Chapter 8
Heat Transfer and Heat Generation

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8.1 Thermal Conductivity and Resistance

Thermal conductivity is a decisive parameter for the sizing of heat sinks for semiconductors or for other elements in electronic circuits. It is defined as

$$\Lambda_W = \frac{Q}{\delta T}, \tag{8.1}$$

where Q is the heat flux through the element and δT is the difference in temperature between both ends. Alternately, the thermal resistance R_W is used, which is simply the inverse of the thermal conductivity:

$$R_W = \frac{1}{\Lambda_W} = \frac{\delta T}{Q}. \tag{8.2}$$

The *heat flux density* $\vec{q}$ in an isotropic continuum is proportional to the temperature gradient:

$$\vec{q} = -\lambda \nabla T, \tag{8.3}$$

where λ is the specific thermal conductivity.

The change in temperature in a homogenous medium is described by the heat equation

$$\rho c \frac{\partial T}{\partial t} = -\operatorname{div} \vec{q} = \lambda \Delta T, \tag{8.4}$$

in which ρ is the density and c is the specific heat capacity of the medium. Using the thermal diffusivity $\alpha = \lambda / \rho c$, Eq. (8.4) can also be written in the form

$$\frac{\partial T}{\partial t} = \alpha \Delta T. \tag{8.5}$$

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In the steady-state case, the temperature distribution must satisfy the Laplace equation

$$\Delta T = 0, \tag{8.6}$$

the solution of which is the next topic of discussion for various boundary conditions. The results will directly show that also heat conductivity problems can be exactly solved within the framework of the method of dimensionality reduction. The mappability is not only limited to the thermal conductivity or resistance, but rather includes also local parameters, such as the temperature distribution on the surface.

8.2 Temperature Distribution for a Point Heat Source on a Conductive Half-Space

We consider a point heat source Q on an isotropic half-space, as shown in Fig. 8.1. With the exception of the location of the point source, let the entire surface be ideally insolated (adiabatic) and at an infinite distance, the temperature T_0 is reached. With these thermal boundary conditions, the solution to the steady-state conduction problem (see, for example [1]) is

$$\delta T(R) := T(R) - T_0 = \frac{Q}{2\pi\lambda R} \quad \text{with } R := \sqrt{x^2 + y^2 + z^2}. \tag{8.7}$$

On the surface of the half-space ($z = 0$), the resulting temperature distribution is

$$\delta T(r) := T(r) - T_0 = \frac{Q}{2\pi\lambda r} \quad \text{with } r := \sqrt{x^2 + y^2}. \tag{8.8}$$

A relationship equivalent to (8.8) appears also in the *elastic* problem, which was shown by Francis [2], among others. The normal surface displacement of an elastic half-space caused by a normal force at the origin is [3]

$$\bar{u}_z(r) = \frac{1 - \nu^2}{\pi E} \frac{F_N}{r}. \tag{8.9}$$

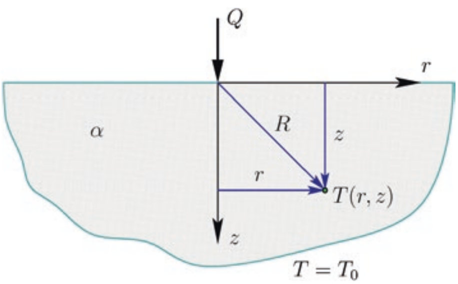

Fig. 8.1 Point heat source Q on a homogeneous half-space with the thermal diffusivity α

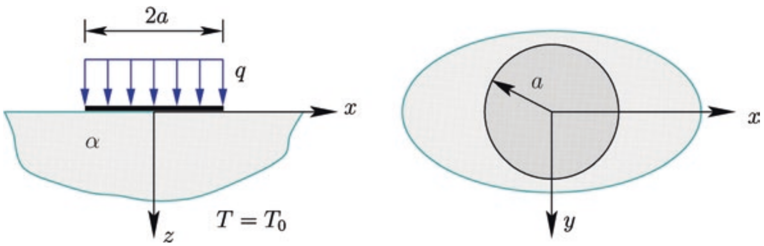

Fig. 8.2 Constant heat flux density from a *circular area* of radius a into the half-space; *cross-sectional view* in the x–z plane (*left*), *top view* (*right*)

