# Chapter 11
# Large Deviations in Turbulence

**Guido Boffetta and Andrea Mazzino**

**Abstract** We give a survey of the use of the multifractal method, as a manifestation of the large deviation theory, to study the scaling behavior in fully developed turbulence. Particular emphasis is reserved to the phenomenon of intermittency, i.e., the most relevant manifestation of the break-down of mean field arguments in turbulence. To explain intermittency, the statistical role of fluctuations are explicitly accounted for by means of the multifractal formalism. Its application to the statistics of velocity gradients and acceleration will be discussed. A remark related to the use of large deviation theory in multifractal formalism will be emphasized. Also, the presentation of the famous Refined Similarity Hypothesis due to Kolmogorov and Obukhov in 1962 to account for the statistical role of fluctuations will be reviewed.

## 11.1 Introduction

The multifractal approach to fully developed turbulence stands, technically speaking, on the shoulders of the large deviation theory and is one of the most fruitful idea which allowed to physically understand the phenomenology of intermittency and anomalous scaling in turbulence [1–3].

From a technical point of view, one can say that multifractal analysis is a large deviation theory of self-similar measure [4]. The so-called multifractal spectrum and structure function, which are related by Legendre transforms, are the analogs of an entropy and a free energy function, respectively.

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These important relationships permitted to gain a rigorous formulation of multifractals, as well as to provide a guide for deriving new results. As pioneering works which anticipated some aspects of the the multifractal approach to turbulence we can cite the lognormal theory of Kolmogorov [5], the contributions of Novikov and Stewart [6] and Mandelbrot [7]. In the lognormal model of Kolmogorov, the anomalous scaling of structure functions was attributed to large fluctuations of the velocities which, in turn, were supposed to be triggered by "intermittent" nature of the coarse grained energy dissipation rate. Since then, a number of models have been proposed to understand the essential features of these fluctuations. Among these models, the multifractal model represents the most general approach to intermittency and anomalous scaling in turbulence.

Our main aim here is to give a survey of the use of the multifractal method, as a manifestation of the large deviation theory, to study the scaling behavior of fully developed turbulence.

The material of the chapter is organized as follows. In Sect. 11.2 we introduce the concept of scale invariance in turbulence and how it is related to the famous $4/5$-th law for fully developed turbulence. In Sect. 11.3 the statistical role of fluctuations are explicitly accounted for by means of the multifractal formalism. A remark related to the use of large deviation theory in multifractal formalism will be also discussed. Sect. 11.4 is devoted to the presentation of the Refined Similarity Hypothesis due to Kolmogorov and Obukhov in 1962. Conclusions are reserved to Sect. 11.5.

## 11.2 Global Scale Invariance and Kolmogorov Theory

Turbulence in fluids is described by the Navier-Stokes equations for an incompressible ($\nabla \cdot \boldsymbol{v} = 0$) velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nu \nabla^2 \boldsymbol{v} + \boldsymbol{f} \tag{11.1}$$

where $p$ represents the pressure, $\nu$ is the kinematic viscosity of the fluid and $\boldsymbol{f}$ is a forcing terms necessary to have a statistically stationary state. Turbulence appears spontaneously as the dimensionless Reynolds number $Re = UL/\nu \to \infty$ ($U$ is a typical velocity in the flow and $L$ a typical scale, e.g. the scale at which the forcing is acting). The nonlinearity of the equation, together with the non-locality (due to the pressure term), implies that in general an analytical treatment of (11.1) is a formidable task, while some special, time-independent solutions, for small $Re$ are known [8]. A confirmation of this difficulty comes from the fact that for the three-dimensional case, and given some initial conditions, mathematicians have not yet proved that smooth solutions always exist, or that if they do exist they have bounded kinetic energy. This is called the Navier-Stokes existence and smoothness problem. The Clay Mathematics Institute in May 2000 made this problem one of its seven Millennium Prize problems in mathematics. It offered a US 1,000,000 prize to the first person providing a solution for a specific statement of the problem.

In two dimensions it is possible to prove that in the deterministic case the solution of the Cauchy problem exists and is unique [9] and, very recently, that in the stochastic case (see, e.g., [10]) the solution is a Markov process exponentially mixing in time and ergodic with a unique invariant (steady state) measure even when the forcing acts only on two Fourier modes [11].

For an inviscid fluid ($\nu = 0$) and in the absence of external forces (i.e. $\boldsymbol{f} = 0$), the evolution of the velocity field (11.1) becomes the Euler equation, which conserves kinetic energy. In such a case, introducing an ultraviolet cutoff $K_{max}$ on the wave numbers, it is possible to build up an equilibrium statistical mechanics simply following the standard approach used in Hamiltonian statistical mechanics. However, because of the so-called dissipative anomaly [3, 12], in $3D$ the limit of zero viscosity is singular and cannot be interchanged with $K_{max} \to \infty$. In other words, given any viscosity as small as possible, there exist a wavenumber $k < K_{max}$ at which the dissipative term in (11.1) is not negligible and the energy dissipation rate reaches a value which is independent on $\nu$. This basic empirical property of turbulent flows implies that the statistical mechanics of an inviscid fluids has a rather limited relevance for the Navier-Stokes equations at very high Reynolds numbers $Re$ (which is equivalent to very small $\nu$).

