \alpha)^{1-k}.$$

For the TAZRP, it is known that the Geometric product measure of parameter $\frac{1}{1+\rho}$ with $\rho \in (0, +\infty)$, that we denote by $\mu_\rho$, is invariant. That is, $\mu_\rho$ is defined on $\Omega_{ZR}$ and for $x \in \mathbb{Z}$ and $k \in \mathbb{N}_0$, $\mu_\rho$ has marginal given by

$$\mu_\rho(\xi : \xi(x) = k) = \left(\frac{\rho}{1+\rho}\right)^k \frac{1}{1+\rho}.$$

In the sequel we will make use of the following measures. For $\alpha, \beta \in [0, 1]$ let $\nu_{\alpha,\beta}$ be the product measure, such that for $x \in \mathbb{Z}$ and $k \in \{0, 1\}$

$$\nu_{\alpha,\beta}(\eta : \eta(x) = k) = \begin{cases} \alpha^k(1-\alpha)^{1-k}, \text{ if } x < 0 \\ \beta^k(1-\beta)^{1-k}, \text{ if } x \geq 0 \end{cases}. \tag{1}$$

Analogously, for $\rho, \lambda \in (0, +\infty)$ let $\mu_{\rho,\lambda}$ be the product measure such that for $k \in \mathbb{N}_0$ and $x \in \mathbb{Z}$

$$\mu_{\rho,\lambda}(\xi : \xi(x) = k) = \begin{cases} \left(\frac{\rho}{1+\rho}\right)^k \frac{1}{1+\rho}, \text{ if } x < 0 \\ \left(\frac{\lambda}{1+\lambda}\right)^k \frac{1}{1+\lambda}, \text{ if } x \geq 0 \end{cases}. \tag{2}$$

Moreover, below we also consider the zero-range process starting from the measure $\mu_{\infty,\lambda}$, with $\lambda \geq 0$. This means that, if a configuration $\xi \in \Omega_{ZR}$ is distributed according to $\mu_{\infty,\lambda}$, then $\xi(x) = \infty$ for $x < 0$ and $\xi(x)$ is distributed according to $\mu_\lambda$ for $x \geq 0$. When $\lambda = 0$, $\mu_{\infty,0}$ gives weight one to the configuration $\tilde{\xi}$, such that $\tilde{\xi}(x) = \infty$ for $x < 0$ and $\tilde{\xi}(x) = 0$ for $x \geq 0$.

## 3 Hydrodynamics

The hydrodynamic limit consist in a LLN for the empirical measure associated to a particle system [9]. For that purpose, given a process $\zeta_t$, let $\pi^n(\zeta_t, du)$ be the empirical measure given by

$$\pi^n(\zeta_t, du) = \frac{1}{n} \sum_{x \in \mathbb{Z}} \zeta_t(x) \delta_{\frac{x}{n}}(du).$$

Here $\delta_u$ denotes the Dirac measure at $u$. Let $\pi_t^n(\zeta, du) = \pi^n(\zeta_t, du)$. A measure $\mu_n$ is said to be associated to a profile $\rho_0 : \mathbb{R} \to \mathbb{R}$, if the $\mu_n$-weak LLN holds for $\pi_0^n(\zeta, du)$, for details we refer the reader to [9]. Here we fix

$$\rho_0(u) = \theta_1 \mathbf{1}\{u < 0\} + \theta_2 \mathbf{1}\{u \geq 0\},$$

with $\theta_1 > \theta_2$. Since the work of [12], it is known that starting the TASEP or the TAZRP from such $\mu_n$, if $\pi_0^n(\zeta, du)$ converges to $\rho_0(u)du$ in probability (with respect to $\mu_n$), as $n \to \infty$, then $\pi_{tn}^n(\zeta, du)$ converges to $\rho(t,u)du$ in probability (with respect to the distribution of the process at time $t$ starting from $\mu_n$), as $n \to \infty$, where $\rho(t,u)$ is the unique entropy solution of the corresponding hydrodynamic equation. For both processes the hydrodynamic equation is given by

$$\partial_t \rho(t,u) + \partial_u j(\rho(t,u)) = 0$$

and with initial condition $\rho(0,u) := \rho_0(u)$ for all $u \in \mathbb{R}$. In fact, the aforementioned result is more general [12], but as it is stated it is sufficient for our purposes. The function $j(\rho)$ above corresponds to the mean (with respect to the invariant measure of the process, that we represent generically by $m_r$, indexed in $r$) of what is called the instantaneous current at the bond $\{0,1\}$. Since jumps are totally asymmetric, this current is simply the jump rate to the right neighboring site. For the TASEP, the instantaneous current is $\eta(0)(1-\eta(1))$ and since $m_r = \nu_\alpha$, we get

$$j(\alpha) := E_{\nu_\alpha}[\eta(0)(1-\eta(1))] = \alpha(1-\alpha)$$

and the hydrodynamic equation becomes

$$\partial_t \rho(t,u) + \partial_u \left(\rho(t,u)(1-\rho(t,u))\right) = 0, \tag{3}$$

which is known as the *inviscid Burgers equation*. For the TAZRP, the instantaneous current is $\mathbf{1}\{\xi(0) \geq 1\}$ and since $m_r = \mu_\rho$, we get

$$j(\rho) := E_{\mu_\rho}[\mathbf{1}\{\xi(0) \geq 1\}] = \frac{\rho}{1+\rho}$$

and the hydrodynamic equation becomes

$$\partial_t \rho(t,u) + \partial_u \left(\frac{\rho(t,u)}{1+\rho(t,u)}\right) = 0. \tag{4}$$

Now, we notice that the solution of (3) under the initial condition $\rho(0,u) = \alpha\mathbf{1}\{u<0\} + \beta\mathbf{1}\{u \geq 0\}$, with $\alpha > \beta$, is given by

$$\rho(t,u) = \begin{cases} \alpha, \text{ if } u < (1-2\alpha)t \\ \beta, \text{ if } u > (1-2\beta)t \\ \frac{t-u}{2t}, \text{ if } (1-2\alpha)t \leq u \leq (1-2\beta)t \end{cases} \tag{5}$$

and the solution of (4) under the initial condition $\rho(0,u) = \rho\mathbf{1}\{u<0\} + \lambda\mathbf{1}\{u \geq 0\}$, with $\rho > \lambda$, is given by

$$\rho(t,u) = \begin{cases} \rho, \text{ if } u < \frac{t}{(1+\rho)^2} \\ \lambda, \text{ if } u > \frac{t}{(1+\lambda)^2} \\ \frac{\sqrt{t}-\sqrt{u}}{\sqrt{u}}, \text{ if } \frac{t}{(1+\rho)^2} \leq u \leq \frac{t}{(1+\lambda)^2}. \end{cases} \tag{6}$$

We will see that these solutions, under a proper renormalization, are the probability density functions of a "special" particle whose dynamics we define below.

## 4 Law of Large Numbers for a Second Class Particle

In this section we describe the LLN for a second class particle added to the TASEP and to the TAZRP. Since the dynamics of this particle is completely different in these processes, we start by describing its motion in the TASEP. In the TASEP, the first class particles see the second class particle as a hole, therefore if a first class particle jumps to a site occupied by a second class particle then they exchange positions. A second class particle can jump only to empty sites. For example, suppose to start the TASEP from a configuration $\eta$ as

$$\bullet\bullet\circ\bullet\underline{\circledast}\circ\circ\bullet \qquad \eta$$

In this case $\circledast$ represents the second class particle and a particle underlined means it stands at the origin. Suppose now, that the clock $T_{-1}$ rings. In spite of the exclusion rule and the fact that the origin being occupied with a second class particle, the jump is performed and the particles exchange positions.

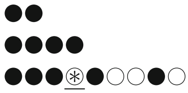
$$\bullet\bullet\circ\bullet\underline{\circledast}\circ\circ\bullet \qquad \eta \longrightarrow \bullet\bullet\circ\circledast\underline{\bullet}\circ\circ\bullet \qquad \eta^{-1,0}$$

Now, if on $\eta^{-1,0}$ the second class particle attempts to jump to its right neighboring site which is occupied by a first class particle, then nothing happens. On the other hand, if on $\eta$ the second class particle jumps to its right, then the jump is performed and it exchanges positions with the hole to its right.

$$\bullet\bullet\circ\bullet\underline{\circledast}\circ\circ\bullet \qquad \eta \longrightarrow \bullet\bullet\circ\bullet\underline{\circ}\circledast\circ\bullet \qquad \eta^{0,1}$$

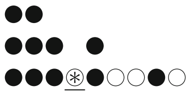
Concluding, in the TASEP, a second class particle can jump backwards and this happens if and only if a first class particle at its left jumps to the right.

In the TAZRP the dynamics of a second class particle is substantially different from the dynamics described above. In the TAZRP, if first and second class particles share the same site, then if the clock rings for that site the second class particle only leaves the site if there is no other first class particle there. For example, consider the TAZRP starting from a configuration $\xi$ as the one given below.



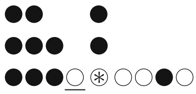
$\xi$

Suppose now, that the clock at the origin rings. Then, the first class particle at the origin jumps to the right and the second class particle remains at the origin.



$\xi^{0,1}$

Now, if the clock at the origin rings again, then the second class particle can jump to the right.



Concluding, in the TAZRP, a second class particle can never jump backwards and it only jumps from a site $x$ to $x+1$, if there is no other first class particle at $x$. Moreover, the jump occurs independently of the number/type of particles at $x+1$.

A second class particle in the TASEP or TAZRP can be obtained considering the 'basic coupling' for those processes. The idea is the following. Consider two TAZRP $\xi_t^0$ and $\xi_t^1$ starting from initial configurations $\xi_0^0$ and $\xi_0^1$, respectively, such that $\xi_0^0(x) \leq \xi_0^1(x)$ for all $x \in \mathbb{Z}$. We couple the two processes so that whenever a particle in the $\xi^0$ configuration moves, a corresponding $\xi^1$ particle makes the same jump. That is, a particle at $x$ in the $\xi^0$ and $\xi^1$ processes jumps to $x+1$ with rate $\mathbf{1}\{\xi^0(x) \geq 1\}$ and also one of the particles at $x$ in the $\xi^1$ process displaces by 1 with rate $\mathbf{1}\{\xi^1(x) \geq 1\} - \mathbf{1}\{\xi^0(x) \geq 1\}$. Then, we can write $\xi_t^1 = \xi_t^0 + Z(t)$, where, $Z(t)(x)$ counts the second-class particles. For the TASEP it is analogous. We notice that under this coupling both processes are attractive. For details we refer the reader to [2]. Now we describe the asymptotic limit for a second class particle in TASEP.

**Theorem 1 ([3, 4, 7, 11]).** *Consider the TASEP starting from $\nu_{\alpha,\beta}$ with $0 \leq \beta < \alpha \leq 1$. At time $t = 0$ put a second class particle at the origin regardless the value of the configuration at this point and let $X_2^{EP}(t)$ denote its position at time $t$. Then*

$$\lim_{t\to\infty} \frac{X_2^{EP}(t)}{t} = \mathcal{U}, \quad \textit{almost surely},$$

*where $\mathcal{U}$ is uniformly distributed on $[1-2\alpha, 1-2\beta]$. That is*

$$F_{\mathcal{U}}(u) := P(\mathcal{U} \leq u) = \frac{\beta - \rho(1,u)}{\beta - \alpha},$$

*where $\rho(t,u)$ is given in (5).*

The proof of last result for convergence in distribution was given in [4] and it was generalized to partial asymmetric jumps in [3]. The almost sure convergence was derived in [7, 11]. The result for the TAZRP is given in the next theorem.

**Theorem 2 ([8]).** *Consider the TAZRP starting from $\mu_{\rho,\lambda}$, with $0 \leq \lambda < \rho \leq \infty$. At time $t = 0$ add a second class particle at the origin and let $X_2^{ZR}(t)$ denote its position at time $t$. Then*

$$\lim_{t\to\infty} \frac{X_2^{ZR}(t)}{t} = \mathcal{V} = \left(\frac{1+\mathcal{U}}{2}\right)^2, \quad \textit{almost surely},$$

*where $\mathcal{U}$ is uniformly distributed on $\left[\frac{1-\rho}{1+\rho}, \frac{1-\lambda}{1+\lambda}\right]$. That is,*

$$F_{\mathcal{V}}(u) := P(\mathcal{V} \leq u) = \frac{1+\lambda}{\rho-\lambda}\left((1+\rho)(1-j(\rho(1,u)))-1\right),$$

*where $\rho(t,u)$ is given in (6) and $j(\cdot)$ is given above (4).*

## 5 Coupling TASEP and TAZRP with a Second Class Particle

In this section we present a coupling between the TASEP and the TAZRP in the presence of one second class particle. It uses the particle to particle coupling introduced in [8] and it relates the TAZRP and TASEP in such a way that the position of the second class particle in the TAZRP corresponds to the flux of holes that crossover the second class particle in the TASEP. Now we explain the relation between the configurations of the two processes. To make easier the exposition we give an example of a initial configuration for the TASEP as below and we denote it by $\eta_0$.

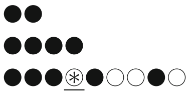
○●●●○●●●○●●⊛●○●○○○●○○

Let $X_2^{EP}(t)$ denote the position at time $t$ of the second class particle in TASEP starting from $\eta_0$. For $\eta_0$ we have $X_2^{EP}(0) = 0$. Initially we label the holes by denoting the position of the $i$-th hole at time $t = 0$ by $x_i(0)$. To simplify notation, we label the leftmost (resp. rightmost) hole at the right (resp. left) hand side of the second class particle at time $t = 0$ by 1 (resp. $-1$). Both processes are related in such a way that basically on the TASEP the distance between two consecutive holes minus one is the number of particles at a site in the TAZRP, but near the second class particle one has to be more careful. At time $t = 0$, we define:

- For $i = X_2^{EP}(0)-1$: $\xi_0(i)$ corresponds to the number of particles between $X_2^{EP}(0)$ and the first hole to its left, therefore, $\xi_0(i) = |X_2^{EP}(0) - x_{-1}(0)| - 1$;

- For $i = X_2^{EP}(0)$: $\xi_0(i)$ has a second class particle plus a number of first class particles that coincides with the number of first class particles between $X_2^{EP}(0)$ and the first hole to its right, therefore, $\xi_0(i)$ has $|x_1(0) - X_2^{EP}(0)| - 1$ first class particles and a second class particle;
- For $i \in \mathbb{Z} \setminus \{X_2^{EP}(0) - 1, X_2^{EP}(0)\}$: $\xi_0(i)$ corresponds to the number of particles between consecutive holes, therefore, for $\kappa > 0$ and for $i = X_2^{EP}(0) + \kappa$, $\xi_0(i) = x_{\kappa+1}(0) - x_\kappa(0) - 1$, similarly for $\kappa < 0$;

We notice that under this mapping $X_2^{EP}(0) = X_2^{ZR}(0)$. Now, for $\eta_0$ we have $x_{-3}(0) = -11$, $x_{-2}(0) = -7$, $x_{-1}(0) = -3$, $X_2^{EP}(0) = 0$, $x_1(0) = 2$, $x_2(0) = 4$, $x_3(0) = 5$, $x_4(0) = 6$, $x_5(0) = 8$, $x_6(0) = 9$, which corresponds in TAZRP to the configuration given below that we denote by $\xi_0$.

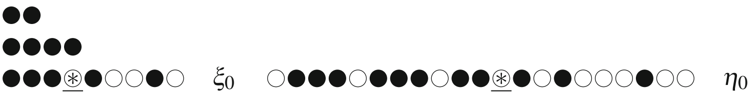


With the established relations we notice that for a positive site (resp. negative site) if in the TAZRP there are $k$ particles at a given site, then for the TASEP there are $k$ particles plus a hole to their right (resp. left). For positive (resp. negative) sites there are $k$ particles at that site with probability $\alpha^k(1-\alpha)$ (resp. $\beta^k(1-\beta)$). For the TAZRP at the site $X_2^{ZR}(t)$ there are $k$ particles, if in the TASEP there are $k$ particles plus a hole to the right of the second class particle. By the definition of the invariant measures for the TAZRP we have that

$$\alpha = \rho/(1+\rho) \quad \text{and} \quad \beta = \lambda/(1+\lambda).$$

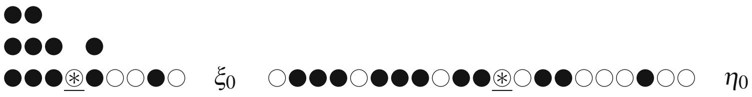
This is the reason why in the statement of Theorem 2 the Uniform random variable is supported on $\left[\frac{1-\rho}{1+\rho}, \frac{1-\lambda}{1+\lambda}\right]$.

On the figure below, we put together $\eta_0$ and its corresponding configuration in TAZRP, namely $\xi_0$, according to the rules stated above.

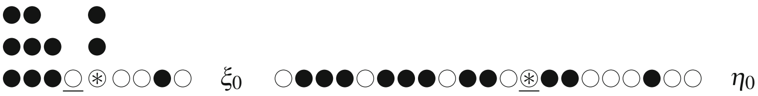



Now, in the presence of the particle to particle coupling, particles are labeled and only the configuration in the TAZRP looks at the clocks. If the clock rings for the $i$-th particle in the TAZRP configuration then the $i$-th particles in both processes jump. For more details on this coupling we refer to [8].

Using this particle to particle coupling we present some possible jumps for the configurations $\xi_0$ and $\eta_0$. For example, if the clock rings for the first class particle at the origin in $\xi_0$, we get



Now, if the clock rings for the second class particle in $\xi_0$, then we get



Now, in $\xi_0$ the second class particle cannot jump since there are two first class particles at its site, neither the second class particle in $\eta_0$. Therefore, from the mapping described above and for any initial configuration with one second class particle, we have that

$$J_2^{EP}(t) = J_2^{ZR}(t) \quad \text{and} \quad H_2^{EP}(t) = X_2^{ZR}(t),$$

where $J_2^{EP}(t)$ (resp. $J_2^{ZR}(t)$) is the process that counts the number of first class particles that jump over the second class particle in the time interval $[0,t]$ in the TASEP (resp. TAZRP) and $H_2^{EP}(t)$ is the process that counts the number of holes that the second class particle jumps over in the time interval $[0,t]$ in the TASEP.

Since these processes have been very well studied in the TASEP, see for example [6] and references therein, from there one can get information about a second class particle in the TAZRP.

## 6 Second and Third Class Particles

In this section we add to the TASEP and to the TAZRP one second class particle and one third class particle. The dynamics of the third class particle is defined as follows. The first class particles and the second class particle see the third class particle as a hole, therefore in the TASEP if a first or a second class particle jumps to a site occupied by a third class particle then the particles exchange positions. In the TAZRP if first, second and third class particles share the same site, then the third class particle only leaves this site if there is no first class particles nor the second class particle there.

In this section we present a simple proof of Theorem 3 and a similar statement for the TAZRP, namely Corollary 1, which is a consequence of Theorem 3 and the coupling described in the previous section.

**Theorem 3 ([1, 3]).** *Consider the TASEP, starting from the configuration $\eta$, such that all the sites $x \in \mathbb{Z}_-$ are occupied by first class particles, the origin is occupied by a second class particle, while the site $x = 1$ is occupied by a third class particle* *and the remaining sites are empty. See the figure below, where the second class particle is represented by* ⊛ *and the third class particle is represented by* ⊖.

●●●●⊛⊖○○○○ $\eta$

*Let* $X_2^{EP}(t)$ *and* $X_3^{EP}(t)$ *denote the position of the second class particle and the position of the third class particle, respectively, at time* $t$. *Then*

$$\lim_{t\to+\infty} P\left(X_2^{EP}(t) > X_3^{EP}(t)\right) = \frac{2}{3}.$$

*Proof.* Denote by $\tilde{\eta}$, the configuration that has a second class particle at the origin, while the negative sites are occupied by first class particles and the rest is empty.

●●●●⊛○○○○ $\tilde{\eta}$

Let $\Xi$ denote the space of configurations of $\{0,1\}^{\mathbb{Z}}$ that have exactly one second class particle. For a configuration $\eta \in \Xi$, let $X_2^{EP}(t,\eta)$ denote the position of the second class particle at time $t$ in the configuration $\eta$.

The process $(\eta_t, X_2^{EP}(t,\eta))$ has generator given on local functions $f : \{0,1\}^{\mathbb{Z}} \times \mathbb{Z} \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{L}^2 f(\eta,z) = &\sum_{x,x+1\neq z} \eta(x)(1-\eta(x+1))\{f(\eta^{x,x+1},z) - f(\eta,z)\}\\ &+ \eta(z-1)\{f(\eta^{z-1,z},z-1) - f(\eta,z)\}\\ &+ (1-\eta(z+1))\{f(\eta^{z,z+1},z+1) - f(\eta,z)\}.\end{aligned} \tag{7}$$

This generator translates the dynamics of the second class particle in the TASEP that we defined above: the second class particle has the same jump rate as the first class particles, but whenever a first class particle jumps to a site occupied by a second class particle they exchange positions and when a second class particle attempts to jump to a site occupied by a first class particle, the jump is forbidden.

For a configuration $\eta \in \Xi$, denote by $J_t^2(\eta)$ the process that counts the number of first class particles that jump from $X_2^{EP}(s,\eta) - 1$ to $X_2^{EP}(s,\eta)$, for $s \in [0,t]$. This current can be formally defined by:

$$J_t^2(\eta) = \sum_{x\geq 0} \eta_t(x + X_2^{EP}(t,\eta)) - \eta_0(x),$$

so that

$$J_t^2(\tilde{\eta}) = \sum_{x \geq X_2^{EP}(t,\tilde{\eta})} \tilde{\eta}_t(x).$$

Then, applying the Kolmogorov backwards equation, we have that

$$\frac{d}{dt}E\left(J_t^2(\tilde{\eta})\right) = E\left(\mathcal{L}^2(J_t^2(\tilde{\eta}))\right) = E\left(J_t^2(\tilde{\eta}^{-1,0})\right) + E\left(J_t^2(\tilde{\eta}^{0,1})\right) - 2E\left(J_t^2(\tilde{\eta})\right), \tag{8}$$

where $\tilde{\eta}^{-1,0}$ corresponds to a jump of the rightmost first class particle in $\eta$ from the site $-1$ to 0 and $\tilde{\eta}^{0,1}$ corresponds to a jump of the second class particle from the site 0 to the site 1 which is occupied by the leftmost hole.

●●●⊛●○○○○ $\tilde{\eta}^{-1,0}$

●●●●○⊛○○○ $\tilde{\eta}^{0,1}$

Analogously, for a configuration $\eta$ in $\Xi$, we denote by $H_t^2(\eta)$ the process that counts the number of holes that jump from $X_2^{EP}(s,\eta)+1$ to $X_2^{EP}(s,\eta)$, for $s \in [0,t]$, formally defined by

$$H_t^2(\eta) = \sum_{x \leq 0} \Big\{(1-\eta_t(x - X_2^{EP}(t,\eta))) - (1-\eta_0(x))\Big\}.$$

Notice that,

$$H_t^2(\tilde{\eta}) = \sum_{x \leq X_2^{EP}(t,\tilde{\eta})} (1-\tilde{\eta}_t(x)).$$

Now, the processes $J_t^2(\eta)$ and $H_t^2(\eta)$ behave symmetrically when starting them from the configurations $\tilde{\eta}^{-1,0}$ and $\tilde{\eta}^{0,1}$, respectively, see Lemma 1. Therefore, by Lemma 1, we can write (8) as

$$\frac{d}{dt}E(J_t^2(\tilde{\eta})) = E(H_t^2(\tilde{\eta}^{0,1})) + E(J_t^2(\tilde{\eta}^{0,1})) - 2E(J_t^2(\tilde{\eta})). \tag{9}$$

On the other hand we also have that $H_t^2(\tilde{\eta}) = J_t^2(\tilde{\eta})$ in distribution, see Lemma 2.

Now, we are in a good position to compute (9) by coupling the TASEP starting from $\tilde{\eta}^{0,1}$ and $\tilde{\eta}$. There are two discrepancies between the configurations $\tilde{\eta}^{0,1}$ and $\tilde{\eta}$ which stand at the sites 0 and 1 as can be seen in the figure below.

●●●●⊛○○○ $\tilde{\eta}$

●●●●○⊛○○ $\tilde{\eta}^{0,1}$

Let $Y_0(t)$ and $Y_1(t)$ denote the position at time $t$ of the discrepancies initially at site 0 and 1, respectively. These discrepancies behave as a second class particle and as a third class particle in the coupled process until the time they meet. The coupled process starts from $\eta$. Then, until this meeting time, we have that

$$X_2^{EP}(t) = Y_0(t),$$

$$X_3^{EP}(t) = Y_1(t).$$

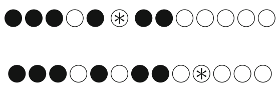
Now, let $A_t = \left\{Y_0(t) < Y_1(t)\right\}$. If $A_t$ happens, then

$$H_t^2(\tilde{\eta}^{0,1}) = H_t^2(\tilde{\eta}) + 1 + \sum_{x=Y_0(t)+1}^{Y_1(t)} (1 - \tilde{\eta}_t(x))$$

and

$$J_t^2(\tilde{\eta}) = J_t^2(\tilde{\eta}^{0,1}) + \sum_{x=Y_0(t)+1}^{Y_1(t)} \tilde{\eta}_t(x),$$

see the figure below.



Otherwise $H_t^2(\tilde{\eta}^{0,1}) = H_t^2(\tilde{\eta})$ and $J_t^2(\tilde{\eta}) = J_t^2(\tilde{\eta}^{0,1})$, since the configurations at time $t$ are equal.

We can partition the space to rewrite (9) as

$$\begin{aligned}\frac{d}{dt}E(J_t^2(\tilde{\eta})) =& E\left(1_{A_t}(H_t^2(\tilde{\eta}^{0,1}) + J_t^2(\tilde{\eta}^{0,1}) - 2J_t^2(\tilde{\eta}))\right) \\ &+ E\left(1_{A_t^c}(H_t^2(\tilde{\eta}^{0,1}) + J_t^2(\tilde{\eta}^{0,1}) - 2J_t^2(\tilde{\eta}))\right).\end{aligned}$$

Using the relations established above, we have that:

$$\frac{d}{dt}E(J_t^2(\tilde{\eta})) = P(A_t) + E\Big(1_{A_t}\Big\{\sum_{x=Y_0(t)+1}^{Y_1(t)} \Big(1 - \tilde{\eta}_t(x)\Big) - \sum_{x=Y_0(t)+1}^{Y_1(t)} \tilde{\eta}_t(x)\Big\}\Big).$$

Now, by symmetry it holds that

$$\sum_{x=Y_0(t)+1}^{Y_1(t)} \Big(1 - \tilde{\eta}_t(x)\Big) =_{law} \sum_{x=Y_0(t)+1}^{Y_1(t)} \tilde{\eta}_t(x).$$

Then, we obtain

$$\frac{d}{dt}E(J_t^2(\tilde{\eta})) = P(A_t) = P\Big(X_2^{EP}(t) < X_3^{EP}(t)\Big). \tag{10}$$

It remains to compute the left hand side of last expression. For the configuration $\tilde{\eta}$ we can label the first class particles from the left to the right, in such a way that $P_i(0,\tilde{\eta})$ denotes the position of the $i$-th first class particle at time 0. Clearly one has $P_i(0,\tilde{\eta}) = -i$. Let $P_i(t,\tilde{\eta})$ denote the position of this particle at time $t$.

Since first class particles preserve their order, it is easy to see that the current through the second class particle, namely $J_t^2(\tilde{\eta})$, can be written as

$$J_t^2(\tilde{\eta}) = \sum_{x=X_2^{EP}(t,\tilde{\eta})}^{P_1(t,\tilde{\eta})} \tilde{\eta}_t(x), \tag{11}$$

see the figure below where the rightmost particle is at $P_1(t,\tilde{\eta}) = 6$, $X_2^{EP}(t,\tilde{\eta}) = -1$ and $J_t^2(\tilde{\eta}) = 3$.

●●●○●●●○ ⊛ ○○○●●○●○○○

By the LLN for $X_2^{EP}(t,\tilde{\eta})$ and for $P_1(t,\tilde{\eta})$ together with the hydrodynamic limit for the empirical measure for the TASEP (see [12]) and since $J_t^2(\tilde{\eta})$ can be written as in (11), in [5,6] it was shown that

$$\frac{J_t^2(\tilde{\eta})}{t} \xrightarrow[t\to+\infty]{} \Big(\frac{1-\mathcal{U}}{2}\Big)^2, \quad \text{almost surely},$$

where $\mathcal{U}$ is the random variable with Uniform distribution on $[-1,1]$ given in Theorem 1 with $\alpha = 1$ and $\beta = 0$. In particular the convergence in distribution also holds.

Using the martingale decomposition of the current it is easy to show, that for any $\epsilon > 0$,

$$\left(\frac{J_t^2(\tilde{\eta})}{t}\right)^{2-\epsilon}$$

is uniformly integrable since its $L^2$-norm is finite. As a consequence, by a well know result on weak convergence of random variables, it holds that

$$\lim_{t\to+\infty} E\left(\frac{J_t^2(\tilde{\eta})}{t}\right) = E\left(\frac{1-\mathcal{U}}{2}\right)^2 = \frac{1}{3}. \tag{12}$$

Moreover,

$$\frac{1}{t}\int_0^t \frac{d}{ds}E(J_s^2(\tilde{\eta}))ds = E\left(\frac{J_t^2(\tilde{\eta})}{t}\right), \tag{13}$$

and by (10), the left hand side of last expression is equal to

$$\frac{1}{t}\int_0^t P(A_s)ds.$$

Now, since $A_s$ are decreasing sets, then $P(A_s)$ decreases and as a consequence the limit, as $s \to +\infty$ exists. By the Césaro theorem

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t P(A_s)ds = \lim_{t\to+\infty} P(A_t).$$

Putting together last result, (12) and (13), we obtain that $\lim_{t\to+\infty} P(A_t) = \frac{1}{3}$, which concludes the proof.

**Lemma 1.** *The process $J_t^2(\tilde{\eta}^{-1,0})$ has the same distribution as the process $H_t^2(\tilde{\eta}^{0,1})$.*

*Proof.* In other words, we have to show that if $L^J$ and $L^H$ represent the generators of the processes $J_t^2(\tilde{\eta}^{-1,0})$ and $H_t^2(\tilde{\eta}^{0,1})$, respectively, then for every local function $f : \{0,1\}^{\mathbb{Z}} \times \mathbb{Z} \to \mathbb{R}$,

$$L^J f(\tilde{\eta}^{-1,0}, z) = L^H f(\tilde{\eta}^{0,1}, z).$$

The easiest way of showing this is to consider the process seen from the position of the second class particle.

For a configuration $\eta \in \Xi$, let $\eta'_t = \tau_{X_2^{EP}(t,\eta)}\eta_t$ be such that for a site $x \in \mathbb{Z}$, $\eta'_t(x) = \eta_t(x + X_2^{EP}(t,\eta))$ be the process whose generator is given on local functions $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ by

$$\begin{aligned} L' f(\eta') = &\sum_{x,x+1\neq 0} \eta'(x)(1-\eta'(x+1))\{f(\sigma_{EP}^{x,x+1}\eta') - f(\eta')\} \\ &+ \eta'(-1)\{f(\tau_{-1}\sigma_{EP}^{-1,0}\eta') - f(\eta')\} \\ &+ (1-\eta'(1))\{f(\tau_1\sigma_{EP}^{0,1}\eta') - f(\eta')\}. \end{aligned}$$

Above $\tau_x\eta$ is the shift in $\eta$ that places the second class particle at the origin. In this process the position of $X_2^{EP}(t,\eta)$ corresponds to the number of shifts of the system, of size $-1$, during the time interval $[0,t]$ and as a consequence, in this process the site 0 is always occupied by a second class particle.

Denote by $N_1(t,\eta')$ the number of particles that jump from the site $-1$–$0$ during the time interval $[0,t]$:

$$N_1(t,\eta') = \sum_{x\geq 0} \left(\eta'_t(x) - \eta'_0(x)\right).$$

Note that $N_1(t,\eta')$ corresponds to the number of particles at the right hand side of $X_2^{EP}(t,\eta)$ at time $t$, and as a consequence one has that $J_t^2(\eta) = N_1(t,\eta')$.

Consider now the process $(\eta'_t, N_1(t,\eta'))$ with generator given on local functions $f : \{0,1\}^{\mathbb{Z}} \times \mathbb{Z} \to \mathbb{R}$ by

$$\begin{aligned} L_1 f(\eta',N) = & \sum_{x,x+1\neq 0} \eta'(x)(1-\eta'(x+1))\{f(\sigma_{EP}^{x,x+1}\eta',N) - f(\eta',N)\} \\ & + \eta'(-1)\{f(\tau_{-1}\sigma_{EP}^{-1,0}\eta',N+1) - f(\eta',N)\} \\ & + (1-\eta'(1))\{f(\tau_1\sigma_{EP}^{0,1}\eta',N) - f(\eta',N)\}. \end{aligned} \tag{14}$$

Analogously, we can consider $N_{-1}(t,\eta')$ as the number of jumps, of size 1, of the second class particle, that is, the number of shifts of the system of size 1. Whenever the second class particle jumps one unit ahead, the hole placed before the jump at site 1 jumps to the site $-1$, then we can write:

$$N_{-1}(t,\eta') = \sum_{x\leq 0} \left((1-\eta'_t(x)) - (1-\eta'_0(x))\right).$$

In this case we also have that $H_t^2(\eta) = N_{-1}(t,\eta')$.

The process $(\eta'_t, N_{-1}(t,\eta'))$ has generator given on local functions $f : \{0,1\}^{\mathbb{Z}} \times \mathbb{Z} \to \mathbb{R}$ by

$$\begin{aligned} L_{-1} f(\eta',N) = & \sum_{x,x+1\neq 0} \eta'(x)(1-\eta'(x+1))\{f(\sigma_{EP}^{x,x+1}\eta',N) - f(\eta',N)\} \\ & + \eta'(-1)\{f(\tau_{-1}\sigma_{EP}^{-1,0}\eta'^{-1,0},N) - f(\eta',N)\} \\ & + (1-\eta'(1))\{f(\tau_1\sigma_{EP}^{0,1}\eta',N+1) - f(\eta',N)\}. \end{aligned}$$

To fix notation, let $\zeta = \tilde{\eta}^{-1,0}$ and $\varsigma = \tilde{\eta}^{0,1}$, as shown below.

●●●⊛●○○○○ $\zeta$

●●●●○⊛○○○ $\varsigma$

As before denote by $\varsigma'$ and $\zeta'$ the configurations $\varsigma$ and $\zeta$ seen from the second class particle, respectively. We couple the processes starting from $\zeta$ and $\varsigma$ under the basic coupling, so that clocks are attached to sites. By the symmetry of the configurations, it is easy to see that $\forall x \neq 0$, $\zeta(x) = 1 - \varsigma(-x)$ and both have a second class particle at the origin. Now simple computations show that

$$L^J f(\tilde{\eta}^{-1,0}, N) = L_1 f(\varsigma', z) = L_{-1} f(\zeta', N) = L^H(\tilde{\eta}^{0,1}, N),$$

which concludes the proof.

We give a sketch of last equality. Let $f : \{0,1\}^{\mathbb{Z}} \times \mathbb{Z} \to \mathbb{R}$ be a local function and $\zeta'$ and $\varsigma'$ as defined above. Then:

$$\begin{aligned} L_1 f(\varsigma', N) = \sum_{x, x+1 \neq 0} & (1 - \zeta'(-x))(1 - (1 - \zeta'(-(x+1))) \\ & \times \{ f(\sigma_{EP}^{-x-(x+1)} \zeta', N) - f(\zeta', N) \} \\ & + (1 - \zeta'(1)) \{ f(\tau_1 \sigma_{EP}^{0,1} \zeta', N+1) - f(\zeta', N) \} \\ & + \zeta'(-1) \{ f(\tau_{-1} \sigma_{EP}^{-1,0} \zeta', N) - f(\zeta', N) \}. \end{aligned}$$

In the first equality we used the fact that $\forall x \neq 0$, $\zeta'(x) = 1 - \varsigma'(-x)$ and notice that last expression is precisely $L_{-1} f(\zeta', N)$.

**Lemma 2.** *The process $J_t^2(\tilde{\eta})$ has the same distribution as the process $H_t^2(\tilde{\eta})$.*

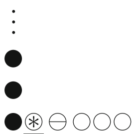
*Proof.* The proof follows the same computations as the ones performed in the proof of last lemma since what we have to show is that for every local function $f : \{0,1\}^{\mathbb{Z}} \times \mathbb{Z} \to \mathbb{R}$,

$$L^J f(\tilde{\eta}, z) = L^H f(\tilde{\eta}, z).$$

This is a consequence of the particle-hole symmetry of the processes for the configuration $\tilde{\eta}$.

As a consequence of Theorem 3 and a simple modification of the coupling described in the previous section (see [8] for details) the following result holds.

**Corollary 1.** *Consider the TAZRP starting from the configuration $\xi$, such that all the sites $x \in \mathbb{Z}_-$ are occupied by infinitely many first class particles, the origin is occupied by a second class particle, the site $x = 1$ is occupied by a third class particle and the remaining sites are empty. See the figure below, where the second class particle is represented by $\circledast$ and the third class particle is represented by $\ominus$.*



*Let* $X_2^{ZR}(t)$ *and* $X_3^{ZR}(t)$ *denote the position of the second class particle and the position of the third class particle, respectively, at time* $t$*. Then*

$$\lim_{t\to+\infty} P(X_2^{ZR}(t) \geq X_3^{ZR}(t)) = \frac{2}{3}.$$

To finish I would like to mention that it would be an interesting problem to derive the previous result without going to the coupling argument. It would also be a very interesting problem to extend the results presented here for more general zero-range processes with a rate function given by a function $g(\cdot)$ and with partially asymmetric jumps. In each case the coupling with TASEP presented in Sect. 5 fails dramatically.