Following this analogy and the interpretation of (8.8) and (8.9) as Green's functions of the corresponding problem, arbitrary heat flux density distributions $q(x, y)$ on the surface of the half-space present no difficulties. In place of the explicit calculation of the integral

$$\delta T(x, y) = \frac{1}{2\pi\lambda} \iint_A \frac{q(\tilde{x}, \tilde{y})}{\sqrt{(x-\tilde{x})^2 + (y-\tilde{y})^2}} \mathrm{d}\tilde{x}\, \mathrm{d}\tilde{y}, \tag{8.10}$$

we can call on the solution of the (equivalent) elastic problem and transfer this directly to the heat transfer problem. For this, we need only undertake the following reassignments:

$$p(x, y) \mapsto q(x, y), \quad \bar{u}_z(x, y) \mapsto \delta T(x, y), \quad \text{and } E/\left(1-\nu^2\right) \mapsto 2\lambda, \tag{8.11}$$

where $q(x, y)$ is the component of the heat flux density that is normal to the surface. Figure 8.2 shows an example of a constant heat flux density (isoflux) on a circular area with the radius a. Determining the corresponding temperature distribution on the surface is the goal of Problem 5.

Let it be mentioned that the equivalence is limited to the surface and is not valid for the field within the media. This does not, however, affect the heat flux Q through the surface, which is calculated by integrating the heat flux density over the surface:

$$Q := \int_A q(x, y) dA. \tag{8.12}$$

In the elastic problem, this is the role of the normal force, which is similarly defined as the integral of the normal stress.

It is known from Chap. 3 that every axially-symmetric elastic contact problem can be mapped exactly to a one-dimensional model. Due to the existing equivalence between the heat transfer and the elastic contact, characterized by the reassignments in (8.11), the dimensionality reduction must also be valid for these problems.

8.3 The Universal Dependence of Thermal Conductivity and Contact Stiffness

If two half-spaces are in an ideal thermal contact by means of a circular area with the radius a and the temperature difference between the two is δT at an infinite distance, then the entire steady-state heat flux through the contact area is

$$Q = 4a\lambda^* \delta T \tag{8.13}$$

and the conductivity of the contact is [1]

$$\Lambda_W := \frac{Q}{\delta T} = 4a\lambda^* \quad \text{with} \quad \frac{1}{\lambda^*} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}. \tag{8.14}$$

Here, λ_1 and λ_2 denote the specific thermal conductivity of the two half-spaces and we can summarize λ^* as a type of *effective* specific thermal conductivity. Comparing this to the contact stiffness of a circular contact with the radius a,

$$k_z := \frac{\mathrm{d}F_N}{\mathrm{d}\delta} = 2aE^*, \tag{8.15}$$

shows that there exists the following relationship between the thermal conductivity and contact stiffness:

$$\Lambda_W = \frac{2\lambda^*}{E^*} k_z. \tag{8.16}$$

Both properties are proportional to the characteristic *length* of the contact. Interestingly, the validity of Eq. (8.16) goes much beyond the circular contact. It is, likewise, valid for individual contacts with arbitrarily formed isothermal contact areas and even remains unchanged for the contact between rough surfaces (Sevostianov and Kachanov [4], Barber [5]). This universal relation has a very important meaning, because with its help, one must not investigate both the thermal and elastic behavior of a contact separately. Contact stiffness and thermal conductivity are connected in a simple way.

It is generally known that thermal conduction and electrical conduction are equivalent problems. If a constant electric potential difference U is applied at a sufficiently large separation distance over the contact between two half-spaces, then a steady-state electric current flows through the contact area. If we once again assume a circular ideal contact (without impurities), then the entire current must flow through this *constriction,* which is characterized by the so-called constriction resistance R_E and can be interpreted as the contact resistance. The entire electrical current I through the equipotential contact area is

$$I = \frac{4a}{\rho_1 + \rho_2} U \tag{8.17}$$

and the corresponding constriction resistance is

$$R_E := \frac{U}{I} = \frac{\rho_1 + \rho_2}{4a}, \tag{8.18}$$

where ρ_1 and ρ_2 are the specific resistances of the two bodies. If instead of the resistances in Eq. (8.18), we use the inverse of the (specific) electrical conductivities, this leads to the electrical contact conductivity

$$\Lambda_E := \frac{I}{\delta V} = 4a\lambda_E^* \quad \text{with } \frac{1}{\lambda_E^*} = \frac{1}{\lambda_{E1}} + \frac{1}{\lambda_{E2}}. \tag{8.19}$$

Completely identically to the thermal contact, the electrical conductivity is proportional to the *contact length*. Except for the form factor, the proportionality is also valid for contact areas of other forms as well as multiple micro-contacts sufficiently far from one another. For the latter, the contact length is the sum of the characteristic diameters for the so-called *a-spots* [6].