In addition, mainly as a consequence of the non-Gaussian statistics, even a systematic statistical approach, e.g. in term of closure approximations, is very difficult [3, 12]. In the fully developed turbulence (FDT) limit, i.e. $Re \to \infty$, and in the presence of forcing at large scale, one has a non equilibrium statistical steady state, with an inertial range of scales, where neither energy pumping nor dissipation acts, which shows strong departures from the equipartition [3, 12].

The main features of FDT are described by the statistical theory of Kolmogorov developed in three papers published in 1941 (now called K41 theory) [13,14]. At the basis of the K41 theory [3, 13] there is the idea of turbulent *cascade* (introduced by Richardson in [15]): energy fluctuations, introduced at large scale by a mechanical forcing, reach the smallest scale (where they are converted into heat) via a scale-by-scale cascade process. As a conseguence, one may expect that small scale turbulence, at sufficiently high Reynolds numbers, is statistically independent on the large scales and can thus locally recover homogeneity and isotropy. This implies that small scale features of turbulence are universal, i.e. independent on the particular flow and forcing mechanism. The concept of homogeneous and isotropic turbulence was already introduced by Taylor [16] for describing grid generated turbulence. The important step made by Kolmogorov in 1941 was to postulate that small scales are statistically isotropic, no matter how turbulence is generated.

This hypothesis is based on intrinsic properties of the dynamics, i.e. the invariance of Navier-Stokes equations (11.1) under space translations, rotations and scaling transformation:

$$\boldsymbol{x} \to \lambda \boldsymbol{x} \quad , \boldsymbol{v} \to \lambda^h \boldsymbol{v} \quad , t \to \lambda^{1-h} t \quad , \quad \nu \to \lambda^{h+1} \nu \ , \tag{11.2}$$

for any $\lambda > 0$ and $h$ (and we have neglected the contribution of forcing). A classical example of scaling symmetry is the so-called similarity principle of fluid mechanics which states that two flows with the same geometry and the same Reynolds number are similar. The similarity principle is at the basis of laboratory modeling of engineering and geophysical flows where, because usually the fluid is water, its application requires $h = -1$ in (11.2) in order to keep the value of $\nu$.

Kolmogorov's treatment of small scale turbulence is based on the hypothesis that, in the limit of high Reynolds numbers and far from boundaries, the symmetries of Navier-Stokes equation are restored for statistical quantities. To be more precise, let us consider the velocity increment $\delta\boldsymbol{v}(\boldsymbol{x},\boldsymbol{\ell}) \equiv \boldsymbol{v}(\boldsymbol{x}+\boldsymbol{\ell}) - \boldsymbol{v}(\boldsymbol{x})$ over the scales $\ell \ll L$. Restoring of homogeneity in statistical sense requires that $\delta\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r},\boldsymbol{\ell}) \overset{law}{=} \delta\boldsymbol{v}(\boldsymbol{x},\boldsymbol{\ell})$, where equality in law means that the PDF of $\delta\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r},\boldsymbol{\ell})$ and $\delta\boldsymbol{v}(\boldsymbol{x},\boldsymbol{\ell})$ are identical. Similarly, statistical isotropy, also used by Kolmogorov in his 1941 papers, requires $\delta\boldsymbol{v}(A\boldsymbol{x},A\boldsymbol{\ell}) \overset{law}{=} \delta A\boldsymbol{v}(\boldsymbol{x},\boldsymbol{\ell})$ where $A$ is a rotation matrix. Because we will consider homogeneous, isotropic turbulence, in the following for simplicity we will use the notation $\delta\boldsymbol{v}(\boldsymbol{\ell})$ for the velocity increment.

In the limit of large Reynolds number, Kolmogorov made the hypothesis that for separation in the inertial range of scales $\ell_D \ll \ell \ll L$ (where the dissipative scale is $\ell_D \simeq LRe^{-3/4}$) the PDF of $\delta\boldsymbol{v}(\boldsymbol{\ell})$ becomes independent on viscosity $\nu$. As a consequence, in this limit and in this range of scales, scaling invariance (11.2) is statistically recovered without fixing the value of the scaling exponent $h$:

$$\delta\boldsymbol{v}(\lambda\boldsymbol{\ell}) \overset{law}{=} \lambda^h \delta\boldsymbol{v}(\boldsymbol{\ell}). \tag{11.3}$$

The values of the scaling exponent, $h$ are now limited only by the requiring that the velocity fluctuations do not break incompressibility, which is equivalent to $h \geq 0$ [3].