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## References

1. Amir, G., Angel, O., Valko, B.: . The TASEP speed process. Ann. Probab. **39**(4), 1205–1242 (2011)
2. Andjel, E.: Invariant measures for the zero range process. Ann. Probab. **10**(3), 525–547 (1982)
3. Ferrari, P., Gonçalves, P., Martin, J.: Collision probabilities in the rarefaction fan of asymmetric exclusion processes. Ann. Inst. H. Poincaré Probab. Stat. **45**(4), 1048–1064 (2009)
4. Ferrari, P., Kipnis, C.: Second class particles in the rarefaction fan. Ann. Inst. H. Poincaré Probab. Stat. **31**(1), 143–154 (1995)
5. Ferrari, P., Martin, J., Pimentel, L.: Roughening and inclination of competition interfaces. Phys. Rev. E **73**, 031602 (2006)
6. Ferrari, P., Martin, J., Pimentel, L.: A phase transition for competition interfaces. Ann. Appl. Probab. **19**(1), 281–317 (2009)
7. Ferrari, P., Pimentel, L.: Competition interfaces and Second class particles. Ann. Probab. **33**(4), 1235–1254 (2005)
8. Gonçalves, P.: On the asymmetric zero-range in the rarefaction fan. J. Stat. Phys. **154**(4), 1074–1095 (2014)
9. Kipnis, C., Landim, C.: Scaling Limits of Interacting Particle Systems. Springer, New York (1999)
10. Liggett, T.: Interacting Particle Systems. Springer, New York (1985)
11. Mountford, T., Guiol, H.: The motion of a second class particle for the TASEP starting from a decreasing shock profile. Ann. Appl. Probab. **15**(2), 1227–1259 (2005)
12. Rezakhanlou, F.: Hydrodynamic limit for attractive particle systems on $\mathbb{Z}^d$. Comm. Math. Phys. **140**(3), 417–448 (1991)

# Microscopic Derivation of an Isothermal Thermodynamic Transformation

**Stefano Olla**

## 1 Introduction

Isothermal transformations are fundamental in thermodynamics, in particular they are one of the components of the Carnot cycle. As often in thermodynamics, they represent *idealized* transformations where the system is maintained at a constant temperature by being in constant contact with a *large* heat reservoir (*heat bath*). An isothermal thermodynamic transformations connects two equilibrium states $A_0$ and $A_1$ at the same temperature $T$, by changing the exterior forces applied. According to the first law of thermodynamics, the change in the internal energy is given by $U_1 - U_0 = W + Q$, where $W$ is the work made by the exterior forces and $Q$ is the heat (energy) exchanged with the thermal reservoir. The second law prescribes that the change of the free energy $F = U - TS$ (where $S$ is the thermodynamic entropy), satisfy the Clausius inequality $F_1 - F_0 \leq W$, with equality satisfied for *reversible quasistatic* transformations. In the quasistatc transformation we can then identify $Q = T(S_1 - S_0)$.

The purpose of this article is to prove mathematically that the thermodynamic behavior of isothermal transformations, as described above, can be obtained by proper space and time scaling of a microscopic dynamics. We consider a one dimensional system, where the equilibrium thermodynamic intensive parameters are given by the temperature $T = \beta^{-1}$ and the tension (or pressure) $\tau$, or by the extensive observables: length (volume) $\mathcal{L}$ and energy $U$. This simplifies the problem as only two parameters are needed to specify the equilibrium thermodynamic state and no phase transitions will appear.

The microscopic model is given by a chain of $N$ anharmonic oscillators, where the first particle is attached to a fix point and on the last particle acts a force

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C. Bernardin and P. Gonçalves (eds.), *From Particle Systems to Partial Differential Equations*, Springer Proceedings in Mathematics & Statistics 75,
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(tension) $\tilde{\tau}$, eventually changing in time. The action of the thermal bath is modeled by independent Langevin processes at temperature $T$, acting on each particle. A mathematically equivalent model for the heat bath is given by random collisions with the environment: at exponentially distributed independent random times, each particle has a new velocity distributed by a centered gaussian with variance $T$.

As a consequence of the action of the thermal bath, the time evolution of the microscopic configuration of the positions and velocities of the particles is stochastic. The distance between the first and the last particle defines the microscopic length of the system, while the energy is given by the sum of the kinetic energies of each particle and the potential energy of each spring.

For each value of the applied tension $\tau$, the system has an equilibrium probability distribution explicitly given by a Gibbs measure, a product measure in this case. The temperature parameter is fixed by the heat bath. Starting the system with an equilibrium given by tension $\tau_0$, and changing the applied tension to $\tau_1$, the system will go out of equilibrium before reaching the new equilibrium state. During this transformation a certain amount of energy is exchanged with the thermostats and mechanical work is done by the force applied. We prove that, under a proper macroscopic rescaling of space and time, all these (random) quantities, converge to *deterministic* values predicted by thermodynamics.

When the system is out of equilibrium, either for a change in the tension applied, or by initial conditions, there is an evolution of the local length (or stretch) on a diffusive macroscopic space-time scale. This is governed by a diffusion equation that describe the inhomogeneity of the system during the isothermal transformation. After an infinite time (in this scale) it reach the new equilibrium state given by a constant value of the local length, corresponding to the value of the tension $\tau_1$. We have obtained, in this diffusive time scale, an *irreversible* thermodynamic transformation, that satisfies a strict Clausius inequality between work and change of the free energy. Under a further rescaling of time, that correspond in a slower change of the applied tension, we obtain a *reversible quasi-static* transformation that satisfies Clausius inequality. In fact, for the irreversible transformation we obtain the following relation between heat and changes of thermodynamic entropy $S$

$$Q = T\Delta S - \mathcal{D}$$

where $\mathcal{D}$ is a strictly positive dissipation term that has an explicit expression in terms of the solution of the diffusive equation that govern macroscopically the transformation (cf. (48)). In the quasi-static limit we prove that $\mathcal{D} \to 0$. A similar interpretation of quasi-static transformations, for thermodynamic systems with one parameter (density), has been proposed in recent works by Bertini et al. [1,2].

In the case of the harmonic chain, the thermodynamic entropy is a function of the temperature, so it remains constant in isothermal transformation. Then heat is equal to the dissipation term $\mathcal{D}$. It means that in the quasistatic limit for the harmonic chain, there is no heat produced, internal energy is changed by work in a perfectly efficient way.

Thermodynamics does not specify the time scale for the transformations, this may depend on the nature of the transformation (isothermal, adiabatic, ...) and the details of the microscopic system and of the exterior agent (heat bath etc.). In this system of oscillators, in adiabatic setting, with also momentum conservation, the relevant space–time scale is hyperbolic (cf. [3]).

The proof of the hydrodynamic limit follows the lines of [5,7], using the relative entropy method (cf. [4,8]). The method has to be properly adapted to deal with the boundary conditions.

## 2 Isothermal Microscopic Dynamics

We consider a chain of $N$ coupled oscillators in one dimension. Each particle has the same mass that we set equal to 1. The position of atom $i$ is denoted by $q_i \in \mathbb{R}$, while its momentum is denoted by $p_i \in \mathbb{R}$. Thus the configuration space is $(\mathbb{R} \times \mathbb{R})^N$. We assume that an extra particle 0 to be attached to a fixed point and does not move, i.e. $(q_0, p_0) \equiv (0, 0)$, while on particle $N$ we apply a force $\tilde{\tau}(t)$ depending on time. Observe that only the particle 0 is constrained to not move, and that $q_i$ can assume also negative values.

Denote by $\mathbf{q} := (q_1, \ldots, q_N)$ and $\mathbf{p} := (p_1, \ldots, p_N)$. The interaction between two particles $i$ and $i-1$ will be described by the potential energy $V(q_i - q_{i-1})$ of an anharmonic spring relying the particles. We assume $V$ to be a positive smooth function which for large $r$ grows faster than linear but at most quadratic, that means that there exists a constant $C > 0$ such that

$$\lim_{|r|\to\infty} \frac{V(r)}{|r|} = \infty. \tag{1}$$

$$\limsup_{|r|\to\infty} V''(r) \le C < \infty. \tag{2}$$

Energy is defined by the following Hamiltonian:

$$\mathcal{H}_N(\mathbf{q}, \mathbf{p}) := \sum_{i=1}^{N} \left( \frac{p_i^2}{2} + V(q_i - q_{i-1}) \right).$$

Since we focus on a nearest neighbor interaction, we may define the distance between particles by

$$r_i = q_i - q_{i-1}, \qquad i = 1, \ldots, N.$$

The chain is immersed in a thermal bath at temperature $\beta^{-1}$ that we model by the action of $N$ independent Langevin processes. The dynamics is defined by the solution of the system of stochastic differential equations

$$
\begin{aligned}
dr_i &= N^2(p_i - p_{i-1})\, dt \\
dp_i &= N^2(V'(r_{i+1}) - V'(r_i))\, dt - N^2\gamma p_i\, dt - N\sqrt{2\gamma\beta^{-1}}dw_i, \qquad i = 1, \dots, N-1, \\
dp_N &= N^2(\tilde{\tau}(t) - V'(r_N))\, dt - N^2\gamma p_N\, dt - N\sqrt{2\gamma\beta^{-1}}dw_N
\end{aligned}
\tag{3}
$$

Here $\{w_i(t)\}_i$ are N-independent standard Wiener processes, $\gamma > 0$ is a parameter of intensity of the interaction with the heat bath, $p_0$ is set identically to 0. We have also already rescaled time according to the diffusive space-time scaling. Notice that $\tilde{\tau}(t)$ changes at this macroscopic time scale.

The generator of this diffusion is given by

$$
\mathcal{L}_N^{\tilde{\tau}(t)} := N^2 A_N^{\tau(t)} + N^2\gamma S_N. \tag{4}
$$

Here the Liouville operator $A_N^\tau$ is given by

$$
\begin{aligned}
A_N^\tau = &\sum_{i=1}^{N}(p_i - p_{i-1})\frac{\partial}{\partial r_i} + \sum_{i=1}^{N-1}\left(V'(r_{i+1}) - V'(r_i)\right)\frac{\partial}{\partial p_i} \\
&+ \left(\tau - V'(r_N)\right)\frac{\partial}{\partial p_N},
\end{aligned}
\tag{5}
$$

while

$$
S = \sum_{i=1}^{N}\left(\beta^{-1}\partial_{p_i}^2 - p_i\partial_{p_i}\right) \tag{6}
$$

For $\tilde{\tau}(t) = \tau$ constant, the system has a unique stationary measure given by the product

$$
d\mu_{\tau,\beta}^N = \prod_{i=1}^{N} e^{-\beta(\mathcal{E}_i - \tau r_i) - \mathcal{G}_{\tau,\beta}}\, dr_i\, dp_i = g_\tau^N\, d\mu_{0,\beta}^N \tag{7}
$$

where we denoted $\mathcal{E}_i = p_i^2/2 + V(r_i)$, the energy we attribute to the particle $i$, and

$$
\mathcal{G}_{\tau,\beta} = \log\left[\sqrt{2\pi\beta^{-1}}\int e^{-\beta(V(r) - \tau r)}\, dr\right]. \tag{8}
$$

Observe that the function $\mathfrak{r}(\tau) = \beta^{-1}\partial_\tau\mathcal{G}_{\tau,\beta}$ gives the average equilibrium length in function of the tension $\tau$, and we denote the inverse by $\boldsymbol{\tau}(\mathfrak{r})$.

We will need also to consider local Gibbs measure (inhomogeneous product), corresponding to profiles of tension $\{\tau(x), x \in [0,1]\}$:

$$d\mu_{\tau,\beta}^{N} = \prod_{i=1}^{N} e^{-\beta(\mathcal{E}_i - \tau(i/N) r_i) - \mathcal{G}_{\tau(i/N),\beta}}\, dr_i\, dp_i = g_{\tau(\cdot)}^{N} \prod_{i=1}^{N} dr_i\, dp_i \tag{9}$$

Given an initial profile of tension $\tau(0,x)$, we assume that initial probability state is given by an absolutely continuous measure (with respect to the Lebesgue measure), whose density is given by $f_0^N$, such that the relative entropy

$$H_N(0) = \int f_0^N \log\left(\frac{f_0^N}{g_{\tau(0,\cdot)}^N}\right) \prod_{i=1}^{N} dr_i\, dp_i \tag{10}$$

satisfies

$$\lim_{N\to\infty} \frac{H_N(0)}{N} = 0 \tag{11}$$

This implies the following convergence in probability with respect to $f_0^N$:

$$\frac{1}{N}\sum_{i=1}^{N} G(i/N) r_i(0) \longrightarrow \int_0^1 G(x)\mathfrak{r}(\tau(0,x))\, dx \tag{12}$$

The macroscopic evolution for the stress will be given by

$$\begin{aligned} \partial_t r(t,x) &= \gamma^{-1}\partial_x^2 \boldsymbol{\tau}(r(x,t)), \quad x \in [0,1] \\ \partial_x r(t,0) &= 0, \quad \boldsymbol{\tau}(r(t,1)) = \tilde{\tau}(t), \quad t > 0 \\ \boldsymbol{\tau}(r(0,x)) &= \tau(0,x), \quad x \in [0,1] \end{aligned} \tag{13}$$

Observe that we do not require that $\tau(r(0,1)) = \tilde{\tau}(0)$, so we can consider initial profiles of equilibrium with tension different than the applied $\tilde{\tau}$.

The main result is the following

**Theorem 1.**

$$\lim_{N\to\infty} \frac{H_N(t)}{N} = 0 \tag{14}$$

*where*

$$H_N(t) = \int f_t^N \log\left(\frac{f_t^N}{g_{\tau(t,\cdot)}^N}\right) \prod_{i=1}^{N} dr_i\, dp_i \tag{15}$$

*with* $\tau(t,x) = \boldsymbol{\tau}(r(t,x))$, *and* $f_t^N$ *the density of the configuration of the system at time* $t$.

A sketch of the proof is postponed to Sect. 5.

*Remark 1.* The proof and the result are identical (up to some constant) if we use a different modelling of the heat bath, where the particles undergo independent random collisions such that after the collision they get a new value distributed by a gaussian distribution with variance $\beta^{-1}$, i.e.

$$Sf(\mathbf{r},\mathbf{p}) = \sum_{i=1}^{N} \int \left( f(\mathbf{r}, p_1, \dots, p_i', \dots) - f(\mathbf{r},\mathbf{p}) \right) \frac{e^{-\beta(p_i')^2/2}}{\sqrt{2\pi\beta^{-1}}} dp_i' \tag{16}$$

## 3 Thermodynamic Consequences

Consider the case where we start our system with a constant tension $\tau(0,x) = \tau_0$ and we apply a tension $\tilde{\tau}(t)$ going smoothly from $\tilde{\tau}(0) = \tau_0$ to $\tilde{\tau}(t) = \tau_1$ for $t \geq t_1$. It follows from standard arguments that

$$\lim_{t\to\infty} \boldsymbol{\tau}(r(t,x)) \;=\; \tau_1, \qquad \forall x \in [0,1] \tag{17}$$

so on an opportune time scale, this evolution represents an isothermal thermodynamic transformation from the equilibrium state $(\tau_0, \beta^{-1})$ to $(\tau_1, \beta^{-1})$. Clearly this is an irreversible transformation and will satisfy a strict Clausius inequality.

The length of the system at time $t$ is given by

$$L(t) = \int_0^1 r(t,x)\, dx \tag{18}$$

and the work done by the force $\tilde{\tau}$:

$$\begin{aligned} W(t) = \int_0^t \tilde{\tau}(s) dL(s) = \gamma^{-1} \int_0^t ds\; \tilde{\tau}(s) \int_0^1 dx\; \partial_x^2 \boldsymbol{\tau}(r(s,x)) \\ = \gamma^{-1} \int_0^t \tilde{\tau}(s) \partial_x \boldsymbol{\tau}(r(s,1)) ds \end{aligned} \tag{19}$$

The free energy of the equilibrium state $(r,\beta)$ is given by the Legendre transform of $\beta^{-1}\mathcal{G}_{\tau,\beta}$:

$$F(r,\beta) = \inf_{\tau} \left\{ \tau r - \beta^{-1} \mathcal{G}_{\tau,\beta} \right\} \tag{20}$$

Since $\beta$ is constant, we will drop the dependencies on it in the following. It follows that $\boldsymbol{\tau}(r) = \partial_r \mathcal{F}$. Thanks to the local equilibrium, we can define the free energy at time $t$ as

$$\mathcal{F}(t) = \int_0^1 F(r(t,x),\beta)\, dx. \tag{21}$$

Its time derivative is (after integration by parts):

$$\frac{d}{dt}\mathcal{F}(t) = -\gamma^{-1}\int_0^1 (\partial_x \boldsymbol{\tau}(r(t,x)))^2\ dx + \gamma^{-1}\tilde{\tau}(t)\partial_x \boldsymbol{\tau}(r(t,x))\Big|_{x=1}$$

i.e.

$$\mathcal{F}(t) - \mathcal{F}(0) = W(t) - \gamma^{-1}\int_0^t ds \int_0^1 (\partial_x \boldsymbol{\tau}(r(s,x)))^2\ dx$$

Because or initial condition, $\mathcal{F}(0) = F(\tau_0)$, and because (17) we have $\mathcal{F}(t) \to F(\tau_1)$, and we conclude that

$$F(\tau_1) - F(\tau_0) = W - \gamma^{-1}\int_0^{+\infty} ds \int_0^1 (\partial_x \boldsymbol{\tau}(r(s,x)))^2\ dx \tag{22}$$

where $W$ is the total work done by the force $\tilde{\tau}$ in the transformation up to reaching the new equilibrium and is expressed by taking the limit in (19) for $t \to \infty$:

$$W = \int_0^\infty \tilde{\tau}(s) dL(s) = \gamma^{-1}\int_0^\infty \tilde{\tau}(s)\partial_x \boldsymbol{\tau}(r(s,1)) ds \tag{23}$$

By the same argument we will use in the proof of Proposition 1 we have that the second term of the righthand side of (22) is finite, that implies the existence of $W$.

Since the second term on right hand side is always strictly positive, we have obtained a strict Clausius inequality. This is not surprizing since we are operating an irreversible transformation.

If we want to obtain a *reversible quasistatic isothermal transformation*, we have introduce another larger time scale, i.e. introduce a small parameter $\varepsilon > 0$ and apply a tension slowly varying in time $\tilde{\tau}(\varepsilon t)$. The diffusive equation becomes

$$\partial_t r_\varepsilon(t,x) = \gamma^{-1}\partial_x^2 \boldsymbol{\tau}(r_\varepsilon(t,x)) \tag{24}$$

with boundary conditions

$$\begin{aligned} &\partial_x r_\varepsilon(t,0) = 0 \\ &\boldsymbol{\tau}(r_\varepsilon(t,1)) = \tilde{\tau}(\varepsilon t) \end{aligned} \tag{25}$$

Then (22) became

$$F(r_1) - F(r_0) = W_\varepsilon - \gamma^{-1} \int_0^\infty ds \int_0^1 \left(\partial_x \boldsymbol{\tau}(r_\varepsilon(s,x))\right)^2 \, dx \tag{26}$$

**Proposition 1.**

$$\lim_{\varepsilon\to 0} \int_0^\infty ds \int_0^1 \left(\partial_x \boldsymbol{\tau}(r_\varepsilon(s,x))\right)^2 \, dx = 0 \tag{27}$$

*Proof.* To simplify notations, let set here $\gamma = 1$. We look at the time scale $\mathfrak{t} = \varepsilon^{-1} t$, then $\tilde r_\varepsilon(\mathfrak{t}, x) = r_\varepsilon(\varepsilon^{-1} t, x)$ satisfy the equation

$$\partial_{\mathfrak{t}} \tilde r_\varepsilon(\mathfrak{t}, x) = \varepsilon^{-1} \partial_x^2 \boldsymbol{\tau}(\tilde r_\varepsilon(\mathfrak{t}, x)) \tag{28}$$

with boundary conditions

$$\begin{aligned} \partial_x r_\varepsilon(\mathfrak{t}, 0) &= 0 \\ \boldsymbol{\tau}(r_\varepsilon(\mathfrak{t}, 1)) &= \tilde\tau(\mathfrak{t}) \end{aligned} \tag{29}$$

$$\begin{aligned} &\frac{1}{2} \int_0^1 \left(\tilde r_\varepsilon(\mathfrak{t}, x) - \mathfrak{r}[\tilde\tau(\mathfrak{t})]\right)^2 d\mathfrak{t} \\ &= \int_0^{\mathfrak{t}} ds \int_0^1 dx \ \left(\tilde r_\varepsilon(s,x) - \mathfrak{r}(\tilde\tau(s))\right) \left(\varepsilon^{-1} \partial_x^2 \boldsymbol{\tau}[\tilde r_\varepsilon(s,x)] - \frac{d}{ds} \mathfrak{r}[\tilde\tau(s)]\right) \\ &= -\,\varepsilon^{-1} \int_0^{\mathfrak{t}} ds \int_0^1 dx \ \left(\partial_x \tilde r_\varepsilon(s,x)\right)^2 \frac{d\boldsymbol{\tau}}{dr} [\tilde r_\varepsilon(s,x)] \\ &\quad - \int_0^t ds \frac{d\mathfrak{r}}{d\tau}(\tilde\tau(s)) \tilde\tau'(s) \int_0^1 dx \ \left(\tilde r_\varepsilon(s,x) - \tilde r_\varepsilon(s,1)\right) \end{aligned} \tag{30}$$

Rewriting

$$\begin{aligned} \left| \int_0^1 dx \ \left(\tilde r_\varepsilon(s,x) - \tilde r_\varepsilon(s,1)\right) \right| &= \left| \int_0^1 dx \ \int_x^1 dy \ \partial_y \tilde r_\varepsilon(s,y) \right| \\ = \left| \int_0^1 dy \ y \partial_y \tilde r_\varepsilon(s,y) \right| &\le \frac{\alpha}{2\varepsilon} \int_0^1 dx \ \left(\partial_x \tilde r_\varepsilon(s,x)\right)^2 + \frac{\varepsilon}{4\alpha} \end{aligned}$$

By our assumption we have $0 < C_- \le \frac{d\mathfrak{r}}{d\tau} \le C_+ < +\infty$, and furthermore we have chosen $\tilde\tau$ such that $|\tilde\tau'(t)| \le 1_{t \le t_1}$. Regrouping positive terms on the left hand side we obtain the bound:

$$\frac{1}{2}\int_0^1 (\tilde{r}_\varepsilon(\mathfrak{t},x) - \mathfrak{r}[\tilde{\tau}(\mathfrak{t})])^2\,dx + \varepsilon^{-1}\left(C_- - \frac{C_+\alpha\mathfrak{t}}{2}\right)\int_0^{\mathfrak{t}} ds \int_0^1 dx\ \ (\partial_x \tilde{r}_\varepsilon(s,x))^2 \le \frac{\varepsilon C_+ \mathfrak{t}}{4\alpha} \tag{31}$$

By choosing $\alpha = \frac{C_-}{C_+\mathfrak{t}}$, we obtain, for any $\mathfrak{t} > t_1$:

$$\frac{1}{C_-}\int_0^1 (\tilde{r}_\varepsilon(\mathfrak{t},x) - \mathfrak{r}[\tilde{\tau}(t_1)])^2\,dx + \varepsilon^{-1}\int_0^{\mathfrak{t}} ds \int_0^1 dx\ \ (\partial_x \tilde{r}_\varepsilon(s,x))^2 \le \frac{\varepsilon}{2} \tag{32}$$

then we can take the limit as $\mathfrak{t} \to \infty$, the first term on the right hand side of (32) will disappear, and we obtain

$$\varepsilon^{-1}\int_0^{+\infty} ds \int_0^1 dx\ \ (\partial_x \tilde{r}_\varepsilon(s,x))^2 \le \frac{\varepsilon}{2} \tag{33}$$

that implies (27).

Consequently we obtain the Clausius identity for the quasistatic reversible isothermal transformation.

Along the lines of the proof above it is also easy to prove that

$$\lim_{\varepsilon\to 0}\int_0^1 (r_\varepsilon(t,x) - \mathfrak{r}[\tilde{\tau}(\varepsilon t)])^2\,dx = 0 \tag{34}$$

that gives a rigorous meaning to the *quasistatic* definition.

The internal energy of the thermodynamic equilibrium state $(r, T)$ is defined as $U = F + TS$, where $S$ is the thermodynamic entropy. The first principle of thermodynamics defines the heat $Q$ transferred as $\Delta U = W + Q$.

The change of internal energy in the isothermal transformation is given by

$$\Delta U = \Delta F + T\Delta S = W - \gamma^{-1}\int_0^{+\infty} ds \int_0^1 dx\ \ (\partial_x \boldsymbol{\tau}(r(s,x)))^2 + T\Delta S \tag{35}$$

Then for the irreversible transformation we have $Q \le T\Delta S$, while equality holds in the quasistatic limit.

The linear case is special, it corresponds to the microscopic harmonic interaction. In this case $S$ is just a function of the temperature ($S \sim \log T$), so $\Delta S = 0$ for any isothermal transformation. Correspondingly the heat exchanged with the thermostat is always negative and given by $Q = -\gamma^{-1}\int_0^{+\infty} ds \int_0^1 dx\ \ (\partial_x r(s,x))^2$, and null in the quasistatic limit.

# 4 Work and Microscopic Heat

The microscopic total length is defined by $q_N = \sum_i r_i$, the position of the last particle. To connect it to the macroscopic space scale we have to divide it by $N$, so we define

$$\mathcal{L}_N(t) = \frac{q_N(t)}{N} = \frac{1}{N}\sum_{i=1}^{N} r_i(t). \tag{36}$$

The time evolution in the scale considered is given by

$$\mathcal{L}_N(t) - \mathcal{L}_N(0) = \int_0^t N p_N(s)\, ds. \tag{37}$$

If we start with the equilibrium distribution with length $r_0$, the law of large numbers guarantees that

$$\mathcal{L}_N(0) \underset{N\to\infty}{\longrightarrow} r_0, \tag{38}$$

in probability.

By Theorem 1, we also have the convergence at time t:

$$\mathcal{L}_N(t) \underset{N\to\infty}{\longrightarrow} L(t) \underset{t\to\infty}{\longrightarrow} r_1 = \mathfrak{r}(\tau_1), \tag{39}$$

where $L(t)$ is defined by (18). Notice that in (37) while $Np_N(s)$ fluctuates wildly as $N \to \infty$, its time integral is perfectly convergent and in fact converges to a deterministic quantity.

The microscopic work done up to time $t$ by the force $\tilde{\tau}$ is given by

$$\mathcal{W}_N(t) = \int_0^t \tilde{\tau}(s) d\mathcal{L}_N(s) = \int_0^t \tilde{\tau}(s) N p_N(s) ds \tag{40}$$

We adopt here the convention that positive work means energy increases in the system. Notice that $\mathcal{W}_N(t)$ defines the actual microscopic work divided by $N$.

It is a standard exercise to show that, since $\tilde{\tau}(t)$ and $L(t)$ are smooth functions of $t$, by (39) it follows that

$$\mathcal{W}_N(t) \underset{N\to\infty}{\longrightarrow} W(t) = \int_0^t \tilde{\tau}(s) dL(s) \tag{41}$$

given by (19).

Microscopically the energy of the system is defined by

$$E_N = \frac{1}{N} \sum_i \mathcal{E}_i \tag{42}$$

Energy evolves in time as

$$\begin{aligned} E_N(t) - E_N(0) &= \mathcal{W}_N(t) + \mathcal{Q}_N(t) \\ \mathcal{Q}_N(t) &= -\gamma \int_0^t N \sum_{i=1}^N \left(p_i^2(s) - T\right) \, ds + \sqrt{2\gamma\beta^{-1}} \sum_{i=1}^N \int_0^t p_i(s) dw_i(s) \end{aligned} \tag{43}$$

where $\mathcal{Q}_N$ is the energy exchanged with the heat bath, what we call *heat*.

The law of large numbers for the initial distribution gives

$$E_N(0) \underset{N\to\infty}{\longrightarrow} U(\beta, \tau_0)$$

in probability. By the hydrodynamic limit, we expect that

$$E_N(t) \underset{N\to\infty}{\longrightarrow} \int_0^1 U(\beta, \boldsymbol{\tau}(r(t,x))) \, dx \underset{t\to\infty}{\longrightarrow} U(\beta, \tau_1). \tag{44}$$

This is not a consequence of Theorem 1, because the relative entropy does not control the convergence of the energy. In the harmonic case it can be proven by using similar argument as in [6] (in fact in this case $f_N(t)$ is a gaussian distribution where we have control of any moments).

Assuming (44), we have that $Q_N(t)$ converges, as $N \to \infty$, to the deterministic value

$$Q(t) = \int_0^1 \left[U(\beta, \boldsymbol{\tau}(r(t,x))) - U(\beta, \tau_0)\right] \, dx - W(t) \tag{45}$$

and as $t \to \infty$:

$$Q = U(\beta, \tau_1) - U(\beta, \tau_0) - W, \qquad \text{(first principle)}. \tag{46}$$

Recalling that the free energy is equal to $F = U - \beta^{-1} S$, then we can compute the variation of the entropy $S$ as

$$\beta^{-1}(S_1 - S_0) = -(F_1 - F_0) + W + Q \tag{47}$$

or also that

$$Q = \beta^{-1}(S_1 - S_0) - \gamma^{-1} \int_0^\infty dt \int_0^1 dx \, (\partial_x \boldsymbol{\tau}(r(t,x)))^2 \tag{48}$$

In the quasi static limit, we have seen that $F_1 - F_0 = W$, and consequently $\beta Q = S_1 - S_0$, in accord to what thermodynamics prescribe for quasistatic transformations.

*Remark 2.* Assume that the distribution of $p_i(t)$ is best approximated by

$$e^{\frac{\beta}{N\gamma}\sum_i \partial_x \tau(t,i/N) p_i} g^N_{\tau(t,\cdot)} \prod_{i=1}^N dr_i \ dp_i$$

properly normalized. Then the average of $p_i$ is $\frac{1}{N\gamma}\partial_x \tau(t, i/N)$, and (43) can be rewritten as

$$N\gamma \sum_{i=1}^N \left( \left( p_i(t) - \frac{1}{N\gamma}\partial_x \tau(t, i/N) \right)^2 - \beta^{-1} \right)$$
$$- \frac{1}{N\gamma} \sum_{i=1}^N \partial_x \tau(t, i/N)^2 + 2 \sum_{i=1}^N \partial_x \tau(t, i/N) p_i(t)$$

Taking expectation, the first term is null (as well as the martingale not written here) while the last two terms converge to $\gamma^{-1} \int_0^1 (\partial_x \tau(t,x))^2 dx$. This is correct only in the harmonic case, i.e. the fluctuation inside the time integral are very important in order to get the changes in entropy $S$.

## 5 Proof of the Hydrodynamic Limit

Define the modified local Gibbs density

$$\tilde{g}^N_{\tau(t,\cdot)} = e^{\frac{\beta}{\gamma N}\sum_i \partial_x \tau(t,i/N) p_i} g^N_{\tau(t,\cdot)} Z^{-1}_{N,t} \tag{49}$$

where $Z_{N,t}$ is a normalization factor. Then define the corresponding relative entropy

$$\tilde{H}_N(t) = \int f_t^N \log\left( \frac{f_t^N}{\tilde{g}^N_{\tau(t,\cdot)}} \right) \prod_{i=1}^N dr_i \ dp_i \tag{50}$$

It is easy to see that $\lim_{N\to\infty} N^{-1} \left( \tilde{H}_N(t) - H_N(t) \right) = 0$.

Computing the time derivative

$$\frac{d}{dt}\tilde{H}_N(t) = \int f_t^N \left[ \mathcal{L}_N^{\tilde{\tau}(t)} \left( \log \frac{f_t^N}{\tilde{g}^N_{\tau(t,\cdot)}} \right) - \partial_t \log \tilde{g}^N_{\tau(t,\cdot)} \right] \prod_{i=1}^N dr_i dp_i \tag{51}$$

Using the inequality

$$f \mathcal{L}_N^{\tilde{\tau}(t)} \log f \leq \mathcal{L}_N^{\tilde{\tau}(t)} f$$

and since $d\mu_{0,\beta}^N$ is stationary for $\mathcal{L}_N^0$, we have

$$\int f_t^N \mathcal{L}_N^{\tilde{\tau}(t)} \log f_t^N \; d\mu_{0,\beta}^N \leq N^2 \tau \int \partial_{p_N} f_t^N \; d\mu_{0,\beta}^N = N^2 \tau \beta \int p_N f_t^N \; d\mu_{0,\beta}^N$$

we obtain

$$\frac{d}{dt} \tilde{H}_N(t) \leq \int f_t^N \frac{\left[ (\mathcal{L}_N^{\tilde{\tau}(t)})^* - \partial_t \right] \tilde{g}_{\tau(t,\cdot)}^N}{\tilde{g}_{\tau(t,\cdot)}^N} \prod_{i=1}^N dr_i \, dp_i$$

By explicit calculations and up to smaller order in $N$, taking into account the cancellation of the boundary terms, we obtain:

$$\frac{d}{dt} \tilde{H}_N(t) \leq \beta \int \sum_i \Big[ \gamma^{-1} \partial_x^2 \tau(t, i/N) \left( V'(r_i) - \tau(t, i/N) \right) + \partial_t \tau(t, i/N)(r_i - r(t, i/N)) \Big] f_t^N \prod_{i=1}^N dr_i \, dp_i + o(N)$$

and the rest of the proof follows by the standard arguments of the relative entropy method (cf. [3, 4, 7, 8]).

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## References

1. Bertini, L., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Thermodynamic transformations of nonequilibrium states, J. Stat. Phys. **149**(5), 773–802 (2012)
2. Bertini, L., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Clausius inequality and optimality of quasistatic transformations for nonequilibrium stationary states. Phys. Rev. Lett. **110**(2), 020601 (2013)
3. Even, N., Olla, S.: Hydrodynamic limit for an hamiltonian system with boundary conditions and conservative noise. http://arxiv.org/pdf/1009.2175
4. Kipnis, C., Landim, C.: Scaling Limits of Interacting Particle Systems. Springer, Berlin (1999)
5. Olla, S., Varadhan, S.: Hydrodynamical limit for Ornstein-Uhlenbeck interacting particles. Commun. Math. Phys. **135**, 355–378 (1991)

6. Simon, M.: Hydrodynamic limit for the velocity-flip model. Stoch. Process. Appl. **123**, 3623–3662 (2013)
7. Tremoulet, C.: Hydrodynamic limit for interacting Ornstein–Uhlenbeck particles. Stoch. Proc. App. **102**(1), 139–158 (2002)
8. Yau, H.T.: Relative entropy and hydrodynamics of Ginzburg-Landau models. Lett. Math. Phys. **22**(1), 63–80 (1991)

# Unique Continuation Property for the Benjamin Equation

**Mahendra Panthee**

## 1 Introduction

In this work, our interest is in studying the following initial value problem (IVP):

$$\eta_t - \beta\eta_{xxx} - \alpha L\eta_x + (\eta + \eta^2)_x = 0, \qquad \eta(x,0) = \eta_0(x), \tag{1}$$

where $x, t \in \mathbb{R}$, $\eta = \eta(x,t)$ is a real valued function, $\alpha,\ \beta$ are positive constants with $\alpha \ll \beta$. The linear symmetric operator $L$ is defined by its symbol $|\xi|$ so that one can write $L = \mathscr{H}\partial_x$, with $\mathscr{H}$ the Hilbert transform given by

$$\mathscr{H} f(x) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy, \tag{2}$$

where p.v. stands for the Cauchy principal value.

This model was introduced by Benjamin [4] which governs approximately the evolution of waves on the interface of a two-fluid system in which surface-tension effects cannot be ignored. More precisely, this model is concerned with an incompressible system that, at rest, consists of a layer of depth $h_1$ of light fluid of density $\rho_1$ bounded above by a rigid plane and resting upon a layer of heavier fluid of density $\rho_2 > \rho_1$ of depth $h_2$, also resting on a rigid plane. Because of the density difference, waves can propagate along the interface between the two fluids. In Benjamin's theory, diffusivity is ignored, but the parameters of the system are such that capillarity cannot be discarded. While deriving (1) in [4], it is assumed

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that the constant $\beta$ satisfies $4\beta \gg 1$. For technical reason (see (27) below), in our analysis we suppose $\beta$ is such that $3\beta > 1$.

After [4] and [5], several authors have studied this model in recent literature, see for example [1–3, 5, 27, 28] and references there in. Existence and stability of solitary waves is studied in [1, 3], and that of periodic traveling wave is addressed in [2]. For the well-posedness of the Cauchy problem associated with (3), we refer to [28] for the global well-posedness in $L^2$ and to [27] for the local well-posedness in $H^s$, $s > -3/4$.

In this work we are concerned with the unique continuation property (UCP) for the Benjamin equation (3). There are various forms of UCP in the literature, see for example [6, 24–26, 33] and references there in. The following is the definition of UCP given in [33], where the first result of the UCP for a dispersive model is proved.

**Definition ([33]).** Let $L$ be an evolution operator acting on functions defined on some connected open set $\Omega$ of $\mathbb{R}^n \times \mathbb{R}_t$. The operator $L$ is said to have unique continuation property (UCP) if every solution $u$ of $Lu = 0$ that vanishes on some nonempty open set $\mathscr{O} \subset \Omega$ vanishes in the horizontal component of $\mathscr{O}$ in $\Omega$.

As far as we know, the first result of the UCP is due to Carleman [8], who used weighted estimates for the associated solution (now widely referred as Carleman-type estimates) to obtain uniqueness theorem for the general linear equations. Later, Hormander and Mizohata [17, 29] extended Carleman's method to address the UCP for parabolic and hyperbolic operators. The result due to Saut-Scheurer [33] is the first one to deal with the UCP for dispersive models. In recent literature much effort has been used in studying the UCP for various models, see for example [6–22, 24–26, 30–36] and references therein. In most cases Carleman type estimates are employed to obtain the UCP results.

Recently, Bourgain in [6] introduced a new method based on complex analysis to prove the UCP for dispersive models. Although, by using Paley-wiener theorem, the UCP for linear dispersive models, with this method, is almost immediate, the same is not so simple when one considers full nonlinear model. Some extra and technical efforts are necessary to address the case of nonlinear model. The structure on the model under consideration demands appropriate choice of the parameters. This can clearly be seen in the proof of Theorem 1 below. This method has been successfully adapted to address the UCP issue for the bi-dimensional models as well, see [13, 30, 31]. In this work we use this method to prove that, if a sufficiently smooth solution to the IVP (3) is supported compactly in a nontrivial time interval then it vanishes identically.

Before stating the main result, we make a change of variables $u(x,t) = -\eta(x - t, -t/c_0)$ to write the IVP (1) in the following form

$$u_t + \beta u_{xxx} + \alpha \mathscr{H} u_{xx} + (u^2)_x = 0, \qquad u(x,0) = u_0(x), \tag{3}$$

$x, t \in \mathbb{R}$. From here onwards, we consider the IVP (3). Note that, as pointed out in the original paper of Benjamin [4], the following functionals

$$M(u) := \frac{1}{2}\int_{\mathbb{R}} |u(x,t)|^2\,dx, \tag{4}$$

$$E(u) := \int_{\mathbb{R}} \frac{\beta}{2}u_x^2 - \frac{\alpha}{2}u\mathscr{H}u_x - \frac{1}{3}u^3\,dx, \tag{5}$$

are the constants of motion for (3). This means that, if $u$ is a smooth solution of the IVP (3) that vanishes at $x = \pm\infty$, then $M(u)$ and $E(u)$ are independent of time.

Now, we state the main result of this work, that reads as follows.

**Theorem 1.** *Let $u \in C(\mathbb{R}; H^s(\mathbb{R}))$ be a solution to the IVP (3) with $s > 0$ large enough. If there exists a non trivial time interval $I = [-T, T]$ such that for some $B > 0$,*

$$\text{supp}\;\, u(t) \subseteq [-B, B], \quad \forall\, t \in I,$$

*then $u \equiv 0$.*

In some sense the result in Theorem 1 is a weak version of the UCP for the Benjamin equation given in the above definition. Although, looking at the structure of the linear part, the UCP for the Benjamin equation seems to be quite natural, it is not reported in the literature so far.

The stronger form of the UCP is concerned in considering the hypothesis of the compact support at two different times as has been the case in the g-KdV equation and the NLS equation [24–26], see also [9, 10] for the higher order NLS equation. In recent times, even stronger versions of the UCP results, where one considers appropriate decay conditions on the solution at two or three different times have also been reported in the literature, see [7, 11, 12, 16, 20–22] and references therein.

