Chapter 7
Contacts with Elastomers

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7.1 Introduction

Rubber and other elastomers play a large role in many tribological applications. They are used where large frictional forces or large deformations are needed. These materials are especially used for tires, transportation rollers, shoes, seals, rubber bands, in electronic devices (e.g., contacts for keyboards) as well as applications for adhesion. When compared to purely elastic contacts, the calculation of elastomer contacts is made more difficult by the fact that they exhibit a time-dependent behavior, which is also normally characterized by a large spectrum of relaxation times. The correct mechanical description must, therefore, take several orders of magnitude in characteristic times into account. The multi-scalar properties of the surface roughness are supplemented here by the multi-scalar character of the relaxation of the material, which makes the numerical simulation of elastomers especially complicated. It is, therefore, important to develop fast simulation methods for the calculation of contact and frictional properties for this class of materials. In this chapter, we will show how the method of dimensionality reduction can be generalized to contacts of elastomers with arbitrary linear rheology.

In the first section, we remind the reader of the fundamental definitions that are necessary for the description of elastomers, for which we follow the presentation of [1]. The general process is then explained using the very simple special case of a linearly viscous material for the purposes of understanding. Only afterwards, we will continue to the treatment of general viscoelastic materials. The detailed derivations can be found in Chap. 19.

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7.2 Stress Relaxation in Elastomers

We consider a rubber block, which is loaded under shear stress (Fig. 7.1). The shear angle is denoted by ε.[1] If the block is deformed quickly by the shear angle of ε_0, then the stress increases initially to a high level $\sigma(0)$ and then relaxes slowly to a much lower level of $\sigma(\infty)$ (Fig. 7.2b). For elastomers, $\sigma(\infty)$ can be three to four orders of magnitude smaller than $\sigma(0)$. The ratio

$$G(t) = \frac{\sigma(t)}{\varepsilon_0} \tag{7.1}$$

is denoted as the *time-dependent shear modulus*. It is easy to see that this function completely describes the mechanical properties of a material, provided that the material behaves *linearly*. Let us assume that the block is deformed according to an arbitrary function in time $\varepsilon(t)$. An arbitrary dependence $\varepsilon(t)$ can always be presented as the sum of temporally shifted step functions, as shown schematically in Fig. 7.3. An elementary step function in this presentation at the time t' has the amplitude $d\varepsilon(t') = \dot{\varepsilon}(t')dt'$. The corresponding stress component is $d\sigma = G(t-t')\dot{\varepsilon}(t')dt'$, and the total stress at every point in time is, therefore, calculated as

$$\sigma(t) = \int_{-\infty}^{t} G(t-t')\dot{\varepsilon}(t')dt'. \tag{7.2}$$

Equation (7.2) shows that the time-dependent shear modulus can be understood as a weighted function with deformation changes in the past leading to current changes in the stress. Due to this, $G(t)$ is sometimes called the *memory function*.

If $\varepsilon(t)$ changes harmonically:

$$\varepsilon(t) = \tilde{\varepsilon}\cos(\omega t), \tag{7.3}$$

then a periodic change in the stress with the same frequency is reached after a transient period. The relationship between the changes in deformation and stress can be very easily presented if the real function $\cos(\omega t)$ is presented as the sum of two complex exponential functions:

$$\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right). \tag{7.4}$$

Due to the superposition principle, one can first calculate the stresses that are caused by the complex oscillations

$$\varepsilon(t) = \tilde{\varepsilon}e^{i\omega t} \quad \text{and} \quad \varepsilon(t) = \tilde{\varepsilon}e^{-i\omega t}. \tag{7.5}$$

[1] Let it be stressed that the defined value ε is equal to *two times* the shear component of the deformation tensor.