Starting from (11.1) Kolmogorov was able to derive an exact relation, known as the "4/5-th law" [3, 13], which, under the assumption of stationarity, homogeneity and isotropy, and in the inertial range of scales $\ell_D \ll \ell \ll L$ states

$$\langle \delta v_{||}^3(\ell) \rangle = -\frac{4}{5}\bar{\varepsilon}\ell\,, \tag{11.4}$$

where $\delta v_{||}(\ell)$ is the longitudinal velocity difference, i.e. $\delta v_{||}(\ell) = \delta\boldsymbol{v}(\ell)\cdot\boldsymbol{\ell}/\ell$ (which, under homogeneity and isotropy, depends on $\ell$ only). Assuming global scaling invariance, i.e. a unique exponent $h$ in (11.3), the 4/5-law (11.4) fixes its value to $h = 1/3$. As a consequence, one expects a power-law behavior in the inertial range for any structure function of velocity difference

$$S^{(p)}(\ell) \equiv \langle \delta v_{||}^p(\ell) \rangle = C_p \bar{\varepsilon}^{p/3} \ell^{p/3} \tag{11.5}$$

where the $C_p$ are dimensionless, universal constant, not determined by the theory except for $C_3 = -4/5$.

We remark that the fact that the third moment of velocity differences does not vanish is a consequence of the directional transfer (from large to small scales) of energy on average. An important consequence, which will be discussed in details, is that the PDF of velocity differences in turbulence cannot be Gaussian.

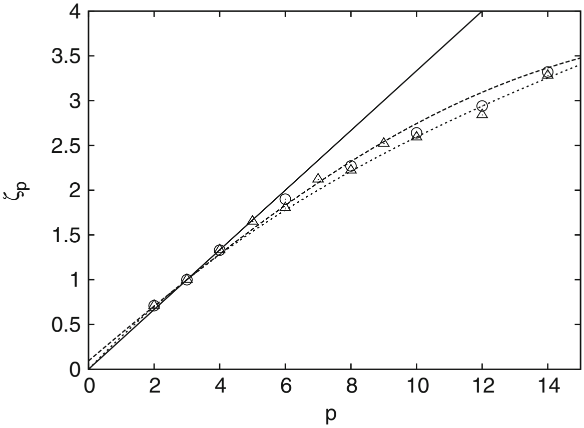


**Fig. 11.1** Structure function scaling exponents $\zeta_p$ plotted vs. $p$. *Circles* and *triangles* correspond to the data of different experiments (Anselmet et al. [17]). The *solid line* corresponds to Kolmogorov scaling $p/3$; the *dashed line* is the random beta model prediction (11.25) with $B = 1/2$ and $x = 7/8$; the *dotted line* is the She-Leveque prediction (11.29) with $\beta = 2/3$

## 11.3 Accounting for the Fluctuations: The Multifractal Model

Kolmogorov 41 theory is not exact because both experiments and numerical simulations show that higher order structure functions display unambiguous departure from the scaling exponents (11.5). Indeed one has

$$S^{(p)}(\ell) \sim \left(\frac{\ell}{L}\right)^{\zeta(p)} \tag{11.6}$$

with $\zeta(p) \neq p/3$. We remark that in (11.6) and in the following we do not include, for notation simplicity, the terms built on $\varepsilon$ and needed in order to make these expressions dimensionally correct. In Fig. 11.1 we report a collection of scaling exponents $\zeta(p)$ extracted from different experimental data [17]. Let us recall that the scaling exponents are not completely free as (11.4) requires $\zeta(3) = 1$. Under very general hypothesis, one can also demonstrate that $\zeta_p$ has to be a concave and nondecreasing function of $p$ [3]. From Fig. 11.1 it is evident that the $\zeta(p)$ exponents are firstly *universal* and secondarily *anomalous*, i.e. they are expressed by a non-linear function of $p$. This also means that the PDF's of velocity differences $\delta \boldsymbol{v}(\ell)$ not only deviate from the Gaussian (as required by (11.4)), but also that at different scales the PDF's are differents and that the skewness of velocity differences increases going to small scales.

The deviation of scaling exponents $\zeta_p$ from $p/3$ goes under the name of intermittency [3], and is physically due to the fact that the turbulent intensity and local energy dissipation $\varepsilon$ are strongly fluctuating in physical space. One consequence is that, for example, $\overline{\varepsilon^{p/3}} \neq \bar{\varepsilon}^{p/3}$ and therefore (11.5) is not justified (apart from $p = 3$, of course).

A simple way to modify the K41 consists in assuming that the energy dissipation $\varepsilon$ is distributed uniformly on a subset $S \subset \mathscr{R}^3$ of fractal dimension $D_F < 3$. This is equivalent to assume that $\delta \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{\ell}) \sim (\ell/L)^h$ with $h = (D_F - 2)/3$ for $\boldsymbol{x}$ on the fractal set $S$ and $\delta \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{\ell})$ non singular otherwise. The relation between $h$ and $D_F$ is obtained by the request that $\zeta_3 = 1$. This assumption leads to the so-called absolute curdling or $\beta$-model for which

$$\zeta_p = \frac{D_F - 2}{3} p + (3 - D_F)\,. \tag{11.7}$$

Such a prediction, with $D_F \simeq 2.83$, is in fair agreement with the experimental data for small values of $p$, but higher order scaling exponents give a clear indication of a non linear behavior in $p$ (see Fig. 11.1).