At this point we would like to record the recent works on the UCP for the Ostrovsky equation [20, 21], KP-II equation [22] and Zakharov-Kuznetsov equation [7], where the authors proved that the sufficiently smooth solutions that have compact support at two different times must vanish identically. It is interesting to note that, to conclude these stronger UCP results, the authors in [7, 20–22] used the weak versions of the UCP results obtained via the complex analysis approach in [30–32]. So, we believe, our result in this article may pave way to get much stronger UCP results for the Benjamin equation like the ones in [7, 11, 12, 16, 20–22]. However, in order to apply these recent techniques developed in [7, 11, 12, 16, 20–22], the solution of the model under consideration must satisfy the exponential decay condition. But, in the case of Benjamin equation, due to the presence of the Hilbert transform, we could not find such decay property of the solution. Therefore, we believe, some extra modification is needed to get the stronger UCP results using these techniques.

Quite recently, the UCP result for the Benjamin-Ono (BO) equation has been reported in [14] showing that the solution that satisfies certain decay condition at three different times mush vanish identically. The result in [14] improves the one obtained in [18]. For the most recent work in this direction for the BO equation we refer to [15] where the authors proved that the uniqueness result proved in [14] cannot be extended to any pair of non-vanishing solutions. Also, they showed that the hypothesis used in [14] on the solution at three different times cannot be relaxed to two different times. Since the Hilbert transform is involved in both the Benjamin equation considered here and the BO equation, they exhibit similar behavior in weighted Sobolev spaces. After completing this work we came to know that, using the techniques developed in [14, 15], Jiménez Urrea proved stronger UCP results for the Benjamin equation in [23].

The plan of this article is as follows. We establish some preliminary estimates in Sect. 2 and we supply the proof of the main result in Sect. 3.

Now, we introduce some notations that will be used throughout this article. The Fourier transform of a function $f$ denoted by $\hat{f}$ is defined as,

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx. \tag{6}$$

Using the definition of the Fourier transform, the Hilbert transform defined in (2) can be written as $\widehat{\mathscr{H} f}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi)$. We use $H^s$ to denote $L^2$-based Sobolev space with index $s$. The various constants whose exact values are immaterial will be denoted by $c$. We use $\operatorname{supp} f$ to denote support of a function $f$ and $f * g$ to denote the usual convolution product of $f$ and $g$. Also, we use the notation $A \lesssim B$ if there exists a constant $c > 0$ such that $A \le cB$.

## 2 Preliminary Estimates

In this section we gather some estimates that play crucial role in the proof of our main result. The details of the proof of these estimates can be found in [6] and the author's previous works [30, 31]. For the sake of clearness we sketch the idea of the proofs.

Let us start by recording the following result.

**Lemma 1.** *Let $u \in C([-T, T]; H^s(\mathbb{R}))$ be a sufficiently smooth solution to the IVP (3). If for some $B > 0$, $\operatorname{supp} u(t) \subseteq [-B, B]$, then for all $\xi, \theta \in \mathbb{R}$, we have*

$$|\widehat{u(t)}(\xi + i\theta)| \lesssim e^{c|\theta|B}. \tag{7}$$

*Proof.* The proof follows using the Cauchy-Schwarz inequality and the conservation law (4). The argument is similar to the 2-dimensional case presented in [30] and [31].

Now, we define

$$u^*(\xi) = \sup_{t\in I} |\widehat{u(t)}(\xi)| \tag{8}$$

and

$$m(\xi) = \sup_{|\xi'|\geq|\xi|} |u^*(\xi')|. \tag{9}$$

Considering $u(0)$ sufficiently smooth and taking into account the well-posedness theory for the IVP (3) (see for e.g., [28]), we have the following result.

**Lemma 2.** *Let $u \in C([-T, T]; H^s(\mathbb{R}))$ be a sufficiently smooth solution to the IVP (3) with* $\operatorname{supp} u(t) \subseteq [-B, B]$, $\forall\, t \in I$, *then for some constant $B_1$, we have*

$$m(\xi) \lesssim \frac{B_1}{1+|\xi|^4}. \tag{10}$$

*Proof.* The proof follows by using Cauchy-Schwarz inequality, conservation law (4) and the well-posedness theory with the similar argument in the author's previous works [30] and [31].

**Proposition 1.** *Let $u(t)$ be compactly supported and suppose that there exists $t \in I$ with $u(t) \neq 0$. Then there exists a number $c > 0$ such that for any large number $Q > 0$ there are arbitrary large $\xi$-values such that*

$$m(\xi) > c(m * m)(\xi) \tag{11}$$

*and*

$$m(\xi) > e^{-\frac{|\xi|}{Q}}. \tag{12}$$

*Proof.* The main ingredient in the proof of this proposition is the estimate (10) in Lemma 2. The argument is similar to the one given in the proof of lemma in page 440 in [6], so we omit it.

Now, using the definition of $m(\xi)$ and Proposition 1 we choose $\xi$ large enough and $t_1 \in I$ such that

$$|\widehat{u(t_1)}(\xi)| = u^*(\xi) = m(\xi) > c(m * m)(\xi) + e^{-\frac{|\xi|}{Q}}. \tag{13}$$

In what follows we prove some derivative estimates for an entire function. We start with the following result whose proof is given in [6].

**Lemma 3.** *Let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire function which is bounded and integrable on the real axis and satisfies*

$$|\phi(\xi + i\theta)| \lesssim e^{|\theta| B}, \quad \xi,\, \theta \in \mathbb{R}.$$

*Then, for* $\xi_1 \in \mathbb{R}^+$ *we have*

$$|\phi'(\xi_1)| \lesssim B\left(\sup_{\xi'\geq\xi_1} |\phi(\xi')|\right)\left[1 + \left|\log\left(\sup_{\xi'\geq\xi_1} |\phi(\xi')|\right)\right|\right]. \tag{14}$$

**Corollary 1.** *Let* $\theta \in \mathbb{R}$ *be such that*

$$|\theta| \leq B^{-1}\left[1 + \left|\log\left(\sup_{\xi'\geq\xi_1>0} |\phi(\xi')|\right)\right|\right]^{-1}. \tag{15}$$

*Then*

$$\sup_{\xi'\geq\xi_1} |\phi(\xi' + i\theta)| \leq 2 \sup_{\xi'\geq\xi_1} |\phi(\xi')| \tag{16}$$

*and*

$$\sup_{\xi'\geq\xi_1} |\phi(\xi' + i\theta)| \lesssim B\left(\sup_{\xi'\geq\xi_1} |\phi(\xi')|\right)\left[1 + \left|\log\left(\sup_{\xi'\geq\xi_1} |\phi(\xi')|\right)\right|\right]. \tag{17}$$

*Proof.* Detailed proof of this corollary can be found in Corollary 2.9 in [6]. So, we omit it.

Now we state the last result of this section whose proof can be found in the author's previous works [30] and [31].

**Corollary 2.** *Let* $t \in I$, $\phi(z) = \widehat{u(t)}(z)$, $\theta$ *be as in Corollary 1 and* $m(\xi)$ *be as in (9). Then, for* $|\theta'| \leq |\theta|$ *fixed, we have*

$$|\phi'(\xi - \xi' + i\theta')| \lesssim B\left[m(\xi) + m(\xi - \xi')\right]\left[1 + |\log m(\xi)|\right]. \tag{18}$$

## 3 Proof of the Main Results

Now we are in position to supply proofs of the main results of this work. The main idea in the proof of Theorem 1 is similar to the one employed in [6, 30] and [31], but the structure of the Fourier symbol associated with the linear part of the IVP (3) demands special attention and some basic modifications.

Before supplying the details of the proof of Theorem 1, we write the IVP (3) in the equivalent integral form

$$u(t) = U(t)u_0 - \int_0^t U(t-t')(u^2)_x(t')\,dt', \tag{19}$$

where $U(t)$ is the unitary group describing the solution to the linear problem

$$u_t + \beta u_{xxx} + \alpha \mathscr{H} u_{xx} = 0, \qquad u(x,0) = u_0(x), \tag{20}$$

and is given by

$$U(t)f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\{x\xi + (\beta\xi^3 - \alpha|\xi|\xi)t\}} \hat{f}(\xi)\, d\xi. \tag{21}$$

*Proof (Proof of Theorem 1).* If possible, suppose that there is some $t \in I$ such that $u(t) \neq 0$. Our goal is to use the estimates derived in the previous section to arrive at a contradiction.

Let $t_1, t_2 \in I$, with $t_1$ as in (13). Now, using the equivalent integral equation (19), we have

$$u(t_2) = U(t_2 - t_1)u(t_1) - c\int_{t_1}^{t_2} U(t_2 - t')(u^2)_x(t')\, dt'. \tag{22}$$

Taking Fourier transform in the space variable in (22), we get

$$\widehat{u(t_2)}(\xi) = e^{i(t_2 - t_1)(\beta\xi^3 - \alpha|\xi|\xi)}\widehat{u(t_1)}(\xi) - ci\xi \int_{t_1}^{t_2} e^{i(t_2 - t')(\beta\xi^3 - \alpha|\xi|\xi)}\widehat{u^2(t')}(\xi)\, dt'. \tag{23}$$

Let $t_2 - t_1 = \Delta t$ and make a change of variables $s = t' - t_1$, to obtain

$$\begin{aligned} \widehat{u(t_2)}(\xi) &= e^{i\Delta t(\beta\xi^3 - \alpha|\xi|\xi)}\widehat{u(t_1)}(\xi) - ci\xi \int_0^{\Delta t} e^{(i\Delta t - s)(\beta\xi^3 - \alpha|\xi|\xi)}\widehat{u^2(t_1 + s)}(\xi)\, ds \\ &= e^{i\Delta t(\beta\xi^3 - \alpha|\xi|\xi)}\left[\widehat{u(t_1)}(\xi) - ci\xi \int_0^{\Delta t} e^{-is(\beta\xi^3 - \alpha|\xi|\xi)}\widehat{u^2(t_1 + s)}(\xi)\, ds\right]. \end{aligned} \tag{24}$$

Since $u(t), t \in I$ has compact support, by Paley-Wiener theorem, $\widehat{u(t)}(\xi)$ has analytic continuation in $\mathbb{C}$, and we have

$$\begin{aligned} \widehat{u(t_2)}(\xi + i\theta) = e^{i\Delta t\left(\beta(\xi + i\theta)^3 - \alpha|\xi + i\theta|(\xi + i\theta)\right)}\Big[\widehat{u(t_1)}(\xi + i\theta) - \\ -ci(\xi + i\theta)\int_0^{\Delta t} e^{-is\left(\beta(\xi + i\theta)^3 - \alpha|\xi + i\theta|(\xi + i\theta)\right)}\widehat{u^2(s + t_1)}(\xi + i\theta)\, ds\Big]. \end{aligned} \tag{25}$$

Since

$$\begin{aligned} \beta(\xi + i\theta)^3 - \alpha|\xi + i\theta|(\xi + i\theta) = &\left(\beta\xi^3 - 3\beta\xi\theta^2 - \alpha\xi|\xi + i\theta|\right) \\ &+ i\left(3\beta\xi^2\theta - \beta\theta^3 - \alpha\theta|\xi + i\theta|\right), \end{aligned}$$

using Lemma 1 we obtain from (25)

$$
\begin{aligned}
&c e^{\Delta t\left(3\beta\xi^2\theta-\beta\theta^3-\alpha\theta|\xi+i\theta|\right)} \geq \\
&\qquad |\widehat{u(t_1)}(\xi+i\theta)| - ci|\xi+i\theta|\int_0^{\Delta t} e^{-s\left(3\beta\xi^2\theta-\beta\theta^3-\alpha\theta|\xi+i\theta|\right)}|\widehat{u^2(s+t_1)}(\xi+i\theta)|\,ds.
\end{aligned}
\tag{26}
$$

Now, let us select $\xi$ very large and $\theta=\theta(\xi)$ such that $|\theta|\sim 0$, and the following hold,

$$
\frac{1}{|\xi|} \ll |\theta| \qquad \text{and} \qquad (3\beta-1)\xi^2 > \alpha|\xi+i\theta|. \tag{27}
$$

Also, let us choose $\theta$ in such a way that

$$
\theta\Delta t < 0. \tag{28}
$$

Now, using these choices we get from (26)

$$
\begin{aligned}
&e^{-|\Delta t||\theta|\left(3\beta\xi^2-\alpha|\xi+i\theta|\right)} \gtrsim \\
&\qquad \left|\widehat{u(t_1)}(\xi+i\theta)\right| - \left|\xi\right|\int_0^{|\Delta t|} e^{-s|\theta|\left(3\beta\xi^2-\alpha|\xi+i\theta|\right)}\left|\widehat{u^2(t_1\pm s)}(\xi+i\theta)\right|\,ds,
\end{aligned}
\tag{29}
$$

where '+' sign corresponds to $\Delta t>0$ and '−' sign to $\Delta t<0$. From here onwards we consider the case $\Delta t>0$, the other case follows similarly. We can write (29) as,

$$
\begin{aligned}
&e^{-\left(3\beta\xi^2-\alpha|\xi+i\theta|\right)|\theta\Delta t|} \gtrsim \\
&\qquad \left|\widehat{u(t_1)}(\xi+i\theta)\right| - |\xi|\int_0^{\Delta t} e^{-s\left(3\beta\xi^2-\alpha|\xi+i\theta|\right)|\theta|}\left|\widehat{u^2(t_1+s)}(\xi+i\theta)\right|\,ds.
\end{aligned}
\tag{30}
$$

Finally, we write the estimate (30) in the following way

$$
\begin{aligned}
e^{-\left(3\beta\xi^2-\alpha|\xi+i\theta|\right)|\theta\Delta t|} &\gtrsim \left|\widehat{u(t_1)}(\xi)\right| - |\xi|\int_0^{\Delta t} e^{-s\left(3\beta\xi^2-\alpha|\xi+i\theta|\right)|\theta|}\left|\widehat{u^2(t_1+s)}(\xi)\right|\,ds \\
&\quad - \left|\widehat{u(t_1)}(\xi+i\theta)-\widehat{u(t_1)}(\xi)\right| \\
&\quad - |\xi|\int_0^{\Delta t} e^{-s\left(3\beta\xi^2-\alpha|\xi+i\theta|\right)|\theta|}\Big|\widehat{u^2(t_1+s)}(\xi+i\theta) \\
&\qquad - \widehat{u^2(t_1+s)}(\xi)\Big|\,ds \\
&:= I_1 - I_2 - I_3.
\end{aligned}
\tag{31}
$$

In sequel we use the preliminary estimates from the previous section to get appropriate estimates for $I_1$, $I_2$ and $I_3$ to arrive at a contradiction in (31).

Now, we use definition of $u^*(\xi)$, the estimate (11) and the choice of $\xi$ and $\theta$ in (27), to obtain

$$\begin{aligned}
|\xi| \int_0^{\Delta t} & e^{-s\left(3\beta\xi^2-\alpha|\xi+i\theta|\right)|\theta|} \left|\widehat{u(t_1+s)}\right| * \left|\widehat{u(t_1+s)}\right| (\xi)\, ds \\
&\le |\xi|(u^* * u^*)(\xi) \int_0^{\Delta t} e^{-s\left(3\beta\xi^2-\alpha|\xi+i\theta|\right)|\theta|} ds \\
&\le |\xi|(m * m)(\xi) \frac{1-e^{-\Delta t\left(3\beta\xi^2-\alpha|\xi+i\theta|\right)|\theta|}}{(3\beta\xi^2-\alpha|\xi+i\theta|)\,|\theta|} \\
&\le \frac{|\xi|(m*m)(\xi)}{\xi^2|\theta|} \lesssim \frac{m(\xi)}{|\xi\theta|}.
\end{aligned}$$

Therefore, we get

$$I_1 \gtrsim m(\xi) - \frac{m(\xi)}{|\xi\theta|} \ge \frac{m(\xi)}{2}. \tag{32}$$

To obtain estimate for $I_2$ we define $\phi(z) = \widehat{u(t_1)}(z)$, for $z \in \mathbb{C}$. Using (13) we get

$$|\phi(z)| = |\widehat{u(t_1)}(\xi)| = \sup_{|\xi'|\ge|\xi|} |\phi(\xi')| = m(\xi). \tag{33}$$

Now, choose $|\theta|$ such that

$$|\theta| \lesssim B^{-1}\left[1+|\log m(\xi)|\right]^{-1}. \tag{34}$$

Using Corollary 1 we obtain

$$\begin{aligned}
I_2 &\lesssim |\theta| \sup_{|\xi'|\ge|\xi|} |\partial\widehat{u(t_1)}(\xi'+i\theta)| \\
&\lesssim |\theta| B\, m(\xi)\,[1+|\log m(\xi)|] \\
&\lesssim m(\xi) \lesssim \frac{1}{15} m(\xi).
\end{aligned} \tag{35}$$

Finally, to get estimate for $I_3$, we use Proposition 1, Corollary 2 and $|\theta|$ as in (34) to obtain

$$\begin{aligned}
&\left|\widehat{u^2(t_1+s)}\;(\xi+i\theta) - \widehat{u^2(t_1+s)}(\xi)\right| \\
&\qquad \le \int_{\mathbb{R}} \left|\widehat{u(t_1+s)}(\xi-\xi'+i\theta) - \widehat{u(t_1+s)}(\xi-\xi')\right| \left|\widehat{u(t_1+s)}(\xi')\right| d\xi'
\end{aligned}$$

$$\begin{aligned}
&\leq |\theta| \int_{\mathbb{R}} \sup_{|\xi'|\leq|\xi|} \left|\partial \widehat{u(t_1+s)}(\xi-\xi'+i\theta')\right| m(\xi')\, d\xi' \\
&\leq \int_{\mathbb{R}} [m(\xi)+m(\xi-\xi')]\, m(\xi')\, d\xi' \\
&\leq m(\xi)c_2 + (m*m)(\xi) \\
&\leq m(\xi)(c_2+c^{-1}) \lesssim m(\xi).
\end{aligned}$$

Therefore, as in $I_2$, using (27), one gets

$$\begin{aligned}
I_3 &\lesssim |\xi| m(\xi) \int_0^{\Delta t} e^{-s\left(3\beta\xi^2-\alpha|\xi+i\theta|\right)|\theta|} ds \\
&= |\xi| m(\xi)\, \frac{1-e^{-\Delta\left(3\beta\xi^2-\alpha|\xi+i\theta|\right)|\theta|}}{(3\beta\xi^2-\alpha|\xi+i\theta|)\,|\theta|} \\
&\leq \frac{|\xi| m(\xi)}{3\xi^2|\theta|} \lesssim \frac{m(\xi)}{|\xi\theta|} \lesssim \frac{1}{15} m(\xi).
\end{aligned} \tag{36}$$

Now, using (32), (35) and (36) in (31) and using the estimate (12) one gets,

$$e^{-\{3\xi^2-l|\xi+i\theta|\}|\theta\Delta t|} \gtrsim \frac{m(\xi)}{2} - \frac{m(\xi)}{15} - \frac{m(\xi)}{15} = \frac{11}{30} m(\xi) \gtrsim e^{-\frac{|\xi|}{Q}}. \tag{37}$$

On the other hand, with the choice of $\xi$ and $\theta$ in (27) we have, $3\beta\xi^2 - \alpha|\xi+i\theta| > \xi^2$ and consequently $\{3\beta\xi^2 - \alpha|\xi+i\theta|\}|\theta| > |\xi||\xi||\theta| > |\xi|$.

Therefore,

$$e^{-\{3\beta\xi^2-\alpha|\xi+i\theta|\}|\theta||\Delta t|} \leq e^{-|\xi||\Delta t|}. \tag{38}$$

Now from (38) and (37), we obtain

$$e^{-|\xi||\Delta t|} \gtrsim e^{-\frac{|\xi|}{Q}}, \tag{39}$$

which is false for $|\xi|$ large if we choose $Q$ large enough such that $\frac{1}{Q} < |\Delta t|$. This contradiction completes the proof of the theorem.

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# References


1. Albert, J.P., Bona, J.L., Restrepo, J.M.: Solitary-wave solutions of the Benjamin equation. SIAM J. Appl. Math. **59**, 2139–2161 (1999)
2. Alvarez Samaniego, B., Angulo Pava, J.: Existence and stability of periodic travelling-wave solutions of the Benjamin equation. Commun. Pure Appl. Anal. **4**, 367–388 (2005)
3. Angulo, J.: Existence and stability of solitary wave solutions of the Benjamin equation. J. Differ. Equ. **152**, 136–159 (1999)
4. Benjamin, T.B.: A new kind of solitary waves. J. Fluid Mech. **245**, 401–411 (1992)
5. Benjamin, T.B.: Solitary and periodic waves of a new kind. Philos. Trans. R. Soc. Lond. Ser. A **354**, 1775–1806 (1996)
6. Bourgain, J.: On the compactness of the support of solutions of dispersive equations. IMRN, Int. Math. Res. Not. **9**, 437–447 (1997)
7. Bustamante, E., Isaza, P., Mejía, J.: On the support of solutions to the Zakharov-Kuznetsov equation. J. Funct. Anal. **264**, 2529–2549 (2013)
8. Carleman, T.: Sur les systèmes aux dérivées partielles du premier ordre à deux variables. C. R. Acad. Sci. Paris **97**, 471–474 (1939)
9. Carvajal, X., Panthee, M.: Unique continuation property for a higher order nonlinear Schrödinger equation. J. Math. Anal. Appl. **303**, 188–207 (2005)
10. Carvajal, X., Panthee, M.: On uniqueness of solution for a nonlinear Schrödinger-Airy equation. Nonlinear Anal. **64**(1), 146–158 (2006)
11. Escauriaza, L., Kenig, C.E., Ponce, G., Vega, L.: On uniqueness properties of solutions of Schrödinger equations. Commun. Partial Differ. Equ. **31**, 1811–1823 (2006)
12. Escauriaza, L., Kenig, C.E., Ponce, G., Vega, L.: On uniqueness properties of solutions of the k-generalized KdV equations. J. Funct. Anal. **244**, 504–535 (2007)
13. Esfahani, A., Pastor, A.: On the unique continuation property for Kadomtsev-Petviashvili-I and Benjamin-Ono-Zakharov-Kuznetsov equations. Bull. London Math. Soc. **43**, 1130–1140 (2011)
14. Fonseca, G., Ponce, G.: The IVP for the Benjamin-Ono equation in weighted Sobolev spaces. J. Funct. Anal. **260**(2), 436–459 (2011)
15. Fonseca, G., Linares, F., Ponce, G.: The IVP for the Benjamin-Ono equation in weighted Sobolev spaces II. J. Funct. Anal. **262**, 2031–2049 (2012)
16. Himonas, A., Misiolek, G., Ponce, G., Zhou, Y.: Persistence properties and unique continuation of solutions of the Camassa-Holm equation. Commun. Math. Phys. **271**(2), 511–522 (2007)
17. Hormander, L.: Linear Partial Differential Operators. Springer, Berlin/Heidelberg/New York (1969)
18. Iório, R.J., Jr.: Unique continuation principles for some equations of Benzamin-Ono Type. In: Nonlinear Equations: Methods, Models and Applications. Progr. Nonlinear Differential Equations Appl. vol. 54, pp. 163–179. Birkhäuser, Basel (2003)
19. Isakov, V.: Carleman type estimates in an anisotropic case and applications. J. Differ. Equ. **105**, 217–238 (1993)
20. Isaza, P., Mejía, J.: On the support of solutions to the Ostrovsky equation with negative dispersion. J. Differ. Equ. **247**(6), 1851–1865 (2009)
21. Isaza, P., Mejía, J.: On the support of solutions to the Ostrovsky equation with positive dispersion. Nonlinear Anal. **72**(11), 4016–4029 (2010)
22. Isaza, P., Mejía, J.: On the support of solutions to the Kadomtsev-Petviashvili (KP-II) equation. Commun. Pure Appl. Anal. **10**, 1239–1255 (2011)
23. Jiménez Urrea, J.: The Cauchy problem associated to the Benjamin equation in weighted Sobolev spaces. J. Differ. Equ. **254**, 1863–1892 (2013)
24. Kenig, C.E., Ponce, G., Vega, L.: On the support of solutions to the generalized KdV equation. Ann. Inst. Henri Poincare **19**(2), 191–208 (2002)
25. Kenig, C.E., Ponce, G., Vega, L.: On unique continuation for nonlinear Schrödinger equation. Commun. Pure Appl. Math. **56**, 1247–1262 (2003)

26. Kenig, C.E., Ponce, G., Vega, L.: On the unique continuation of solutions to the generalized KdV equation. Math. Res. Lett. **10**, 833–846 (2003)
27. Kozono, H., Ogawa, T., Tanisaka, H.: Well-posedness for the Benjamin equations. J. Korean Math. Soc. **38**, 1205–1234 (2001)
28. Linares, F.: $L^2$ global well-posedness of the initial value problem associated to the Benjamin equation. J. Differ. Equ. **152**, 377–393 (1999)
29. Mizohata, S.: Unicité du prolongement des solutions pour quelques operateurs differentials paraboliques. Mem. Coll. Sci. Univ. Kyoto A **31**, 219–239 (1958)
30. Panthee, M.: A note on the unique continuation property for Zakharov-Kuznetsov equation, Nonlinear Anal. TMA **59**, 425–438 (2004)
31. Panthee, M.: Unique continuation property for the Kadomtsev-Petviashvili (KP-II) equation. Electron. J. Differ. Equ. **2005**, 1–12 (2005)
32. Panthee, M.: On the compact support of solutions to a nonlinear long internal waves model. Nepali Math. Sci. Rep. **24**, 4958 (2005)
33. Saut, J-C., Scheurer, B.: Unique continuation for some evolution equations. J. Differ. Equ. **66**, 118–139 (1987)
34. Tataru, D.: Carleman type estimates and unique continuation for the Schrödinger equation. Differ. Integral Equ. **8**(4), 901–905 (1995)
35. Zhang, B.Y.: Unique continuation for the Korteweg-de Vries equation. SIAM J. Math. Anal. **23**, 55–71 (1992)
36. Zhou, Y.: Infinite propogation speed for a shallow water equation, preprint. http://www.fim.math.ethz.ch/preprints/2005/zhou.pdf

# On the Kinetic Systems for Simple Reacting Spheres: Modeling and Linearized Equations

**Filipe Carvalho, Jacek Polewczak, and Ana Jacinta Soares**

## 1 Introduction

The problem of polyatomic reactive mixtures, within kinetic theory, was first investigated by Prigogine and Xhrouet [13] in 1949. They treated the reactive contributions as perturbations of the elastic terms. This approach is only valid if the reactive cross sections are much smaller than the elastic cross sections. In 1959, Present gave another important contribution to this problem [12]. Although, in some aspects different from the work by Prigogine and Xhrouet, the Present's theory is also based on the assumption that the reactive terms are small perturbations of the elastic terms. Ross and Mazur, in 1961, as well as Shizgal and Karplus, in 1970, see papers [14, 15] respectively, used the Chapman-Enskog method in the spatial homogeneous case with the aim of investigating the non-equilibrium effects induced by the chemical reactions and deducing, in particular, the explicit expression of the reaction rate specifying the chemical production of each constituent of the mixture. The works of Moreau [9], in 1975, and Xystris and Dahler [17], in 1978, used the method of Grad in both spatial homogeneous and inhomogeneous cases with the aim of deducing, again, explicit expressions for the reaction rate.

The kinetic theory of the simple reacting spheres was first proposed by Marron in 1970, see [8], and then developed by Xystris and Dahler in 1978, see [18]. Within this theory, both elastic and reactive collisions are of hard-sphere type. This feature reduces the micro-reversibility principle to a simpler condition.

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In 2000, Polewczak proved, in his work [10], the existence of global in time, spatially inhomogeneous, and $L^1$-renormalized solution for the model of simple reacting spheres, under the assumption of finite initial mass, momentum and energy. The existence result refers to a four component mixture with a chemical bimolecular reaction in which there was neither mass nor diameters exchange. In this paper we consider a more general situation where the mass and the diameter exchange is allowed. In the dilute-gas limit, this constitutes an interesting kinetic model of chemical reactions that has not yet been studied in detail.

The paper is organized as follows. First, in Sect. 2, we present the mathematical aspects of the kinetic modeling within the SRS theory. In Sects. 3 and 4 we introduce the relevant properties of the collisional operators that are essential to assure the mathematical and physical consistency of the model, and study the tendency of the mixture to approach the equilibrium. In Sect. 5 we define the macroscopic variables and derive the connection of the SRS model to the macroscopic framework in terms of hydrodynamic equations. In Sects. 6 and 7 we introduce the linearized SRS system, state its main properties and provide explicit representations for the kernels of the linearized integral operators. Finally, in Sect. 8 we include a brief discussion about our ongoing research in progress.

## 2 Kinetic Modeling

We consider a gas mixture with four constituents, say $A_1, \ldots, A_4$, with masses $m_1, \ldots, m_4$ and formation energies $E_1, \ldots, E_4$, respectively. We restrict our analysis to particles without internal degrees of freedom, which can interact through binary elastic collisions and reactive collisions obeying to the reversible chemical law

$$A_1 + A_2 \rightleftharpoons A_3 + A_4. \tag{1}$$

The mass is conserved during the chemical reaction, so that $m_1 + m_2 = m_3 + m_4$. The constituents' indexes are chosen in such a way that the reaction heat, defined by $Q_R = E_3 + E_4 - E_1 - E_2$, verifies the condition $Q_R > 0$. This means that the reverse chemical reaction, $A_3 + A_4 \rightharpoonup A_1 + A_2$, is exothermic.

**Elastic collisions.** An elastic collision between particles $A_i$ and $A_s$ with velocities $c_i$ and $c_s$, respectively, results in a change of velocities of both constituents, $(c_i, c_s) \to (c_i', c_s')$, with $i, s = 1, \ldots, 4$. The conservation laws of linear momentum and kinetic energy of the colliding particles are specified by

$$m_i c_i + m_s c_s = m_i c_i' + m_s c_s', \qquad m_i c_i^2 + m_s c_s^2 = m_i c_i'^2 + m_s c_s'^2. \tag{2}$$

In our model we consider elastic cross sections of hard-spheres type, given by

$$\sigma_{is}^2 = \frac{1}{4}(d_i + d_s)^2, \tag{3}$$

where $d_i$ and $d_s$ denote the diameters of the particle constituents $A_i$ and $A_s$, respectively. This model of cross sections is one of the most important and frequently used model, mainly due to its simplicity. Conditions (2), together with assumption (3), imply that the elastic post-collisional velocities corresponding to the pre-collisional velocities $c_i$ and $c_s$ are given by

$$c_i' = c_i - 2\frac{\mu_{is}}{m_i}\epsilon\langle\epsilon, c_i - c_s\rangle \qquad \text{and} \qquad c_s' = c_s + 2\frac{\mu_{is}}{m_s}\epsilon\langle\epsilon, c_i - c_s\rangle. \tag{4}$$

**Reactive collisions.** A reactive collision between particles $A_i$ and $A_j$ with velocities $c_i$ and $c_j$, respectively, results in a transition of the reactants $A_i$ and $A_j$ into products $A_k$ and $A_l$ and a consequent change of velocities to $c_k^*$ and $c_l^*$, with $(i, j, k, l) \in \{(1, 2, 3, 4), (2, 1, 4, 3), (3, 4, 1, 2), (4, 3, 2, 1)\}$. In addition, a reactive collision results in a rearrangement of masses and a redistribution of formation energies. Besides the mass conservation, also the linear momentum and total energy of the colliding particles are preserved, so that the following conditions hold

$$m_i c_i + m_j c_j = m_k c_k^* + m_l c_l^*, \tag{5}$$

$$E_i + \frac{1}{2}m_i c_i^2 + E_j + \frac{1}{2}m_j c_j^2 = E_k + \frac{1}{2}m_k c_k^{*2} + E_l + \frac{1}{2}m_l c_l^{*2}. \tag{6}$$

In what follows, we use the following notation for the relative velocities of the colliding particles participating in reactive collisions, $\xi_1 = \xi_2 = \xi = c_1 - c_2$, $\xi_3 = \xi_4 = \xi' = c_3 - c_4$. In the SRS model, reactive collisions are treated as hard-spheres like collisions, with the particularity that a reactive collision between particles $A_i$ and $A_j$ occurs if the kinetic energy associated with the relative motion along the line of centers exceeds the activation energy. Accordingly, reactive cross sections are assumed in the form

$$\sigma_{12}^{*2} = \begin{cases} \beta_{12}\sigma_{12}^2, & \langle\epsilon, c_1 - c_2\rangle \geq \Gamma_{12}, \\ 0, & \langle\epsilon, c_1 - c_2\rangle < \Gamma_{12}, \end{cases} \qquad \sigma_{34}^{*2} = \begin{cases} \beta_{34}\sigma_{34}^2, & \langle\epsilon, c_3 - c_4\rangle \geq \Gamma_{34}, \\ 0, & \langle\epsilon, c_3 - c_4\rangle < \Gamma_{34}, \end{cases} \tag{7}$$

for the direct and reverse chemical reaction, respectively. Above, $\beta_{ij}$ is the steric factor for the collision between constituents $A_i$ and $A_j$, with $0 \leq \beta_{ij} \leq 1$. Moreover, $\Gamma_{ij}$ is a threshold velocity given by $\Gamma_{ij} = \sqrt{2\gamma_i/\mu_{ij}}$, where $\gamma_i$ is the activation energy for the constituent $A_i$, and $\mu_{ij} = \frac{m_i m_j}{m_i + m_j}$ is a reduced mass. The notation $\langle\cdot,\cdot\rangle$ is used for the inner product in $\mathbb{R}^3$ and $\epsilon$ is the unit vector along the line passing through the centers of the spheres at the moment of impact,

$$\epsilon \in \left\{\epsilon \in \mathbb{R}^3 : |\epsilon| = 1 \wedge \langle\epsilon, c_i - c_j\rangle \geq 0\right\} \equiv \mathbb{S}_+^2.$$

Notice that, for the chemical reaction defined in Eq. (1), the activation energies verify the conditions $\gamma_2 = \gamma_1$, $\gamma_3 = \gamma_1 - Q_R$, $\gamma_4 = \gamma_3$ and the steric factors are such that $\beta_{ij} = 0$, for $(i, j) \notin \{(1,2), (2,1), (3,4), (4,3)\}$. Furthermore, since the reaction heat $Q_R$ is positive, we must have $\gamma_1 > Q_R$.

In our SRS model, the post-collisional velocities for the direct chemical reaction are given by

$$c_3^* = \frac{1}{M}\left[m_1 c_1 + m_2 c_2 + m_4 \sqrt{\frac{\mu_{12}}{\mu_{34}}}\left\{\xi - \epsilon\langle\epsilon, \xi\rangle + \epsilon\alpha^-\right\}\right], \tag{8}$$

$$c_4^* = \frac{1}{M}\left[m_1 c_1 + m_2 c_2 - m_3 \sqrt{\frac{\mu_{12}}{\mu_{34}}}\left\{\xi - \epsilon\langle\epsilon, \xi\rangle + \epsilon\alpha^-\right\}\right], \tag{9}$$

whereas, the post-collisional velocities for the reverse chemical reaction are

$$c_1^* = \frac{1}{M}\left[m_3 c_3 + m_4 c_4 + m_2 \sqrt{\frac{\mu_{34}}{\mu_{12}}}\left\{\xi' - \epsilon\langle\epsilon, \xi'\rangle + \epsilon\alpha^+\right\}\right] \tag{10}$$

$$c_2^* = \frac{1}{M}\left[m_3 c_3 + m_4 c_4 - m_1 \sqrt{\frac{\mu_{34}}{\mu_{12}}}\left\{\xi' - \epsilon\langle\epsilon, \xi'\rangle + \epsilon\alpha^+\right\}\right], \tag{11}$$

where $\alpha^- = \sqrt{(\langle\epsilon, \xi\rangle)^2 - 2Q_R/\mu_{12}}$ and $\alpha^+ = \sqrt{(\langle\epsilon, \xi'\rangle)^2 + 2Q_R/\mu_{34}}$. Velocities $c_i, c_j, c_k^*, c_l^*$, with $(i, j, k, l) \in \{(1,2,3,4), (2,1,4,3), (3,4,1,2), (4,3,2,1)\}$ and $c_k^*, c_l^*$ given by expressions (8)–(11), verify the conservation laws expressed in Eqs. (5) and (6).

**Kinetic equations.** In the absence of external forces, the kinetic equations, describing the time-space evolution of the one-particle distribution functions $f_i(x, c_i, t)$, $i = 1, \ldots, 4$, can be written in the form

$$\frac{\partial f_i}{\partial t} + \sum_{l=1}^{3} c_l^i \frac{\partial f_i}{\partial x_l} = \mathcal{Q}_i^E + \mathcal{Q}_i^R, \quad i = 1, \ldots, 4, \tag{12}$$

where $\mathcal{Q}_i^E$ and $\mathcal{Q}_i^R$ represent the elastic and reactive collisional operators. Following paper [11], the collisional operators have the form

$$\mathcal{Q}_i^E = \sum_{s=1}^{4}\left\{\sigma_{is}^2 \int_{\mathbb{R}^3}\int_{\mathbb{S}_+^2}\left[f_i' f_s' - f_i f_s\right]\langle\epsilon, c_i - c_s\rangle d\epsilon dc_s\right\} \tag{13}$$
$$-\beta_{ij}\sigma_{ij}^2 \int_{\mathbb{R}^3}\int_{\mathbb{S}_+^2}\left[f_i' f_j' - f_i f_j\right]\Theta(\langle\epsilon, \xi_i\rangle - \Gamma_{ij})\langle\epsilon, \xi_i\rangle d\epsilon dc_j,$$

where $\Theta$ is the Heaviside step function and $(i, j) \in \{(1,2), (2,1), (3,4), (4,3)\}$, and

$$\mathcal{Q}_i^R = \beta_{ij}\sigma_{ij}^2 \int_{\mathbb{R}^3}\int_{\mathbb{S}_+^2} \left[ \left(\frac{\mu_{ij}}{\mu_{kl}}\right)^2 f_k^* f_l^* - f_i f_j \right] \Theta(\langle \epsilon, \xi_i \rangle - \Gamma_{ij}) \langle \epsilon, \xi_i \rangle d\epsilon dc_j, \quad (14)$$

where $(i, j, k, l) \in \{(1, 2, 3, 4), (2, 1, 4, 3), (3, 4, 1, 2), (4, 3, 2, 1)\}$, and $f_k^* = f_k(x, c_k^*, t)$, $f_l^* = f_l(x, c_l^*, t)$. As explained in papers [11, 18], the second term in the expression of $\mathcal{Q}_i^E$ is a correction term for the occurrence of reactive collisions and prevent a double counting of the contributions in the collisional operators. In fact, those encounters between $A_i$ and $A_j$ particles which are sufficiently energetic in the sense that $\langle \epsilon, \xi_i \rangle \geq \Gamma_{ij}$ result in chemical reaction and should not be counted as elastic encounters.

## 3 Properties of the Collisional Operators

The consistency of the model is assured when the collisional operators have some important properties. We begin with the following fundamental results, concerning the elastic and reactive operators.