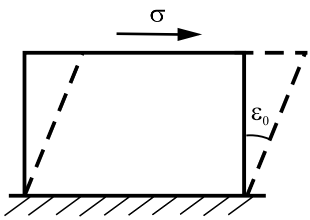


Fig. 7.1 Shear deformation of a rubber block. ε_0 is the shear angle

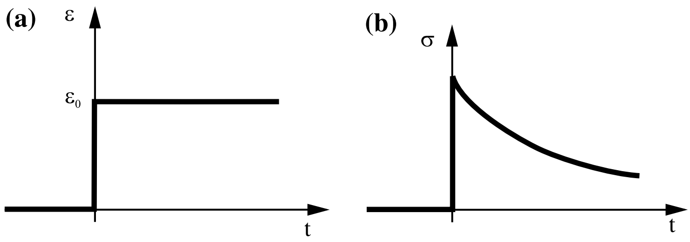
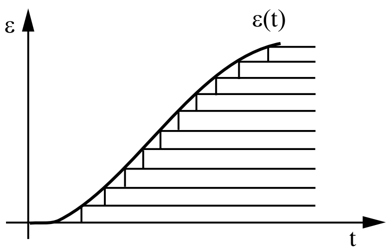


Fig. 7.2 If a rubber block is quickly deformed at time $t = 0$ by ε_0, then the stress increases initially to a high level and then relaxes with time to a much lower stress

Fig. 7.3 Presentation of a time-dependent function as a superposition of multiple temporally displaced step functions



Subsequently, these may be summed. If we insert $\varepsilon(t) = \tilde{\varepsilon} e^{i\omega t}$ into (7.2), then we obtain a stress of

$$\sigma(t) = \int_{-\infty}^{t} G(t - t') i\omega \tilde{\varepsilon} e^{i\omega t'} dt'. \tag{7.6}$$

By substituting $\xi = t - t'$, we bring the integral into the following form:

$$\sigma(t) = i\omega \tilde{\varepsilon} e^{i\omega t} \int_{0}^{\infty} G(\xi) e^{-i\omega \xi} d\xi \tag{7.7}$$

or

$$\sigma(t) = \hat{G}(\omega)\tilde{\varepsilon}e^{i\omega t} = \hat{G}(\omega)\varepsilon(t). \tag{7.8}$$

For a harmonic excitation in the form of a complex exponential function $e^{i\omega t}$, the stress is proportional to the deformation. The proportionality coefficient

$$\hat{G}(\omega) = i\omega \int_0^\infty G(\xi)e^{-i\omega\xi}d\xi \tag{7.9}$$

is generally a complex value and is called the *complex shear modulus*. Its real component $G'(\omega) = \mathrm{Re}\,\hat{G}(\omega)$ is called the *storage modulus*, and its imaginary component $G''(\omega) = \mathrm{Im}\,\hat{G}(\omega)$ is called the *loss modulus*. Further details as to the definition and measurement methods of the time-dependent shear modulus and the complex shear modulus can be found in the book [1].

7.3 Application of the Method of Dimensionality Reduction in Viscoelastic Media: The Basic Idea

If the indentation or slip speed for the dynamic loading of an elastomer is lower than the lowest speed of sound (which is related to the smallest relevant modulus of elasticity), then the contact can be considered to be quasi-static. If this condition is met and an area of the elastomer is excited with an angular frequency of ω, then the material exhibits a linear relationship between stress and deformation, and therefore, between force and displacement. Thereby, the medium can be considered to be an elastic body with the effective shear modulus of $G(\omega)$. All theorems that are valid for elastic bodies must also be valid for harmonically excited viscoelastic media. More importantly, the incremental stiffness is proportional to the diameter of the contact area, which forms the mathematical basis for the applicability of the method of dimensionality reduction. Because of this, an elastomer can be mapped to a one-dimensional system, for which the individual springs can be chosen according to (3.5):

$$\Delta k_z = E^*\Delta x. \tag{7.10}$$

The only difference to the elastic contact is the fact that the effective modulus of elasticity is now a function of frequency. Elastomers can be often considered to be incompressible media. In this case, $\nu = 1/2$ and

$$\Delta k_z = E^*(\omega)\Delta x = \frac{E(\omega)}{1-\nu^2}\Delta x = \frac{2G(\omega)}{1-\nu}\Delta x \approx 4G(\omega)\Delta x. \tag{7.11}$$