One generalization of the (fractal) $\beta$-model is the multifractal model of turbulence [1, 3, 18]. The multifractal model relaxes the assumption of global scale invariance for a more general local invariance, i.e. the existence of a continuous set of exponents $h$ such that $\delta v(\ell) \sim (\ell/L)^h$ where, as in the $\beta$-model, each exponent is realized on a different fractal set of dimension $D(h)$. More precisely one assumes that in the inertial range of scales $\ell$ one has

$$\delta \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{\ell}) \sim \left(\frac{\ell}{L}\right)^h\,, \tag{11.8}$$

if $\boldsymbol{x} \in S_h$, where $S_h$ is a fractal set with dimension $D(h)$ and $h \in (h_{min}, h_{max})$. The probability to observe a given scaling exponent $h$ at the scale $\ell$ is determined by the codimension $3 - D(h)$ of the fractal set as $P_\ell(h) \sim \ell^{3-D(h)}$ and therefore

$$S_p(\ell) \sim \int_{h_{min}}^{h_{max}} \ell^{hp} \ell^{3-D(h)} \mathrm{d}h \sim \ell^{\zeta_p}\,. \tag{11.9}$$

For $\ell \ll 1$, a steepest descent estimation gives the scaling exponent

$$\zeta_p = \min_h \{hp + 3 - D(h)\} = h^* p + 3 - D(h^*) \tag{11.10}$$

where $h^* = h^*(p)$ is the solution of the equation $D'(h^*(p)) = p$. The Kolmogorov 4/5-th law (11.4) imposes $\zeta_3 = 1$ which implies that

$$D(h) \le 3h + 2\,, \tag{11.11}$$

with the equality realized by $h^*(3)$. We remark that the Kolmogorov similarity theory $\zeta_p = p/3$ corresponds to the case of only one singularity exponent $h = 1/3$ with $D(h = 1/3) = 3$.

It is important to remark that the multifractal model is not predictive in a strict sense as it depends on an infinite set of parameters (the function $D(h)$) which are not derived from the Navier-Stokes equations. Nonetheless, it is able to reproduce the set of scaling exponents $\zeta_p$ on the basis of simple phenomenological arguments, as it will be discussed in the next section. Moreover, once $D(h)$ has been obtained from a model or from experimental data, the multifractal model can be used to make predictions on other statistical quantities in turbulence [19].

Let us now discuss an important issue related to the use of large deviation theory in the multifractal formalism. To obtain the scaling behavior of $S_p(\ell) \sim (\ell/L)^{\zeta_p}$ given by (11.9) with $\zeta_p$ obtained from (11.10), one has to assume that the exponent $ph + 3 - D(h)$ has a minimum, $\zeta_p$, which is a function of $h$, and that such an exponent behaves quadratically with $h$ in the vicinity of the minimum. This is the basic assumption to apply the Laplace's method of steepest descent. The point we would like to discuss here is that, for small separations $\ell$, indeed $S_p(\ell) \sim (\ell/L)^{\zeta_p}$ but with a logarithmic prefactor:

$$S_p(\ell) \sim \left[-\ln\left(\frac{\ell}{L}\right)\right]^{-1/2} \left(\frac{\ell}{L}\right)^{\zeta_p} . \tag{11.12}$$

Such a prefactor is usually not considered in the naive application of Laplace method leading to (11.10). Moreover, the presence of logarithmic correction would clearly invalidate the $4/5$-th law (11.4), which is an exact results obtained from the Navier-Stokes equations.

The problem to reconcile logarithmic corrections in the multifractal model with the $4/5$-th law has been quantitatively addressed by Frisch et al. [20]. There, exploiting the refined large-deviations theory, the Authors were able to show in which way logarithmic contributions cancel out thus giving a prediction compatible with the naive (and a priory not justified) procedure to extract the scaling behavior (11.9). The key point is that the leading order large deviation result for the probability $P_\ell(h)$ to be within a distance $\ell$ of the set $S_h$ carrying singularities of scaling exponent $h$ must be extended to take into account next subleading order. As a result one obtains [20]

$$P_\ell(h) \sim \left(\frac{\ell}{L}\right)^{3-D(h)} \left[-\ln\left(\frac{\ell}{L}\right)\right]^{1/2} , \tag{11.13}$$

which contains subleading logarithmic correction. It is worth observing that despite the multiplicative character of the logarithmic correction one speaks of "subleading correction". This is justified by the fact that the correct statement of the large-deviations leading-order result involves the logarithm of the probability divided by the logarithm of the scale. The correction is then a subleading additive term.