**Proposition 1.** *If we assume that* $\beta_{ij} = \beta_{ji}$ *then, for* $\psi_i$ *measurable on* $\mathbb{R}^3$ *and* $f_i \in C_0(\mathbb{R}^3)$, $i = 1, \ldots, 4$, *we have*

$$\begin{aligned}\int_{\mathbb{R}^3} \psi_i \mathcal{Q}_i^E dc_i = \frac{1}{4}\sum_{s=1}^{4} \left\{ \sigma_{is}^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}_+^2} [\psi_i + \psi_s - \psi_i' - \psi_s'] [f_i' f_s' - f_i f_s] \langle \epsilon, c_i - c_s \rangle d\epsilon dc_s dc_i \right\} \\ -\frac{1}{4}\beta_{ij}\sigma_{ij}^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}_+^2} \left[\psi_i + \psi_j - \psi_i' - \psi_j'\right]\left[f_i' f_j' - f_i f_j\right] \\ \times \Theta(\langle \epsilon, \xi_i \rangle - \Gamma_{ij}) \langle \epsilon, \xi_i \rangle \, d\epsilon dc_j dc_i \, .\end{aligned}$$

**Proposition 2.** *If* $\beta_{ij} = \beta_{ji}$ *and* $\beta_{12}\sigma_{12}^2 = \beta_{34}\sigma_{34}^2$, *then we have*

$$\begin{aligned}\sum_{i=1}^{4}\int_{\mathbb{R}^3} \psi_i \mathcal{Q}_i^R dc_i &= \beta_{12}\sigma_{12}^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}_+^2} \left[\psi_1 + \psi_2 - \psi_3^* - \psi_4^*\right] \\ &\times \left[ \left(\frac{\mu_{12}}{\mu_{34}}\right)^2 f_3^* f_4^* - f_1 f_2 \right] \Theta(\langle \epsilon, \xi \rangle - \Gamma_{12}) \langle \epsilon, \xi \rangle d\epsilon dc_2 dc_1 \\ &= \beta_{34}\sigma_{34}^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}_+^2} \left[\psi_3 + \psi_4 - \psi_1^* - \psi_2^*\right] \\ &\times \left[ \left(\frac{\mu_{34}}{\mu_{12}}\right)^2 f_1^* f_2^* - f_3 f_4 \right] \Theta(\langle \epsilon, \xi' \rangle - \Gamma_{34}) \langle \epsilon, \xi' \rangle d\epsilon dc_4 dc_3 .\end{aligned}$$

Propositions 1 and 2 can be proven considering the symmetry properties of the collisional operators, and constitute the basis of the proof of the following results.

**Proposition 3.** *The elastic collisional operators are such that*

$$\int_{\mathbb{R}^3} \mathcal{Q}_i^E \, dc_i = 0, \qquad i = 1, \ldots, 4. \tag{15}$$

Proposition 3 states that elastic encounters do not change the number of particles of each constituent.

**Proposition 4.** *The reactive collisional operators satisfy the following properties*

$$\int_{\mathbb{R}^3} \mathcal{Q}_1^R \, dc_1 = \int_{\mathbb{R}^3} \mathcal{Q}_2^R \, dc_2 = -\int_{\mathbb{R}^3} \mathcal{Q}_3^R \, dc_3 = -\int_{\mathbb{R}^3} \mathcal{Q}_4^R \, dc_4. \tag{16}$$

Proposition 4 states that the variation of the number of particles of constituent $A_1$ is the same as that of constituent $A_2$ and symmetric to the variation of constituents $A_3$ and $A_4$. It assures the correct chemical exchange rates of the constituents predicted by the reaction mechanism.

**Definition 1.** A function $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ is a collisional invariant in the velocity space, for the SRS model, if

$$\sum_{i=1}^{4} \int_{\mathbb{R}^3} \psi_i \left( \mathcal{Q}_i^E + \mathcal{Q}_i^R \right) dc_i = 0. \tag{17}$$

The following Proposition 5 presents the collisional invariants of model and establishes the consistency of the model from the physical point of view.

**Proposition 5.** *Functions* $\psi = (1, 0, 1, 0)$, $\psi = (1, 0, 0, 1)$, $\psi = (0, 1, 1, 0)$, *and functions* $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ *defined by* $\psi_i = m_i c_1^i$, $\psi_i = m_i c_2^i$, $\psi_i = m_i c_3^i$ *and* $\psi_i = E_i + \frac{1}{2} c_i^2 m_i$ *are collisional invariants.*

The first three invariants assure the conservation of the partial number density of a pair of constituents, one reactant and one product of the chemical reaction, namely $A_1$ and $A_3$, $A_1$ and $A_4$, and $A_2$ and $A_3$, respectively. They also assure the conservation of the total number density of the reactive mixture. The next three invariants assure the conservation of the linear momentum components of the mixture, whereas the last invariant assures the conservation of the total energy of the reactive mixture.

## 4 Equilibrium Distributions and the Boltzmann H-Theorem

When the gas reaches the equilibrium the elastic and reactive collisions do not stop, they become balanced. This means that, when the mixture is at equilibrium conditions, the collisional process does not modify the one-particle distributions $f_i$.

In particular, the number of particles that enter a volume element in the phase space per unit time is the same as the number of particles that leave the volume element in the phase space per unit time.

**Definition 2.** The gas mixture is in thermodynamical equilibrium when the elastic and reactive collisional operators are such that

$$\mathscr{Q}_i^E + \mathscr{Q}_i^R = 0, \qquad i = 1, \ldots, 4. \tag{18}$$

In particular, in our model, condition (18) implies the vanishing of the elastic collisional operators,

$$\mathscr{Q}_i^E = 0, \qquad i = 1, \ldots, 4. \tag{19}$$

Condition (19), in absence of reactive terms, is usually called a state of *mechanical equilibrium*.

The following Proposition 6 is well known in the case on one-single component gas and a formal proof can be found in many books, see for example [3].

**Proposition 6.** *If all constituents are at the same temperature, the only distribution function that assures the mechanical equilibrium is the Maxwellian distribution*

$$f_i^M(x, c_i, t) = n_i \left(\frac{m_i}{2\pi kT}\right)^{\frac{3}{2}} \exp\left[-\frac{m_i (c_i - v)^2}{2kT}\right], \quad i = 1, \ldots, 4, \tag{20}$$

where $n_i$, $T$ and $v$ are the number density of constituent $A_i$, temperature and mean velocity of the whole mixture, respectively, and $k$ is the Boltzmann constant.

The above Maxwellian distributions (20) do not assure, in general, the vanishing of the reactive collisional operators and thus do not define a state of thermodynamical equilibrium for the reactive mixture.

**Proposition 7.** *If all constituents are at the same temperature, the only distribution function that assures the thermodynamical equilibrium is the thermodynamical Maxwellian distribution given by*

$$M_i(x, c_i, t) = n_i \left(\frac{m_i}{2\pi kT}\right)^{3/2} \exp\left[-\frac{m_i (c_i - v)^2}{2kT}\right], \quad i = 1, \ldots, 4, \tag{21}$$

*with the number densities $n_i$ constrained to the condition*

$$n_1 n_2 = n_3 n_4 \sqrt{\frac{\mu_{12}}{\mu_{34}}} \exp\left(\frac{Q_R}{kT}\right). \tag{22}$$

Equation (22) represents the law of mass action for the SRS model.

The important physical feature of trend to equilibrium is now presented in the specific case of spatial domain $\Omega = \mathbb{R}^3$, proving the existence of an H-function (Liapunov functional) of the SRS system (12)–(14).

**Proposition 8 ($\mathscr{H}$-theorem).** *If the steric factors $\beta_{ij}$ and cross sections $\sigma_{ij}$ are such that $\beta_{ij} = \beta_{ji}$ and $\beta_{12}\sigma_{12}^2 = \beta_{34}\sigma_{34}^2$, the convex function $\mathsf{H}(t)$, defined by*

$$\mathsf{H}(t) = \sum_{i=1}^{4} \int_{\Omega} \int_{\mathbb{R}^3} f_i \log\left(\frac{f_i}{\mu_{ij}}\right) dc_i\, dx, \tag{23}$$

*where $(i, j) \in \{(1,2), (2,1), (3,4), (4,3)\}$ and $f_i \in L^1(\Omega \times \mathbb{R}^3)$ for all $i = 1, \ldots, 4$, is an $H$-function (Liapunov functional) for the SRS system (12)–(14), that is*

$$\frac{d\mathsf{H}}{dt}(t) \le 0 \quad \textit{for all} \quad t \ge 0, \quad \textit{and}$$

$$\frac{d\mathsf{H}}{dt}(t) = 0 \quad \textit{if and only if} \quad f_i = M_i \quad \textit{for all} \quad i = 1, \ldots, 4.$$

In the case of a spatial homogeneous evolution, the domain $\Omega$ is irrelevant for the behavior of the corresponding $H$-function, $\mathscr{H}(t) = \sum_{i=1}^{4} \int_{\mathbb{R}^3} f_i \log\left(\frac{f_i}{\mu_{ij}}\right) dc_i$. In the general case considered in Proposition 8, there exists a limited range of known situations for which the result is still valid. Some of them correspond to consider $\Omega$ as a box with boundary conditions of periodic type or boundary conditions of specular reflection at the walls, see for instance Refs. [3, 16].

The result expressed in Proposition 8 states that the reactive mixture evolves to a thermodynamical equilibrium state. In particular, in the proof of this proposition one shows that both elastic and reactive collisions contribute, independently, to this tendency to equilibrium. The spatial homogeneous version of an $\mathscr{H}$-theorem, similar to Proposition 8, is proven in paper [7], for a kinetic model for a quaternary reactive mixture undergoing a reversible bimolecular reaction. In comparison to the SRS model studied in our paper, the kinetic model considered in paper [7] has two major differences. First, the reactive cross sections of paper [7] follow the line-of-centers model, in contrast to those considered in our paper which are of hard-sphere type. Second, contrarily to the SRS system, the model of paper [7] does not consider any correction term in the collisional operators for preventing a double counting of the contributions. See the explanations at the end of Sect. 2.

In paper [7], the authors use their $\mathscr{H}$-theorem to prove the strong convergence in $L^1$-sense of the solution of their kinetic system to a Maxwellian distribution of thermodynamical equilibrium, under the assumption of uniformly boundedness and equicontinuity of the distribution functions for the spatial homogeneous case. Thus, in our opinion, Proposition 8 should constitute a central result in the convergence analysis of the solution of the kinetic equations (12) to a thermodynamical Maxwellian distribution.

## 5 Macroscopic Framework

It is well known that the Boltzmann equation constitutes a fundamental model in the kinetic theory of gases, that describes the dynamics of the gas particles. At the same time, in the hydrodynamic limit, it leads to a description in terms of physically meaningful macroscopic quantities and related balance equations, see for instance Refs. [3, 6]. The same happens with the SRS model studied in this paper. The mathematical and physical properties of the collisional operators, stating the consistency of the SRS system, are fundamental for the validity of the model as well as for the passage to the hydrodynamic limit. From a formal point of view, the connection between the microscopic variables and the macroscopic framework is based on the idea that all measurable macroscopic quantities can be expressed in terms of microscopic averages of the distribution functions. We now define the macroscopic quantities of the SRS model and provide the evolution equations for the most relevant macroscopic quantities.

**Macroscopic variables.** As usual, we define the macroscopic variables as suitable moments of the distribution functions $f_i$. We use the index $i$ for those quantities associated to the constituent $A_i$, $i = 1, \dots, 4$, and denote with plain symbols the macroscopic variables referred to the whole mixture. Moreover, indexes $l$ and $j$ are used to represent spatial components of vectorial quantities in $\mathbb{R}^3$.

*Number density* $$n_i = \int_{\mathbb{R}^3} f_i \, d\mathbf{c}_i \quad \text{and} \quad n = \sum_{i=1}^{4} n_i$$

*Mass density* $$\varrho_i = \int_{\mathbb{R}^3} m_i f_i \, d\mathbf{c}_i \quad \text{and} \quad \varrho = \sum_{i=1}^{4} \varrho_i$$

*Momentum density* $$\varrho_i v_i = \int_{\mathbb{R}^3} m_i c_i f_i \, d\mathbf{c}_i \quad \text{and} \quad \varrho v = \sum_{i=1}^{4} \varrho_i v_i$$

*Diffusion velocity* $$u_i = \frac{1}{\varrho_i} \int_{\mathbb{R}^3} m_i \zeta_i f_i \, d\mathbf{c}_i$$

*Pressure* $$p_i = \frac{1}{3} \int_{\mathbb{R}^3} m_i \zeta_i^2 f_i \, d\mathbf{c}_i \quad \text{and} \quad p = \sum_{i=1}^{4} p_i$$

*Pressure tensor components* $$p_{lj}^i = \int_{\mathbb{R}^3} m_i \zeta_l^i \zeta_j^i f_i \, d\mathbf{c}_i \quad \text{and} \quad p_{lj} = \sum_{i=1}^{4} p_{lj}^i$$

*Temperature* $$T_i = \frac{p_i}{n_i k} \quad \text{and} \quad T = \sum_{i=1}^{4} \frac{n_i}{n} T_i = \frac{p}{nk}$$

*Heat flux components* $$q_l^i = \int_{\mathbb{R}^3} \frac{1}{2} m_i \zeta_i^2 \zeta_l^i f_i \, dc_i \quad \text{and} \quad q_l = \sum_{i=1}^{4} \left( q_l^i + n_i E_i u_l^i \right)$$

where $\zeta_i = c_i - v$ is the peculiar velocity, and $\zeta_l^i$ or $\zeta_j^i$ represent its spatial components. Moreover, the term $n_i E_i u_l^i$ in the definition of $q_l$ refers to the formation energy transfer of the constituent $A_i$ due to diffusion.

**Balance equations.** By multiplying the SRS equations (12) by suitable functions $\psi_i$, and then integrating over $c_i \in \mathbb{R}^3$, one can derive the balance equations for each constituent $A_i$. Omitting here the details, the balance equations for the number density, linear momentum components and total energy of each constituent $A_i$, $i = 1, \ldots, 4$, have the form

$$\frac{\partial n_i}{\partial t} + \sum_{l=1}^{3} \frac{\partial}{\partial x_l} \left(n_i u_l^i + n_i v_l\right) = \int_{\mathbb{R}^3} \left(\mathscr{Q}_i^E + \mathscr{Q}_i^R\right) d\mathbf{c}_i, \tag{24}$$

$$\frac{\partial}{\partial t}(\varrho_i v_l^i) + \sum_{r=1}^{3} \frac{\partial}{\partial x_r} \left(p_{lr}^i + \varrho_i u_l^i v_r + \varrho_i u_r^i v_l + \varrho_i\right) = \int_{\mathbb{R}^3} m_i c_l^i \left(\mathscr{Q}_i^E + \mathscr{Q}_i^R\right) d\mathbf{c}_i, \tag{25}$$

$$\begin{aligned} \frac{\partial}{\partial t}\left[\frac{3}{2}p_i + n_i E_i + \varrho_i u_l^i v_l + \frac{1}{2}\varrho_i v^2\right] + \sum_{l=1}^{3} \frac{\partial}{\partial x_l}\Bigg[q_l^i + p_{lr}^i v_r + n_i E_i u_l^i + \frac{1}{2}\varrho_i u_l^i v^2 \\ + \left(\frac{3}{2}p_i + n_i E_i + \varrho_i u_l^i v_l + \frac{1}{2}\varrho_i v^2\right) v_i\Bigg] \\ = \int_{\mathbb{R}^3} \left(\frac{1}{2} m_i c_i^2 + E_i\right) \left(\mathscr{Q}_i^E + \mathscr{Q}_i^R\right) d\mathbf{c}_i. \end{aligned} \tag{26}$$

In particular, Eq. (24) constitutes the *reaction rate equation* of the SRS system and specifies the production rate of each constituent of the gas mixture.

**Conservation laws.** The conservation equations for partial number densities are obtained from the balance equations (24) by summing over one reactant ($i = 1, 2$) and one product ($i = 3, 4$) of the chemical reaction. Moreover, the conservation equations for mass density, linear momentum components and total energy of the whole mixture are obtained from the balance equations (24)–(26) by summing over all constituents. They can be written in the form

$$\frac{\partial}{\partial t}(n_i + n_j) + \sum_{l=1}^{3} \frac{\partial}{\partial x_l}\left[n_i u_l^i + n_j u_l^j + (n_i + n_j) v_l^i\right] = 0, \quad i = 1, 2, \quad j = 3, 4, \tag{27}$$

$$\frac{\partial \varrho}{\partial t} + \sum_{l=1}^{3} \frac{\partial}{\partial x_l}(\varrho v_l) = 0, \tag{28}$$

$$\frac{\partial}{\partial t}(\varrho v_l) + \sum_{k=1}^{3} \frac{\partial}{\partial x_k}(p_{lk} + \varrho v_l v_k) = 0, \qquad l = 1, 2, 3, \tag{29}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}nkT + \sum_{i=1}^{4} n_i E_i + \frac{1}{2}\varrho v^2\right) + \sum_{l=1}^{3} \frac{\partial}{\partial x_l}\left[q_l + \sum_{l=1}^{3} p_{lk} v_k \right. \\ \left. + \left(\frac{3}{2}nkT + \sum_{i=1}^{4} n_i E_i + \frac{1}{2}\varrho v^2\right) v_l\right] = 0. \tag{30}$$

## 6 Linearized SRS System

The linearized formulation of the SRS system around thermodynamical equilibrium arises as a simplification of the full system, which is valid when the reactive mixture is close to the thermodynamical equilibrium. In this section we construct the linearized SRS kinetic system and state its fundamental properties.

**Linearized SRS system.** To obtain the linearized equations, first the distribution function $f_i$ is expanded around the thermodynamical Maxwellian distribution $M_i$ with zero drifting velocity ($v = 0$), in the form

$$f_i(x, c_i, t) = M_i(x, c_i, t)\,[1 + h_i(x, c_i, t)], \qquad i = 1, \ldots, 4, \tag{31}$$

where $h_i$ represents the deviation of the distribution function from the equilibrium. Then, expansions (31) are inserted into the SRS system (12)–(14) and the conservation laws (2), (5) and (6) associated to elastic and reactive collisions, respectively, are used, together with the law of mass action (22). In the sequel we introduce the notation $\underline{w} = (w_1, w_2, w_3, w_4)^T$.

**Proposition 9.** *If we neglect quadratic and higher order terms in the deviations $h_i$, the linearized SRS system takes the form*

$$\frac{\partial h_i}{\partial t} + \sum_{l=1}^{3} c_l^i \frac{\partial h_i}{\partial x_l} = \mathscr{L}_i^E(\underline{h}) + \mathscr{L}_i^R(\underline{h}) \equiv \mathscr{L}_i(\underline{h}), \qquad i = 1, \ldots, 4, \tag{32}$$

*with*

$$\mathscr{L}_i^E(\underline{h}) = \sum_{s=1}^{4} \sigma_{is}^2 \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} M_s \left[h_i' + h_s' - h_i - h_s\right] \langle \epsilon, c_i - c_s \rangle \, d\epsilon \, dc_s \tag{33}$$

$$-\beta_{ij}\sigma_{ij}^2 \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} M_j \left[h_i' + h_j' - h_i - h_j\right] \Theta(\langle \epsilon, \xi_i \rangle - \Gamma_{ij}) \langle \epsilon, \xi_i \rangle \, d\epsilon \, dc_j,$$

*for* $(i, j) \in \{(1,2), (2,1), (3,4), (4,3)\}$, *and*

$$\mathscr{L}_i^R(\underline{h}) = \beta_{ij}\sigma_{ij}^2 \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} M_j \left[ h_k^* + h_l^* - h_i - h_j \right] \Theta(\langle \epsilon, \xi_i \rangle - \Gamma_{ij}) \langle \epsilon, \xi_i \rangle d\epsilon dc_j, \tag{34}$$

*for* $(i, j, k, l) \in \{(1,2,3,4), (2,1,4,3), (3,4,1,2), (4,3,2,1)\}$.

**Properties of the linearized SRS system.** Some important mathematical properties of the linearized SRS system (32)–(34) will be presented in the sequel. In order to easily compare our results with previous ones existing in literature for inert gases, we consider the following weighted distribution function and weighted operator,

$$\hat{f}_i = M_i^{1/2} f_i \qquad \text{and} \qquad \hat{\mathscr{L}_i}(\hat{\underline{h}}) = M_i^{1/2} \mathscr{L}_i(\underline{h}), \qquad i = 1, \dots, 4. \tag{35}$$

We can easily verify that $h_i$ defines a solution of the linearized SRS system (32)–(34) if and only if $\hat{h}_i$ defines a solution of the following weighted linearized system

$$\frac{\partial \hat{h}_i}{\partial t} + \sum_{l=1}^{3} c_l^i \frac{\partial \hat{h}_i}{\partial x_l} = \hat{\mathscr{L}_i}(\hat{\underline{h}}), \qquad i = 1, \dots, 4, \tag{36}$$

where the weighted linearized operator $\hat{\mathscr{L}_i}(\hat{\underline{h}})$ can be split in its elastic and reactive parts, given by

$$\hat{\mathscr{L}}_i^E(\hat{\underline{h}}) = M_i^{1/2} \mathscr{L}_i^E(h) \quad \text{and} \quad \hat{\mathscr{L}}_i^R(\hat{\underline{h}}) = M_i^{1/2} \mathscr{L}_i^R(h). \tag{37}$$

Moreover, we introduce the space $Y = L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ and consider the Maxwellian weighted velocity $L^2$-space, $Y^4$, endowed with the inner product defined by

$$\langle \underline{F}, \underline{G} \rangle = \sum_{i=1}^{4} \int_{\mathbb{R}^3} F_i(c_i) G_i(c_i) dc_i. \tag{38}$$

The weighted linearized collisional operator satisfies the following property.

**Proposition 10.** *If the steric factors $\beta_{ij}$ and cross sections $\sigma_{ij}$ are such that $\beta_{ij} = \beta_{ji}$ and $\beta_{12}\sigma_{12}^2 = \beta_{34}\sigma_{34}^2$, the weighted linearized collisional operator $\hat{\mathscr{L}}$ is symmetric and non-positive semi-definite, that is*

*(a)* $\langle \hat{\underline{g}}, \hat{\mathscr{L}}(\hat{\underline{h}}) \rangle = \langle \hat{\underline{h}}, \hat{\mathscr{L}}(\hat{\underline{g}}) \rangle$, *for all* $\underline{g}, \underline{h} \in Y^4$;

*(b)* $\langle \hat{\underline{h}}, \hat{\mathscr{L}}(\hat{\underline{h}}) \rangle \leq 0$, *for all* $\underline{h} \in Y^4$, *and* $\langle \hat{\underline{h}}, \hat{\mathscr{L}}(\hat{\underline{h}}) \rangle = 0$ *if and only if* $\underline{h}$ *is a collisional invariant.*

## 7 Kernels of the Linearized Integral Operators

For a one component inert gas, the explicit expression of the kernel of the linearized collisional operator, as well as the techniques used to compute the kernel, are detailed in many works, in particular in paper [4]. However, this is not the case for a reactive gas mixture. In fact, the computations for the case of a reactive gas mixture are long and very technical. Concerning, in particular, the SRS system, we were able to obtain the explicit representation of the kernels of the linearized elastic and reactive operators, in the general case of arbitrary molecular masses. The computations have been done by the first author of the present work and are part of his PhD thesis, see [1]. See also Ref. [2] for the details about the computations of the kernels.

**Kernels of the linearized elastic operators.** The operator $\hat{\mathscr{L}}_i^E(\underline{\hat{h}})$ can be split into several contributions as follows

$$\hat{\mathscr{L}}_i^E(\underline{\hat{h}}) = -\nu_i \hat{h}_i - Q_i^{(1)}(\underline{\hat{h}}) + Q_i^{(2)}(\underline{\hat{h}}) + Q_i^{(3)}(\underline{\hat{h}}), \tag{39}$$

where

$$\nu_i \hat{h}_i = \hat{h}_i \sum_{s=1}^{4} \sigma_{is}^2 \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} M_s \langle \epsilon, c_i - c_s \rangle \, d\epsilon \, dc_s, \tag{40}$$

$$Q_i^{(1)}(\underline{\hat{h}}) = \sum_{s=1}^{4} \sigma_{is}^2 \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} M_i^{1/2} M_s^{1/2} \hat{h}_s \langle \epsilon, c_i - c_s \rangle \, d\epsilon \, dc_s, \tag{41}$$

$$Q_i^{(2)}(\underline{\hat{h}}) = \sum_{s=1}^{4} \sigma_{is}^2 \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} M_i^{1/2} M_s M_i'^{-1/2} \hat{h}_i' \langle \epsilon, c_i - c_s \rangle \, d\epsilon \, dc_s, \tag{42}$$

$$Q_i^{(3)}(\underline{\hat{h}}) = \sum_{s=1}^{4} \sigma_{is}^2 \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} M_i^{1/2} M_s M_s'^{-1/2} \hat{h}_s' \langle \epsilon, c_i - c_s \rangle \, d\epsilon \, dc_s. \tag{43}$$

The multiplication operator $\nu_i$ defined in (40) can be identified as a mean collision frequency. Concerning the integral operators (41)–(43), the full representation of the kernels, in the general case of arbitrary molecular masses, is omitted here due to space limitations. In fact, some of the expressions are very long. We only include the expressions of the kernels in the particular case of a reactive mixture with equal molecular masses. They are given by the following expressions, for $i = 1, 2, 3, 4$,

$$N(Q_i^{(1)})(u, w) = \pi \sigma_{is}^2 \|u - w\| \sqrt{n_i n_s} \left( \frac{m}{2\pi k T} \right)^{3/2} \exp\left( -\frac{m(u^2 + w^2)}{4kT} \right), \tag{44}$$

$$N(Q_i^{(2)})(u,w) = \sigma_{is}^2 n_s \left(\frac{m}{2\pi kT}\right)^{1/2} \frac{1}{\|u-w\|} \times \exp\left[-\frac{m}{8kT}\frac{(u^2-w^2)^2}{\|u-w\|^2} - \frac{m}{8kT}(u-w)^2\right], \tag{45}$$

$$N(Q_i^{(3)})(u,w) = \sigma_{is}^2 \sqrt{n_i n_s} \left(\frac{m}{2\pi kT}\right)^{1/2} \frac{1}{\|u-w\|} \times \exp\left[-\frac{m}{8kT}\frac{(u^2-w^2)^2}{\|u-w\|^2} - \frac{m}{8kT}(u-w)^2\right]. \tag{46}$$

Note that, in the case of one component gas, expressions (44)–(46) reduce to those presented by Grad in paper [5], for the intermolecular potential of hard-sphere type.

**Kernels of the linearized reactive operators.** The procedure adopted to obtain the representation of the kernels of the reactive operators is similar to the one used for the elastic operators. The starting point is the decomposition of the operator $\hat{\mathscr{L}}_i^R(\underline{\hat{h}})$ into several contributions, in the form

$$\hat{\mathscr{L}}_i^R(\underline{\hat{h}}) = -\nu_i^R(u)\hat{h}_i(u) - \mathscr{R}_i^{(1)}(\underline{\hat{h}}) + \mathscr{R}_i^{(2)}(\underline{\hat{h}}) + \mathscr{R}_i^{(3)}(\underline{\hat{h}}),$$

where, for $i = 1, 2, 3, 4$,

$$\nu_i^R(u)\hat{h}_i(u) = \hat{h}_i \beta_{ij}\sigma_{ij}^2 \int_{\mathbb{R}^3}\int_{\mathbb{S}_+^2} M_j \Theta(\langle\epsilon,\xi_i\rangle - \Gamma_{ij})\,\langle\epsilon,\xi_i\rangle\, d\epsilon\, dc_j, \tag{47}$$

$$\mathscr{R}_i^{(1)}(\underline{\hat{h}}) = \beta_{ij}\sigma_{ij}^2 \int_{\mathbb{R}^3}\int_{\mathbb{S}_+^2} M_i^{1/2} M_j \hat{h}_j \Theta(\langle\epsilon,\xi_i\rangle - \Gamma_{ij})\,\langle\epsilon,\xi_i\rangle\, d\epsilon\, dc_j, \tag{48}$$

$$\mathscr{R}_i^{(2)}(\underline{\hat{h}}) = \beta_{ij}\sigma_{ij}^2 \int_{\mathbb{R}^3}\int_{\mathbb{S}_+^2} M_j^{1/2} M_l^{*1/2} \hat{h}_k^* \Theta(\langle\epsilon,\xi_i\rangle - \Gamma_{ij})\,\langle\epsilon,\xi_i\rangle\, d\epsilon\, dc_j, \tag{49}$$

$$\mathscr{R}_i^{(3)}(\underline{\hat{h}}) = \beta_{ij}\sigma_{ij}^2 \int_{\mathbb{R}^3}\int_{\mathbb{S}_+^2} M_j^{1/2} M_k^{*1/2} \hat{h}_l^* \Theta(\langle\epsilon,\xi_i\rangle - \Gamma_{ij})\,\langle\epsilon,\xi_i\rangle\, d\epsilon\, dc_j. \tag{50}$$

Again, the multiplication operator $\nu_i^R$ defined in (47) can be identified as a mean reactive collision frequency. Concerning the integral operators (48)–(50), the full representation of their kernels, for the general case of arbitrary molecular masses, is omitted here due to space limitations. The reader is addressed to Refs. [1, 2] for the details. Here we present the particular case of equal molecular masses, and only present the kernels of the operators $\mathscr{R}_i^{(1)}(\underline{\hat{h}})$, for $i = 1, 2, 3, 4$, and $\mathscr{R}_1^{(2)}(\underline{\hat{h}})$, $\mathscr{R}_1^{(3)}(\underline{\hat{h}})$,

$$N(\mathscr{R}_i^{(1)})(u,w) = \frac{1}{2}\,\beta_{ij}\sigma_{ij}^2\sqrt{n_i n_j}\left(\frac{m}{2\pi kT}\right)^{3/2}\;\frac{\|u-w\|^2-\Gamma_{ij}^2}{\|u-w\|} \times \exp\left(-\frac{m(u^2+w^2)}{4kT}\right), \quad i=1,2,3,4, \tag{51}$$

$$N(\mathscr{R}_1^{(2)})(u,w) = \frac{1}{2}\,\beta_{12}\,\sigma_{12}^2\sqrt{n_2 n_4}\left(\frac{m}{2\pi kT}\right)^{3/2} \tag{52}$$

$$\times \frac{||u-w||^2}{\left(||u-w||+\frac{Q_R}{m||u-w||}\right)^3}\;\Theta\left(||u-w||+\frac{Q_R}{m||u-w||}-\Gamma_{12}\right)$$

$$\times\left[\frac{2Q_R}{m\left(||u-w||+\frac{Q_R}{m||u-w||}\right)^2\sqrt{1-\frac{4Q_R}{m\left(||u-w||+\frac{Q_R}{m||u-w||}\right)^2}}}-\frac{||u-w||}{||u-w||+\frac{Q_R}{m||u-w||}}\right]$$

$$\times\int_{L\perp(w-u)}\exp\left\{-\frac{m}{4kT}\left[\left(u-L+\frac{2(w-u)}{1-\sqrt{1-\frac{4Q_R}{m\left(||u-w||+\frac{Q_R}{m||u-w||}\right)^2}}}\right)^2 + \left(2u-L-w+\frac{2(w-u)}{1-\sqrt{1-\frac{4Q_R}{m\left(||u-w||+\frac{Q_R}{m||u-w||}\right)^2}}}\right)^2\right]\right\}dL,$$

$$N(\mathscr{R}_1^{(3)})(u,w) = \beta_{12}\,\sigma_{12}^2\sqrt{n_2 n_3}\;\left(\frac{m}{2\pi kT}\right)^{3/2}\int_{D_L}\exp\left\{-\frac{m}{4kT}\left[\right.\right. \tag{53}$$

$$\left.\left.\times\left(u+\frac{1}{2}\left(-1+\sqrt{1-\frac{4Q_R}{mL^2}}\right)L\right)^2+\left(w+\frac{1}{2}\left(-1+\sqrt{1-\frac{4Q_R}{mL^2}}\right)L\right)^2\right]\right\}$$

$$\times\,\Theta\,(\,||L||-\Gamma_{12}\,)\,\frac{1}{||w-u-\frac{1}{2}\left(1+\sqrt{1-\frac{4Q_R}{mL^2}}\right)L||}\;dL,$$

where the integration domain $D_L$ is defined by

$$D_L = \left\{ L \in \mathbb{R}^3 : \quad \langle L, v - w \rangle = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4Q_R}{mL^2}}\right) L^2 \right\}.$$

## 8 Final Remarks

The content of this paper is the first part of a work in progress on the SRS model for a quaternary mixture with no restriction on the molecular masses of the constituents. The properties of the linearized SRS system presented here, and especially the explicit representations of its kernels, are essential in obtaining detailed spectral analysis of the system. This in turn provides the asymptotic behavior of its evolution operator that is used in existence and stability of close to equilibrium solutions for the SRS system and the rigorous treatment of the hydrodynamical limits at the Euler and Navier-Stokes level.

## References

1. Carvalho, F.: Mathematical methods for the Boltzmann equation in the context of chemically reactive gases. Doctoral thesis. http://hdl.handle.net/1822/24430 (2013). Repositorium of the University of Minho
2. Carvalho, F., Polewczak, J., Soares, A.J.: Representation of the kernels of the simple reacting spheres system. Computations available at http://w3.math.uminho.pt/~ajsoares/Kernels_SRS_Model (2013)
3. Cercignani, C.: Theory and Application of the Boltzmann Equation. Scottish Academic Press, Edinburgh (1975)
4. Grad, H.: Asymptotic theory of the Boltzmann equation II. In: Laurmann, J.A. (ed.) Rarefied Gas Dynamics (Paris, 1962), vol. 1, pp. 26–59. Academic, New York (1963)
5. Grad, H.: Asymptotic theory of the Boltzmann equation. Phys. Fluids **6**, 147–181 (1963)
6. Kremer, G.: Introduction to the Boltzmann Equation and Transport Processes in Gases. Springer, Berlin (2010)
7. Kremer, G., Oliveira F., Soares, A.J.: $\mathscr{H}$-theorem and trend to equilibrium of chemically reactive mixture of gases. Kinet. Relat. Models **2**, 333–343 (2009)
8. Marron, M.T.: Simple collision theory of reactive hard spheres. J. Chem. Phys. **52**, 4060–4061 (1970)
9. Moreau, M.: Formal study of a chemical reaction by Grad expansion of the Boltzmann equation. Physica A **79**, 18–51 (1975)
10. Polewczak, J.: The kinetic theory of simple reacting spheres: I. Global existence result in a dilute-gas case. J. Stat. Phys. **100**, 327–362 (2000)
11. Polewczak, J., Soares, A.J.: Kinetic Theory of Simple Reacting Spheres I. In: 27th International Symposium on Rarefied Gas Dynamics, 2010, Pacific Grove. AIP Conference Proceedings, vol. 1333, pp. 117–122 (2011)
12. Present, R.D.: On the velocity distribution in a chemically reacting gas. J. Chem. Phys. **31**, 747–797 (1959)
13. Prigogine, I., Xhrouet, E.: On the perturbation of Maxwell distribution function by chemical reaction in gases. Physica **XV**, 913–932 (1949)

14. Ross, J., Mazur, P.: Some deductions from a formal statistical mechanical theory of chemical kinetics. J. Chem. Phys. **35**, 19–28 (1961)
15. Shizgal, B., Karplus, M.: Nonequilibrium contributions to the rate of reaction. I. Perturbation of the velocity distribution function. J. Chem. Phys. **52**, 4262–4278 (1970)
16. Villani, C.: A review of mathematical topics in collisional kinetic theory. In: Friedlander, S., Serre, D. (eds.) Handbook of Mathematical Fluid Dynamics, vol. 1, pp. 71–305. Elsevier, North-Holland (2002)
17. Xystris, N., Dahler, J.S.: Mass and momentum transport in dilute reacting gases. J. Chem. Phys. **68**, 354–373 (1978)
18. Xystris, N., Dahler, J.S.: Kinetic theory of simple reactive spheres. J. Chem. Phys. **68**, 387–401 (1978)

# Hydrodynamic Limit for the Velocity-Flip Model

**Marielle Simon**

## 1 Introduction

We consider a Hamiltonian system of $N$ coupled oscillators with the same mass that we set equal to 1. Since the ergodic properties of Hamiltonian dynamics are poorly understood, especially when the size of the system goes to infinity, we perturb it by an additional conservative mixing noise, as it has been proposed for the first time by Olla, Varadhan and Yau [11] in the context of gas dynamics, and then in [6] in the context of Hamiltonian lattice dynamics.

We are interested in the macroscopic behavior of this system as $N$ goes to infinity, after rescaling space and time. The system is considered under periodic boundary conditions, more precisely we work on the one-dimensional discrete torus $\mathbb{T}_N := \{0, \dots, N-1\}$. A typical configuration is given by $\omega = (p_x, r_x)_{x\in\mathbb{T}_N}$ where $p_x$ stands for the velocity of the oscillator at site $x$, and $r_x$ represents the distance between oscillator $x$ and oscillator $x + 1$. The deterministic dynamics is described by the harmonic Hamiltonian

$$\mathscr{H}_N = \sum_{x=0}^{N-1} \left[ \frac{p_x^2 + r_x^2}{2} \right] . \tag{1}$$

The stochastic perturbation is added only to the velocities, in such a way that the energy of particles is still conserved. Nevertheless, the momentum conservation is no longer valid, so that we can hope for a normal diffusion of energy.[1] The added

[1] If the momentum is conserved, abnormal behaviors can emerge, see for example [1], or [3].

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noise can be easily described: each particle independently waits an exponentially distributed time interval and then flips the sign of velocity. The strength of the noise is regulated by the parameter $\gamma > 0$. The total deformation $\sum r_x$ and the total energy $\sum \left(p_x^2 + r_x^2\right)/2$ are the only two conserved quantities. Thus, the Gibbs states are parametrized by two potentials, temperature and tension: for $\beta > 0$ and $\lambda \in \mathbb{R}$, the equilibrium Gibbs measures $\mu_{\beta,\lambda}^N$ on the configuration space $\Omega^N := (\mathbb{R} \times \mathbb{R})^{\mathbb{T}_N}$ are products of Gaussians (see (10)).

The goal is to prove that the two empirical profiles associated to the conserved quantities converge in the thermodynamic limit $N \to \infty$ to the macroscopic profiles $\mathbf{r}(t,\cdot)$ and $\mathbf{e}(t,\cdot)$, which satisfy an autonomous system of coupled parabolic equations. More precisely, let $\mathbf{r}_0 : \mathbb{T} \to \mathbb{R}$ and $\mathbf{e}_0 : \mathbb{T} \to \mathbb{R}$ be respectively the initial macroscopic deformation profile and the initial macroscopic energy profile defined on the one-dimensional torus $\mathbb{T} = [0,1]$ and denote by $\mu_0^N$ the Gibbs local equilibrium associated to $\mathbf{r}_0$ and $\mathbf{e}_0$ (see (14) for the explicit formula). If the initial law of the process is $\mu_0^N$, then the law of the process in the diffusive scale, namely at time $tN^2$, is close in the large $N$ limit, to the Gibbs local equilibrium associated to the functions $\mathbf{r}(t,q)$ and $\mathbf{e}(t,q)$ (defined on $\mathbb{R}_+ \times \mathbb{T}$), which are solutions of

$$\begin{cases} \partial_t \mathbf{r} = \dfrac{1}{\gamma}\partial_q^2 \mathbf{r}\ , \\[2ex] \partial_t \mathbf{e} = \dfrac{1}{2\gamma}\partial_q^2\left(\mathbf{e} + \dfrac{\mathbf{r}^2}{2}\right)\ , \end{cases} \qquad q \in \mathbb{T},\ t \in \mathbb{R}_+\ , \tag{2}$$

with the initial conditions $\mathbf{r}(0,\cdot) = \mathbf{r}_0(\cdot)$ and $\mathbf{e}(0,\cdot) = \mathbf{e}_0(\cdot)$.