In the case of rubber, the stiffness of the individual "springs" of the linearly elastic foundation is equal to four-fold the shear modulus multiplied by the discretization step size. In the one-dimensional equivalent system, we obtain the following spring force for a harmonic excitation:

$$f_N(x,\omega) = E^*(\omega)\Delta x \cdot u_z(x,\omega) \approx 4G(\omega)\Delta x \cdot u_z(x,\omega). \tag{7.12}$$

The reverse transformation into the time domain results in the force law

$$f_N(x,t) = 4\Delta x \int_{-\infty}^{t} G(t-t')\dot{u}_z(x,t')\mathrm{d}t'. \tag{7.13}$$

In the next section, we will explain this general, but somewhat formally described, idea by using the simplest example of the linearly viscous medium. We discuss how the viscoelastic contact problem can be reduced to the elastic contact problem and, subsequently, how this can be mapped to a one-dimensional system.

7.4 Radok's Method of the Functional Equations

In 1955, Lee [2] published a method for solving viscoelastic contact problems by reducing them to elastic problems. This procedure is advantageous because contacts between elastic bodies are comparatively simpler to solve and the solution to many problems can already be found in many textbooks. The procedure was later generalized by Radok [3] and entered the literature as the *principle of the functional equations*.

The basic idea of the method is conceivably simple. Beginning with a given viscoelastic problem, the material properties are replaced by those of an elastic body. However, all other influences, such as geometric configuration, remain unchanged. Subsequently, the elastic problem is solved. One obtains the solution to the viscoelastic problem by once again replacing the elastic properties in the elastic solution by the viscoelastic properties. This substitution takes place in the Laplace domain and takes the most effort. The entire algorithm is presented schematically in Fig. 7.4.

We will explain the procedure by using a concrete example. Let us consider a linearly viscous, incompressible body with the viscosity η, which is also large

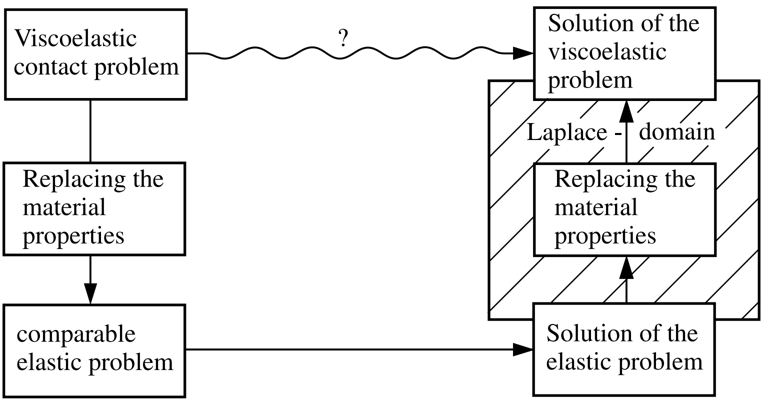


Fig. 7.4 Principle of the functional equations (schematic presentation)

enough that the half-space approximation is valid. The body is loaded on the surface by a constant point-loaded normal force. How will the surface of the body be deformed? The comparable elastic problem is simply the loading of a linearly elastic, incompressible half-space by a constant normal force. The solution to the problem can be found in many textbooks on elasticity theory or contact mechanics (e.g., [4] or [1]). If G is the shear modulus, F_N is the normal force, and r is the distance to the point of force application, then the normal displacement of the surface can be given by the expression[2]