Once the expression (11.13) is plugged in the integral

$$S_p(\ell) \sim \int dh P_\ell(h) \left(\frac{\ell}{L}\right)^{ph} \tag{11.14}$$

and the saddle point estimation is carried out according to [21], logarithms disappear and the $4/5$-th law is correctly recovered.

It is worth mentioning that the presence of a square root of a logarithm correction in the multifractal probability density had already been discussed in [22] on the basis of a normalization requirement. In that paper, the Authors also pointed out that a similar correction has been proposed by [23] in connection with the measurement of generalized Renyi dimensions.

We conclude this section by observing that anomalous scaling for the velocity differences implies that the local dissipative scale, $\ell_D$, does not take a unique value. The latter scale is indeed determined by imposing the effective Reynolds number to be of order unity:

$$Re(\ell_D) = \frac{\delta v_D \ell_D}{\nu} \sim 1 \, , \tag{11.15}$$

therefore the dependence of $\ell_D$ on $h$ is thus

$$\ell_D(h) \sim L Re^{-\frac{1}{1+h}} \tag{11.16}$$

where $Re = Re(L)$ is the large scale Reynolds number [18]. The fluctuation of the dissipative scale has important consequences on the statistics of small scale quantities, such as velocity gradients and acceleration, which will be discussed in the next sections. Another consequence is that it predicts the existence of an *intermediate dissipation range* at the lower bound of the inertial range, where the inertial range contributions of the various scaling exponents $h$ are successively turned off [24].

### *11.3.1 The Statistics of Velocity Gradient*

Let us denote by $s$ the longitudinal velocity gradient, e.g. $s = \partial u_x/\partial x$. On the basis of the above considerations, this quantity can be expressed in terms of the singularity exponents $h$ as

$$s \sim \frac{\delta v_D}{\ell_D} = v_0 \ell_D^{h-1} = v_0^{\frac{2}{1+h}} \nu^{\frac{h-1}{h+1}} \tag{11.17}$$

where we used the fact that $\delta v_D \simeq v_0 (\ell_D/L)^h$ and we have exploited (11.16). From (11.17) we realize that we can easily express the probability density function (PDF) of $s$ (for a fixed $h$), $P_h(s)$, in terms of the PDF, $\Pi(V_0)$, of the large-scale velocity differences $V_0$, with $v_0 \equiv |V_0|$. The latter PDF is indeed known to be in general well described by a Gaussian distribution [25]. The link between the two PDFs is given by the standard relation:

$$P_h(s) = \Pi(V_0) \left| \frac{dV_0}{ds} \right| \tag{11.18}$$

from which one immediately gets:

$$P_h(s) \sim \frac{\nu}{|s|}^{\frac{1-h}{2}} e^{-\frac{\nu^{1-h}|s|^{1+h}}{2\langle V_0^2 \rangle}} . \tag{11.19}$$

The K41 theory corresponds to $h = 1/3$, therefore (11.19) predicts a stretched exponential form for the PDF with an exponent, $1 + h$, larger than one. Experimental data (see e.g. [26, 27]) are not consistent with this prediction and indicate for the tail of the PDF a stretched exponential with exponent smaller than one. In this respect, the multifractal description has been used to describe correctly these experimental evidences (see, e.g., [19] for a derivation).

### *11.3.2 The Statistics of Acceleration*

Acceleration in fully developed turbulence is an extremely intermittent quantity which displays fluctuations up to 80 times its root mean square [28]. These extreme events generate very large tails in the PDF of acceleration which is therefore expected to be very far from Gaussian.

We remark that even within non-intermittent, Kolmogorov scaling turbulence, acceleration PDF is expected to be non-Gaussian. Indeed acceleration can be estimated from velocity fluctuations at the Kolmogorov scale as

$$a = \frac{\delta v(\tau_D)}{\tau_D} \tag{11.20}$$

where $\tau_D = \ell_D/\delta v_D$ and the Kolmogorov scale $\ell_D$ is given by the condition $\ell_D \delta v_D/\nu = 1$. Using the relation $\delta v(\ell) \simeq v_0(\ell/L)^h$ (with $h = 1/3$ for Kolmogorov scaling) one obtains

$$\frac{\ell_D}{L} \sim \left( \frac{v_0 L}{\nu} \right)^{-\frac{1}{1+h}} \tag{11.21}$$

and finally

$$a = \frac{v_0^2}{L} \left( \frac{v_0 L}{\nu} \right)^{\frac{1-2h}{1+h}} \tag{11.22}$$

Similarly to the derivation of the velocity gradient, assuming a Gaussian distribution for large scale velocity fluctuations $v_0$, and taking $h = 1/3$, one obtains for the PDF of $a$ a stretched exponential tail $p(a) \sim \exp(-Ca^{8/9})$.