We approach this problem by using the relative entropy[2] method introduced for the first time by H. T. Yau [14] for a gradient[3] diffusive Ginzburg-Landau dynamics. Roughly speaking, we measure the distance between the Gibbs local equilibrium[4] $\mu_{\mathbf{e}(t,\cdot),\mathbf{r}(t,\cdot)}^N$ and the state $\mu_t^N$ by their relative entropy $H_N(t)$ (see (28)). The strategy consists in proving that $\lim_{N\to\infty} H_N(t)/N = 0$ and deducing that the hydrodynamic limit holds. In the context of diffusive systems, the relative entropy method works if the following conditions are satisfied.

- First, the dynamics has to be *ergodic*: the only time and space invariant measures for the infinite system, with finite local entropy, are given by mixtures of the Gibbs measures in infinite volume $\mu_{\beta,\lambda}$ (see (16)). From [6], we know that the velocity-flip model is ergodic in the sense above (see Theorem 3).
- Next, we need to establish the so-called *fluctuation-dissipation equations* in the mathematics literature (for example, in [9]). Such equations express the

[2]The relative entropy of the probability measure $\mu$ with respect to the probability measure $\nu$ is denoted by $H(\mu|\nu)$ and is defined in (15).

[3]A conservative system is called gradient if the currents corresponding to the conserved quantities are gradients.

[4]For the sake or readability, in the following sections we will denote it by $\mu_{\beta_t(\cdot),\lambda_t(\cdot)}^N$ (see (14)).

microscopic current of energy (which here is not a discrete gradient) as the sum of a discrete gradient and a fluctuating term. More precisely, the microscopic current of energy, denoted by $j_{x,x+1}$, is defined by the local energy conservation law: $\mathscr{L} e_x = \nabla j_{x-1,x}$, where $\mathscr{L}$ is the generator of the infinite dynamics. The standard approach consists in proving that there exist local functions $f_x$ and $h_x$ such that the following decomposition holds:

$$j_{x,x+1} = \nabla f_x + \mathscr{L} h_x \ . \tag{3}$$

Equation (3) is called a microscopic fluctuation-dissipation equation. The term $\mathscr{L} h_x$, when integrated in time, is a martingale. Roughly speaking, $\mathscr{L} h_x$ represents rapid fluctuation, whereas $\nabla f_x$ represents dissipation. Gradient models are systems for which $h_x = 0$ with the previous notations.
In general, these equations are not explicit but we are able to compute them in our model (see [12], Appendix A).

- Finally, since we observe the system on a diffusive scale and the system is non-gradient, we need second order approximations. If we want to obtain the entropy estimate of order $o(N)$, we can not work with the measure $\mu^N_{\mathbf{e}(t,\cdot),\mathbf{r}(t,\cdot)}$: we have to correct the Gibbs local equilibrium state with a small term. This idea was first introduced in [7] and then used in [13] for interacting Ornstein-Uhlenbeck processes, and in [10] for the asymmetric exclusion process. However, as far as we know, it is the first time that this is applied for a system with several conservation laws.

Up to present, the derivation of hydrodynamic equations for the harmonic oscillators perturbed by the velocity-flip noise is not rigorously achieved (see e.g. [5]), because the control of large energies has not been considered so far. Along the proof, we need to control all the following moments,

$$\int \left[ \frac{1}{N} \sum_{x \in \mathbb{T}_N} |p_x|^k \right] \mathrm{d}\mu_t^N \ , \tag{4}$$

uniformly in time and with respect to $N$. In fact, the only first moments are necessary to cut-off large energies and we need all the others to obtain the Taylor expansion that appears in the relative entropy method (Proposition 1). Usually, the following entropy inequality (true for any $\alpha > 0$ and any positive measurable function $f$)

$$\int f \, d\mu \leqslant \frac{1}{\alpha} \left\{ \log \left( \int e^{\alpha f} \, d\nu \right) + H(\mu|\nu) \right\} \tag{5}$$

reduces the control of (4) to the estimate of the following equilibrium exponential moments

$$\int \exp(\delta |p_x|^k)\, \mathrm{d}\mu_{1,0}^N\,, \quad \text{with } \delta > 0 \text{ small.} \tag{6}$$

Unfortunately, in our model, these integrals are infinite for all $k \geqslant 3$ and all $\delta > 0$.

Bernardin [2] deals with a harmonic chain perturbed by a stochastic noise which is different from ours but has the same motivation: energy is conserved, momentum is not. He derives the hydrodynamic limit for a particular value of the intensity of the noise. In this case the hydrodynamic equations are simply given by two decoupled heat equations. The author highlights that good energy bounds are necessary to extend his work to other values of the noise intensity. In fact, in [2], only the following weak form is proved:

$$\lim_{N\to+\infty} \int \left[ \frac{1}{N^2} \sum_{x\in\mathbb{T}_N} p_x^4 \right] \mathrm{d}\mu_t^N = 0\,. \tag{7}$$

In [12], we get uniform control of (4) for our model (Theorem 2). Let us notice that the harmonicity of the chain is crucial to get this result: roughly speaking, it ensures that the set of mixtures of Gaussian probability measures is left invariant during the time evolution. The article is divided into two parts: after the main results being stated, we give the ideas of proof. All the results discussed here are in [12] to which we refer for the details.

## 2 The Velocity-Flip Model

We consider the unpinned harmonic chain perturbed by the momentum-flip noise. Each particle has the same mass that we set equal to 1. A typical configuration is $\omega = (\mathbf{r}, \mathbf{p}) \in \Omega^N := (\mathbb{R} \times \mathbb{R})^{\mathbb{T}_N}$, where $\mathbf{r} = (r_x)_{x\in\mathbb{T}_N}$ and $\mathbf{p} = (p_x)_{x\in\mathbb{T}_N}$.

The generator of the dynamics is given by $\mathscr{L}_N := \mathscr{A}_N + \gamma \mathscr{S}_N$, where for any continuously differentiable function $f : \Omega^N \to \mathbb{R}$,

$$\mathscr{A}_N(f)(\mathbf{r},\mathbf{p}) := \sum_{x\in\mathbb{T}_N} \left[ (p_{x+1} - p_x)\, \partial_{r_x} f(\mathbf{r},\mathbf{p}) + (r_x - r_{x-1})\, \partial_{p_x} f(\mathbf{r},\mathbf{p}) \right], \tag{8}$$

$$\mathscr{S}_N(f)(\mathbf{r},\mathbf{p}) := \frac{1}{2} \sum_{x\in\mathbb{T}_N} \left[ f(\mathbf{r},\mathbf{p}^x) - f(\mathbf{r},\mathbf{p}) \right]. \tag{9}$$

Here $\mathbf{p}^x$ is the configuration obtained from $\mathbf{p}$ by the flip of $p_x$ into $-p_x$. The parameter $\gamma > 0$ regulates the strength of the random flip of momenta.

The operator $\mathscr{A}_N$ is the Liouville operator of a chain of harmonic oscillators, and $\mathscr{S}_N$ is the generator of the stochastic part of the dynamics that flips at random time the velocity of one particle. The dynamics conserves two quantities: the total deformation of the lattice $\mathscr{R} := \sum_{x\in\mathbb{T}_N} r_x$ and the total energy $\mathscr{E} := \sum_{x\in\mathbb{T}_N} e_x$, where $e_x = \left(p_x^2 + r_x^2\right)/2$. Observe that the total momentum is no longer conserved.

The deformation and the energy define a family of invariant measures depending on two parameters. For $\beta > 0$ and $\lambda \in \mathbb{R}$, we denote by $\mu^N_{\beta,\lambda}$ the Gaussian product measure on $\Omega^N$ given by

$$\mu^N_{\beta,\lambda}(\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{p}) := \prod_{x \in \mathbb{T}_N} \frac{e^{-\beta e_x - \lambda r_x}}{Z(\beta, \lambda)} \mathrm{d}r_x \mathrm{d}p_x \;, \tag{10}$$

where $Z(\beta, \lambda)$ is the partition function.

In the following, we shall denote by $\mu[\cdot]$ the expectation with respect to the measure $\mu$. The thermodynamic relations between the averaged conserved quantities $\bar{\mathbf{r}} \in \mathbb{R}$ and $\bar{\mathbf{e}} \in (0, +\infty)$, and the potentials $\beta \in (0, +\infty)$ and $\lambda \in \mathbb{R}$ are given by

$$\begin{cases} \bar{\mathbf{e}}(\beta, \lambda) := \mu^N_{\beta,\lambda}[e_x] = \dfrac{1}{\beta} + \dfrac{\lambda^2}{2\beta^2} \;, \\ \bar{\mathbf{r}}(\beta, \lambda) := \mu^N_{\beta,\lambda}[r_x] = -\dfrac{\lambda}{\beta} \;. \end{cases} \tag{11}$$

Notice that

$$\forall\, \beta \in (0, +\infty), \forall\, \lambda \in \mathbb{R},\ \bar{\mathbf{e}}(\beta, \lambda) > \frac{\bar{\mathbf{r}}^2(\beta, \lambda)}{2} \;. \tag{12}$$

*Remark 1.* There exists a bijection between the two sets $\{(\beta, \lambda) \in \mathbb{R}^2 \;;\; \beta > 0\}$ and $\{(\mathbf{e}, \mathbf{r}) \in \mathbb{R}^2 \;;\; \mathbf{e} > \mathbf{r}^2/2\}$. The equations above can be inverted according to the functional

$$\begin{array}{rcl} \Psi : \{(\mathbf{e}, \mathbf{r}) \in \mathbb{R}^2 \;;\; \mathbf{e} > \mathbf{r}^2/2\} & \to & \{(\beta, \lambda) \in \mathbb{R}^2 \;;\; \beta > 0\} \\ (\mathbf{e}, \mathbf{r}) & \mapsto & \left( \dfrac{1}{\mathbf{e} - \mathbf{r}^2/2}, -\dfrac{\mathbf{r}}{\mathbf{e} - \mathbf{r}^2/2} \right) \;. \end{array}$$

We assume that the system is initially close to a *local equilibrium*.

**Definition 1.** A sequence $(\mu^N)_N$ of probability measures on $\Omega^N$ is a *local equilibrium* associated to a deformation profile $\mathbf{r}_0 : \mathbb{T} \to \mathbb{R}$ and an energy profile $\mathbf{e}_0 : \mathbb{T} \to (0, +\infty)$ if for every continuous function $G : \mathbb{T} \to \mathbb{R}$ and for every $\delta > 0$, we have

$$\begin{cases} \displaystyle\lim_{N \to \infty} \mu^N \left[ \left| \frac{1}{N} \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) r_x - \int_{\mathbb{T}} G(q) \mathbf{r}_0(q) \mathrm{d}q \right| > \delta \right] = 0 \;, \\ \displaystyle\lim_{N \to \infty} \mu^N \left[ \left| \frac{1}{N} \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) e_x - \int_{\mathbb{T}} G(q) \mathbf{e}_0(q) \mathrm{d}q \right| > \delta \right] = 0 \;. \end{cases} \tag{13}$$

*Example 1.* For any integer $N$ we define the probability measures

$$\mu^N_{\beta_0(\cdot),\lambda_0(\cdot)}(\mathrm{d}\mathbf{r},\mathrm{d}\mathbf{p}) := \prod_{x\in\mathbb{T}_N} \frac{\exp(-\beta_0(x/N)e_x - \lambda_0(x/N)r_x)}{Z(\beta_0(x/N),\lambda_0(x/N))}\mathrm{d}r_x\mathrm{d}p_x\ , \tag{14}$$

where the two profiles $\beta_0$ and $\lambda_0$ are related to $\mathbf{e}_0$ and $\mathbf{r}_0$ by (11). Then, this sequence of probability measures is a local equilibrium, and it is called the *Gibbs local equilibrium state* associated to the macroscopic profiles $\beta_0$, $\lambda_0$. Both profiles are assumed to be continuous.

To establish the hydrodynamic limits, we look at the process with generator $N^2\mathscr{L}_N$, namely in the diffusive scale. The configuration at time $tN^2$ is denoted by $\omega^N_t$, and the law of the process $(\omega^N_t)_{t\geqslant 0}$ is denoted by $\mu^N_t$.

## 2.1 Hydrodynamic Equations

Let $\mu$ and $\nu$ be two probability measures on the same measurable space $(X,\mathscr{F})$. We define the relative entropy $H(\mu|\nu)$ of the probability measure $\mu$ with respect to the probability measure $\nu$ by

$$H(\mu|\nu) := \sup_f \left\{ \int_X f\ \mathrm{d}\mu - \log\left(\int_X e^f\ \mathrm{d}\nu\right)\right\}\ , \tag{15}$$

where the supremum is carried over all bounded measurable functions $f$ on $X$. The Gibbs states in infinite volume are the probability measures $\mu_{\beta,\lambda}$ on $\Omega = (\mathbb{R}\times\mathbb{R})^{\mathbb{Z}}$ given by

$$\mu_{\beta,\lambda}(\mathrm{d}\mathbf{r},\mathrm{d}\mathbf{p}) := \prod_{x\in\mathbb{Z}} \frac{e^{-\beta e_x - \lambda r_x}}{Z(\beta,\lambda)}\mathrm{d}r_x\mathrm{d}p_x\ . \tag{16}$$

We denote by $\tau_x\varphi$ the shift of $\varphi$: $(\tau_x\varphi)(\omega) = \varphi(\tau_x\omega) = \varphi(\omega(x+\cdot))$. Hereafter, all statements involving time $t$ assume that $t$ belongs to a compact set $[0,T]$. In [12] the following theorem is proved.

**Theorem 1.** *Let $(\mu^N_0)_N$ be a sequence of probability measures on $\Omega^N$ which is a local equilibrium associated to a deformation profile $\mathbf{r}_0$ and an energy profile $\mathbf{e}_0$ such that $\mathbf{e}_0 > \mathbf{r}_0^2/2$. We denote by $\beta_0$ and $\lambda_0$ the potential profiles associated to $\mathbf{r}_0$ and $\mathbf{e}_0$: $(\beta_0,\lambda_0) := \Psi(\mathbf{e}_0,\mathbf{r}_0)$.*

*We assume that the initial profiles are continuous, and that*

$$H\left(\mu^N_0 | \mu^N_{\beta_0(\cdot),\lambda_0(\cdot)}\right) = o(N). \tag{17}$$

*We also assume that the energy moments are bounded: let us suppose that there exists a positive constant C which does not depend on N and t, such that*

$$\forall\, k \geqslant 1,\ \mu_t^N \left[ \sum_{x \in \mathbb{T}_N} e_x^k \right] \leqslant (Ck)^k \times N \,. \tag{18}$$

*Let G be a continuous function on the torus $\mathbb{T}$ and $\varphi$ be a local function which satisfies the following property: there exists a finite subset $\Lambda \subset \mathbb{Z}$ and a constant $C > 0$ such that, for all $\omega \in \Omega^N$, $\varphi(\omega) \leqslant C\left(1 + \sum_{i \in \Lambda} e_i(\omega)\right)$. Then,*

$$\mu_t^N \left[ \left| \frac{1}{N} \sum_x G\left(\frac{x}{N}\right) \tau_x \varphi - \int_{\mathbb{T}} G(q)\, \tilde{\varphi}(\mathbf{e}(t,q), \mathbf{r}(t,q)) dq \right| \right] \xrightarrow[N\to\infty]{} 0 \tag{19}$$

*where $\tilde{\varphi}$ is the grand-canonical expectation of $\varphi$: in other words, for any $(\mathbf{e}, \mathbf{r}) \in \mathbb{R}^2$, if $(\beta, \lambda) = \Psi(\mathbf{e}, \mathbf{r})$ then*

$$\tilde{\varphi}(\mathbf{e}, \mathbf{r}) = \mu_{\beta,\lambda}[\varphi] = \int_{(\mathbb{R}\times\mathbb{R})^{\mathbb{Z}}} \varphi(\omega)\, d\mu_{\beta,\lambda}(\omega) \,. \tag{20}$$

*Besides, $\mathbf{e}$ and $\mathbf{r}$ are defined on $\mathbb{R}_+ \times \mathbb{T}$ and are solutions of*

$$\begin{cases} \partial_t \mathbf{r} = \dfrac{1}{\gamma} \partial_q^2 \mathbf{r}, \\ \partial_t \mathbf{e} = \dfrac{1}{2\gamma} \partial_q^2 \left( \mathbf{e} + \dfrac{\mathbf{r}^2}{2} \right), \end{cases} \qquad q \in \mathbb{T},\ t \in \mathbb{R}_+ \,, \tag{21}$$

*with the initial conditions $\mathbf{r}(\cdot, 0) = \mathbf{r}_0(\cdot)$ and $\mathbf{e}(\cdot, 0) = \mathbf{e}_0(\cdot)$.*

*Remark 2.* Let us notice that the functions $\mathbf{e}, \mathbf{r}, \beta$ and $\lambda$ are smooth when $t > 0$, since the system of partial differential equations is parabolic.

In Sect. 4, we will see that the hypothesis on moments bounds (18) holds for a large class of initial local equilibrium states. Before stating the theorem, we give some definitions. We denote by $\mathfrak{S}_N(\mathbb{R})$ the set of real symmetric matrices of size $N$. The correlation matrix $C \in \mathfrak{S}_{2N}(\mathbb{R})$ of a probability measure $\nu$ on $\Omega^N$ is the symmetric matrix $C = (C_{i,j})_{1 \leqslant i,j \leqslant 2N}$ defined by

$$C_{i,j} := \begin{cases} \nu[r_i r_j] & i, j \in \{1, \dots, N\} \,, \\ \nu[r_i p_j] & i \in \{1, \dots, N\},\ j \in \{N+1, \dots, 2N\} \,, \\ \nu[p_i r_j] & i \in \{N+1, \dots, 2N\},\ j \in \{1, \dots, N\} \,, \\ \nu[p_i p_j] & i, j \in \{N+1, \dots, 2N\} \,. \end{cases} \tag{22}$$

Let us denote by $\Sigma_N$ the subset of $\mathbb{R}^{2N} \times \mathfrak{S}_{2N}(\mathbb{R})$ defined by the following condition:

$$(m, C) \in \Sigma_N \Leftrightarrow \begin{cases} m_k = 0 & \text{for all } k = N + 1 \dots 2N \ , \\ C_{i,j} = 0 & \text{for all } i \neq j \ , \\ C_{i,i} > 0 & \text{for all } i = 1 \dots 2N \ , \\ C_{i,i} - m_i^2 = C_{i+N,i+N} & \text{for all } i = 1 \dots N \ . \end{cases} \tag{23}$$

Precisely, it means that $m$ is written as $m = (m_1, \dots, m_N, 0, \dots, 0)$, and $C$ is a diagonal matrix whose components are $(m_1^2 + \alpha_1, \dots, m_N^2 + \alpha_N, \alpha_1, \dots, \alpha_N)$, where $\alpha_i > 0$ for all $i = 1 \dots N$. For $(m, C) \in \Sigma_N$, we denote by $G_{m,C}(\cdot)$ the Gaussian measure with mean $m$ and correlations given by the matrix $C$. The covariance matrix of $G_{m,C}(\cdot)$ is thus $C - m^t m$. In [12] the following lemma is proved.

**Lemma 1.** *Let $\lambda$ and $\beta$ be two functions of class $\mathscr{C}^1$ defined on $\mathbb{T}$, and $\mu^N_{\beta(\cdot),\lambda(\cdot)}$ be the Gibbs local equilibrium defined by* (14)*. If we denote by $m_{\beta(\cdot),\lambda(\cdot)}$ and $C_{\beta(\cdot),\lambda(\cdot)}$ respectively the mean vector and the correlation matrix of the probability measure $\mu^N_{\beta(\cdot),\lambda(\cdot)}$, then we have*

$$(m_{\beta(\cdot),\lambda(\cdot)}, C_{\beta(\cdot),\lambda(\cdot)}) \in \Sigma_N \quad \textit{and} \quad \mu^N_{\beta(\cdot),\lambda(\cdot)} = G_{m_{\beta(\cdot),\lambda(\cdot)}, C_{\beta(\cdot),\lambda(\cdot)}} \ . \tag{24}$$

Now we state our second main theorem.

**Theorem 2.** *We assume that the initial probability measure $\mu_0^N$ is a Gibbs local equilibrium state, defined by* (14)*.*

*Then,* (18) *holds, and the conclusions of Theorem 1 are valid.*

In the following, we will denote by $\mathbf{e}_t(\cdot)$, $\mathbf{r}_t(\cdot)$, $\lambda_t(\cdot)$ and $\beta_t(\cdot)$ respectively the functions $q \to \mathbf{e}(t, q)$, $q \to \mathbf{r}(t, q)$, $q \to \lambda(t, q)$, and $q \to \beta(t, q)$ defined on $\mathbb{T}$.

## *2.2 Ergodicity of the Infinite Volume Velocity-Flip Model*

We conclude this part by giving the ergodicity theorem, which is proved in [3], Sects. 2.2 and 2.4.2, by following the ideas of [6]. We define, for all finite subsets $\Lambda \subset \mathbb{Z}$, and for two probability measures $\nu$ and $\mu$ on $\Omega = (\mathbb{R} \times \mathbb{R})^{\mathbb{Z}}$, the restricted relative entropy $H_\Lambda(\nu|\mu) := H(\nu_\Lambda|\mu_\Lambda)$ where $\nu_\Lambda$ and $\mu_\Lambda$ are the marginal distributions of $\nu$ and $\mu$ on $\Omega$. The Gibbs states in infinite volume are the probability measures $\mu_{\beta,\lambda}$ on $\Omega$ given by (16). The formal generator of the infinite dynamics is denoted by $\mathscr{L}$ (respectively $\mathscr{A}$ and $\mathscr{S}$ for the antisymmetric and the symmetric part).

**Theorem 3.** *Let $\nu$ be a probability measure on the configuration space $\Omega$ such that*

1. *$\nu$ has finite density entropy: there exists $C > 0$ such that for all finite subsets $\Lambda$ of $\mathbb{Z}$, $H_\Lambda(\nu|\mu_*) \leqslant C|\Lambda|$, with $\mu_* := \mu_{1,0}$ a reference Gibbs measure on $(\mathbb{R}\times\mathbb{R})^{\mathbb{Z}}$;*
2. *$\nu$ is translation invariant;*
3. *$\nu$ is stationary: for any compactly supported and differentiable function $F(\mathbf{r},\mathbf{p})$,*

$$\int \mathscr{A}(F)\,\mathrm{d}\nu = 0\,; \tag{25}$$

4. *The conditional probability distribution of $\mathbf{p}$ given the probability distribution of $\mathbf{r}$, denoted by $\nu(\mathbf{p}|\mathbf{r})$, is invariant by any flip $\mathbf{p}\to\mathbf{p}^x$, with $x\in\mathbb{Z}$.*

*Then, $\nu$ is a mixture of infinite Gibbs states.*

**Corollary 1.** *If $\nu$ is a probability measure on $\Omega$ satisfying 1, 2 and if $\nu$ is stationary in the sense that: for any compactly supported and differentiable function $F(\mathbf{r},\mathbf{p})$,*

$$\int \mathscr{L}(F)\,\mathrm{d}\nu = 0\,, \tag{26}$$

*then $\nu$ is a mixture of infinite Gibbs states.*

## 3 The Relative Entropy Method

For the sake of simplification, we denote all couples of the form $(\beta(\cdot),\lambda(\cdot))$ by $\chi(\cdot)$. First, we introduce the corrected local Gibbs state $\nu^N_{\chi_t(\cdot)}$ defined by

$$\frac{\mathrm{d}\nu^N_{\chi_t(\cdot)}}{\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{p}} := \frac{1}{Z(\chi_t(\cdot))}\prod_{x\in\mathbb{T}_N}\exp\left(-\beta_t\left(\frac{x}{N}\right)e_x - \lambda_t\left(\frac{x}{N}\right)r_x + \frac{1}{N}F\left(t,\frac{x}{N}\right)\cdot\tau_x h(\mathbf{r},\mathbf{p})\right) \tag{27}$$

where $Z(\chi_t(\cdot))$ is the partition function. Functions $F$ and $h$ should be judiciously chosen, and are explicitly defined in [12].

We are going to use the relative entropy method, with the corrected local Gibbs state $\nu^N_{\chi_t(\cdot)}$ instead of the usual one $\mu^N_{\chi_t(\cdot)}$. We define

$$H_N(t) := H\left(\mu_t^N|\nu^N_{\chi_t(\cdot)}\right) = \int_{\Omega^N} f_t^N(\omega)\log\frac{f_t^N(\omega)}{\phi_t^N(\omega)}\mathrm{d}\mu^N_{1,0}(\omega)\,, \tag{28}$$

where $f_t^N$ is the density of $\mu_t^N$ with respect to the reference measure $\mu^N_{1,0}$. In the same way, $\phi_t^N$ is the density of $\nu^N_{\chi_t(\cdot)}$ with respect to $\mu^N_{1,0}$ (which here is easily computable). The objective is to prove a Gronwall estimate of the entropy production of the form

$$\partial_t H_N(t) \leqslant C\ H_N(t) + o(N)\,, \tag{29}$$

where $C > 0$ does not depend on $N$. In order to prove Theorem 1, we show in [12] that $H_N(t) = o(N)$ and this implies the existence of the hydrodynamic limit in the sense given in the theorem, by using the relative entropy inequality (5). For a proof of this last step, we refer the reader to [3], Proposition 3.3.2. and [8]. Thus, our purpose now is to prove (29).

We begin with the following lemma, proved in [8], Chap. 6, Lemma 1.4 and [4], Sect. 3.2. The operator $\mathscr{L}_N^* = -\mathscr{A}_N + \gamma S_N$ is the adjoint of $\mathscr{L}_N$ in $\mathbb{L}^2(\mu_{1,0}^N)$.

**Lemma 2.**

$$\partial_t H_N(t) \leqslant \int \frac{1}{\phi_t^N} \left(N^2 \mathscr{L}_N^* \phi_t^N - \partial_t \phi_t^N\right) f_t^N \mathrm{d}\mu_{1,0} = \mu_t^N \left[\frac{1}{\phi_t^N} \left(N^2 \mathscr{L}_N^* \phi_t^N - \partial_t \phi_t^N\right)\right].$$

We define $\xi_x := (e_x, r_x)$ and $\eta(t,q) := (\mathbf{e}(t,q), \mathbf{r}(t,q))$. If $f$ is a vectorial function, we denote its differential by $Df$. In [12], we prove that we can choose the correction term to obtain the following technical result.

**Proposition 1.** *The term $(\phi_t^N)^{-1}\left(N^2 \mathscr{L}_N^* \phi_t^N - \partial_t \phi_t^N\right)$ is given by the sum of five terms in which a microscopic expansion up to the first order appears.*

*In other words,*

$$\begin{aligned}(\phi_t^N)^{-1} &\left(N^2 \mathscr{L}_N^* \phi_t^N - \partial_t \phi_t^N\right) \\ &= \sum_{k=1}^{5} \sum_{x \in \mathbb{T}_N} v_k\left(t, \frac{x}{N}\right)\left[J_x^k - H_k\left(\eta\left(t,\frac{x}{N}\right)\right) - (DH_k)\left(\eta\left(t,\frac{x}{N}\right)\right)\cdot\left(\xi_x - \eta\left(t,\frac{x}{N}\right)\right)\right] \\ &\quad + o(N) \qquad (30)\end{aligned}$$

*where*

| $k$ | $J_x^k$ | $H_k(\mathbf{e},\mathbf{r})$ | $v_k(t,q)$ |
|---|---|---|---|
| 1 | $p_x^2 + r_x r_{x-1} + 2\gamma p_x r_{x-1}$ | $\mathbf{e} + \mathbf{r}^2/2$ | $-(2\gamma)^{-1}\partial_q^2\beta(t,q)$ |
| 2 | $r_x + \gamma p_x$ | $\mathbf{r}$ | $-\gamma^{-1}\partial_q^2\lambda(t,q)$ |
| 3 | $p_x^2\,(r_x + r_{x-1})^2$ | $(2\mathbf{e} - \mathbf{r}^2)\left(\mathbf{e} + 3\mathbf{r}^2/2\right)$ | $(4\gamma)^{-1}[\partial_q\beta(t,q)]^2$ |
| 4 | $p_x^2\,(r_x + r_{x-1})$ | $\mathbf{r}\,(2\mathbf{e} - \mathbf{r}^2)$ | $\gamma^{-1}\partial_q\beta(t,q)\,\partial_q\lambda(t,q)$ |
| 5 | $p_x^2$ | $\mathbf{e} - \mathbf{r}^2/2$ | $\gamma^{-1}[\partial_q\lambda(t,q)]^2$ |

(31)

*Remark 3.* Along the proof, the so-called fluctuation-dissipation equations will play a crucial role, in particular for the choice of functions $F, h$.

A priori the first term on the right-hand side of (30) is of order $N$, but we want to take advantage of these microscopic Taylor expansions. First, we need to cut-off large energies in order to work with bounded variables only. Second, the strategy consists in performing a one-block estimate: we replace the empirical truncated current which is averaged over a microscopic box centered at $x$ by its mean with respect to a Gibbs measure with the parameters corresponding to the microscopic averaged profiles. This is achieved thanks to the ergodicity of the dynamics (see Theorem 3). A one-block estimate is performed for each term of the form

$$\sum_{x\in\mathbb{T}_N} v_k\left(t,\frac{x}{N}\right)\left[J_x^k - H_k\left(\eta\left(t,\frac{x}{N}\right)\right) - (DH_k)\left(\eta\left(t,\frac{x}{N}\right)\right)\cdot\left(\xi_x - \eta\left(t,\frac{x}{N}\right)\right)\right]. \tag{32}$$

We deal with error terms by taking advantage of the following equality

$$H_k\left(\eta\left(t,\frac{x}{N}\right)\right) = \nu^N_{\chi_t(x/N)}(J_0^k) \tag{33}$$

and by using the large deviation properties of the probability measure $\nu^N_{\chi_t(\cdot)}$, that locally is almost homogeneous. Along the proof, we will need to control, uniformly in $N$, the quantity

$$\int \sum_{x\in\mathbb{T}_N} \exp\left(\frac{e_x}{N}\right)\, d\mu_t^N. \tag{34}$$

In fact, to get the convenient estimate, it is not difficult to see that it is sufficient to prove (18). For all the details, we refer the reader to [12], where the proof is written following the lines of [3], Sect. 3.3 and inspired from [11].

## 4 Proof of Theorem 2: Moments Bounds

Now we review how to prove the two conditions on the moments bounds for a class of local equilibrium states. Hereafter we assume that the initial law $\mu_0^N$ is the Gibbs local equilibrium state $\mu^N_{\beta_0(\cdot),\lambda_0(\cdot)}$.

We need to control the moments $\mu_t^N\left[\sum_x e_x^k\right]$ for all $k \geqslant 1$. The first two bounds would be sufficient to justify the cut-off of the currents, but here we need more bounds because of the Taylor expansion (Proposition 1). Precisely, the moments bounds are necessary to get the term of order $o(N)$ in the right hand-side of (32). Since the chain is harmonic, Gibbs states are gaussian. Remarkably, all Gaussian moments can be expressed in terms of variances and covariances. We start with an other representation of the dynamics of the process, and then we prove the bounds and describe their dependence on $k$. Let us highlight that, from now on, we consider the process with generator $\mathscr{L}_N$: it is not accelerated any more. The law of this new process $(\tilde{\omega}_t)_{t\geqslant 0}$ is denoted by $\tilde{\mu}_t^N$. Theorem 2 will be easily deduced since all estimates will not depend on $t$, and the following equality still holds: $\mu_t^N = \tilde{\mu}^N_{tN^2}$.

*Remark 4.* 1. In the following, we always respect the decomposition of the space $\Omega^N = \mathbb{R}^N \times \mathbb{R}^N$. Let us recall that the first $N$ components stand for $\mathbf{r}$ and the last $N$ components stand for $\mathbf{p}$. All vectors and matrices are written according to this decomposition. Let $\nu$ be a measure on $\Omega^N$. We denote by $m \in \mathbb{R}^{2N}$ its mean vector and by $C \in \mathfrak{M}_{2N}(\mathbb{R})$ its correlation matrix (see (22)). There exist $\rho := \nu[\mathbf{r}] \in \mathbb{R}^N$ , $\pi := \nu[\mathbf{p}] \in \mathbb{R}^N$ and $U, V, Z \in \mathfrak{M}_N(\mathbb{R})$ such that

$$m = (\rho, \pi) \in \mathbb{R}^{2N} \quad \text{and} \quad C = \begin{pmatrix} U & Z^* \\ Z & V \end{pmatrix} \in \mathfrak{S}_{2N}(\mathbb{R}) \, . \tag{35}$$

Hereafter, we denote by $Z^*$ the transpose of the matrix $Z$.

2. Thanks to the convexity inequality $(a+b)^k \leqslant 2^{k-1}\,(a^k + b^k)$ $(a, b > 0$, $k$ a positive integer), we have

$$e_x^k \leqslant \frac{1}{2}\,\left(p_x^{2k} + r_x^{2k}\right) \, . \tag{36}$$

Thus, instead of proving (18) we can show

$$\mu_t^N\left[\sum_{x\in\mathbb{T}_N} p_x^{2k}\right] \leqslant (Ck)^k \times N \quad \text{and} \quad \mu_t^N\left[\sum_{x\in\mathbb{T}_N} r_x^{2k}\right] \leqslant (Ck)^k \times N \, . \tag{37}$$

### 4.1 Poisson Process and Gaussian Measures

We start by giving a graphical representation of the process $(\tilde{\omega}_t)_{t\geqslant 0}$. Let us define

$$A := \begin{pmatrix}
0 & \cdots & \cdots & 0 & -1 & 1 & & (0) \\
\vdots & & & \vdots & 0 & \ddots & \ddots & \\
\vdots & & & \vdots & 0 & & \ddots & 1 \\
0 & \cdots & \cdots & 0 & 1 & 0 & 0 & -1 \\
1 & 0 & 0 & -1 & 0 & \cdots & \cdots & 0 \\
-1 & \ddots & & 0 & \vdots & & & \vdots \\
 & \ddots & \ddots & 0 & \vdots & & & \vdots \\
(0) & & -1 & 1 & 0 & \cdots & \cdots & 0
\end{pmatrix} \in \mathfrak{M}_{2N}(\mathbb{R}) \, . \tag{38}$$

We consider the Markov process $(m_t, C_t)_{t\geqslant 0}$ on $\mathbb{R}^{2N} \times \mathfrak{S}_{2N}(\mathbb{R})$ defined by its generator $\mathscr{G}$, which can be written as follows.

Take $m = (\rho, \pi) \in \mathbb{R}^{2N}$ and $C = \begin{pmatrix} U & Z^* \\ Z & V \end{pmatrix} \in \mathfrak{S}_{2N}(\mathbb{R})$, with two vectors $\rho, \pi \in \mathbb{R}^N$ and three matrices $U, V \in \mathfrak{S}_N(\mathbb{R})$, $Z \in \mathfrak{M}_N(\mathbb{R})$. The generator $\mathscr{G}_N$ is given by

$$(\mathscr{G}_N v)(m, C) := (\mathscr{K}_N v)(m, C) + \gamma\, (\mathscr{H}_N v)(m, C)\,, \tag{39}$$

where

$$\mathscr{K}_N := \sum_{i,j\in\mathbb{T}_N} (-AC + CA)_{i,j}\ \partial_{C_{i,j}} + \sum_{i\in\mathbb{T}_N} \left\{(\pi_{i+1} - \pi_i)\partial_{\rho_i} + (\rho_i - \rho_{i-1})\partial_{\pi_i}\right\}\,, \tag{40}$$

$$(\mathscr{H}_N v)(m, C) := \frac{1}{2}\sum_{k\in\mathbb{T}_N} [v(m^k, C^k) - v(m, C)]\,. \tag{41}$$

In these formulas, we define $m^k := (\rho, \pi^k)$ and $C^k := \Sigma_k^* \cdot C \cdot \Sigma_k = \begin{pmatrix} U & Z^{k*} \\ Z^k & V^k \end{pmatrix}$, where $\pi^k$ is the vector obtained from $\pi$ by the flip of $\pi_k$ into $-\pi_k$, and $\Sigma_k$ is

$$\Sigma_k := \begin{pmatrix} I_n & 0_n \\ 0_n & I_n - 2E_{k,k} \end{pmatrix}\,. \tag{42}$$

Here, $E_{i,j}$ denotes the $(n,n)$-matrix which has only one non-zero entry, the component $(i, j)$, equal to 1.

We denote by $\mathbb{P}_{m_0,C_0}$ the law of the process $(m_t, C_t)_{t\geqslant 0}$ starting from $(m_0, C_0)$, and by $\mathbb{E}_{m_0,C_0}[\cdot]$ the expectation with respect to $\mathbb{P}_{m_0,C_0}$. For $t \geqslant 0$ fixed, let $\theta^t_{m_0,C_0}(\cdot,\cdot)$ be the law of the random variable $(m_t, C_t) \in \mathbb{R}^{2N} \times \mathfrak{S}_{2N}(\mathbb{R})$, knowing that the process starts from $(m_0, C_0)$.

The following lemma, which is proved in [12], gives the link between the two Markov processes defined in this paper. The proof is based on the Harris description.