$$u(r) = \frac{F_N}{4\pi G r}. \tag{7.14}$$

This equation is the solution to the comparable elastic problem. The solution now undergoes a Laplace transformation. For the description of the viscoelastic problem, contrary to elastic problems, it is necessary to specify the history of the normal force. We assume that the force begins loading at the time $t = 0$ and is then constant with the magnitude F_N:

$$F(t) = F_N H(t), \tag{7.15}$$

where

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{7.16}$$

is the Heaviside function. The application of the Laplace transformation, the replacement of the material properties, and the reverse transformation are carried out in Chap. 19. We obtain the following surface deformation as the solution to the viscous contact problem:

$$u(r,t) = \frac{F_N t}{4\pi \eta r}. \tag{7.17}$$

If one differentiates (7.17) with respect to time, the velocity is obtained with which the surface is deformed in response to the external force:

$$\dot{u}(r) = \frac{F_N}{4\pi \eta r}. \tag{7.18}$$

If we consider (7.18) as a solution and compare this with that of the elastic problem, then it is easy to recognize how the two equations are related. Apparently, the elastic solution (7.14) switches to the viscous solution (7.18) when the shear modulus G is replaced by the shear viscosity η and the deformation u, by the deformation velocity $\dot{u}$. We would like to stress that the transition to this form is only valid for linearly viscous materials using the force law (7.15). Figure 7.5 presents this process schematically.

[2] Let us once again remember that we are dealing with an incompressible, viscous medium; therefore, the corresponding elastic medium is also incompressible and it is assumed that $\nu = 1/2$.

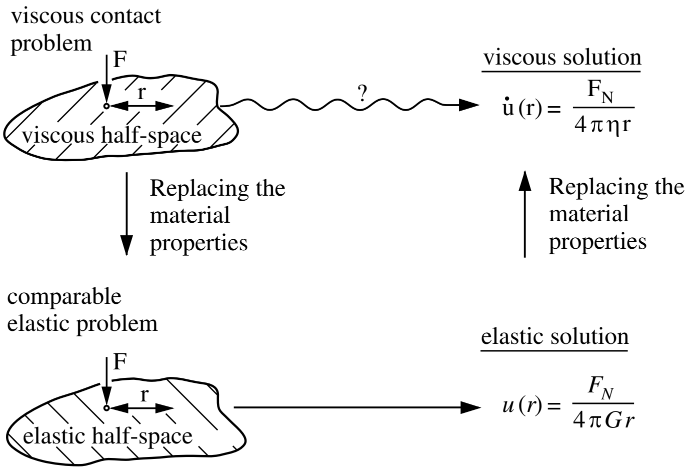


Fig. 7.5 Solution of the viscous contact problem with Radok's principle of the functional equations

One could also derive Eq. (7.18) without the Laplace transformation directly from the following analogy. The deformation of the surface of an elastic body is uniquely determined due to the equilibrium equation

$$G\Delta\vec{u} + (\lambda + G)\nabla(\nabla \cdot \vec{u}) = 0 \tag{7.19}$$

as well as the stress distribution on the surface. In this equation, the first Lamé coefficient is $\lambda = 2\nu G/(1-2\nu)$ [4]. The corresponding "equilibrium equation" for the creeping flow of a linearly viscous fluid (Navier-Stokes equation without the inertial term) is [5]

$$\eta\Delta\dot{\vec{u}} + (\xi + \eta)\nabla(\nabla \cdot \dot{\vec{u}}) = 0. \tag{7.20}$$

By integrating this equation once with respect to time and assuming that the medium is in a non-deformed state at time $t = 0$, we obtain

$$\eta\Delta\vec{u} + (\xi + \eta)\nabla(\nabla \cdot \vec{u}) = 0. \tag{7.21}$$

With the exception of the constants, this equation corresponds identically with Eq. (7.19) for an elastic continuum. If a displacement now occurs in a particular contact area, then the displacement fields in the elastic and viscous case will be *identical*.[3] This implication is exact and is not only valid for a linearly viscous fluid, but also for a medium with an arbitrary linear rheology. It was Lee and Radok that first came to this conclusion and based upon this, developed the study of contact mechanics of viscoelastic media [2].