Of course, the conductivity for arbitrary contacts can also be determined from the incremental contact stiffness, because Eq. (8.16) remains absolutely valid when replacing the thermal properties by the analogous electrical properties.

8.4 The Implementation of the Steady-State Current Flow Within the Framework of the Reduction Method

The contact stiffness of arbitrary axially-symmetric bodies and rough contact is correctly mapped using the method of dimensionality reduction. A simple way for calculating the thermal and electrical conductivity of a (rough) contact consists of first determining the contact stiffness using the method of dimensionality reduction and subsequently calculating the conductivity using Eq. (8.16). Alternatively, we can look at every element of the linearly elastic foundation as having a (specific) conductivity of

$$\Delta\Lambda = 2\lambda^* \cdot \Delta x. \tag{8.20}$$

The latter is imperative, when mapping contacts with arbitrary thermal or electrical boundary conditions.[1] Due to the analogy with the elastic problem in the form of the reassignments in Eq. (8.11), both the global relations and the local parameters on the surface can be correctly mapped. According to Eq. (8.11), the thermal flow density $q(r)$ takes over the role of the normal stress σ_{zz} and temperature, the role of the normal surface displacement.

As an example, we want to investigate the thermal contact between two half-spaces. At an infinite distance, there exists a temperature difference of δT. The non-contacting surface is adiabatic and the contact area has a radius of a. We would like to determine the heat flux Q, the thermal resistance R_W, and the distribution of the heat flux density q within the contact area. In the three-dimensional

[1] In the following, we constrict ourselves to the mapping of thermal contacts, because these can be directly transferred to electrical contacts.

problem, there is a so-called isothermal contact area. This means that every point on the contact area has the same temperature. The equivalent elastic problem is the indentation of a flat cylinder, the equivalent profile of which remains the same. This leads to the fact that the temperature in all of the elements of the foundation is the same and also that the heat flux through every element ΔQ is independent of the coordinate:

$$\Delta Q(x) = \Delta\Lambda \cdot \delta T(x) = 2\lambda^* \cdot \Delta x \cdot \delta T. \tag{8.21}$$

The flux density j (*per unit length in the one-dimensional system*) is equal to

$$j(x) = \frac{\Delta Q(x)}{\Delta x} = 2\lambda^* \cdot \delta T \tag{8.22}$$

and the entire flux is found by integration of the one-dimensional flux density over the contact area:

$$Q := \int_{-a}^{a} j(x)\mathrm{d}x = 2\int_{0}^{a} 2\lambda^*\delta T\,\mathrm{d}x = 4a\lambda^*\delta T, \tag{8.23}$$

which corresponds to the three-dimensional result (8.13). The same is true for the thermal resistance

$$R_W := \frac{\delta T}{Q} = \frac{1}{4a\lambda^*}. \tag{8.24}$$

Analogously to the elastic contact, we can calculate the three-dimensional heat flux density $q(r)$ by using the Abel transformation (3.37) of the one-dimensional flux density $j(x)$:

$$q(r) := -\frac{1}{\pi}\frac{1}{r}\frac{d}{dr}\int_{r}^{a}\frac{x \cdot j(x)}{\sqrt{x^2 - r^2}}\mathrm{d}x = -\frac{1}{\pi}\int_{r}^{a}\frac{j'(x)}{\sqrt{x^2 - r^2}}\,\mathrm{d}x + \frac{1}{\pi}\frac{j(a)}{\sqrt{a^2 - r^2}}. \tag{8.25}$$

In the present case of a constant, one-dimensional flux density according to (8.22), the integral on the right-hand side of (8.25) disappears so that only the three-dimensional flux density remains:

$$q(r) = \frac{1}{\pi}\frac{2\lambda^*\delta T}{\sqrt{a^2 - r^2}}. \tag{8.26}$$

Also this result corresponds exactly to the three-dimensional distribution. In the thermal contact considered, we assume an *isothermal* contact surface. In the case of an axially-symmetric, spatial temperature distribution, we must transfer the three-dimensional to a one-dimensional temperature distribution. The respective transformation takes place in the familiar way (3.27):

$$\delta T_{1D}(x) = \delta T_{3D}(0) + |x|\int_{0}^{|x|}\frac{\delta T'_{3D}(r)}{\sqrt{x^2 - r^2}}\mathrm{d}r. \tag{8.27}$$