**Lemma 3.** *Let $\mu_0^N := \mu^N_{\beta_0(\cdot),\lambda_0(\cdot)}$ be the Gibbs equilibrium state defined by (14), where $\lambda_0(\cdot)$ and $\beta_0(\cdot)$ are two macroscopic potential profiles.*

*Then,*

$$\tilde{\mu}_t^N = \int G_{m,C}(\cdot)\, d\theta^t_{m_0,C_0}(m, C)\,, \tag{43}$$

*where the components of $(m_0, C_0) \in \Sigma_N$ can be explicitly expressed (and depend on $\lambda_0$ and $\beta_0$).*

*Remark 5.* Observe that we have, from (43),

$$\tilde{\mu}_t^N[p_x] = \int G_{m,C}(p_x)\, d\theta^t_{m_0,C_0}(m, C) = \int \pi_x\, d\theta^t_{m_0,C_0}(m, C) = \mathbb{E}_{m_0,C_0}[\pi_x(t)]\,, \tag{44}$$

$$\tilde{\mu}_t^N[r_x] = \int G_{m,C}(r_x)\, d\theta^t_{m_0,C_0}(m, C) = \int \rho_x\, d\theta^t_{m_0,C_0}(m, C) = \mathbb{E}_{m_0,C_0}[\rho_x(t)]\,. \tag{45}$$

Finally, thanks to the Harris description and Lemma 3, it is proved in [12] that we can control the quantities $\pi_y(t)$ and $\rho_y(t)$ for all $t > 0$. More precisely,

**Lemma 4.** *Let $(m_t, C_t)_{t\geqslant 0}$ be the Markov process defined above. As previously done, we introduce $\rho(t), \pi(t) \in \mathbb{R}^N$ and $U(t), V(t), Z(t) \in \mathfrak{M}_N(\mathbb{R})$ such that*

$$m_t = (\rho(t), \pi(t)) \ \textit{and} \ C_t = \begin{pmatrix} U(t) & Z^*(t) \\ Z(t) & V(t) \end{pmatrix} . \tag{46}$$

*Then,*

$$\mathbb{P}_{m_0,C_0} \text{ - a. s. }, \ \forall\, t \geqslant 0, \ \begin{cases} \pi_y^2(t) \leqslant V_{y,y}(t) \, , \\ \rho_y^2(t) \leqslant U_{y,y}(t) \, . \end{cases} \tag{47}$$

## 4.2 Evolution of the Process $(m_t, C_t)_{t\geqslant 0}$

According to Lemma 4, the problem is reduced to estimate $U_{y,y}(t)$ and $V_{y,y}(t)$ for $t > 0$. Thanks to the regularity of $\beta_0$ and $\lambda_0$, we know that there exists a constant $K$ which does not depend on $N$ such that

$$\frac{1}{N} \sum_i \left[ (U_{i,i})^k(0) + (V_{i,i})^k(0) \right] \leqslant K^k, \text{ for all } k \geqslant 1 \, . \tag{48}$$

It is easy to see that the above inequality is uniform in $t > 0$, in the sense that

$$\frac{1}{N} \mathbb{E}_{m_0,C_0} \left[ \sum_i \left[ (U_{i,i})^k(t) + (V_{i,i})^k(t) \right] \right] \leqslant K^k, \text{ for all } k \geqslant 1 \, . \tag{49}$$

We are going to see how this last inequality can be used to show (18). We denote by $u_k(t)$ and $v_k(t)$ the two quantities

$$u_k(t) := \mathbb{E}_{m_0,C_0} \left[ \sum_{i \in \mathbb{T}_N} U_{i,i}^k(t) \right], \quad v_k(t) := \mathbb{E}_{m_0,C_0} \left[ \sum_{i \in \mathbb{T}_N} V_{i,i}^k(t) \right] . \tag{50}$$

Let us make the link with (18). We are going to focus on $u_k(t)$. The same ideas work for $v_k(t)$. In view of (43), we can write

$$\tilde{\mu}_t^N \left[ p_y^{2k} \right] = \int G_{m,C} \left[ p_y^{2k} \right] \, \mathrm{d}\theta_{m_0,C_0}^t(m, C) \, . \tag{51}$$

We use the convexity inequality to get

$$\tilde{\mu}_t^N \left[ p_y^{2k} \right] = \int G_{m,C} \left[ (p_y - \pi_y + \pi_y)^{2k} \right] \, \mathrm{d}\theta_{m_0,C_0}^t(m, C)$$
$$\leqslant 2^{2k-1} \left[ \int G_{m,C} \left[ (p_y - \pi_y)^{2k} \right] \, \mathrm{d}\theta_{m_0,C_0}^t(m, C) + \int \pi_y^{2k} \, \mathrm{d}\theta_{m_0,C_0}^t(m, C) \right] .$$

We deal with the two terms of the sum separately. First, observe that Gaussian centered moments are easily computable:

$$G_{m,C} \left[ (p_y - \pi_y)^{2k} \right] = \left( V_{y,y} - \pi_y^2 \right)^k \frac{(2k)!}{k!\, 2^k} . \tag{52}$$

Hence,

$$\sum_{y \in \mathbb{T}_N} \int \left( V_{y,y} - \pi_y^2 \right)^k \frac{(2k)!}{k!\, 2^k} \, d\theta_{m_0,C_0}^t(m, C) \leqslant \frac{(2k)!}{k!\, 2^k} \left( v_k(t) + \mathbb{E}_{m_0,C_0} \left[ \sum_{y \in \mathbb{T}_N} \pi_y^{2k}(t) \right] \right) . \tag{53}$$

Lemma 4 shows that

$$\mathbb{E}_{m_0,C_0} \left[ \sum_{y \in \mathbb{T}_N} \pi_y^{2k}(t) \right] \leqslant \mathbb{E}_{m_0,C_0} \left[ \sum_{y \in \mathbb{T}_N} V_{y,y}^k(t) \right] = v_k(t) . \tag{54}$$

As a result,

$$\sum_y \tilde{\mu}_t^N \left[ p_y^{2k} \right] \leqslant \frac{(2k)!}{k!} \, v_k(t) \sim 2 \left( \frac{4}{e} \right)^k k^k \, v_k(t) . \tag{55}$$

As a result, in order to get (18) we need to estimate the two quantities $u_k(t)$ and $v_k(t)$, which are related to $C_t$. In [12], the following final technical lemma is proved.

**Lemma 5.** *For any integer $k$ not equal to 0, there exists a positive constant $K$ which does not depend on $N$ and $t$ such that*

$$\begin{cases} v_k(t) \leqslant K^k \, N \, , \\ u_k(t) \leqslant K^k \, N \, . \end{cases} \tag{56}$$

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## References

1. Basile, G., Bernardin, C., Olla, S.: Thermal conductivity for a momentum conservative model. Commun. Math. Phys. **287**, 67–98 (2009)
2. Bernardin, C.: Hydrodynamics for a system of harmonic oscillators perturbed by a conservative noise. Stoch. Process. Appl. **117**, 487–513 (2007)
3. Bernardin, C., Olla, S.: Non-equilibrium macroscopic dynamics of chains of anharmonic oscillators. Available at http://www.ceremade.dauphine.fr/olla (2011, in preparation)
4. Bernardin, C., Olla, S.: Transport properties of a chain of anharmonic oscillators with random flip of velocities. J. Stat. Phys. **145**, 1124–1255 (2011)
5. Bernardin, C., Kannan, V., Lebowitz, J.L., Lukkarinen, J.: Harmonic systems with bulk noises. Eur. Phys. J. B. **84**, 685–689 (2011)
6. Fritz, J., Funaki, T., Lebowitz, J.L.: Stationary states of random Hamiltonian systems. Probab. Theory Relat. Fields **99** 211–236 (1994)
7. Funaki, T., Uchiyama, K., Yau, H.T.: Hydrodynamic limit for lattice gas reversible under Bernoulli measures, Nonlinear stochastic PDEs. IMA Vol. Math. Appl. **77**, 1–40 (1996)
8. Kipnis, L., Landim, C.: Scaling limits of interacting particle systems. Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences). Springer, Berlin (1999)
9. Landim, C., Yau, H.T.: Fluctuation-dissipation equation of asymmetric simple exclusion processes. Probab. Theory Relat. Fields **108**, 321–356 (1997)
10. Landim, C., Sued, M., Valle, G.: Hydrodynamic limit of asymmetric exclusion processes under diffusive scaling in $d \geq 3$. Commun. Math. Phys. **249**, 215–247 (2004)
11. Olla, S., Varadhan, S.R.S., Yau, H.T.: Hydrodynamical limit for a Hamiltonian system with weak noise. Commun. Math. Phys. **155**, 523–560 (1993)
12. Simon, M.: Hydrodynamic limit for the velocity-flip model. Stoch. Process. Appl. **123**, 3623–3662 (2013)
13. Tremoulet, C.: Hydrodynamic limit for interacting Ornstein-Uhlenbeck particles. Stoch. Process. Appl. **102**, 139–158 (2002)
14. Yau, H.T.: Relative entropy and hydrodynamics of Ginzburg-Landau models. Lett. Math. Phys. **22**, 63–80 (1991)

# Large Number Asymptotics for Two-Component Systems with Self-Consistent Coupling

**Valeria Ricci**

## 1 Introduction

The large number limit of systems with particle components involving a self-consistent structure is one of the most common problems met in applications when dealing with approximations of equations by particle systems.

In this paper we shall shortly describe two models for two-component mixtures where the self-consistent structure arises and we shall illustrate how different techniques can be applied in order to derive macroscopic limits for the associated particle systems. The models have been studied in [1, 7] and they are on purpose chosen as simple as possible, although they are inspired by more complex ones, used in a wide set of contexts (porous media, radiative transfer or, more in general, various systems in the presence of chemical reactions).

The first model is a semi-deterministic system (with respect to the interaction among particles) where at the microscopic scale both components are particle-like.

The second one is a system having only one particle-like component, while the other component is in a fluid state (i.e. somehow the macroscopic limit for this component has already been performed): this is a deterministic system, in the sense that a unique stochastic element comes from the initial distributions of the particle-like component.

The macroscopic limit will be obtained using different techniques, because of the different nature of components; both techniques are nevertheless of some interest for the asymptotics of the coupling term.

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## 2 The Semi-Deterministic Particle System

The first system we shall consider, in dimension $d \geq 2$, is a binary semi-deterministic system consisting in $n$ point like (light) particles, with initial position $x_i \in \mathbb{R}^d$ and velocity $v_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, moving uniformly among fixed, spherical obstacles of radius $\varepsilon$ whose centres $c_k \in \mathbb{R}^d$, $k = 1, 2, \ldots$, are Poisson distributed with parameter $\mu_\varepsilon$. The particle system is linear, in the sense that particles not belonging to the same species do not interact among themselves, and particles of different species interact in the following way: a light particle is removed from the system at the first time it meets an obstacle and an obstacle is removed from the system in a stochastic interval of time whose size is connected, through a local mean-field type interaction, to the number of light particles travelling within the area of detection (range) of the obstacle.

As far as the absorption self-consistent coupling is concerned, the system combines two type of interactions which have been analysed separately for one species systems in reaction–diffusion equations (see [10] for the deterministic case, [8] for the local mean-field type interaction), and which give here a system where different species of particles interact in a non symmetric way.

More precisely, denoting by $t \in \mathbb{R}_+$ the time variable, by $B_r(p)$ the ball of radius $r$ centred in $p \in \mathbb{R}^d$, and, for a given $z = (x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, by $T^t(z) = (T_1^t(z), T_2^t(z)) = (x + vt, v)$, $\quad t \in \mathbb{R}_+$, the free flow associated to the light particle with initial position in the phase space $z$, while $x(t) = T_1^t(z)$, $t \in \mathbb{R}_+$, is its trajectory, we may describe the evolution of the system through the following (interdependent) functions (defined for $n$ light particles with initial positions in the phase space $z_i$, $i = 1, \ldots, n$, moving among $M$ obstacles).

The lifetime of an obstacle located in $c$ is fixed through the *risk function* at a given position $c$, which defines the local mean-field type interaction between light particles and obstacles:

$$V_{n,\varepsilon,M}(t, c) = \frac{1}{n} \sum_{i=1}^{n} \int_0^t ds q_n(x_i(s) - c)\xi_{n,\varepsilon,M}(s, z_i), \tag{1}$$

where $q_n(x) = a_n^d q(a_n x)$, is an (at least continuous) approximant, up to a multiplicative constant, of the Dirac delta distribution ($q \geq 0$ is a radial function s.t. $\int_{\mathbb{R}^d} q(y)dy = \Theta > 0$ and $\lim_{n\to\infty} a_n = \infty$) and $\xi_{n,\varepsilon,M}$ is the stochastic function associated to the light particles defined below.

The evolution of both particles components is described through their *life functions*, resp. for a light particle with initial position and velocity $(x, v)$

$$\xi_{n,\varepsilon,M}(t, z) = I_{\{x(s) \notin \bigcup_{h\in\{1,\ldots,M\}:\eta_{n,\varepsilon,M}(s,c_h)=1} B_\varepsilon(c_h) \quad \forall s\in[0,t)\}} \qquad M \geq 1 \tag{2}$$

(in the absence of obstacles, $\xi_{n,\varepsilon,0}(t,z) = 1$) and for an obstacle centred in $c_k \in \{c_1,\dots,c_M\}$,

$$\eta_{n,\varepsilon,M}(t,c_k) = I_{\{V_{n,\varepsilon,M}(t,c_k)<\tau_k\}}, \tag{3}$$

where $\tau_1, \tau_2, \dots$ are independently distributed exponential variables.

We want to study this system in the large $n$ limit, when the mean free path of the light particles is kept finite (so that the interaction between light particles and obstacles does not disappear in the limit), and in particular we want to analyse the Markovian limit, i.e. the one where correlations are absent (cf. [9] for a short review concerning linear particle systems).

As discussed in [1], the proper scaling in this case, when in addition the radius $\varepsilon$ of the obstacles vanishes, is selected by choosing the Poisson parameter such that $\mu_\varepsilon = \mu\varepsilon^{1-d}$, $\mu > 0$, together with $a_n^d n^{-\frac{1}{2}} < C$ (here and in what follows, $C$ will denote constants whose value is not relevant to the estimates), so to keep the mean free path finite; moreover, in order to avoid the presence of correlations, we have to assume $\varepsilon^\zeta = o(a_n^{-d})$ when $n \to \infty$ for some $\zeta \in (0, \frac{1}{2} - \frac{1}{2d})$, so connecting the scalings of $\varepsilon$ and $n$, together with $\lim_{n\to\infty} \frac{a_n^d}{n^\kappa} = 0$ for some $\kappa \in (0, \frac{1}{2})$, as a technical requirement due to the singularity in the limit of the risk function.

The quantities associated to the densities of the two species of particles are then ($\delta_x$ denotes the Dirac delta distribution)

$$\mu_n(t, dx, dv; \mathbf{z}_n, \mathbf{c}_M) = \frac{1}{n}\sum_{i=1}^{n} \delta_{T^t(z_i)}(dx, dv)\xi_{n,\varepsilon,M}(t, z_i) \tag{4}$$

and

$$\sigma_n(t, dx; \mathbf{z}_n, \mathbf{c}_M) = \varepsilon^{d-1}\sum_{k=1}^{M} \delta_{c_k}(dx)\eta_{n,\varepsilon,M}(t, c_k). \tag{5}$$

The guessed limit system of partial differential equations for the two densities $f = f(t,x,v)$ and $\sigma = \sigma(t,x)$, $t \in [0,T]$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$, associated resp. to the light particles and the obstacles, assuming the before mentioned requirements are fulfilled, is given by the system:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = -C_d|v|\sigma f \\ \partial_t\sigma = -\Theta(\int dv f)\sigma \\ f(0,x,v) = f_0(x,v) \\ \sigma(0,x) = \mu, \end{cases} \tag{6}$$

where $C_d = \int_{\{\omega\in\mathbb{R}^d:|\omega|=1\}} |N\cdot\omega|d\omega$ is a constant (here $N$ is a generic unit vector, $|N| = 1$, and, as can be easily checked, $C_d$ is independent of it). As discussed in [1], we choose $f_0 \geq 0$ such that $f_0 \in L^1(\mathbb{R}^d_v; W^{1,\infty}(\mathbb{R}^d_x))$, $v f_0 \in L^1(\mathbb{R}^d_v; W^{1,\infty}(\mathbb{R}^d_x)) \cap L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_v)$, $v^2 f_0 \in L^1(\mathbb{R}^d_v; L^\infty(\mathbb{R}^d_x))$, so to have existence and uniqueness for the solution to the system.

In spite of the extreme simplicity of the limit system, the rigorous proof of this limit is quite complicate, because of the self-consistent structure of (1), (2), (3), which does not allow to express in explicit form, or in a form easy to compare with other functions, the functions describing the particle system.

Nevertheless, since one of the components interacts with the other through a (stochastic) mean-field type interaction, although a sufficiently careful analysis of the light particles trajectories in the style of [2, 4, 5] is still needed, we can apply a strategy similar to the one used in [8].

The steps in the procedure are then the following:

## 2.1 Step 1: Elimination of the Self-Consistent Structure

In order to eliminate the self-consistent structure of (6) and (1), (2), (3), we define two sequences of linear systems.

The first one, approximating (6), is defined as

$$f^{(0)}(t,x,v) = f_0(x - vt, v), \quad \sigma^{(0)}(t,x) = \mu$$

$$\begin{cases} \partial_t f^{(k)} + v \cdot \nabla_x f^{(k)} = -C_d |v| \sigma^{(k-1)} f^{(k)} \\ \partial_t \sigma^{(k)} = -(\Theta \int_{\mathbb{R}^d} dv f^{(k-1)}) \sigma^{(k)} \\ f(0,x,v) = f_0(x,v) \\ \sigma(0,x) = \mu \end{cases} \qquad k = 1 \dots \ , \tag{7}$$

and we have $\lim_{k\to\infty} \| f - f^{(k)} \|_{L^\infty([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)} = 0$, $\lim_{k\to\infty} \|\sigma - \sigma^{(k)}\|_{L^\infty([0,T]\times\mathbb{R}^d)} = 0$.

The second one, (formally) approximating the particle system (1), (2), (3), is defined resp., for integers $M, k \geq 1$, $j = 1, \dots, n$ and $\ell = 1, \dots, M$, as:

$$V^{(0)}_{n,\varepsilon,M}(t, c_\ell) = 0, \qquad V^{(k)}_{n,\varepsilon,M}(t, c_\ell) = \frac{1}{n} \sum_{i=1}^{n} \int_0^t ds q_n(x_i(s) - c_\ell) \xi^{(k-1)}_{n,\varepsilon,M}(s, z_i) \tag{8}$$

$$\xi^{(0)}_{n,\varepsilon,M}(t, z_j) = 1, \qquad \xi^{(k)}_{n,\varepsilon,M}(t, z_j) = I_{\{x_j(s) \notin \bigcup_{h\in\{1,\dots,M\}:\eta^{(k-1)}_{n,\varepsilon,M}(s,c_h)=1} B_\varepsilon(c_h)\ \forall \in [0,t)\}} \tag{9}$$

and

$$\eta^{(0)}_{n,\varepsilon,M}(t, c_\ell) = 1, \qquad \eta^{(k)}_{n,\varepsilon,M}(t, c_\ell) = I_{\{V^{(k)}_{n,\varepsilon,M}(t,c_\ell) < \tau_\ell\}} \tag{10}$$

(in the absence of obstacles $\xi^{(0)}_{n,\varepsilon,0}(t,z_j) = \xi^{(k)}_{n,\varepsilon,0}(t,z_j) = 1$), and is a linearization of the particle system (1), (2), (3) in the same spirit of the linearization (7) of (6), i.e. where, for each $k$, the two components evolve in the field associated to the functions defined at the previous step in the sequence, $k-1$, which is given.

The relevant property of the approximating system defined by (8), (9), (10) is the so called *sandwiching* property ([8]), i.e., for $k = 1, 2 \ldots$,

$$\begin{aligned}
\xi^{(2k-1)}_{n,\varepsilon,M} \le \xi^{(2k+1)}_{n,\varepsilon,M} \le \xi_{n,\varepsilon,M} \le \xi^{(2k)}_{n,\varepsilon,M} \le \xi^{(2k-2)}_{n,\varepsilon,M}\\
V^{(2k-2)}_{n,\varepsilon,M} \le V^{(2k)}_{n,\varepsilon,M} \le V_{n,\varepsilon,M} \le V^{(2k+1)}_{n,\varepsilon,M} \le V^{(2k-1)}_{n,\varepsilon,M}\\
\eta^{(2k-1)}_{n,\varepsilon,M} \le \eta^{(2k+1)}_{n,\varepsilon,M} \le \eta_{n,\varepsilon,M} \le \eta^{(2k)}_{n,\varepsilon,M} \le \eta^{(2k-2)}_{n,\varepsilon,M}.
\end{aligned} \tag{11}$$

This property, once fixed a given asymptotics for $n$ and $\varepsilon$, has the following implications for the expectation values with respect to the centres distribution (restricted to a volume including all light particle trajectories up to time $T$) and the exponential variables $\tau_1, \tau_2 \ldots$ denoted as $\mathbb{E}^n$:

$$\begin{aligned}
&\mathbb{E}^n |V^{(k)}_{n,\varepsilon,M} - \bar{V}^{(k)}| \to 0 &&\Longrightarrow \mathbb{E}^n |V_{n,\varepsilon,M} - V_L| \to 0\\
&\mathbb{E}^n |\xi^{(k)}_{n,\varepsilon,M} - \bar{\xi}^{(k)}| \to 0 &&\Longrightarrow \mathbb{E}^n \mathbb{E}_{\tau_p} |\xi_{n,\varepsilon,M} - \xi_L| \to 0\\
&\mathbb{E}^n \mathbb{E}_{\tau} |\eta^{(k)}_{n,\varepsilon,M} - \bar{\eta}^{(k)}| \to 0 &&\Longrightarrow \mathbb{E}^n \mathbb{E}_{\tau} |\eta_{n,\varepsilon,M} - \eta_L| \to 0\\
&\mathbb{E}^n |\textstyle\int dz \mu_n \phi(\xi^{(k)}_{n,\varepsilon,M} - \bar{\xi}^{(k)})| \to 0 &&\Longrightarrow \mathbb{E}^n \mathbb{E}_{\tau_p} |\textstyle\int dz \mu_n \phi(\xi_{n,\varepsilon,M} - \xi_L)| \to 0,\\
& && \quad \times \phi \in C_b(\mathbb{R}^d \times \mathbb{R}^d)
\end{aligned} \tag{12}$$

where the risk function $V_L(t,c) = \Theta \int_0^t ds \int_{\mathbb{R}^d} dv\, f(s,c,v)$ and the life functions $\xi_L(t,z) = I_{\{C_d |T_2^t(z)| \int_0^t ds \sigma(s,T_1^s(z)) < \tau_p\}}$, $\eta_L(t,c) = I_{\{V_L(t,c)<\tau\}}$ (for $\tau$ and $\tau_p$ exponentially distributed variables) and, with analogous definitions, the risk function $\bar{V}^{(k)}$ and the life functions $\bar{\xi}^{(k)}$, $\bar{\eta}^{(k)}$ are resp. the risk functions and the life functions associated to the limit system (6) and to its approximating system (7). We recall that the solution to the system (6) can be expressed as $f(t,x,v) = f_0(x - vt, v)\mathbb{E}_{\tau_p}[\xi_L(t, T^{-t}(x,v))]$, $\sigma(t,x) = \mu \mathbb{E}_\tau[\eta_L(t,x)]$, and analogous expressions hold for the solution to (7).

These implications allow to bypass the direct evaluation of quantities related to the particle system (1), (2), (3).

## 2.2 Step 2: Reduction of Correlations

Once eliminated the self-consistent structure, still the system defined by (8), (9), (10) keeps strong correlations for $\varepsilon$ finite between light particles and obstacles.

We can further reduce correlations by defining an intermediary system which can be proved to be asymptotically equivalent to both (7) and (8), (9), (10).

For integers $M, k \geq 1$, $j = 1, \dots, n$ and $\ell = 1, \dots, M$, the intermediate system is defined in the following way

$$\hat{A}^{(k)}_{n,\varepsilon,M}(t, c_\ell) = \frac{1}{n}\sum_{i=1}^{n}\int_0^t ds\, q_n(x_i(s) - c_\ell)\hat{\xi}^{(k-1)}_{n,\varepsilon,M}(s, z_i) \tag{13}$$

$$\hat{\xi}^{(0)}_{n,\varepsilon,M}(t, z_j) = 1, \quad \hat{\xi}^{(k)}_{n,\varepsilon,M}(t, z_j) = I_{\{x_j(s)\notin \bigcup_{h\in\{1,\dots,M\}: I_{\{\bar{V}^{(k-1)}(s,c_h)<\tau_h\}}=1} B_\varepsilon(c_h) \quad \forall s\in[0,t)\}} \tag{14}$$

$$\hat{\eta}^{(k)}_{n,\varepsilon,M}(t, c_\ell) = I_{\{\hat{A}^{(k)}_{n,\varepsilon,M}(t,c_\ell)<\tau_\ell\}} \tag{15}$$

(and in the absence of obstacles $\hat{\xi}^{(0)}_{n,\varepsilon,0}(t, z_j) = \hat{\xi}^{(k)}_{n,\varepsilon,0}(t, z_j) = 1$) where $\tau_k$, $1 \leq k \leq M$, are independent exponentially distributed times.

Here, the life functions of light particles at level $k$ are defined through fictitious obstacle life functions corresponding to the life functions at level $k-1$ associated to the system (7), while the life functions of the obstacles at level $k$ is defined through the life functions of particles for the same system but at the level $k-1$

This system with reduced correlations can be proved to converge in quadratic mean with respect to $\mathbb{E}^n$ to both (7) and (8), (9), (10): the first convergence can be proved separately for the particle component and the obstacle component, while the second convergence is more tricky, requiring estimates based on proving that deleting or adding a finite number of particles or obstacles does not modify the limit.

We resume schematically the relations among the five systems introduced thus far in Fig. 1.

## 2.3 Step 3: Convergence of (8), (9), (10) to (7)

Once we obtain the convergence of the low-correlated system defined by (13), (14), (15) on one side to (8), (9), (10) and on the other to (7), we can easily prove, by triangular inequalities, the convergence of the quantities on the left-hand side of (12). We can then write:

$$\xi_{n,\varepsilon,M} = (\xi_{n,\varepsilon,M} - \xi_L) + (\xi_L - \bar{\xi}^{(k)}) + \bar{\xi}^{(k)}$$

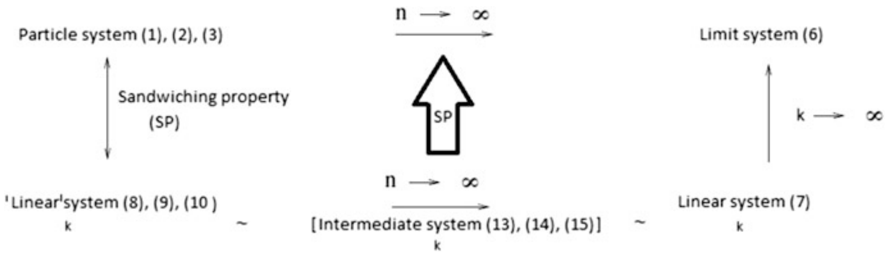


**Fig. 1** Schematic connection between the original systems, (1), (2), (3) and (6), and their asymptotic relation and the auxiliary systems defined in Step 1, (8), (9), (10) and (7), and Step 2, (13), (14), (15)

and

$$\eta_{n,\varepsilon,M} = (\eta_{n,\varepsilon,M} - \eta_L) + (\eta_L - \bar{\eta}^{(k)}) + \bar{\eta}^{(k)}$$

and of course $f = (f - f^{(k)}) + f^{(k)}$. and $\sigma = (\sigma - \sigma^{(k)}) + \sigma^{(k)}$, and prove, thanks to the previously obtained convergences, that for all $\phi \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, $\psi \in C_K(\mathbb{R}^d)$ and for all $k \geq 1$

$$\lim_{n\to\infty} \mathbb{E}^n \left[ \Big| \frac{1}{n} \sum_{i=1}^{n} \phi(T^t(z_i))\xi_{n,\varepsilon,M}(t, z_i) - \int_{\mathbb{R}^d \times \mathbb{R}^d} dxdv\phi(x, v) f(t, x, v) \Big| \right] = 0$$

and

$$\lim_{n\to\infty} \mathbb{E}^n \left[ \Big| \varepsilon^{d-1} \sum_{i=1}^{M} \psi(c_i)\eta_{n,\varepsilon,M}(t, c_i) - \int_{\mathbb{R}^d} dx\psi(x)\sigma(t, x) \Big| \right] = 0$$

so that, under some technical hypothesis and denoting $\mathscr{L}^d$ the Lebesgue measure in $\mathbb{R}^d$, we can prove the following weak law of large numbers for (4) and (5) (Theorem 2.1 in [1]):

**Theorem 1.** *Consider the non-negative functions $f_0$ and $q$ and assume*

- $f_0 \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d) \bigcap L^1(\mathbb{R}^d_v; W^{1,\infty}(\mathbb{R}^d_x))$ *is a probability density such that* $\int_{\mathbb{R}^d} dv f_0 \in \mathscr{S}(\mathbb{R}^d)$ *with* $\int_{\mathbb{R}^d} dv f_0(0, v) > 0$, *and* $v f_0 \in L^1(\mathbb{R}^d_v; W^{1,\infty}(\mathbb{R}^d_x))$, $v^2 f_0 \in L^1(\mathbb{R}^d_v; L^\infty(\mathbb{R}^d_x))$;
- $q$ *is a radial function s.t.* $q \in \mathscr{S}(\mathbb{R}^d)$ *and* $\int_{\mathbb{R}^d} dx q(x) = \Theta > 0$;
- $\{a_n\}_{n=1}^\infty$ *is such that* $a_n > 0$, $\lim_{n\to\infty} a_n = \infty$ *and there exists some* $\kappa \in (0, \frac{1}{2})$ *such that*

$$\lim_{n\to\infty} \frac{a_n^d}{n^\kappa} = 0; \tag{16}$$

- $\{\varepsilon\}_{n=1}^{\infty} = \{\varepsilon_n\}_{n=1}^{\infty}$ *s.t.* $\varepsilon_n > 0$ *and*

$$\lim_{n\to\infty} a_n^d \varepsilon^{\zeta} = 0 \tag{17}$$

*for some* $\zeta \in (0, \frac{1}{2} - \frac{1}{2d})$.

*Then, denoting as* $\mathscr{P}$ *the (infinite product) probability measure defined on the space of infinite sequences* $(\mathbb{R}^d \times \mathbb{R}^d)^{\infty}$ *by* $f_0$*,* $\mathscr{P}$*-almost everywhere w.r.t. sequences of initial data* $\{z_i\}_{i=1}^{\infty}$ *and in probability w.r.t. the Poisson distribution of centres and to the probability distribution associated to* $\tau_1, \tau_2, \ldots$*, when* $n \to \infty$

$$\mu_n(t, dx, dv; \mathbf{z}_n, \boldsymbol{c}_M) \rightharpoonup f(t, x, v)\mathscr{L}^{2d} \tag{18}$$

$$\sigma_n(t, dx; \mathbf{z}_n, \boldsymbol{c}_M) \overset{*}{\rightharpoonup} \sigma(t, x)\mathscr{L}^{d}, \tag{19}$$

*where* $(f, \sigma)$ *is the unique solution to*

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = -C_d |v| \sigma f \\ \partial_t \sigma = -\Theta(\int dv f)\sigma \\ f(0, x, v) = f_0(x, v) \\ \sigma(0, x) = \mu. \end{cases} \tag{20}$$

Here $\mathscr{S}(\mathbb{R}^n)$ denotes the space of $C^{\infty}(\mathbb{R}^n)$ functions of rapid decay at infinity, $\rightharpoonup$ the weak convergence, or convergence in law, in the space of finite measures, while $\overset{*}{\rightharpoonup}$ denotes the *-weak, or vague, convergence on the space of Radon measures.

We pass now to describe a two-component system in a different setting, and other techniques needed to study its asymptotic behaviour.

## 3 The Fluid-Particle System

The second system we shall consider is rather different from the previous one.

In an open bounded domain $\Omega \subset \mathbb{R}^3$ (but, with suitable modifications in the arguments, proofs can be adapted to the case of unbounded domains), we consider a two-phase system, where a background material (continuous component) surrounds $N$ very small spherical inclusions of radius $\epsilon$ (discrete component) located at the positions $x_1, \ldots, x_N \in \mathbb{R}^3$: the variable describing the system satisfies the heat equation in the volume occupied by the continuous component, while each small sphere has its own temperature, constant on the volume of the sphere but varying with time. The interactions between the two components takes place through the boundary conditions on the border of the inclusions.

Using the same notation as in the previous section, the system is defined by the following problem

$$\begin{cases} \partial_t T_\epsilon(t,x) = \sigma \Delta_x T_\epsilon(t,x)\,, & x \in \Omega \setminus \bigcup_{i=1}^N B_\epsilon(x_i)\,,\ t>0\,, \\ \dfrac{\partial T_\epsilon}{\partial n}(t,x) = 0\,, & x \in \partial\Omega\,,\ t>0\,, \\ T_\epsilon(t,x) = T_{i,\epsilon}(t)\,, & x \in \partial B_\epsilon(x_i)\,,\ t>0\,,\ 1 \le i \le N\,, \\ \dot{T}_{i,\epsilon}(t) = \dfrac{\sigma'}{\epsilon}\displaystyle\int_{\partial B_\epsilon(x_i)} \dfrac{\partial T_\epsilon}{\partial n}(t,x)dS(x)\,, & t>0\,,\ 1 \le i \le N\,, \\ T_\epsilon(0,x) = T_\epsilon^{in}(x)\,, & x \in \Omega\,. \end{cases} \tag{21}$$

where $\sigma$ and $\sigma'$ are positive constants, $dS$ is the surface element on $\partial B$ and

$$T_\epsilon^{in} \in \mathscr{H}_N = \Big\{ u \in L^2(\Omega) | \\ u(x) = \frac{1}{|B_\epsilon(x_i)|}\int_{B_\epsilon(x_i)} u(y)dy \text{ for a.e. } x \in B_\epsilon(x_i)\,,\ i=1,\dots,N \Big\}\,.$$

(This problem can be seen as the model governing the exchange of heat between the spherical inclusions and the continuous component when the heat conductivity of the small inclusions is so large to be considered infinite.)

The fourth equation in (21) says that the time derivative of the temperature $T_{i,\epsilon}$ in each ball is determined by the flux of $\nabla T_\epsilon$ across the surface $\partial B_\epsilon(x_i)$.

We want to analyse the limit in which the volume fraction of the inclusions is negligible while the heat capacity of each inclusion is very large, for a sufficiently general distribution of inclusions. We shall assume then the distribution of centres to be such that

$$\frac{1}{N}\sum_{i=1}^N \delta_{x_i} \rightharpoonup \rho \mathscr{L}^3$$

(weakly in the sense of measures), with $\rho$ sufficiently regular and where $\mathscr{L}^3$ denotes, as before, the Lebesgue measure in $\mathbb{R}^3$.

The guessed limit system, assuming that

$$\int_{\Omega \setminus \bigcup_{i=1}^N B_\epsilon(x_i)} T_\epsilon^{in}(x)^2 dx + \frac{\sigma}{\sigma'}\epsilon \sum_{i=1}^N |T_{i,\epsilon}^{in}|^2 \le C \qquad \text{for all } \epsilon > 0\,. \tag{22}$$

and that

$$T_\epsilon^{in} \rightharpoonup T^{in} \qquad \text{in } L^2(\Omega) \text{ weak as } \epsilon \to 0$$

$$\frac{1}{N}\sum_{i=1}^{N} T_{i,\epsilon}^{in}\delta_{x_i} \rightharpoonup \vartheta^{in} \text{ in } \mathscr{M}_b(\Omega) \text{ weak-* as } \epsilon \to 0\,,$$

is the following system of partial differential equations (corresponding to models proposed on the basis of macroscopic arguments)

$$\begin{cases} (\partial_t - \sigma\Delta_x)T(t,x) + 4\pi\sigma(\rho(x)T(t,x) - \vartheta(t,x)) = 0\,, & x \in \Omega\,,\ t > 0\,, \\ \partial_t\vartheta(t,x) + 4\pi\sigma'(\vartheta(t,x) - \rho(x)T(t,x)) = 0\,, & x \in \Omega\,,\ t > 0\,, \\ \dfrac{\partial T}{\partial n}(t,x) = 0\,, & x \in \partial\Omega\,,\ t > 0\,, \\ T(0,x) = T^{in}(x)\,, \quad \vartheta(0,x) = \vartheta^{in}(x)\,, & x \in \Omega\,. \end{cases} \tag{23}$$

where $T$ and $\frac{\vartheta}{\rho}$ are resp. the temperatures of the background material and of the inclusions and $\rho$ represents the limit density of the inclusions.

We shall consider for the convergence proof weak solutions to this system, and in particular, assuming $\rho, \frac{1}{\rho} \in C_b(\bar{\Omega})$, we have $T \in L^\infty([0,+\infty); L^2(\Omega)) \cap L^2(0,\tau; H^1(\Omega))$ and $\vartheta \in L^\infty([0,+\infty); L^2(\Omega))$ for all $\tau > 0$, such that

$$\frac{d}{dt}\int_\Omega T\phi dx + \sigma\int_\Omega \nabla_x T\cdot\nabla\phi dx + 4\pi\sigma\int_\Omega(\rho T - \vartheta)\phi dx = 0 \tag{24}$$

$$\frac{d}{dt}\int_\Omega \vartheta\psi dx + 4\pi\sigma'\int_\Omega(\vartheta - \rho T)\psi dx = 0 \tag{25}$$

in the sense of distributions on $(0,+\infty)$ for each $\phi \in H^1(\Omega)$ and $\psi \in L^2(\Omega)$, together with the initial conditions.

In order to obtain the limit, we choose the number of inclusions to scale as $N = \frac{1}{\epsilon}$, which can be heuristically justified by the fact that in this way the energy identity associated to (21)

$$\begin{aligned} &\tfrac{1}{2}\left[\int_{\Omega\setminus\bigcup_{i=1}^N B_\epsilon(x_i)} T_\epsilon^2 dx + \frac{\sigma}{\sigma'}\epsilon\sum_{i=1}^N T_{i,\epsilon}^2\right] + \sigma\int_0^t\int_{\Omega\setminus\bigcup_{i=1}^N B_\epsilon(x_i)} |\nabla_x T_\epsilon|^2 dxds \\ &\qquad = \tfrac{1}{2}\left[\int_{\Omega\setminus\bigcup_{i=1}^N B_\epsilon(x_i)} |T_\epsilon^{in}|^2 dx + \frac{\sigma}{\sigma'}\epsilon\sum_{i=1}^N |T_{i,\epsilon}^{in}|^2\right] \end{aligned} \tag{26}$$

keeps in the limit a finite, non vanishing contributions of the inclusions. We shall assume moreover that the inclusions are far apart one of each other, i.e. that $|x_i - x_j| > 2r_\epsilon$, with $r_\epsilon = \epsilon^{\frac{1}{3}}$, for all $i, j = 1, \ldots, N$.

This system is on one hand, i.e. from the probability point of view, simpler than the previous one, since one of the component is continuous; on the other hand, from the point of view of the PDE analysis, it is defined by a mixed, time dependent boundary condition problem on which it may be tricky to perform the limit in the needed asymptotics.

Indeed, the procedure to be used in this case is quite different from the one used in the previous case: we adopt to prove the limit homogenization techniques already used in ([6]) and originating from [3], which are specific for the homogenization of boundary value problems.

First, we start by giving the variational formulation of (21), which we shall use to pass to the limit.

This is, for each $\Phi_\epsilon \in \mathscr{V}_N = \mathscr{H}_N \cap H^1(\Omega)$,

$$\frac{d}{dt}\left(\int_{\Omega\setminus\bigcup_{i=1}^N B_\epsilon(x_i)} T_\epsilon \Phi_\epsilon dx + \tfrac{3\sigma}{4\pi\sigma'}\frac{1}{\epsilon^2}\int_{\bigcup_{i=1}^N B_\epsilon(x_i)} T_\epsilon \Phi_\epsilon dx\right) + \sigma\int_{\Omega\setminus\bigcup_{i=1}^N B_\epsilon(x_i)} \nabla_x T_\epsilon\cdot\nabla\Phi_\epsilon dx = 0 \tag{27}$$

for a.e. $t \in [0, +\infty)$. As described in [7], the solution $T_\epsilon \in C([0, +\infty); \mathscr{H}_N) \cap L^2(0, \tau, \mathscr{V}_N)$ for all $\tau > 0$ to this problem (together with the initial condition) exists and is unique.