In the presence of intermittency the above argument has to be modified by taking into account the fluctuations of the scaling exponent and of the dissipative scale. In the recent years, several models have been proposed for describing turbulent acceleration statistics, on the basis of different physical ingredients. In the following we show that the multifractal model of turbulence, when extended to describe fluctuation at the dissipative scale, is able to predict the PDF of acceleration observed in simulations and experiments with high accuracy [29]. Moreover the model does not require the introduction of new parameters, besides the set of Eulerian scaling exponents. In this sense, multifractal model become a *predictive* model for the statistics of the acceleration.

Accounting for intermittency in the above argument is simply obtained by weighting (11.22) with both the distribution of $v_0$ (still assumed Gaussian, as intermittency is not expected to affect large scale statistics) and the distribution of scaling exponent $h$ which can be rewritten, using (11.21), as

$$p(h) \sim \left(\frac{\ell_D}{L}\right)^{3-D(h)} \sim \left(\frac{v_0 L}{\nu}\right)^{\frac{D(h)-3}{1+h}} \tag{11.23}$$

The final prediction, when written for the dimensionless acceleration $\tilde{a} = a/\langle a^2\rangle^{1/2}$, becomes [29]

$$p(\tilde{a}) \sim \int_h \tilde{a}^{[h-5+D(h)]/3} R_\lambda^{y(h)} \exp\left(-\frac{1}{2}\tilde{a}^{2(1+h)/3} R_\lambda^{z(h)}\right) dh \tag{11.24}$$

where $y(h) = \chi(h-5+D(h))/6 + 2(2D(h)+2h-7)/3$ and $z(h) = \chi(1+h)/3 + 4(2h-1)/3$. $R_\lambda = v_{rms}\lambda/\nu$ is the Reynolds number based on the Taylor scale $\lambda = v_{rms}/\langle(\partial_x v_x)^2\rangle^{1/2}$. The coefficient $\chi$ is the scaling exponent for the Reynolds dependence of the acceleration variance, $\langle a^2\rangle \sim R_\lambda^\chi$. Its expression is given by $\chi = \sup_h (2(D(h)-4h-1)/(1+h))$. For the non-intermittent Kolmogorov scaling ($h = 1/3$ and $D(1/3) = 3$) one obtains $\chi = 1$ and (11.24) recovers the stretched exponential prediction discussed above.

We note that (11.24) may show an unphysical divergence for $a \to 0$ for many multifractal models of $D(h)$ at small $h$. This is not a real problem for two reasons. First, the multifractal formalism cannot be extended to very small velocity and acceleration increments because it is based on arguments valid only to within a constant of order one. Thus, it is not suited for predicting precise functional forms for the core of the PDF. Second, small values of $h$ correspond to very intense velocity fluctuations which have never been accurately tested in experiments or DNS. The precise functional form of $D(h)$ for those values of $h$ is therefore unknown.

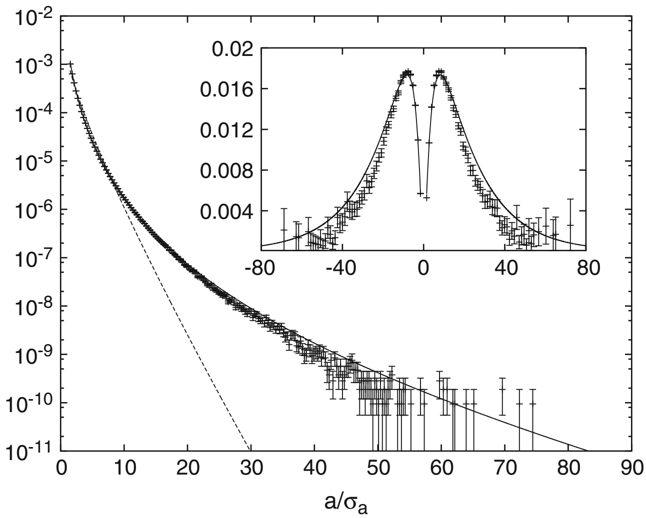


**Fig. 11.2** Log-linear plot the PDF of the acceleration. Points are obtained from Direct Numerical Simulations of homogeneous-isotropic turbulence at $R_\lambda \simeq 280$ [29] with the statistics of $10^{10}$ events. The *dashed line* represents the K41 prediction $p(a) \sim exp(-Ca^{8/9})$. The *continuous line* is the multifractal prediction. *Inset*: $\tilde{a}^4 p(\tilde{a})$ for the DNS data (*crosses*) and the multifractal prediction

Figure 11.2 shows the comparison between the PDF of the acceleration obtained from high-resolution Direct Numerical Simulations [29] together with the theoretical prediction obtained from K41 and the multifractal models. The figure clearly shows that the multifractal model is able to capture accurately the shape of the PDF. It is remarkable is that (11.24) agrees with the DNS data over a wide range of fluctuations – from the order of one standard deviation $\sigma_a$ up to order $70\sigma_a$. We emphasize that the only free parameter in the multifractal formulation of $p(\tilde{a})$ is the minimum value of the acceleration, $\tilde{a}_{\min}$.