The constant term on the right-hand side disappeared in the equivalent elastic problem by choosing the appropriate coordinates,[2] the second term expresses the same relationship as that in Eq. (3.27). As it will be seen in the next section, the inverse question is also interesting: How can we determine the three-dimensional temperature distribution from the one-dimensional distribution? Referring to [7], the inverse transformation is

$$\delta T_{3D}(r) = \frac{2}{\pi} \int_0^r \frac{\delta T_{1D}(x)}{\sqrt{r^2 - x^2}} \, \mathrm{d}x. \tag{8.28}$$

In the mentioned literature, the transformation is given as well as the physical interpretation that allows for the calculation of the one-dimensional flux density distribution from the three-dimensional distribution:

$$j(x) = 2 \int_x^a \frac{r \cdot q(r)}{\sqrt{r^2 - x^2}} \, \mathrm{d}r. \tag{8.29}$$

We would like to clarify its application using a simple example. In this example, we assume that a stable constant thermal flux density is given on the surface of the half-space within a circle of radius a (see Fig. 8.2) of

$$q(r) = q_0 \quad \text{for} \quad 0 < r < a, \tag{8.30}$$

and the rest of the surface is adiabatic. We want to find the one-dimensional flux density and the one-dimensional and three-dimensional temperature distribution. When taking Eq. (8.30) into account, Eq. (8.29) provides the one-dimensional flux density

$$j(x) = 2 \int_x^a \frac{r q_0}{\sqrt{r^2 - x^2}} \, \mathrm{d}r = 2 q_0 \sqrt{a^2 - x^2}, \tag{8.31}$$

which of course leads to the entire flux of the original contact after integrating over the contact length:

$$Q = \int_{-a}^a j(x) \mathrm{d}x = 4 q_0 \int_0^a \sqrt{a^2 - x^2} \, \mathrm{d}x = 4 q_0 a^2 \int_0^{\pi/2} \cos^2 \varphi \, \mathrm{d}\varphi = q_0 \pi a^2. \tag{8.32}$$

In the one-dimensional model, the temperature of the element is proportional to the flux density at that point (Eq. 8.22). For this example, it is

$$\delta T_{1D}(x) = \frac{1}{2\lambda^*} j(x) = \frac{q_0}{\lambda^*} \sqrt{a^2 - x^2}. \tag{8.33}$$

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[2] The point of the indenter is the origin of the coordinate system used for the indenter profile.

With the help of Eq. (8.28), it follows that

$$\delta T_{3D}(r) = \frac{2q_0}{\pi\lambda^*}\int_0^r \frac{\sqrt{a^2-x^2}}{\sqrt{r^2-x^2}}\,\mathrm{d}x = \frac{2q_0 a}{\pi\lambda^*}\int_0^{\pi/2}\sqrt{1-(r/a)^2\sin^2\varphi}\,\mathrm{d}\varphi = \frac{2q_0 a}{\pi\lambda^*}\mathrm{E}\left(\frac{r}{a}\right), \tag{8.34}$$

for which the complete elliptical integral of the second kind is shortened to E. Comparing this to the expressions found in literature [8] verifies it to be correct. Further applications of the transformation formulas are handled in the problems at the end of this chapter.

8.5 Heat Generation and Temperature in the Contact of Elastic Bodies

Until now, we have only investigated cases with no relative motion between the bodies. Furthermore, steady-state thermal states have been assumed. We would like to continue to respect the latter, but now allow for relative motion between the bodies. For this, we consider a stationary point source Q under which a half-space moves with a constant speed of v in the x-direction; this is sketched in Fig. 8.3.

While the x, y, z coordinate system is stationary, the $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ system moves with the body. To describe the temperature distribution (measured in the stationary system), the Laplace Eq. (8.6) must be supplemented by a convective term:

$$\Delta T = \frac{v}{\alpha}\frac{\partial T}{\partial x}, \tag{8.35}$$

the steady-state solution of which is [1]

$$\delta T(x,y,z) = T(x,y,z) - T_0 = \frac{Q}{2\pi\lambda R}e^{\frac{-v(R-x)}{2\alpha}} \quad \text{with } R := \sqrt{x^2+y^2+z^2}. \tag{8.36}$$