We associate to $T_\epsilon$ the empirical measure $\mu_\epsilon(t, dxd\theta) := \frac{1}{N}\sum_{i=1}^N \delta_{x_i} \otimes \delta_{T_{i,\epsilon}(t)}$, denoting by $\mu_\epsilon^{in}$ the same quantity with $T_{i,\epsilon} = T_{i,\epsilon}^{in}$. In particular, note that its first moment $\int_{\mathbb{R}} \theta\mu_\epsilon(t, \cdot, d\theta)$ with resp. to $\theta$ describes the temperature of the inclusions.

The variational formulation of the problem can be then rewritten as

$$\frac{d}{dt}\left(\int_\Omega T_\epsilon\Phi_\epsilon dx - \tfrac{4\pi}{3}\epsilon^2\int_{\Omega\times\mathbb{R}} \Phi_\epsilon\theta\mu_\epsilon(t, dxd\theta) + \frac{\sigma}{\sigma'}\int_{\Omega\times\mathbb{R}} \Phi_\epsilon\theta\mu_\epsilon(t, dxd\theta)\right) + \sigma\int_\Omega \nabla_x T_\epsilon\cdot\nabla\Phi_\epsilon dx = 0. \tag{28}$$

As a consequence of the energy identity (26), the quantities $\|T_\epsilon(t, \cdot)\|^2_{L^2(\Omega)}$, $\int_0^t \|\nabla_x T_\epsilon\|^2_{L^2(\Omega\setminus\bigcup_{i=1}^N B_\epsilon(x_i))} ds$ and $\epsilon\sum_{i=1}^N T_{i,\epsilon}(t)^2 = \int_{\Omega\times\mathbb{R}} \theta^2\mu_\epsilon$ are uniformly bounded.

These bounds imply, (together with the hypothesis on the distribution of centres), that the sequence $T_\epsilon$ is relatively compact in $L^\infty([0, +\infty); L^2(\Omega))$ weak-* and in $L^2(0, \tau; H^1(\Omega))$ weak for all $\tau > 0$, and the sequence $(1 + |x|^2 + \theta^2)\mu_\epsilon$ is relatively compact in $L^\infty([0, +\infty); \mathscr{M}_b(\Omega \times \mathbb{R}))$. The limit points $T$ and $\mu$ resp. of $T_\epsilon$ and $\mu_\epsilon$ are such that $T \in L^\infty([0, +\infty); L^2(\Omega)) \cap L^2(0, \tau; H^1(\Omega))$ and $\vartheta = \int_{\mathbb{R}} \theta\mu(t, \cdot, d\theta) \in L^\infty([0, +\infty); L^2(\Omega))$.

On another side, $|\frac{4\pi}{3}\epsilon^2 \int_{\Omega\times\mathbb{R}} \Phi_\epsilon \theta \mu_\epsilon(t, dxd\theta)| \le \epsilon^2 C$, so that the second term in (28) vanishes in the $\epsilon \to 0$ limit.

In order to pass to the limit in (27) we proceed in the following way: we consider two classes of test functions, each capturing one part of the behaviour of the system and covering in the $\epsilon \to 0$ limit $C_c^1(\overline{\Omega})$ (and, by a density argument, the desired class of test functions) and we shall then obtain two limit equations which are equivalent to the system (23).

For each $\psi \in C(\overline{B(0,\epsilon)})$, define $\chi[\psi]$ to be the solution to

$$\begin{cases} \Delta\chi[\psi](z) = 0\,, & \epsilon < |z| < r_\epsilon\,, \\ \chi[\psi](z) = \psi(z)\,, & |z| \le \epsilon\,, \\ \chi[\psi](z) = 0\,, & |z| = r_\epsilon\,. \end{cases} \tag{29}$$

and

$$\mathscr{Q}_\epsilon(x) := \sum_{i=1}^{N} \chi[\phi(x_i + \cdot) - \phi(x_i)](x - x_i) \tag{30}$$

$$\mathscr{P}_\epsilon(x) := \sum_{i=1}^{N} \chi[\phi(x_i + \cdot)](x - x_i) \tag{31}$$

We shall consider first the class of test functions of the form

$$\Phi_\epsilon(x) := \phi(x) - \mathscr{Q}_\epsilon(x)\,,$$

capturing the variation in time on the boundary of the solution to (21) (in general, $\Phi_\epsilon|_{B(x_i,\epsilon)} = \phi(x_i) \neq 0$). Since, by construction, $\mathscr{Q}_\epsilon \to 0$ strongly in $H^1(\Omega)$ when $\epsilon \to 0$, the same is true for the convergence $\Phi_\epsilon \to \phi$, so that, thanks to the a priori estimates on the solution, we have

$$\int_\Omega \nabla_x T_\epsilon \cdot \nabla \Phi_\epsilon dx \to \int_\Omega \nabla_x T(t,x) \cdot \nabla\phi(x) dx \text{ weakly in } L^2([0,+\infty)),$$

$$\int_\Omega T_\epsilon \Phi_\epsilon dx \to \int_\Omega T\phi dx \text{ in } L^\infty([0,+\infty))\text{weak-*},$$

$$\int_{\Omega\times\mathbb{R}} \Phi_\epsilon(x)\theta\mu_\epsilon(t, dxd\theta) \to \int_{\Omega\times\mathbb{R}} \phi\theta\mu(t, dxd\theta) \text{ in } L^\infty([0,+\infty)) \text{ weak-*}.$$

We obtain therefore as a limit of (27)

$$\frac{d}{dt}\left(\int_\Omega T\phi dx + \frac{\sigma}{\sigma'}\int_{\Omega\times\mathbb{R}} \phi\theta\mu(t, dxd\theta)\right) + \sigma\int_\Omega \nabla_x T \cdot \nabla\phi dx = 0 \tag{32}$$

in $L^2_{loc}([0,+\infty))$ for each $\phi \in C_c^1(\overline{\Omega})$.

Then we shall choose

$$\Psi_\epsilon(x) := \phi(x) - \mathscr{P}_\epsilon(x).$$

Now, $\Psi_\epsilon \in H^1(\Omega)$ is such that $\Psi_\epsilon|_{B(x_i,\epsilon)} = 0$ for all $i = 1, \ldots, N$, so that this class of test functions "sees" (21) as if the boundary conditions would be constant with respect to time, and the variational formulation of the problem simplifies as

$$\frac{d}{dt}\int_\Omega T_\epsilon \Psi_\epsilon dx + \sigma \int_\Omega \nabla_x T_\epsilon \cdot \nabla \Psi_\epsilon dx = 0. \tag{33}$$

The asymptotic behaviour of the equation is now more complicate to evaluate: due to the requirement that the test function vanishes on the boundary of the spheres, we only have $\mathscr{P}_\epsilon \rightharpoonup 0$ weakly in $H^1(\Omega)$, so, while for the first term it is still valid the convergence

$$\int_\Omega T_\epsilon(t,x)\Psi_\epsilon(x)dx \to \int_\Omega T(t,x)\phi(x)dx \text{ in } L^\infty([0,+\infty)) \text{ weak-*}$$

as $\epsilon \to 0$ (since still $\mathscr{P}_\epsilon \to 0$ strongly in $L^2(\Omega)$), the second term in the variational formulation cannot be evaluated directly.

As well explained in [3], although for a different structure of the test functions, the asymptotic behaviour of this term can be determined by taking advantage of the strong convergence in $H^{-1}(\Omega)$ of a part of $\triangle \Psi_\epsilon$, being the bad part such that its integral on $H^1$ functions vanishing on the inclusions is zero (we recall that, although we cannot discuss here this issue, this behaviour is related to the structure of the inclusions and can be described on the basis of more general principles). On another side, on the inclusions, where all involved functions are constant, the $H^{-1}(\Omega)$ strong convergence will not be needed, so that thanks also to the assumption on the distribution of centres, the convergence properties in the given scaling of all involved functions will be good enough to get the limit. We have therefore to decompose the solution into a term vanishing on the inclusions plus a residual term, in order to perform the computation.

We write then

$$T_\epsilon(t,x) = \Theta_\epsilon(t,x) + \mathscr{S}_\epsilon(t,x)$$

where $\mathscr{S}_\epsilon(t,x) := \sum_{i=1}^N \chi[T_{i,\epsilon}(t)](x - x_i)$, so that $\Theta_\epsilon|_{B(x_i,\epsilon)} = 0$ and $\mathscr{S}_\epsilon \rightharpoonup 0$ in $L^\infty([0,+\infty); H^1(\Omega))$ weak-*.

We may then rewrite

$$\begin{aligned}\int_\Omega \nabla_x T_\epsilon \cdot \nabla \Psi_\epsilon dx &= \int_\Omega \nabla_x T_\epsilon \cdot \nabla(\phi - \mathscr{Q}_\epsilon)dx \\ &\quad - \int_\Omega \nabla_x \mathscr{S}_\epsilon \cdot \nabla(\mathscr{P}_\epsilon - \mathscr{Q}_\epsilon)dx - \int_\Omega \nabla_x \Theta_\epsilon \cdot \nabla(\mathscr{P}_\epsilon - \mathscr{Q}_\epsilon)dx,\end{aligned} \tag{34}$$

where $\mathscr{Q}_\epsilon$ is defined in (30): doing so, we split $\mathscr{P}_\epsilon$ into two components, vanishing respectively strongly and weakly in $H^1$:

Notice that the dependence on $\phi$ of the function $(\mathscr{P}_\epsilon - \mathscr{Q}_\epsilon)$ is reduced to the dependence on its values in the centres of the inclusions, $\phi(x_i)$, $i = 1, \dots, N$, so that, writing this dependence as $(\mathscr{P}_\epsilon - \mathscr{Q}_\epsilon)[\phi(x_i), i = 1, \dots, N]$, we have $\mathscr{S}_\epsilon = (\mathscr{P}_\epsilon - \mathscr{Q}_\epsilon)[T_{i,\epsilon}(t), i = 1, \dots, N]$.

Directly from the convergence properties of $\mathscr{S}_\epsilon$ and $\nabla_x T_\epsilon$ we get

$$\int_\Omega \nabla_x T_\epsilon \cdot \nabla(\phi - \mathscr{Q}_\epsilon)dx \to \int_\Omega \nabla_x T \cdot \nabla\phi$$

in $L^2([0, \tau])$ weak for all $\tau > 0$.

The last two terms on the second line of (34) have to be evaluated explicitly, and their values are determined by the distribution of holes [3, 6, 7].

For the first term, we get (cf. Lemma 6.3 in [7])

$$\int_\Omega \nabla_x \mathscr{S}_\epsilon \cdot \nabla(\mathscr{P}_\epsilon - \mathscr{Q}_\epsilon)dx \to 4\pi \int_{\Omega\times\mathbb{R}} \phi(x)\theta\mu(t, dxd\theta)$$

in $L^\infty(\mathbb{R}_+)$ weak-*. This term, which is related to the analysis on the inclusions and on their nearest neighbourhood, can be easily evaluated by observing that

$$\int_\Omega \nabla_x \mathscr{S}_\epsilon \cdot \nabla(\mathscr{P}_\epsilon - \mathscr{Q}_\epsilon)dx = \|\nabla\chi[1]\|^2_{L^2(\mathbb{R}^N)} \sum_{i=1}^N T_{i,\epsilon}(t)\phi(x_i)$$
$$= \frac{4\pi r_\epsilon}{r_\epsilon - \epsilon} \int_{\Omega\times\mathbb{R}} \phi(x)\theta\mu_\epsilon(t, dxd\theta).$$

The second term is the one where the $H^{-1}(\Omega)$ strong convergence of $\triangle\Psi_\epsilon$ is needed, and we get (cf. Lemma 6.4 in [7])

$$\int_\Omega \nabla_x \Theta_\epsilon \cdot \nabla(\mathscr{P}_\epsilon - \mathscr{Q}_\epsilon)dx \to -4\pi \int_\Omega \rho T\phi dx \text{ in } L^2([0, \tau]) \text{ weak},$$

where the required $H^{-1}(\Omega)$ convergence is proved by proving $\sum_{i=1}^N \phi(x_i) r_\epsilon \delta_{\partial B(x_i, r_\epsilon)} \to 4\pi\rho\phi$ strongly in $H^{-1}(\mathbb{R}^3)$ as $\epsilon \to 0$ (cf. [3, 6]).

Collecting all the terms, we finally get in the limit $\epsilon \to 0$ the equation

$$\frac{d}{dt}\int_\Omega T\phi dx + \sigma \int_\Omega \nabla_x T \cdot \nabla\phi dx + 4\pi\sigma \int_\Omega \rho T\phi dx$$
$$-4\pi\sigma \int_{\Omega\times\mathbb{R}} \phi(x)\theta\mu(t, dxd\theta) = 0 \tag{35}$$

in $L^2_{loc}([0, +\infty))$ for each $\phi \in C^1_c(\overline{\Omega})$.

The convergence for the initial conditions can be obtained from the uniform convergence in $[0, \tau]$ of the terms at the interior of the time derivative in (27) and (33), resp. evaluated on the first and the second class of test functions.

The two equations (32), (35) can be recombined (and extended by a density argument) to get the equivalent system (24), i.e. the weak formulation of (23), so that we prove finally the convergence of the whole sequence $(T_\epsilon, \vartheta_\epsilon)$ to the unique weak solution to (23).

The theorem we are able to prove is then the following one (Theorem 2.6 in [7]):

**Theorem 2.** *Let $N = \frac{1}{\epsilon}$. Assume that, when $\epsilon \to 0$, the distribution of inclusion centres satisfies $\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} \rightharpoonup \rho\mathscr{L}^3$, with $\rho, \frac{1}{\rho} \in C_b(\bar{\Omega})$, and that the initial datum $T_\epsilon^{in}$ satisfies the bound (22). Assume further that $T_\epsilon^{in} \to T^{in}$ in $L^2(\Omega)$ weak as $\epsilon \to 0$, while $\frac{1}{N}\sum_{i=1}^{N} T_{i,\epsilon}^{in}\delta_{x_i} \to \vartheta^{in}$ in $\mathscr{M}_b(\Omega)$ weak-* as $\epsilon \to 0$.*

*Let $T_\epsilon \in C([0, +\infty); \mathscr{H}_N) \cap L^2(0, \tau, \mathscr{V}_N)$ for all $\tau > 0$ be the weak solution to the scaled infinite heat conductivity problem (21). Then, in the limit $\epsilon \to 0$,*

$$T_\epsilon \to T$$

*in $L^2(0, \tau; H^1(\Omega))$ weak for all $\tau > 0$ and in $L^\infty([0, +\infty); L^2(\Omega))$ weak-*, and*

$$\vartheta_\epsilon := \frac{1}{N}\sum_{i=1}^{N} T_{i,\epsilon}\delta_{x_i} \to \vartheta$$

*in $L^\infty([0, +\infty); \mathscr{M}_b(\Omega))$ weak-*, where $T_{i,\epsilon} := \frac{3}{4\pi\epsilon^3}\int_{B(x_i,\epsilon)} T_\epsilon(t, x)dx$.*

*Besides*

$$T \in C_b([0, +\infty); L^2(\Omega)) \times L^2(0, \tau; H^1(\Omega)) \text{ for each } \tau > 0$$

*while*

$$\vartheta \in C_b([0, +\infty); L^2(\Omega))\,.$$

*Finally, the pair $(T, \vartheta)$ is the unique weak solution to the homogenized system (23) with initial condition*

$$T|_{t=0} = T^{in}\,, \qquad \vartheta|_{t=0} = \vartheta^{in}\,.$$

## 4 Conclusion

We have presented two examples of systems with at least one particle component leading in the large number asymptotics to sets of partial differential equations modelling two-component systems with self-consistent coupling. The two examples are different in nature, one of them having a prevalent stochastic nature, the other being more on the analytical setting. We have sketched how to obtain the rigorous proof of the convergence of the microscopic system to the macroscopic limit equations in a mean-field type scaling in the two cases. The details of the proofs are presented in [1] and [7].

## References

1. Bernardin, C., Ricci, V.: A simple particle model for a system of coupled equations with absorbing collision term. Kinet. Relat. Models **4**(3), 633–668 (2011)
2. Bourgain, J., Golse, F., Wennberg, B.: On the distribution of free path lengths for the periodic Lorentz gas. Commun. Math. Phys. **190**, 491–508 (1998)
3. Cioranescu, D., Murat, F.: Un terme étrange venu d'ailleurs. In: Nonlinear Partial Differential Equations and Their Applications, Collège de France Seminar, vol. 2 (Paris 1979–1980). Research Notes in Mathematics, vol. 60, pp. 98–138. Pitman, Boston/London (1982)
4. Desvillettes, L., Ricci, V.: A rigorous derivation of a linear kinetic of the Fokker-Planck type in the limit of grazing collisions. J. Stat. Phys. **104**, 1173–1189 (2001)
5. Desvillettes, L., Ricci, V.: Nonmarkovianity of the Boltzmann–Grad limit of a system of random obstacles in a given force field. Bull. Sci. Math. **128**, 39–46 (2004)
6. Desvillettes, L., Golse, F., Ricci, V.: The mean-field limit for solid particles in a Navier-Stokes flow. J. Stat. Phys. **131**, 941–967 (2008)
7. Desvillettes, L., Golse, F., Ricci, V.: Derivation of a Homogenized two-temperature model from the heat equation. Preprint hal–00827912, arXiv:1305.6920, pp. 1–32 (2013), to be published in Mathematical Modelling and Numerical Analysis
8. Nappo, G., Orlandi, E., Rost, H.: A reaction-diffusion model for moderately interacting particles. J. Stat. Phys. **55**, 579–600 (1989)
9. Ricci, V.: Non Markovian behavior of the Boltzmann–Grad limit of linear stochastic particle systems. Commun. Math. Sci. **5**(Suppl 1), 95–105 (2007)
10. Sznitman, A.S.: Propagation of chaos for a system of annihilating Brownian spheres. CPAM **50**, 663–690 (1987)

# On a Stochastic Coupled System of Reaction-Diffusion of Nonlocal Type

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## 1 Introduction

In this paper, we study the following initial-boundary value problem involving a stochastic nonlinear parabolic system of nonlocal type

$$\begin{cases} u_t - a\left(\int_D u\,dx\right)\Delta u = g_1(v) + f_1(u,v)\dfrac{\partial W_1}{\partial t} \quad \text{on} \quad D\times ]0,\infty[\,, \\ v_t - a\left(\int_D v\,dx\right)\Delta v = g_2(u) + f_2(u,v)\dfrac{\partial W_2}{\partial t} \quad \text{on} \quad D\times ]0,\infty[\,, \\ (u(x,0),v(x,0)) = (u_0(x),v_0(x)) \quad \text{in} \quad D, \\ (u,v) = (0,0) \quad \text{on} \quad \partial D\times ]0,\infty[\ , \end{cases} \tag{1}$$

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where $D$ is an open bounded subset of $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\partial D$. On the nonlocal function $a = a(s)$, we assume that:

$$a \text{ is a Lipschitz-continuous function with Lipschitz constant denoted by } L; \tag{2}$$

$$0 < p \leq a(s) \leq P\,, \quad \text{where } p \text{ and } P \text{ are constants.} \tag{3}$$

$(W_1(t), W_2(t))_{t\in[0,\infty[}$ is a two-dimensional real Wiener process, the maps $f_i : L^2(D) \times L^2(D) \to L^2(D)$, $g_i : L^2(D) \to L^2(D)$, with $i = 1, 2$ satisfy the following conditions:

$$\| f_i(z_1, x_1) - f_i(z_2, x_2)\|^2 \leq J \left(\|z_1 - z_2\|^2 + \|x_1 - x_2\|^2\right) \forall\, z_1, z_2,\ x_1, x_2 \in L^2(D); \tag{4}$$

$$f_i(0, 0) = 0; \tag{5}$$

$$\|g_i(z_1) - g_i(z_2)\|^2 \leq K \left(\|z_1 - z_2\|^2\right)\ \forall\, z_1,\ z_2 \in L^2(D); \tag{6}$$

$$g_i(0) = 0\,, \quad \text{where } J \text{ and } K \text{ are positive constants.} \tag{7}$$

Moreover, the multiplicative white noise, represented by the random forcing term $g(t, u)\frac{\partial W}{\partial t}$ in the model, describes a state dependent random noise.

*Remark 1.* For simplicity we are considering reaction terms $g_1$ and $g_2$ depending only on one quantity: $v$ or $u$, respectively. But we may as well consider their dependence on both quantities $u$ and $v$, as long as they satisfy to the corresponding version of the assumption (4). The consideration of cross reactions $g_1(v)$ and $g_2(u)$ is only to emphasize the existence of coupling also on the reaction terms.

For the last several decades, various types of equations have been employed as some mathematical model describing physical, chemical, biological and ecological systems. Among them, the most successful and crucial one is the following model of semilinear parabolic partial differential equation

$$\frac{\partial u}{\partial t} - A\Delta u - f(u) = 0, \tag{8}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function, and $A$ is an $n \times n$ real matrix. In [15] it was considered the reaction-diffusion equation (8), where $A$ is an $n \times n$ real matrix and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^2$ function. There, it was studied the exponential decay for some cases. In the literature, most authors assume that the diffusion matrix $A$ is diagonal, so that the coupling between the equations is present only on the nonlinearity of the reaction term $f$. However, cross-diffusion phenomena are not uncommon (see e.g. [3, 13, 14] and the references cited therein) and can be treated as equations in which $A$ is not even diagonalizable. In [6] the authors studied the existence and uniqueness of solutions for non local problems of the form

$$\begin{cases} u_t - a\left(\int_D u\,dx\right)\Delta u = f(x,t) \quad \text{in } D \times ]0,T[\,, \\ u(x,t) = 0 \quad \text{on} \quad \partial D \times ]0,T[\,, \\ u(x,0) = u_0(x) \quad \text{in} \quad D\,. \end{cases} \tag{9}$$

where $\partial D$ is a smooth boundary, $T$ is some arbitrary time, and $a$ is some function from $\mathbb{R}$ into $(0,+\infty)$. This problem arises in various situations. For instance, $u$ could describe the density of a population (for instance, of bacteria) subject to spreading. The diffusion coefficient $a$ is then supposed to depend on the entire population in the domain rather than on the local density, i.e., moves are guided by considering the global state of the medium.

It was given in [10] an extension of the result obtained in [6], considering $a = a(l(u))$ where $l : L^2(D) \to \mathbb{R}$ is a continuous linear form and $f = f(x,u)$ a continuous functions. Indeed, in [10] the authors improved the results in [6–8] by considering both stationary and evolution situations where the nonlinearity appears not only in the operator $u \to a\left(\int_\Omega u dx\right)\Delta u$ but also in the right-hand side in which one has the nonlinear function $f$.

The problem studied in [6, 10] is nonlocal in the sense that the diffusion coefficient is determined by a global quantity. These kind of problems, besides its mathematical motivation, arise from physical situations related to migration of a population of bacteria in a container in which the velocity of migration $V = a\nabla u$ depends on the global population in a subdomain $D' \subset D$ given by $a = a\left(\int_D u dx\right)$ (see [4, 5] and reference therein). In [9] the authors studied the existence and uniqueness of weak solutions for a random version of class of nonlinear parabolic problems of nonlocal type and with additive noise

$$\begin{cases} u_t - a\left(\int_D u\,dx\right)\Delta u = \gamma u + f + \dfrac{\partial g}{\partial t} \quad \text{in } D \times ]0,T[\,, \\ u(x,t) = 0 \quad \text{on} \quad \partial D \times ]0,T[\,, \\ u(x,0) = u_0(x) \quad \text{in} \quad D\,. \end{cases} \tag{10}$$

As a motivation for our problem, let us consider an island with two types of species: Rabbits and Foxes. Clearly one plays the role of predator while the other the role of a prey. If we are interested to model the population growth of both species, then we have to keep in mind that if, for example, the population of the foxes increases, then the rabbit population will be affected. So, the rate of change of the population of one type will depend on the actual population of the other type. For example, in the absence of the rabbit population, the fox population will decrease (and fast) to face a certain extinction. Something that most of us would like to avoid. In this case, if the difference among the two populations tends to zero, then there is one natural control for the species. In our case, $u$ and $v$ could describe the densities of two population that interact in a common atmosphere. Does this system type have a solution? If affirmative, is the solution stabilized or not controllable? In case that it is stabilized, at which rate? We intend to answer these questions.

In this paper, we study the existence and uniqueness of weak solutions for the problem (1). Using some techniques explored in [2], we improve the results obtained by the authors in [9] for coupled systems. In [6] the authors have proved some results for the problem (1) including the analysis of the asymptotic behavior of its solution, without the term $g(t,u)\frac{\partial W}{\partial t}$.

This paper is organized as follows. Before our main results, in Sect. 2 we briefly outline the notation and terminology to be used subsequently. In Sect. 3, we prove the existence and uniqueness of weak solutions. Section 4 is devoted to establish the asymptotic behavior of the solutions. Finally, we present the Appendix with useful properties that shall be used in the sequel.

## 2 Notation and Formulation of the Problem

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space and let $(\mathscr{F})_{t\in\mathbb{R}_0^+}$ be a right-continuous filtration such that $\mathscr{F}_0$ contains all $\mathscr{F}-$null sets. $E(X)$ denotes the mathematical expectation of the random variable $X$. We abbreviate a.s. for *almost surely* $\omega \in \Omega$ and we write $\mathscr{L}$ for the Lebesgue measure on $\mathbb{R}_0^+ := [0, \infty[$.

Let $B$ be a Banach space with norm $\|\cdot\|_B$. Then $\mathscr{B}(B)$ denotes the Borel $\sigma-$algebra of $B$. The space $L^2(\Omega \times \mathbb{R}_0^+; B)$ is the set of all $\mathscr{F} \otimes \mathscr{B}(\mathbb{R}_0^+)-$measurable process $u : \Omega \times \mathbb{R}_0^+ \to B$ which are $\mathscr{F}_t-$ adapted and $E(\int_0^\infty \|u\|_B^2 dt) < \infty$. Analogously, the space $L^\infty(\Omega \times \mathbb{R}_0^+; \mathbb{R})$ is the set of all $\mathscr{F} \otimes \mathscr{B}(\mathbb{R}_0^+)-$measurable process $u : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}$ which are $\mathscr{F}_t-$ adapted and for almost everywhere $(\omega, t) \in \Omega \times \mathbb{R}_0^+$ bounded. In this work, $(W_1(t), W_2(t))_{t\in\mathbb{R}_0^+}$ is a Wiener process $\mathscr{F}_t-$adapted.

Throughout this work $D$ is an open bounded and connected subset of $\mathbb{R}^n$ with a smooth boundary $\partial D$. We denote the Lebesgue measure of $D$ by $|D| := \int_D dx$. Let $H^s(D)$ denote the usual Sobolev space of order $s$ with norm denoted by $\|\cdot\|_s$, and inner product by $(\cdot,\cdot)_s$. In particular, $H^0(D) = L^2(D)$, $\|\cdot\| := \|\cdot\|_0$ and $(\cdot,\cdot) := (\cdot,\cdot)_0$. $H_0^1(D)$ denotes the Sobolev space of order 1 of functions with zero traces on the boundary, and its dual space is denoted by $H^{-1}(D)$. The duality product between $H^{-1}(D)$ and $H_0^1(D)$ is denoted by $\langle\cdot,\cdot\rangle$. We assume that $(\nabla u, \nabla u) = \|u\|_1^2$ for $u \in H_0^1(D)$.

We define the map $\mathscr{A} : H_0^1(D) \to H^{-1}(D)$ by $\langle \mathscr{A}u, \eta\rangle = a\left(\int_D u\, dx\right)(\nabla u, \nabla \eta)$ for $\eta \in H_0^1(D)$.

Let $u_0$, $v_0$ be random variables $L^2(D)-$valued, $\mathscr{F}_0-$measurable such that $E(\|u_0\|^2 + \|v_0\|^2) < \infty$.

In this work, we mean that the stochastic process $(u, v)$ is a weak solution of the problem (1) in the following sense:

**Definition 1.** The stochastic process $(u(t), v(t))_{t\in\mathbb{R}_0^+} \in L^2(\Omega \times \mathbb{R}_0^+; H_0^1(D)) \times L^2(\Omega \times \mathbb{R}_0^+; H_0^1(D))$, with a.s. sample paths continuous in $L^2(D) \times L^2(D)$, is a weak solution of the problem (1) if satisfies to:

$$(u(t), \eta) + \int_0^t \langle \mathscr{A} u, \eta\rangle (s)ds = (u_0, \eta) + \int_0^t (f_1(u(s), v(s)), \eta)dW_1(s) + \int_0^t (g_1(v(s)), \eta)ds \tag{11}$$

$$(v(t), \xi) + \int_0^t \langle \mathscr{A} v, \xi\rangle (s)ds = (v_0, \xi) + \int_0^t (f_2(u(s), v(s)), \xi)dW_2(s) + \int_0^t (g_2(u(s)), \xi)ds \tag{12}$$

a.s. for all $\eta,\ \xi \in H_0^1(D)$ and for all $t \in \mathbb{R}_0^+$, where the stochastic integral is considered in the Itô sense.

We address the reader to the monographs [1, 11, 12, 16] for an explanation on the mathematical foundations of the stochastic equations, its applications and for more details on the used tools.

## 3 Existence of Solution

**Theorem 1.** *Assume that (2)–(7) are fulfilled and suppose that*

$$p > \frac{C_{\mathscr{P}}\,(2J + K + 1)}{2}, \tag{13}$$

*where $C_{\mathscr{P}}$ is the Poincaré inequality's constant and $J$ and $K$ are the constants from the assumptions (4) and (6). Then the problem (1) has a solution, which is unique and has a.s. sample paths continuous in $L^2(D) \times L^2(D)$.*

Note that, here, we mean uniqueness in the sense of indistinguishability.

*Proof.* We shall split the proof of Theorem 1 into several steps.

**Part 1 – Approximate problem:** If $\{\alpha_j; w_j\}_{j=1}^{\infty}$ is the eigensystem of $-\Delta$, with domain $H_0^1(D) \cap H^2(D)$, then $0 < \alpha_1 \leq \alpha_2 \leq \ldots$. We remark that $\{w_j\}_{j=1}^{\infty}$ is an orthonormal set in $L^2(D)$ and is orthogonal in $H_0^1(D)$. For each $n \in \mathbb{Z}^+$, let $u_{0n} = \sum_{i=1}^n (u_0, w_i)\, w_i(x) \to u_0$, $v_{0n} = \sum_{i=1}^n (v_0, w_i)\, w_i(x) \to v_0$ be in $L^2(\Omega; L^2(D))$, and let $u_n := \sum_{i=1}^n \lambda_{in} w_i$, $v_n := \sum_{i=1}^n \beta_{in} w_i$, where $(\lambda_{in}, \beta_{in})_{i=1}^n$ are solutions to the approximate problem given by the following system of the stochastic differential equations

$$(u_n(t), w_k) = (u_{0n}, w_k) - \int_0^t \langle \mathscr{A} u_n(s), w_k \rangle \, ds + \int_0^t (g_1(v_n(s)), w_k) \, ds$$
$$+ \int_0^t (f_1(u_n(s), v_n(s)), w_k) \, dW_1(s) \,, \tag{14}$$

$$(v_n(t), w_k) = (v_{0n}, w_k) - \int_0^t \langle \mathscr{A} v_n(s), w_k \rangle (s) ds + \int_0^t (g_2(u_n(s)), w_k) \, ds$$
$$+ \int_0^t (f_2(u_n(s), v_n(s)), w_k) \, dW_2(s) \tag{15}$$

a.s., for $k = 1, 2, \ldots, n$ and for all $t \in \mathbb{R}_0^+$. Since (14)–(15) is a finite dimensional Itô system of equations, we may apply a well-known result (see [12, Theorem 3, p. 45] or [1, Corollary 6.3.1, p. 112]) to prove the existence of a unique solution $(u_n, v_n)$ such that $(u_n, v_n) \in L^2(\Omega \times \mathbb{R}_0^+; H_0^1(D)) \times L^2(\Omega \times \mathbb{R}_0^+; H_0^1(D))$ with a.s. sample paths continuous in $L^2(D) \times L^2(D)$ and $\mathscr{F}_t$−adapted.

Then, we apply Itô's formula to the system (14)–(15) to obtain

$$\|u_n(t)\|^2 = \|u_n(0)\|^2 + \int_0^t \| f_1 (u_n(s), v_n(s)) \|^2 ds + 2 \int_0^t (g_1(v_n(s)), u_n(s)) \, ds$$
$$+2 \int_0^t (f_1(u_n(s), v_n(s)), u_n(s)) \, dW_1(s) - 2 \int_0^t \langle \mathscr{A} u_n, u_n \rangle (s) ds \,,$$

$$\|v_n(t)\|^2 = \|v_n(0)\|^2 + \int_0^t \| f_2 (u_n(s), v_n(s)) \|^2 ds + 2 \int_0^t (g_2(u_n(s)), v_n(s)) \, ds$$
$$+2 \int_0^t (f_2(u_n(s), v_n(s)), v_n(s)) \, dW_2(s) - 2 \int_0^t \langle \mathscr{A} v_n, v_n \rangle (s) ds$$

a.s. and for all $t \in \mathbb{R}_0^+$. Now, adding up the above equations, using (3) together with the assumptions (4)–(7) and finally using Poincaré's inequality, we obtain

$$\|u_n(t)\|^2 + \|v_n(t)\|^2 + 2p \int_0^t \left(\|u_n(s)\|_1^2 + \|v_n(s)\|_1^2\right) ds \le$$
$$2JC_{\mathscr{P}} \int_0^t \left(\|u_n(s)\|_1^2 + \|v_n(s)\|_1^2\right) ds + (K+1)C_{\mathscr{P}} \int_0^t \left(\|u_n(s)\|_1^2 + \|v_n(s)\|_1^2\right) ds$$
$$+ 2 \int_0^t (f_1(u_n(s), v_n(s)), u_n(s)) \, dW_1(s) + 2 \int_0^t (f_2(u_n(s), v_n(s)), v_n(s)) \, dW_2(s)$$
$$+ \|u_n(0)\|^2 + \|v_n(0)\|^2 \,,$$

where $C_{\mathscr{P}}$ is the Poincaré inequality's constant and $J$ and $K$ are the constants from the assumptions (4) and (6). Hence,

$$
\begin{aligned}
&E\left(\|u_n(t)\|^2+\|v_n(t)\|^2\right)\\
&\quad+[2p-(2J+K+1)C_{\mathscr{P}}]\,E\left(\int_0^t\left(\|u_n(s)\|_1^2+\|v_n(s)\|_1^2\right)ds\right)\\
&\quad\le\|u_n(0)\|^2+\|v_n(0)\|^2\le\|u(0)\|^2+\|v(0)\|^2.
\end{aligned}
\tag{16}
$$

Then, owe to the assumption (13), (16) and (3) yield that

$$
E\left(\int_0^\infty\|\mathscr{A}u_n(t)\|_{-1}^2dt\right)\le P\;E\left(\int_0^\infty\|u_n(t)\|_1^2dt\right)<\infty\,, \tag{17}
$$

$$
E\left(\int_0^\infty\|\mathscr{A}v_n(t)\|_{-1}^2dt\right)\le P\;E\left(\int_0^\infty\|v_n(t)\|_1^2dt\right)<\infty\,, \tag{18}
$$

where $P$ is the constant upper bound from the assumption (3). Moreover, from the assumptions (4)–(7), and by using Poincaré's inequality, it can also be proved that

$$
E\left(\int_0^\infty\|f_i(u_n(t),v_n(t))\|^2dt\right)\le JC_{\mathscr{P}}E\left(\int_0^\infty\left(\|u_n(t)\|_1^2+\|v_n(t)\|_1^2\right)dt\right)<\infty\,,
$$

for $i=1,2$, and

$$
E\left(\int_0^\infty\|g_1(v_n(t))\|^2dt\right)\le KC_{\mathscr{P}}E\left(\int_0^\infty\|v_n(t)\|_1^2dt\right)<\infty\,, \tag{19}
$$

$$
E\left(\int_0^\infty\|g_2(u_n(t))\|^2dt\right)\le KC_{\mathscr{P}}E\left(\int_0^\infty\|u_n(t)\|_1^2dt\right)<\infty\,, \tag{20}
$$

Thus, using (17)–(20) and by means of reflexivity, there exists a subsequence, still denoted by $\{u_n,v_n\}$, and there exist $\hat{u},\hat{v}\in L^2(\Omega\times\mathbb{R}_0^+;H_0^1(D))$, $a_1^*,\ a_2^*\in L^2(\Omega\times\mathbb{R}_0^+;H^{-1}(D))$, $g_1^*,\ g_2^*\ f_1^*,\ f_2^*\in L^2(\Omega\times\mathbb{R}_0^+;L^2(D))$ such that, letting $n\to\infty$, we have

$$
\begin{aligned}
&u_n\rightharpoonup\hat{u}\,,\quad v_n\rightharpoonup\hat{v}\qquad\text{in}\quad L^2(\Omega\times\mathbb{R}_0^+;H_0^1(D))\,,\\
&\mathscr{A}u_n\rightharpoonup a_1^*\,,\quad\mathscr{A}v_n\rightharpoonup a_2^*\qquad\text{in}\quad L^2(\Omega\times\mathbb{R}_0^+;H^{-1}(D))\,,\\
&g_1(v_n)\rightharpoonup g_1^*\,,\quad g_2(u_n)\rightharpoonup g_2^*\qquad\text{in}\quad L^2(\Omega\times\mathbb{R}_0^+;L^2(D))\,,\\
&f_1(u_n,v_n)\rightharpoonup f_1^*\,,\quad f_2(u_n,v_n)\rightharpoonup f_2^*\qquad\text{in}\quad L^2(\Omega\times\mathbb{R}_0^+;L^2(D))\,.
\end{aligned}
\tag{21}
$$

**Part 2 – Passing to the limit:** Using the convergence results (21) together with Proposition 2 (see Appendix), we can pass to the limit $n\to\infty$ in (14)–(15) to obtain

$$
(\hat{u}(t),\eta)=(u_0,\eta)-\int_0^t\langle a_1^*,\eta\rangle\,ds+\int_0^t\left(g_1^*,\eta\right)+\int_0^t\left(f_1^*,\eta\right)dW_1(s)\,, \tag{22}
$$

$$(\hat{v}(t),\zeta) = (v_0,\zeta) - \int_0^t \langle a_2^*,\zeta\rangle\, ds + \int_0^t \left(g_2^*,\zeta\right) + \int_0^t \left(f_2^*,\zeta\right) dW_2(s) \qquad (23)$$

for almost all $(\omega,t) \in \Omega \times \mathbb{R}_0^+$ and for every $\eta,\zeta \in H_0^1(D)$.