### *11.3.3 Multiplicative Processes for the Multifractal Model*

We have seen in the previous section that the knowledge of the function $D(h)$ allows to predicts several features of a turbulent flow. An analytic computation of $D(h)$, or equivalently $\zeta_p$, from the Navier-Stokes equations is a prohibitive task. In the past years a different approach has been developed, based on a phenomenological approach which gives closed expression for $D(h)$ on the basis of multiplicative processes. The use of multiplicative processes is inspired again from the Richardson cascade picture and the log-normal theory of Kolmogorov.

Let us briefly remind the so-called random $\beta$-model [2], generalization of the $\beta$-model discussed at the beginning of Sect. 11.3. This model describes the energy cascade in real space looking at eddies of size $\ell_n = 2^{-n}L$, with $L$ the length at which the energy is injected. At the $n$-th step of the cascade a mother eddy of size $\ell_n$ splits into daughter eddies of size $\ell_{n+1}$, and the daughter eddies cover a fraction $\beta_j$ ($0 < \beta_j < 1$) of the mother volume. The $\beta_j$'s are independent, identically distributed random variables (the probabilistic nature of $\beta_j$ reflects the complex dynamics generated by the Navier-Stokes equations). Therefore the velocity fluctuations $v_n = \delta v_{\ell_n}$ at the scale $\ell_n$ receive contributions only on a fraction of volume $\prod_j \beta_j$. Taking into account the fact that the energy flux must be constant throughout the cascade (i.e. the $4/5$-th law), one has

$$v_n = v_0 \ell_n^{1/3} \prod_{j=1}^{n} \beta_j^{-1/3} . \tag{11.25}$$

As stated above, all the physics is contained in the distribution of the coefficients $\beta_j$. A simple, and somehow phenomenologically motivated choice is to take $\beta_j = 1$ with probability $x$ and $\beta_j = B = 2^{-(1-3h_{min})}$ with probability $1-x$ (we remark that the distribution is independent on the scale). This multiplicative process generates a two-scale Cantor set, which is a common structure in chaotic systems. The resulting scaling exponents are given by

$$\zeta_p = \frac{p}{3} - \ln_2[x + (1-x)B^{1-p/3}] \tag{11.26}$$

corresponding to

$$D(h) = 3 + \big(3h - 1\big)\Big[1 + \ln_2\Big(\frac{1-3h}{1-x}\Big)\Big] + 3h \ln_2\left(\frac{x}{3h}\right) . \tag{11.27}$$

The two limit cases of interest are $x = 1$, i.e. K41 theory with $\zeta_p = p/3$, and $x = 0$ which gives the $\beta$-model with $D_F = 2 + 3h_{min}$. Using $x = 7/8$, $h_{min} = 0$ (i.e. $B = 1/2$) one has a good fit for the $\zeta_p$ of the experimental data (see Fig. 11.1).

There are many others different models which fit well the experimental scaling exponents, all based on some physical arguments. A popular model is the so-called She-Leveque model [30] where vortex filaments are a fundamental ingredient for intermittency. In terms of the multifractal model, the She-Leveque model is obtained by taking

$$D(h) = 1 + \frac{2\beta - 3h - 1}{\ln \beta}\left[1 - \ln\left(\frac{2\beta - 1 - 3h}{2 \ln \beta}\right)\right] \tag{11.28}$$

and gives for the scaling exponents

$$\zeta_p = \frac{2\beta - 1}{3} p + 2(1 - \beta^{p/3}) . \tag{11.29}$$

The set of exponents given by (11.29) are close to the experimental data for $\beta = 2/3$ (see Fig. 11.1). Another important model, which was introduced by Kolmogorov himself without reference to the multifractal model, is the recalled log-normal model which will be discussed in the next section.

## 11.4 Fluctuations of the Energy Dissipation Rate

On September 1961, Kolmogorov gave a famous talk at a turbulence colloquium organized in Marseille. In this talk he presented new hypotheses, due to himself and to Obukhov which constitute the basis of what is known as the Kolmogorov-Obukhov 62 (KO62) theory [5].

At that time, there were not strong experimental motivations to call for an improvement of the K41 theory. The main criticisms were based on a theoretical ground and were due to a remark by Landau. The Landau's remark, as reported for instance by Frisch [3], states that the constants in (11.5), for example the constant $C_2$ for the second order longitudinal structure function, cannot be universal. As $\overline{\varepsilon^{2/3}}$ differs from $\overline{\varepsilon}^{2/3}$, the former depends from the distribution of $\varepsilon$ at large scales, close to the integral one, which cannot be universal, as it depends on the forcing mechanism. This remark applies to all the structure functions and implies that $C_p$ (a part $C_3$ which depends only on the average $\overline{\varepsilon}$) cannot be universal. The point that Kolmogorov emphasized, starting from this remark, is that dissipation is concentrated on very tiny regions of the flow. This may lead to anomalous values for the scaling exponents of velocity structure functions.