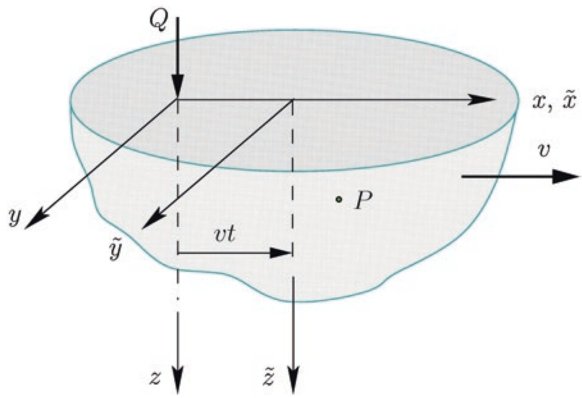

Fig. 8.3 Stationary point source under which a half-space moves at a constant speed of v in the x-direction

In order to calculate the temperature distribution for a distributed thermal flux density of the surface, Eq. (8.36) must be used as Green's function. This is especially essential for the investigation of frictional contacts, for which the (entire) frictional energy is transformed into heat. However, this is only necessary for one part of the solution. For the body on which the stationary frictional position is located, we can simply use the solution for the stationary case (8.7). Only in the special case of very low speeds or very small *Péclet* numbers

$$Pe := \frac{va}{2\alpha} \ll 1, \tag{8.37}$$

can we add the approximation for the other body and, therefore, take advantage of all equivalencies for the entirety of both surfaces (a is the contact radius). We will constrict ourselves in the following to such cases.

We will now consider a frictional contact with the frictional coefficient μ, for which the contact partners move with a relative speed of v with respect to one another. For the heat generated on the contact surface, the following is valid:

$$q(x, y) = \mu v p(x, y) \Rightarrow Q = \mu v F_N \,, \tag{8.38}$$

for which $p(x, y)$ denotes the normal stress distribution and F_N, the normal force distribution. The heat flows into both half-spaces respectively according to

$$q_1(x, y) = \beta \cdot \mu v p(x, y) \quad \text{and} \quad q_2(x, y) = (1 - \beta) \cdot \mu v p(x, y). \tag{8.39}$$

The distribution between the two sometimes causes difficulties, because the weighted function β is generally dependent on x and y in order not to violate the continuity of the temperature within the contact area [9]. We circumvent the problem by assuming that one of the contacts is non-conductive, so that the entire heat flows into the other body. We would now like to determine the temperature distribution on the surface of this body by using the reduction method; its specific thermal conductivity is λ. We consider an element of the linearly elastic foundation with the coordinate x that is indented by $u_z(x)$. The known force acting on this element is then $f_N = E^* \Delta x \cdot u_z(x)$. The frictional power of the element is $\Delta Q(x) = \mu v f_N(x) = \mu v E^* \Delta x \cdot u_z(x)$, for which the resulting temperature difference of the element is

$$\delta T_{1D}(x) = \frac{\Delta Q(x)}{2\lambda \cdot \Delta x} = \frac{E^*}{2\lambda} \mu \cdot v \cdot u_z(x). \tag{8.40}$$

The temperature difference for the three-dimensional model at the point r on the surface within the contact area can be obtained using Eq. (8.28). The temperature can be calculated even outside of the contact surface. For this, we must simply change the upper boundary of the integral in Eq. (8.28):

$$\delta T_{3D}(r) = \frac{2}{\pi} \int_0^a \frac{\delta T_{1D}(x)}{\sqrt{r^2 - x^2}} \, dx \quad \text{for } r > a. \tag{8.41}$$

Applying this classical transformation to the classical example of a parabolic frictional contact is the topic of Problem 1. It is possible that the reader may not see the benefits of the method of dimensionality reduction compared to other methods because of the complicated transformations. Therefore, we would like to emphasize the fact that the reduction method maps *global* parameters such as normal force, indentation depth, contact area/length, total heat flow rate, and maximum surface temperature as well the contact stiffness and resistance seemingly effortlessly and exactly. These relationships are at the forefront of the investigation of rough contacts. If only information about *local* parameters is of interest, then this can also be reconstructed using the transformation rules from the one-dimensional model.

8.6 Heat Generation and Temperature in the Contact of Viscoelastic Bodies

Heat can not only be generated on the surface, but also directly in the material of the contacting bodies, assuming that they exhibit viscoelastic properties. One can qualitatively approximate the temperature distribution as follows. Let us consider an element in a viscoelastic foundation at the point x and assume that it is deformed in the vertical direction with the speed $\dot{u}_z(x,t)$. Thereby, the force produced is given by

$$f_N(x,t) = 4\Delta x \int_0^t G(t-t')\dot{u}_z(x,t')\mathrm{d}t', \tag{8.42}$$

for which an incompressible material is assumed (see Chap. 7).