Now, let $(u(t),v(t))_{t\in\mathbb{R}_0^+}$ denote the $L^2(D)\times L^2(D)$-valued process which is $\mathscr{F}_t$-adapted and equal to $(\hat{u}(t),\hat{v}(t))$ for $\mathbb{P}\times\mathscr{L}$ a.e. $(\omega,t)\in\Omega\times\mathbb{R}_0^+$ which has a.s. sample paths continuous in $L^2(D)\times L^2(D)$ (see [16, Theorem 2, p. 73]). Thus, from (22)–(23), it follows

$$(u(t),\eta) = (u_0,\eta) - \int_0^t \langle a_1^*,\eta\rangle\, ds + \int_0^t (g_1^*,\eta) ds + \int_0^t (f_1^*,\eta) dW_1(s) \qquad (24)$$

$$(v(t),\zeta) = (u_0,\zeta) - \int_0^t \langle a_2^*,\zeta\rangle\, ds + \int_0^t (g_2^*,\zeta) ds + \int_0^t (f_2^*,\zeta) dW_2(s) \qquad (25)$$

a.s., for all $\eta,\zeta\in H_0^1(D)$, and for all $t\in\mathbb{R}_0^+$.

Then, we consider the stopping time $\mathscr{T}_M := \mathscr{T}_M^{u,v}$ (see Appendix). We claim that $(u(t),v(t))_{t\in\mathbb{R}_0^+}$ satisfies to

$$\lim_{n\to\infty} E\int_0^{\mathscr{T}_M} \|u(s)-u_n(s)\|_1^2 ds = 0 \quad \text{and} \quad \lim_{n\to\infty} E\|u(\mathscr{T}_M)-u_n(\mathscr{T}_M)\|^2 = 0\,, \qquad (26)$$

$$\lim_{n\to\infty} E\int_0^{\mathscr{T}_M} \|v(s)-v_n(s)\|_1^2 ds = 0 \quad \text{and} \quad \lim_{n\to\infty} E\|v(\mathscr{T}_M)-v_n(\mathscr{T}_M)\|^2 = 0\,. \qquad (27)$$

In fact, from (24)–(25) and (14)–(15), we have

$$((u-u_n)(t),w_k) = \int_0^t \langle(\mathscr{A}u_n - a_1^*)(s),w_k\rangle\, ds$$
$$+ \int_0^t (f_1^*(s) - f_1(u_n(s),v_n(s)),w_k) dW_1(s) + \int_0^t (g_1^* - g_1(v_n(s)),w_k) ds\,,$$

$$((v-v_n)(t),w_k) = \int_0^t \langle(\mathscr{A}v_n - a_2^*)(s),w_k\rangle\, ds$$
$$+ \int_0^t (f_2^*(s) - f_2(u_n(s),v_n(s)),w_k) dW_2(s) + \int_0^t (g_2^* - g_2(u_n(s)),w_k) ds$$

a.s., for all $t\in\mathbb{R}_0^+$ and all $k=1,\dots,n$. For each $n\in\mathbb{N}$, let $\tilde{u}_n(t) := \sum_{i=1}^n (u(t),w_i)w_i$ and $\tilde{v}_n(t) := \sum_{i=1}^n (v(t),w_i)w_i$. Thanks to Itô's formula, we obtain

$$\|(\tilde{u}_n - u_n)(t)\|^2 = \int_0^t 2\,\langle(\mathscr{A}u_n - a_1^*)(s), (\tilde{u}_n - u_n)(s)\rangle\, ds$$
$$+ 2\int_0^t \big(f_1^*(s) - f_1(u_n(s), v_n(s)), (\tilde{u}_n - u_n)(s)\big)\, dW_1(s)$$
$$+ \int_0^t \|f_1^*(s) - f_1(u_n(s), v_n(s))\|^2 ds + \int_0^t 2(g_1^* - g_1(v_n), (\tilde{u}_n - u_n)(s))ds\,,$$

$$\|(\tilde{v}_n - v_n)(t)\|^2 = \int_0^t 2\,\langle(\mathscr{A}v_n - a_2^*)(s), (\tilde{v}_n - v_n)(s)\rangle\, ds$$
$$+ 2\int_0^t \big(f_2^*(s) - f_2(u_n(s), v_n(s)), (\tilde{v}_n - v_n)(s)\big)\, dW_2(s)$$
$$+ \int_0^t \|f_2^*(s) - f_2(u_n(s), v_n(s))\|^2 ds + \int_0^t 2(g_2^* - g_2(u_n), (\tilde{v}_n - v_n)(s))ds$$

a.s. and for all $t \in \mathbb{R}_0^+$.

Now, let

$$e(t) = e^{\int_0^t h(s)\,ds - C\,t} \quad \text{a.s.} \quad \text{and} \quad \text{for all } t \in \mathbb{R}_0^+\,,$$

where the function $h(s)$ and the constant $C$ are to be specified later on in (35). Using the Itô formula, we have

$$e(t)\|(\tilde{u}_n - u_n)(t)\|^2 = 2\int_0^t e(s)\,\langle(\mathscr{A}u_n - a_1^*)(s), (\tilde{u}_n - u_n)(s)\rangle\, ds$$
$$+2\int_0^t e(s)(f_1^*(s) - f_1(u_n(s), v_n(s)), (\tilde{u}_n - u_n)(s))dW_1(s)$$
$$+ \int_0^t e(s)\|f_1^*(s) - f_1(u_n(s), v_n(s))\|^2 ds$$
$$+2\int_0^t e(s)(g_1^* - g_1(v_n), (\tilde{u}_n - u_n)(s))ds$$
$$+ \int_0^t e(s)h(s)\|(\tilde{u}_n - u_n)(s)\|^2 ds - C\int_0^t e(s)\|(\tilde{u}_n - u_n)(s)\|^2 ds\,, \tag{28}$$

$$e(t)\|(\tilde{v}_n - v_n)(t)\|^2 = 2\int_0^t e(s)\,\langle(\mathscr{A}v_n - a_2^*)(s), (\tilde{v}_n - v_n)(s)\rangle\, ds$$
$$+2\int_0^t e(s)(f_2^*(s) - f_2(u_n(s), v_n(s)), (\tilde{v}_n - v_n)(s))dW_2(s)$$
$$+ \int_0^t e(s)\|f_2^*(s) - f_2(u_n(s), v_n(s))\|^2 ds$$

$$+2\int_0^t e(s)(g_2^* - g_2(u_n), (\tilde{v}_n - v_n)(s))ds$$
$$+\int_0^t e(s)h(s)\|(\tilde{v}_n - v_n)(s)\|^2 ds - C\int_0^t e(s)\|(\tilde{v}_n - v_n)(s)\|^2 ds. \tag{29}$$

To proceed with, we observe that, for $i = 1,\ 2$, it holds

$$\begin{aligned}&\|f_i^*(s) - f_i(u_n(s), v_n(s))\|^2 = \|f_i(u(s), v(s)) - f_i(u_n(s), v_n(s))\|^2 + \\ &2\left(f_i^*(s) - f_i(u(s), v(s)),\ f_i^*(s) - f_i(u_n(s), v_n(s))\right) - \|f_i^*(s) - f_i(u(s), v(s))\|^2\end{aligned} \tag{30}$$

and, by using (6) together with the inequalities of Hölder, Cauchy and Minkowski, it can be proved that

$$\begin{aligned}&\left(g_1^* - g_1(v_n), \tilde{u}_n - u_n\right) + \left(g_2^* - g_2(u_n), \tilde{v}_n - v_n\right) \le \\ &\left(g_1^* - g_1(v), \tilde{u}_n - u_n\right) + \left(g_2^* - g_2(u), \tilde{v}_n - v_n\right) \\ &+ \left(K + \frac{1}{2}\right)\left(\|\tilde{u}_n - u_n\|^2 + \|\tilde{v}_n - v_n\|^2\right) + K\left(\|u - \tilde{u}_n\|^2 + \|v - \tilde{v}_n\|^2\right).\end{aligned} \tag{31}$$

Then, adding up (28) and (29) and using (30)–(31) together with (3), we obtain

$$\begin{aligned}&e(t)\left(\|(\tilde{u}_n - u_n)(t)\|^2 + \|(\tilde{v}_n - v_n)(t)\|^2\right) \\ &+2p\int_0^t e(s)\left(\|(\tilde{u}_n - u_n)(s)\|_1^2 + \|(\tilde{v}_n - v_n)(s)\|_1^2\right) ds \\ &+\sum_{i=1}^{2}\int_0^t e(s)\|f_i^*(s) - f_i(u(s), v(s))\|^2 ds \ \le \\ &2\int_0^t e(s)\left[a\left(\int_D u_n dx\right)(\nabla\tilde{u}_n(s), \nabla(\tilde{u}_n - u_n)(s)) - \langle a_1^*(s), (\tilde{u}_n - u_n)(s)\rangle\right] ds \\ &+2\int_0^t e(s)\left[a\left(\int_D v_n dx\right)(\nabla\tilde{v}_n(s), \nabla(\tilde{v}_n - v_n)(s)) - \langle a_2^*(s), (\tilde{v}_n - v_n)(s)\rangle\right] ds \\ &+2\sum_{i=1}^{2}\int_0^t e(s)\left(f_i^*(s) - f_i(u_n(s), v_n(s)), (\tilde{u}_n - u_n)(s)\right) dW_i(s) \\ &+2\sum_{i=1}^{2}\int_0^t e(s)\left(f_i^*(s) - f_i(u(s), v(s)),\ f_i^*(s) - f_i(u_n(s), v_n(s))\right) ds \\ &+\sum_{i=1}^{2}\int_0^t e(s)\|f_i(u(s), v(s)) - f_i(u_n(s), v_n(s))\|^2 ds\end{aligned}$$

$$+2\int_0^t e(s)\left[(g_1^* - g_1(v(s)), (\tilde{u}_n - u_n)(s)) + (g_2^* - g_2(u(s)), (\tilde{v}_n - v_n)(s))\right] ds$$
$$+(2K+1-C)\int_0^t e(s)\left(\|(\tilde{u}_n - u_n)(s)\|^2 + \|(\tilde{v}_n - v_n)(s)\|^2\right) ds$$
$$+2K\int_0^t e(s)\left(\|(u-\tilde{u}_n)(s)\|^2 + \|(v-\tilde{v}_n)(s)\|^2\right) ds$$
$$+\int_0^t e(s)h(s)\left(\|(\tilde{u}_n - u_n)(s)\|^2 + \|(\tilde{v}_n - v_n)(s)\|^2\right) ds\,. \tag{32}$$

Now, we observe that by using, in this order, Hölder's inequality, assumptions (2)–(3) and Cauchy's inequality with $\epsilon = \frac{p}{4}$, we can prove that

$$a\left(\int_D u_n\,dx\right)(\nabla\tilde{u}_n, \nabla(\tilde{u}_n - u_n)) - \langle a_1^*, \tilde{u}_n - u_n\rangle \;\le \langle \mathscr{A}(u) - a_1^*, \tilde{u}_n - u_n\rangle$$
$$+\frac{P^2}{p}\|\tilde{u}_n - u\|_1^2 + \frac{p}{4}\|\tilde{u}_n - u_n\|_1^2 + \frac{L^2|D|}{p}\|u_n - u\|^2\|\tilde{u}_n\|_1^2 + \frac{p}{4}\|\tilde{u}_n - u_n\|_1^2\,, \tag{33}$$

$$a\left(\int_D v_n\,dx\right)(\nabla\tilde{v}_n, \nabla(\tilde{v}_n - v_n)) - \langle a_2^*, \tilde{v}_n - v_n\rangle \le \langle \mathscr{A}(v) - a_2^*, \tilde{v}_n - v_n\rangle$$
$$+\frac{P^2}{p}\|\tilde{v}_n - v\|_1^2 + \frac{p}{4}\|\tilde{v}_n - v_n\|_1^2 + \frac{L^2|D|}{p}\|v_n - v\|^2\|\tilde{v}_n\|_1^2 + \frac{p}{4}\|\tilde{v}_n - v_n\|_1^2\,. \tag{34}$$

Then we proceed by applying (33)–(34) to the first two terms on the right-hand side of (32) and by using assumption (4) together with Minkowski's and Cauchy's inequalities in the fifth term of the same right-hand side. Next, we use Minkowski's and Cauchy's inequalities on the last term of the right-hand side. This procedure leads us to

$$e(t)\left(\|(\tilde{u}_n - u_n)(t)\|^2 + \|(\tilde{v}_n - v_n)(t)\|^2\right)$$
$$+\,p\int_0^t e(s)\left(\|(\tilde{u}_n - u_n)(s)\|_1^2 + \|(\tilde{v}_n - v_n)(s)\|_1^2\right) ds$$
$$+\sum_{i=1}^{2}\int_0^t e(s)\|f_i^*(s) - f_i(u(s), v(s))\|^2 ds \;\le$$
$$2\int_0^t e(s)\left(\langle \mathscr{A}(u(s)) - a_1^*(s), (\tilde{u}_n - u_n)(s)\rangle + \langle \mathscr{A}(v(s)) - a_2^*(s), (\tilde{v}_n - v_n)(s)\rangle\right) ds$$

$$+ \frac{2P^2}{p} \int_0^t e(s) \left( \|(\tilde{u}_n - u)(s)\|_1^2 + \|(\tilde{v}_n - v)(s)\|_1^2 \right) ds$$
$$+ \frac{2L^2|D|}{p} \int_0^t e(s) \left( \|(u_n - u)(s)\|^2 + \|(v_n - v)(s)\|^2 \right) \left( \|\tilde{u}_n(s)\|_1^2 + \|\tilde{v}_n(s)\|_1^2 \right) ds$$
$$+ 2 \sum_{i=1}^{2} \int_0^t e(s) \left( f_i^*(s) - f_i(u_n(s), v_n(s)), (\tilde{u}_n - u_n)(s) \right) dW_i(s)$$
$$+ 2 \sum_{i=1}^{2} \int_0^t e(s) \left( f_i^*(s) - f_i(u(s), v(s)), f_i^*(s) - f_i(u_n(s), v_n(s)) \right) ds$$
$$+ (4J + 2K) \int_0^t e(s) \left( \|(u - \tilde{u}_n)(s)\|^2 + \|(v - \tilde{v}_n)(s)\|^2 \right) ds$$
$$+ 2 \int_0^t e(s) \left[ (g_1^* - g_1(v(s)), (\tilde{u}_n - u_n)(s)) + (g_2^* - g_2(u(s)), (\tilde{v}_n - v_n)(s)) \right] ds$$
$$+ (4J + 2K + 1 - C) \int_0^t e(s) \left( \|(\tilde{u}_n - u_n)(s)\|^2 + \|(\tilde{v}_n - v_n)(s)\|^2 \right) ds$$
$$+ 2 \int_0^t e(s)h(s) \left( \|(\tilde{u}_n - u)(s)\|^2 + \|(\tilde{v}_n - v)(s)\|^2 \right) ds$$
$$+ 2 \int_0^t e(s)h(s) \left( \|(u_n - u)(s)\|^2 + \|(v_n - v)(s)\|^2 \right) ds .$$

Then, choosing

$$h(s) = -\frac{L^2|D|}{p} \left( \|\tilde{u}_n(s)\|_1^2 + \|\tilde{v}_n(s)\|_1^2 \right) \quad \text{and} \quad C = 4J + 2K + 1 , \tag{35}$$

we can see that the third, eighth and the last two terms on the right-hand side of the above inequality disappear. As a consequence, we obtain

$$\lim_{n\to\infty} E \left( e(\mathcal{T}_M) \|(\tilde{u}_n - u_n)(\mathcal{T}_M)\|^2 \right) = 0 = \lim_{n\to\infty} E \left( e(\mathcal{T}_M) \|(\tilde{v}_n - v_n)(\mathcal{T}_M)\|^2 \right) ,$$
$$\lim_{n\to\infty} E \int_0^{\mathcal{T}_M} e(s) \|(\tilde{u}_n - u_n)(s)\|_1^2 ds = 0 = \lim_{n\to\infty} E \int_0^{\mathcal{T}_M} e(s) \|(\tilde{v}_n - v_n)(s)\|_1^2 ds \tag{36}$$

and

$$E \int_0^{\mathcal{T}_M} e(s) \| f_i^*(s) - f_i(u(s), v(s)) \|^2 ds = 0 \text{ with } i = 1, 2. \tag{37}$$

From (36) and due to the properties of $\mathscr{T}_M$ over $[0, \mathscr{T}_M]$ for fixed $M \in \mathbb{Z}^+$, we obtain

$$\lim_{n\to\infty} E\|(\tilde{u}_n - u_n)(\mathscr{T}_M)\|^2 = \lim_{n\to\infty} E\int_0^{\mathscr{T}_M} \|(\tilde{u}_n - u_n)(s)\|_1^2 ds = 0\,,$$

$$\lim_{n\to\infty} E\|(\tilde{v}_n - v_n)(\mathscr{T}_M)\|^2 = \lim_{n\to\infty} E\int_0^{\mathscr{T}_M} \|(\tilde{v}_n - v_n)(s)\|_1^2 ds = 0\,.$$

Now, (26) and (27) is an immediate consequence.

**Part 3 – Convergence of the non-linear term:** From (37), we obtain that

$$I_{[0,\mathscr{T}_M]} f_i(u(s), v(s)) = I_{[0,\mathscr{T}_M]} f_i^*(s) \text{ a.}e.(\omega, t) \in \Omega \times \mathbb{R}_0^+ \quad \text{for } i = 1,\ 2. \tag{38}$$

We claim that

$$\lim_{n\to\infty} E\left(\int_0^{\mathscr{T}_M} (\langle \mathscr{A} u(s), \eta(s)\rangle - \langle \mathscr{A} u_n(s), \eta(s)\rangle) ds\right) = 0, \tag{39}$$

for all $\eta \in L^2(\Omega \times \mathbb{R}_0^+; H_0^1(D))$. Indeed,

$$\left(a\left(\int_D u\, dx\right)\nabla u - a\left(\int_D u_n\, dx\right)\nabla u_n, \nabla\eta\right) =$$
$$\left(\left(a\left(\int_D u\, dx\right) - a\left(\int_D u_n\, dx\right)\right)\nabla u, \nabla\eta\right) + \left(a\left(\int_D u_n\, dx\right)(\nabla u - \nabla u_n), \nabla\eta\right)$$

implies

$$E\left(\int_0^{\mathscr{T}_M} (\langle \mathscr{A} u(s), \eta(s)\rangle - \langle \mathscr{A} u_n(s), \eta(s)\rangle)\, ds\right) \leq$$
$$L|D|\left(E\left(\int_0^{\mathscr{T}_M} \|(u-u_n)(s)\|^2 ds\right)\right)^{1/2}\left(E\left(\int_0^{\mathscr{T}_M} \|u(s)\|_1^2\|\nabla\eta(s)\|^2 ds\right)\right)^{1/2} +$$
$$P\left(E\left(\int_0^{\mathscr{T}_M} \|(u-u_n)(s)\|_1^2 ds\right)\right)^{1/2}\left(E\left(\int_0^{\mathscr{T}_M} \|\eta(s)\|_1^2 ds\right)\right)^{1/2}.$$

Analogously, we obtain

$$\lim_{n\to\infty} E\left(\int_0^{\mathscr{T}_M} (\langle \mathscr{A} v(s), \eta(s)\rangle - \langle \mathscr{A} v_n(s), \eta(s)\rangle) ds\right) = 0 \tag{40}$$

for all $\eta \in L^2(\Omega \times \mathbb{R}_0^+; H_0^1(D))$

From (39) and (40) and since $\mathscr{A}(u_n) \rightharpoonup a_1^*$, $\mathscr{A}(v_n) \rightharpoonup a_2^*$ in $L^2(\Omega \times \mathbb{R}_0^+; H^{-1}(D))$, we have

$$I_{[0,\mathscr{T}_M]}(s)a_1^*(s) = I_{[0,\mathscr{T}_M]}(s)\mathscr{A}(u(s)), \quad I_{[0,\mathscr{T}_M]}(s)a_2^*(s) = I_{[0,\mathscr{T}_M]}(s)\mathscr{A}(v(s)) \tag{41}$$

for almost everywhere $(\omega, t) \in \Omega \times \mathbb{R}_0^+$. Since

$$\lim_{n\to\infty} E\left(\int_0^{\mathscr{T}_M} (g_1(v(s)), \eta(s)) - (g_1(v_n(s)), \eta(s))ds\right) = 0,$$

$$\lim_{n\to\infty} E\left(\int_0^{\mathscr{T}_M} (g_2(u(s)), \eta(s)) - (g_2(u_n(s)), \eta(s))ds\right) = 0$$

for all $\eta \in L^2(\Omega \times \mathbb{R}_0^+; H_0^1(D))$, we have, for a.e. $(\omega, t) \in \Omega \times \mathbb{R}_0^+$,

$$I_{[0,\mathscr{T}_M]}g_1(v(s)) = I_{[0,\mathscr{T}_M]}g_1^*(s), \qquad I_{[0,\mathscr{T}_M]}g_2(u(s)) = I_{[0,\mathscr{T}_M]}g_2^*(s).$$

Owing to this, we obtain, by using (38), (41) in (24)–(25), that

$$(u(t \wedge \mathscr{T}_M), \eta) = (u_0, \eta) - \int_0^{t\wedge\mathscr{T}_M} \langle \mathscr{A}u(s), \eta\rangle\, ds + \int_0^{t\wedge\mathscr{T}_M} (g_1(v(s)), \eta)ds + \int_0^{t\wedge\mathscr{T}_M} (f_1(u(s), v(s)), \eta)\, dW_1(s), \tag{42}$$

$$(v(t \wedge \mathscr{T}_M), \zeta) = (u_0, \zeta) - \int_0^{t\wedge\mathscr{T}_M} \langle \mathscr{A}v(s), \zeta\rangle\, ds + \int_0^{t\wedge\mathscr{T}_M} (g_2(u(s)), \zeta)ds + \int_0^{t\wedge\mathscr{T}_M} (f_2(u(s), v(s)), \zeta)\, dW_2(s), \tag{43}$$

a.s. $\forall \eta,\ \zeta \in H_0^1(D),\ t \in \mathbb{R}_0^+$.

Next, we observe that, by the properties of $\mathscr{T}_M$ and due to Proposition 1, $\mathbb{P}(\bigcup_{M=1}^{\infty}\{\mathscr{T}_M \le T\}) = 1$.

Now, let $\Omega' := \left\{\omega \in \Omega : \omega \in \bigcup_{M=1}^{\infty}\{\mathscr{T}_M \le T\} \text{ and } u(\omega, t) \text{ satisfies (42)–(43)}\right\}$. We have $\mathbb{P}(\Omega') = 1$. For $\omega \in \Omega'$, there exists $M_0 \in \mathbb{Z}^+$ such that $\mathscr{T}_M(\omega) \ge T$ for all $M \ge M_0$. From (42)–(43), one gets

$$(u(t), \eta) = (u_0, \eta) - \int_0^t \langle \mathscr{A}u(s), \eta\rangle\, ds + \int_0^t (g_1(v(s)), \eta)ds + \int_0^t (f_1(u(s), v(s)), \eta)\, dW_1(s),$$

$$(v(t),\zeta) = (u_0,\zeta) - \int_0^t \langle \mathscr{A} v(s),\zeta\rangle\, ds + \int_0^t (g_2(u(s)),\zeta) ds$$
$$+ \int_0^t (f_2(u(s),v(s)),\zeta)\, dW_2(s),$$

a.s. for all $\eta,\ \zeta \in H_0^1(D)$, $t \in \mathbb{R}_0^+$. Thus $(u(t), v(t))$ is a weak solution of (1).

**Part 4 – Uniqueness:** Let $(u_1, v_1),\ (u_2, v_2) \in L^2(\Omega \times \mathbb{R}_0^+; H_0^1(D)) \times L^2(\Omega \times \mathbb{R}_0^+; H_0^1(D))$ be two weak solutions of the problem (1). We fix the notation

$$\delta(t) = e^{\int_0^t i(s)\, ds} \quad \text{a.s.} \quad \text{and} \quad \text{for all } t \in \mathbb{R}_0^+ ,$$

where the function $i(s)$ is to be defined later on in (44). Thanks to Itô's formula, we obtain

$$\delta(t)\|(u_1-u_2)(t)\|^2 = -2\int_0^t a\left(\int_D u_1\, dx\right) \delta(s)(\nabla(u_1-u_2)(s), \nabla(u_1-u_2)(s))\, ds$$
$$+2\int_0^t \left[a\left(\int_D u_2\, dx\right) - a\left(\int_D u_1\, dx\right)\right] \delta(s)\, (\nabla u_2(s), \nabla(u_1-u_2)(s))\, ds$$
$$+2\int_0^t \delta(s)(g_1(v_1(s)) - g_1(v_2(s)), u_1-u_2) ds$$
$$+\int_0^t \delta(s)\| f_1(u_1(s), v_1(s)) - f_1(u_2(s), v_2(s))\|^2 ds$$
$$+2\int_0^t \delta(s)\, (f_1(u_1(s), v_1(s)) - f_1(u_2(s), v_2(s)), (u_1-u_2)(s))\, dW_1(s)$$
$$+\int_0^t \delta'(s)\|(u_1-u_2)(s)\|^2 ds\,,$$

$$\delta(t)\|(v_1-v_2)(t)\|^2 = -2\int_0^t a\left(\int_D v_1\, dx\right) \delta(s)(\nabla(v_1-v_2)(s), \nabla(v_1-v_2)(s))\, ds$$
$$+2\int_0^t \left[a\left(\int_D v_2\, dx\right) - a\left(\int_D v_1\, dx\right)\right] \delta(s)\, (\nabla v_2(s), \nabla(v_1-v_2)(s))\, ds$$
$$+2\int_0^t \delta(s)(g_2(u_1(s)) - g_2(u_2(s)), (v_1-v_2)(s)) ds$$
$$+\int_0^t \delta(s)\| f_2(u_1(s), v_1(s)) - f_2(u_2(s), v_2(s))\|^2 ds$$
$$+2\int_0^t \delta(s)\, (f_2(u_1(s), v_1(s)) - f_2(u_2(s), v_2(s)), (v_1-v_2)(s))\, dW_2(s)$$
$$+\int_0^t \delta'(s)\|(v_1-v_2)(s)\|^2 ds\,.$$

Then we add up the above two equations and we use assumption (3) in the first two terms on the right-hand side of the resulting equation. In the second two terms, we use assumption (2) and Hölder's inequality. In the terms with $f_i$ we use the assumption (4) and for the terms with $g_i$ we use (6) together with Hölder's and Cauchy's inequalities. This procedure leads us to

$$\begin{aligned}
&\delta(t)\left(\|(u_1-u_2)(t)\|^2+\|(v_1-v_2)(t)\|^2\right)\\
&+2p\int_0^t \delta(s)\left(\|(u_1-u_2)(s)\|_1^2+\|(v_1-v_2)(s)\|_1^2\right)ds \leq\\
&+2\int_0^t \delta(s)L|D|^{\frac{1}{2}}\|(u_2-u_1)(s)\|\,\|\nabla u_2(s)\|\,\|\nabla(u_1-u_2)(s)\|\,ds\\
&+2\int_0^t \delta(s)L|D|^{\frac{1}{2}}\|(v_2-v_1)(s)\|\,\|\nabla v_2(s)\|\,\|\nabla(v_1-v_2)(s)\|\,ds\\
&+(2J+K+1)\int_0^t \delta(s)\left(\|(u_1-u_2)(s)\|^2+\|(v_1-v_2)(s)\|^2\right)ds\\
&+2\int_0^t \delta(s)\left(f_1(u_1(s),v_1(s))-f_1(u_2(s),v_2(s)),(u_1-u_2)(s)\right)dW_1(s)\\
&+2\int_0^t e(s)\left(f_2(u_1(s),v_1(s))-f_2(u_2(s),v_2(s)),(v_1-v_2)(s)\right)dW_2(s)\\
&+\int_0^t \delta'(s)\left(\|(u_1-u_2)(s)\|^2+\|(v_1-v_2)(s)\|^2\right)ds\,.
\end{aligned}$$

By assumption (13), we may consider the positive constant

$$q := p-\frac{C_{\mathscr{P}}\,(2J+K+1)}{2}\,.$$

Next we use Hölder's and Cauchy's inequalities, the later with $\epsilon=\frac{q}{2}$, in the second two terms and Poincaré's inequality in the third term, both on the right-hand side of the previous inequality. We thus achieve to

$$\begin{aligned}
&\delta(t)\left(\|(u_1-u_2)(t)\|^2+\|(v_1-v_2)(t)\|^2\right)\\
&+q\int_0^t \delta(s)\left(\|(u_1-u_2)(s)\|_1^2+\|(v_1-v_2)(s)\|_1^2\right)ds \leq\\
&+\frac{L^2|D|}{q}\int_0^t \delta(s)\left(\|(u_1-u_2)(s)\|^2+\|(v_1-v_2)(s)\|^2\right)\left(\|u_2(s)\|_1^2+\|v_2(s)\|_1^2\right)ds\\
&+2\int_0^t \delta(s)\left(f_1(u_1(s),v_1(s))-f_1(u_2(s),v_2(s)),(u_1-u_2)(s)\right)dW_1(s)
\end{aligned}$$

$$+2\int_0^t \delta(s)\,(f_2(u_1(s),v_1(s))-f_2(u_2(s),v_2(s)),(v_1-v_2)(s))\,dW_2(s)$$
$$+\int_0^t \delta'(s)\left(\|(u_1-u_2)(s)\|^2+\|(v_1-v_2)(s)\|^2\right)ds\,.$$

Finally, choosing

$$i(s)=\frac{-L^2|D|}{q}\left(\|u_2(s)\|_1^2+\|v_2(s)\|_1^2\right), \tag{44}$$

we can see that the first and the last term, on the right-hand side of the above inequality, cancel each other. Thus, from Gronwall's Lemma, we obtain

$$\delta(t)\left(\|(u_1-u_2)(t)\|^2+\|(v_1-v_2)(t)\|^2\right)=0 \quad \text{for all } t\in\mathbb{R}_0^+.$$

Then $\mathbb{P}\left((u_1(t),v_1(t))=(u_2(t),v_2(t))\right)=1$ for all $t\in\mathbb{R}_0^+$. Since $u_1$, $v_1$ and $u_2$, $v_2$ are a.s. continuous in $L^2(D)$, we prove that $\mathbb{P}\big((u_1(t),v_1(t))=(u_2(t),v_2(t))\,,\ \forall t\in\mathbb{R}_0^+\big)=1$. □

## 4 Asymptotic Behavior

We conclude this paper by proving a result on the asymptotic behavior of the weak solution to the problem (1).

**Theorem 2.** *Suppose that (13) holds and denote by $\gamma$ the largest constant such that*

$$\gamma\left(\|u(t)\|^2+\|v(t)\|^2\right)\le\|\nabla u(t)\|^2+\|\nabla v(t)\|^2 \quad \textit{a.s. and for all } t\in\mathbb{R}_0^+$$

*for the solution $(u(t),v(t))_{t\in\mathbb{R}_0^+}$ obtained in Theorem 1. Then*

$$E\left(\|u(t)\|^2+\|v(t)\|^2\right)\le E\left(\|u_0\|+\|v_0\|\right)e^{-(2p\gamma-(K+1)-2J)t} \quad \textit{for all } t\in\mathbb{R}_0^+\,. \tag{45}$$

*Remark 2.* Note that if the initial data $(u_0,v_0)$ is chosen such that the equality (45) is realized in the Poincaré's inequality constant, then the largest constant $\gamma$ is obviously $C_{\mathscr{P}}^{-1}$. Consequently, under assumption (13), the argument of the exponential $e^{-(2p\gamma-(K+1)-2J)t}$ is non-positive.

*Proof.* Let $u_n(t):=\sum_{j=1}^n\left(u(t),w_j\right)w_j$ and $v_n(t):=\sum_{j=1}^n\left(v(t),w_j\right)w_j$. From (11)–(12) and Itô's formula, we obtain

$$\|u_n(t)\|^2 = \sum_{j=1}^{n}(u_0, w_j)^2 - \int_0^t 2\,\langle \mathscr{A}u(s), u_n(s)\rangle\, ds + \int_0^t 2\,(g_1(v(s)), u_n(s))\, ds$$

$$+ \int_0^t \sum_{j=1}^{n}(f_1(u(s), v(s)), w_j)^2 ds + \int_0^t 2\,(f_1(u(s), v(s)), u_n(s))\, dW_1(s)\,,$$

$$\|v_n(t)\|^2 = \sum_{j=1}^{n}(v_0, w_j)^2 - \int_0^t 2\,\langle \mathscr{A}v(s), v_n(s)\rangle\, ds + \int_0^t 2\,(g_2(u(s)), v_n(s))\, ds$$

$$+ \int_0^t \sum_{j=1}^{n}(f_2(u(s), v(s)), w_j)^2 ds + \int_0^t 2\,(f_2(u(s), v(s)), v_n(s))\, dW_2(s)\,.$$

Hence,

$$E\left(\|u_n(t)\|^2\right) = E\left(\sum_{j=1}^{n}(u_0, w_j)^2\right) - E\left(\int_0^t 2\,\langle \mathscr{A}u(s), u_n(s)\rangle\, ds\right)$$

$$+E\left(\int_0^t 2\,(g_1(v(s)), u_n(s))\, ds\right) + E\left(\int_0^t \sum_{j=1}^{n}\left(f_1(u(s), v(s)), w_j\right)^2 ds\right)\,,$$

$$E\left(\|v_n(t)\|^2\right) = E\left(\sum_{j=1}^{n}(v_0, w_j)^2\right) - E\left(\int_0^t 2\,\langle \mathscr{A}v(s), v_n(s)\rangle\, ds\right)$$

$$+E\left(\int_0^t 2\,(g_2(u(s)), v_n(s))\, ds\right) + E\left(\int_0^t \sum_{j=1}^{n}\left(f_2(u(s), v(s)), w_j)^2\right) ds\right)\,.$$

Letting $n \to \infty$, we get

$$E\left(\|u(t)\|^2\right) = E\left(\|u_0\|^2\right) - E\left(\int_0^t 2\,\langle \mathscr{A}u(s), u(s)\rangle\, ds\right)$$

$$+E\left(\int_0^t 2\,(g_1(v(s)), u(s))\, ds\right) + E\left(\int_0^t \|f_1(u(s), v(s))\|^2 ds\right)\,,$$

$$E\left(\|v(t)\|^2\right) = E\left(\|v_0\|^2\right) - E\left(\int_0^t 2\,\langle \mathscr{A}v(s), v(s)\rangle\, ds\right)$$

$$+E\left(\int_0^t 2\,(g_2(u(s)), v(s))\, ds\right) + E\left(\int_0^t \|f_2(u(s), v(s))\|^2 ds\right)\,.$$

Using the properties of $a$ and Hölder's inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(E\left(\|u(t)\|^2+\|v(t)\|^2\right)\right)+2p\gamma\left(E\left(\|u(t)\|^2+\|v(t)\|^2\right)\right)\\
&\quad\le E\left(2\|g_1(v(t))\|\|u(t)\|\right)\\
&\quad+E\left(\|f_1(u(t),v(t))\|^2\right)+E\left(2\|g_2(u(t))\|\|v(t)\|\right)+E\left(\|f_2(u(t),v(t))\|^2\right).
\end{aligned}$$

Using (4)–(7), we have

$$\frac{d}{dt}\left(E\left(\|u(t)\|^2+\|v(t)\|^2\right)\right)+(2p\gamma-(K+1)-2J)\,E\left(\|u(t)\|^2+\|v(t)\|^2\right)\le 0$$

and (45) follows. □

## Appendix

Let $(Q(t),R(t))$ be a $H_0^1(D)\times H_0^1(D)-$valued process with

$$\int_0^{+\infty}\|Q(s)\|_1^2+\|R(s)\|_1^2ds<\infty \quad \text{a.s.}$$

For each $M\in\mathbb{N}$, we introduce the following stopping times

$$\mathscr{T}_M^{Q,R}:=\begin{cases}\inf\left\{t\ge 0:\displaystyle\int_0^t\|Q(s)\|_1^2+\|R(s)\|_1^2ds\ge M\right\}\\ +\infty,\quad \text{if}\displaystyle\int_0^{+\infty}\|Q(s)\|_1^2+\|R(s)\|_1^2ds<M\,.\end{cases}$$

**Proposition 1.** $\mathbb{P}(\mathscr{T}_M^{Q,R}<+\infty)\to 0$ *and* $\mathscr{T}_M^{Q;R}\to+\infty$ *a.s. when* $M\to\infty$

*Proof.* The proof follows immediately from [2, Lemma 3.2-(i)]. □

**Proposition 2.** *Let* $\{x_n\}\in L^2(\Omega,\mathbb{R}_0^+;B)$ *a sequence such that* $x_n\rightharpoonup x$ *in* $L^2(\Omega\times\mathbb{R}_0^+;B)$. *Then*

$$\int_0^t x_n(s)dW(s)\rightharpoonup\int_0^t x(s)dW(s)\,,\quad\int_0^t x_n(s)ds\rightharpoonup\int_0^t x(s)ds \ \text{ in }\ L^2(\Omega\times\mathbb{R}_0^+;B).$$

*Proof.* See [2, Corollary 4.2]. □

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## References


1. Arnold, L.: Stochastic Differential Equations: Theory and Applications. Wiley, New York (1974)
2. Breckner, H. (Lisei): Galerkin approximation and the strong solution of the stochastic NavierStokes equation. J. Appl. Math. Stoch. Anal. **13**(3), 239–259 (2000)
3. Capasso, V., Di Libdo, A.: Global attractivity for reaction-diffusion system. The case of nondiagonal matrices. J. Math. Anal. Appl. **177**, 510–529 (1993)
4. Chipot, M.: The diffusion of a population partly driven by its preferences. Arch. Ration. Mech. Anal. **155**, 237–259 (2000)
5. Chipot, M., Corrêa, F.J.S.A.: Boundary layer solutions to functional elliptic equations. Bull. Braz. Math. Soc. New Ser. **40**(3), 381–393 (2009)
6. Chipot, M., Lovat, B.: Some remarks on nonlocal elliptic and parabolic problems. Nonlinear Anal. **30**(7), 4619–4627 (1997)
7. Chipot, M., Lovat, B.: On the asymptotic behaviour of some nonlocal problems. Positivity **3**, 65–81 (1999)
8. Chipot, M., Rodrigues, J.F.: On a class of nonlocal nonlinear problems. Math. Model. Numer. Anal. **26**(3), 447–468 (1992)
9. Coayla-Teran, E.A., Ferreira, J., Magalhães, P.M.D.: Weak solutions for random nonlinear parabolic equations of nonlocal type. Random Oper. Stoch. Equ. **16**, 213–222 (2008)
10. Corrêa, F.J.S.A., Menezes, S.D.B., Ferreira, J.: On a class of problems involving a nonlocal operator. Appl. Math. Comput. **147**, 475–489 (2004)
11. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions. Encyclopedia of Mathematics and Its Applications, vol. 44. Cambridge University Press, Cambridge (1992)
12. Gihman, I.I., Skorohod, A.V.: Stochastic Differential Equations. Springer, Berlin (1974)
13. Jüngel, A.: Diffusive and nondiffusive population models. In: Mathematical Modeling of Collective Behavior in Socio-economic and Life Sciences. Modeling and Simulation in Science, Engineering and Technology, pp. 397–425. Birkhäuser, Boston (2010)
14. Lepoutre, T., Pierre, M., Rolland, G.: Global well-posedness of a conservative relaxed cross diffusion system. SIAM J. Math. Anal. **44**(3), 1674–1693 (2012)
15. Oliveira, L.A.F.: On reaction-diffusion system. Electron. J. Differ. Equ. **24**, 1–10 (1998)
16. Rozovskii, B.L.: Stochastic Evolution Systems. Kluwer, Dordrecht (1990)