To take into account this point, Kolmogorov introduced the coarse grained energy dissipation on a ball of radius $\ell$ centered on **x**

$$\varepsilon_\ell(\boldsymbol{x}, t) = \frac{1}{4/3\pi\ell^3} \int_{|y|<\ell} d\,\boldsymbol{y}\; \varepsilon(\boldsymbol{x} + \boldsymbol{y}, t) \tag{11.30}$$

and postulated that the dimensionless quantity

$$\frac{\delta v(\ell)}{\varepsilon_\ell^{1/3} \ell^{1/3}} \tag{11.31}$$

has a probability distribution independent of the local Reynolds number $Re_\ell = \delta v(\ell)\ell/\nu$ in the limit $Re_\ell \to \infty$. This is what is called the Refined Similarity Hypothesis (RSH). This hypothesis links the scaling laws of velocity structure functions with the scaling properties of the energy dissipation:

$$S_p(\ell) \sim \langle \delta v^p(\ell) \rangle \sim C_p \overline{\varepsilon_\ell^{p/3}} \ell^{p/3} \tag{11.32}$$

Kolmogorov then introduces a simple multiplicative model for the statistics of $\varepsilon_\ell$. This leads to a Gaussian distribution for the logarithm of $\varepsilon_\ell$ with variance (for $\ln \varepsilon_\ell$)

$$\sigma_\ell^2 = A + 9\mu \ln(L/\ell) \tag{11.33}$$

The lognormal model leads to a parabolic prediction for scaling exponents

$$\zeta_p = \frac{p}{3} + \frac{\mu}{18} p(3-p) \tag{11.34}$$

in which the value of the free parameter can be fixed by experimental data as $\mu \simeq 0.025$.

The lognormal model KO62 can be described within the general framework of multifractal model by taking a quadratic $D(h)$

$$D(h) = -\frac{9}{2\mu}h^2 + 3\frac{2+\mu}{2}h - \frac{4-20\mu+\mu^2}{8\mu} \tag{11.35}$$

which, inserted in (11.10), leads to (11.34).

It can be useful to highlight the relationship between the multifractal model for fully developed turbulence and the description of singular measures (e.g. in chaotic attractors) based on the so-called $f(\alpha)$ spectrum [12]. For this purpose, let us introduce the measure $\mu(\mathbf{x}) = \varepsilon(\mathbf{x})/\bar{\varepsilon}$, based on the local energy dissipation rate, a partition of non overlapping cells $\Lambda_\ell$ of size $\ell$ and the coarse graining probability

$$P_i(\ell) = \int_{\Lambda_\ell(\mathbf{x}_i)} d\mu(\mathbf{x}) \tag{11.36}$$

where $\Lambda_\ell(\mathbf{x}_i)$ is a cube of edge $\ell$ centered in $\mathbf{x}_i$.

The coarse grained energy dissipation averaged over $\Lambda_\ell$ is given by $\varepsilon_\ell \sim \bar{\varepsilon}\ell^{-3}P(\ell)$. Denoting by $\alpha$ the scaling exponent of $P_\ell$ and with $f(\alpha)$ the fractal dimension of the subfractal with scaling exponent $\alpha$, we can introduce the Renyi dimensions [18] $d_p$:

$$\sum_i P_i(\ell)^p \sim \ell^{(p-1)d_p} \tag{11.37}$$

where the sum is over the non empty boxes. A simple computation gives

$$(p-1)d_p = \min_\alpha[p\alpha - f(\alpha)] \ . \tag{11.38}$$

Noting that from the definition

$$\langle \varepsilon_\ell^p \rangle = \ell^3 \sum \varepsilon_\ell^p \tag{11.39}$$

we finally have

$$\langle \varepsilon_\ell^p \rangle \sim \ell^{(p-1)(d_p-3)} \tag{11.40}$$

In conclusion, we have the following correspondence between the multifractal model and the $f(\alpha)$ spectrum

$$h \leftrightarrow \frac{\alpha - 2}{3} \ , \ \ D(h) \leftrightarrow f(\alpha) \ , \ \ \zeta_p = \frac{p}{3} + \left(\frac{p}{3} - 1\right)\left(d_{\frac{p}{3}} - 3\right) . \tag{11.41}$$

which, as it should, gives $\zeta_3 = 1$ independently on the form of $f(\alpha)$.

## 11.5 Conclusions

The experimental study of fully developed turbulence led to the introduction of large deviation theory, in the form of the multifractal model, in order to describe the intermittent nature of the turbulent flow. The multifractal model has been successfully used to describe many features of turbulent flows: from scaling exponents of the structure functions to the statistics of the velocity gradients and acceleration, to the scaling of Lagrangian quantities. Despite the fact that the model is not predictive, once the function $D(h)$ is given, or measured from experimental data, all the other quantities are given without free parameters. In this sense we can see at the multifractal model as a tool which provides, within the general framework of large deviations, a general and consistent comprehension of different aspects of turbulence.

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