The heat generation in the element is

$$\Delta Q(x,t) = f_N(x,t)\cdot\dot{u}_z(x,t) = \dot{u}_z(x,t)\cdot 4\Delta x\int_0^t G(t-t')\dot{u}_z(x,t')\mathrm{d}t'. \tag{8.43}$$

If we interpret this heat generation as that produced in the frictional contact, then we obtain the temperature in the element according to (8.40):

$$\delta T_{1D}(x,t) = \frac{\Delta Q(x,t)}{2\lambda^*\cdot\Delta x} = \frac{2}{\lambda^*}\dot{u}_z(x,t)\int_0^t G(t-t')\dot{u}_z(x,t')\mathrm{d}t'. \tag{8.44}$$

As an example, we consider a simple viscoelastic medium (Kelvin body). In this case, the normal force is given by

$$f_N(x,t) = (4Gu_z(x,t) + 4\eta\dot{u}_z(x,t))\Delta x \tag{8.45}$$

and the temperature by

$$\delta T_{1D}(x,t) = \frac{2}{\lambda^*}(Gu_z(x,t) + \eta\dot{u}_z(x,t))\dot{u}_z(x,t). \tag{8.46}$$

The temperature is, therefore, dependent on the time and can generally either increase or decrease (adiabatic cooling).

8.7 Problems

Problem 1 A non-conducting, rigid body with a smooth surface slides over an elastic half-space with a parabolically curved surface of radius R with a speed of v_0. The modulus of elasticity E, Poisson's ratio ν, and the thermal conductivity λ of the half-space are given. Determine the temperature distribution of the surface of the half-space using the reduction method and assuming steady-state conditions.

Solution We have already solved the purely elastic problem multiple times and carry over several intermediate results. After converting the three-dimensional profile to a one-dimensional equivalent profile using the *rule of Popov* and calculating the indentation depth into a linearly elastic foundation, we obtain the displacement in the one-dimensional system:

$$u_z(x) = d - \frac{x^2}{R} \quad \text{with } d = \frac{a^2}{R}, \tag{8.47}$$

for which the relationship between the indentation depth d and the normal force F_N is given by the Hertzian relation $F_N = \frac{4}{3}E^* R^{1/2} d^{3/2}$. According to (8.40), this leads to the one-dimensional temperature difference

$$\delta T_{1D}(x) = \frac{E^*}{2\lambda}\mu \cdot v_o \cdot u_z(x) = \frac{E^*}{2\lambda}\mu \cdot v_o \cdot \left(d - \frac{x^2}{R}\right) \quad \text{with } E^* = \frac{E}{1-\nu^2}. \tag{8.48}$$

Insertion of (8.48) into (8.28) results in the three-dimensional distribution of the surface temperature within the contact area:

$$\begin{aligned}
\delta T_{3D}(r) &= \frac{2}{\pi}\int_0^r \frac{\delta T_{1D}(x)}{\sqrt{r^2 - x^2}}\,\mathrm{d}x = \frac{\mu v_o E^*}{\pi \lambda R}\int_0^r \frac{a^2 - x^2}{\sqrt{r^2 - x^2}}\,\mathrm{d}x \\
&= \frac{\mu v_o E^*}{\pi \lambda R}\left[a^2 \arcsin\left(\frac{x}{r}\right)\Big|_0^r + x\sqrt{r^2 - x^2}\Big|_0^r - r^2 \int_0^{\pi/2} \cos^2\varphi\,\mathrm{d}\varphi\right]. \\
&= \frac{\mu v_o E^*}{4\lambda R}\left(2a^2 - r^2\right)
\end{aligned} \tag{8.49}$$

By using the Hertzian relationship between normal force and contact radius and taking (8.38) into account, we obtain

$$\delta T_{3D}(r) = \frac{3Q}{16\lambda a^3}\left(2a^2 - r^2\right) \quad \text{for } 0 < r < a. \tag{8.50}$$

We obtain the distribution outside of the contact area from (8.41). Because the formula differs from that valid in the contact area only by the upper bound of the integral, we can simply carry over the antiderivative in (8.49). After rearrangement, we obtain

$$\delta T_{3D}(r) = \frac{3Q}{8\pi\lambda a^3}\left[\left(2a^2 - r^2\right)\arcsin\left(\frac{a}{r}\right) + a\sqrt{r^2 - a^2}\right] \quad \text{for } r > a. \tag{8.51}$$

The reader may be convinced of the validity of the results by the usage of equivalency [2].

Problem 2 Determine the thermal resistance of the contact from Problem 1. Assume a one-dimensional model.

Solution The thermal resistance, as defined in (8.2), presumes an isothermal contact that is not present here. Therefore, we refer here to the *maximum* surface temperature that is present in the middle of the contact. This takes the role of the indentation depth in the elastic contact, where the indentation depth is the same for both the one-dimensional and three-dimensional models. Therefore, the maximum temperature in the middle of the contact is also the same in both models. From (8.48), we obtain

$$\delta T_{\max} = \delta T_{1D}(0) = \frac{E^* \mu v_o a^2}{2\lambda R} = \frac{3Q}{8\lambda a} \tag{8.52}$$

and with it, the thermal resistance

$$R_W := \frac{\delta T_{\max}}{Q} = \frac{3}{8\lambda a}. \tag{8.53}$$

This result initially appears to conflict with the universal formula (8.16), because this would result in

$$R_W := \frac{E^*}{2\lambda \cdot k_z} = \frac{E^*}{2\lambda \cdot 2E^* a} = \frac{1}{4\lambda a}. \tag{8.54}$$

This is indeed the thermal resistance for a round contact, however, this relationship is only (!) valid for isothermal contact areas. Even redefining the thermal resistance with respect to the average temperature instead of the maximum temperature does not help. In this case, the thermal resistance is

$$\frac{\overline{\delta T_{3D}}}{Q} = \frac{9}{32\lambda a}, \tag{8.55}$$

although the deviation is not very large. The proportionality to the contact length is of course always present.

Problem 3 A half-space with a parabolically curved surface having a radius of curvature of R is pressed into a second half-space with a flat surface. Before

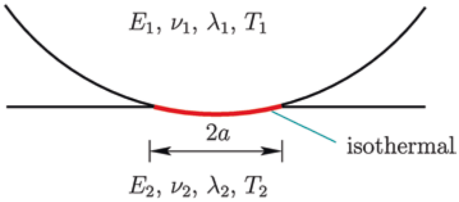

Fig. 8.4 Qualitative presentation of a Hertzian contact with steady-state heat conduction

contact, the bodies exhibit the temperatures T_1 and T_2. Upon bringing the bodies together, a heat flux flows through the contact area. If the temperatures far from the contact surface are held constant, then a steady-state flow will occur after some time. Let it be mentioned that the contact area is isothermal and temperature related deformations are neglected. Calculate the dependence of the thermal resistance on the normal force in the case of an *elastic* contact, which is qualitatively shown in Fig. 8.4.

Solution We can solve the elastic problem and heat conduction problem *separately* with the help of the method of dimensionality reduction. The solution of the elastic problem can be found in Chap. 3. The dependence of the normal force on the contact radius was

$$F_N(a) = \frac{4}{3}E^*\frac{a^3}{R} \quad \text{with} \quad \frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}. \tag{8.56}$$

For a round contact with an isothermal contact area Eq. (8.14) is valid with which we further express the conductivity by means of the resistance:

$$R_W = \frac{1}{4a\lambda^*} \quad \text{with} \quad \frac{1}{\lambda^*} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}. \tag{8.57}$$

We have also already derived this relationship with the reduction method. By solving (8.56) with respect to the contact radius and inserting this into (8.57), we obtain the desired dependence:

$$R_W = \frac{(E^*)^{1/3}}{\lambda^*(48RF_N)^{1/3}} \sim F_N^{-1/3}. \tag{8.58}$$

In conclusion, let it be noted that for the complete plastic contact, the result is

$$F_N \sim a^2 \Rightarrow R_W \sim F_N^{-1/2}. \tag{8.59}$$

Problem 4 Determine the total current, the constriction resistance, and the current density distribution for the electrical contact between two half-spaces with the specific resistances ρ_1 and ρ_2 within a circular area (radius a). It should be assumed that far from the contact, there exist equipotential surfaces within the half-spaces having a difference in potential of U. Furthermore, determine the radius b of the partial contact area through which half of the total current flows.

Solution We have already discussed the equivalent heat conduction problem in Sect. 8.4. First, we reduce the electrical contact between two bodies to the steady-state flow through one body whose effective specific conductivity is

$$\lambda^* = \frac{1}{\rho_1 + \rho_2}. \tag{8.60}$$

Between the circular equipotential surface, and another at infinity (or at a sufficiently large distance), there exists the potential difference U. Because a *constant* three-dimensional potential difference exists, no modification whatsoever is needed and it can be carried over to the one-dimensional system. Every element in the linearly elastic foundation obtains the specific conductivity $\Delta\Lambda = 2\lambda^* \cdot \Delta x$ and the following partial current flows through each:

$$\Delta I(x) = \Delta\Lambda \cdot \delta V(x) = 2\lambda^* \cdot \Delta x \cdot U. \tag{8.61}$$

By summation of the partial currents through all of the elements in the foundation, we obtain the total current:

$$I = \int_{-a}^{a} 2\lambda^* \delta V(x)\,\mathrm{d}x = 4a\lambda^* U \tag{8.62}$$

and from this, the constriction resistance

$$R_E := \frac{U}{I} = \frac{1}{4a\lambda^*} = \frac{\rho_1 + \rho_2}{4a}. \tag{8.63}$$

The three-dimensional distribution of the flux density within the contact area is calculated using (8.25), which is trivial due to the *constant* one-dimensional current density:

$$q(r) = \frac{1}{\pi} \frac{2\lambda^* U}{\sqrt{a^2 - r^2}} = \frac{I}{2\pi a} \frac{1}{\sqrt{a^2 - r^2}}. \tag{8.64}$$

Of course, all results correspond to those in the three-dimensional problem. For this solution of the supplemental problem, we may not assume a one-dimensional current density, but must use the determined three-dimensional current density. For this, we integrate (8.64) over the three-dimensional contact area with the upper radial boundary b and require that the result corresponds to half of the current:

$$\frac{I}{2\pi a} \int_{0}^{b} \frac{r}{\sqrt{a^2 - r^2}} 2\pi\,\mathrm{d}r \overset{!}{=} \frac{I}{2}. \tag{8.65}$$

Elementary integration and a few rearrangements lead to

$$b = \frac{1}{2}\sqrt{3}\,a \approx 0.866\,a. \tag{8.66}$$

Although the radius b divides the surface by the ratio 3:1, meaning that the outer ring is only a quarter of the total area, half of the total current flows through it.

Problem 5 Determine the temperature distribution on the surface as well as the thermal resistance for the conduction problem (*isoflux*) shown in Fig. 8.2. Use the analogy to the elastic problem, the solution of which is considered to be known. According to this, the loading of an elastic half-space by a constant stress p over a circular area with the radius a leads to the following normal surface displacements (see, for example [3]):

$$\bar{u}_z(r) = \begin{cases} \frac{4(1-\nu^2)pa}{\pi E}\mathrm{E}(r/a) & \text{for } r < a \\ \frac{4(1-\nu^2)pr}{\pi E}\left[\mathrm{E}(r/a) - \left(1 - \frac{a^2}{r^2}\right)\mathrm{K}(a/r)\right] & \text{for } r > a \end{cases}, \tag{8.67}$$

where K and E are the complete elliptical integrals of the second kind:

$$\mathrm{K}(k) := \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \varphi}} \mathrm{d}\varphi \quad \text{and } \mathrm{E}(k) := \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \varphi}\, \mathrm{d}\varphi. \tag{8.68}$$

Solution According to Eq. (8.11), we must only replace the displacement with the temperature, the normal stress with the heat flux density, and the effective modulus of elasticity with double the conductivity:

$$\delta T(r) = \begin{cases} \frac{2qa}{\pi\lambda}\mathrm{E}(r/a) & \text{for } r < a \\ \frac{2qr}{\pi\lambda}\left[\mathrm{E}(r/a) - \left(1 - \frac{a^2}{r^2}\right)\mathrm{K}(a/r)\right] & \text{for } r > a \end{cases}. \tag{8.69}$$

In order to calculate the thermal resistance, we need the maximum surface temperature. This is given in the center and has a value of

$$\delta T_{\max} = \delta T(0) = \frac{qa}{\lambda}, \tag{8.70}$$

where we have taken $\mathrm{E}(0) = \pi/2$ into account. The resulting thermal resistance is then

$$R_W := \frac{\delta T_{\max}}{Q} = \frac{qa}{\lambda\pi qa^2} = \frac{1}{\pi\lambda a}. \tag{8.71}$$

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