[3] We stress, thereby, that we use the "non-penetration" boundary conditions and gravitation and capillary effects are completely neglected.

If we are additionally looking for the relationship between the forces and displacements, then we must take into account the fact that the stress in an elastic continuum is a linear function of the gradient of the displacement field $\vec{u}$, while in a fluid, it is a linear function of the gradient of the velocity field $\dot{\vec{u}}$. The fact that the equilibrium equations and the expressions for stress have the same form means that all relations that are valid for the relationship between force and displacement for a given stress distribution in the case of an elastic body are also valid for force and velocity in the case of a fluid. From this, it directly follows that the solution (7.18) for the velocity field in a fluid is obtained from the solution (7.14) for the displacement in an elastic continuum by replacing $u \to \dot{u}$ and $G \to \eta$.

7.5 Formulation of the Reduction Method for Linearly Viscous Elastomers

In this section, the results thus far will be used to demonstrate the application of the reduction method on elastomers. As in the previous section, the procedure will be first shown using a concrete example, the indentation of a rigid indenter into a linearly viscous incompressible half-space. The comparable elastic problem was closely examined in the previous chapter using the reduction method for an elastic half-space. The elastic half-space is mapped to a chain of independent linear spring elements, whose stiffness is

$$\Delta k_z = 4G\Delta x, \tag{7.22}$$

where incompressibility has already been taken into account. The corresponding force law for the ith element of the linearly elastic foundation is

$$f_{N,i} = 4G\Delta x \cdot u_{z,i}. \tag{7.23}$$

This equation can be seen as the solution of the comparable elastic problem. It must now be transferred to the viscous solution by replacing the material properties. The detour by way of the Laplace transformation is not necessary here. We can, as in the previous section, simply conduct the substitution $u \to \dot{u}$, $G \to \eta$ in Eq. (7.23) and obtain the solution

$$f_{N,i} = 4\eta\Delta x \cdot \dot{u}_{z,i}. \tag{7.24}$$

Obviously, this describes the force law of a linear damper with a damping coefficient of

$$\Delta d_z = 4\eta\Delta x. \tag{7.25}$$

The geometry of the indenter must be scaled as in the elastic case [6].

With this, the following may be summarized. The formulation of the reduction method for a linearly viscous material is obtained from that of the elastic material by replacing the springs by dampers with a damping coefficient of Δd_z. In Fig. 7.6, the procedure is schematically presented.

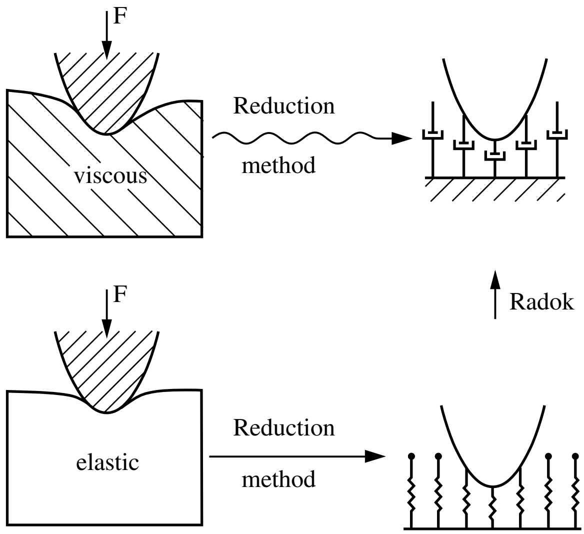


Fig. 7.6 Formulation of the reduction method for a linearly viscous incompressible material

Transferring this to a real viscoelastic material model (i.e., with viscous and elastic components) is done completely analogously and will be shortly explained in the next section and in detail in Chap. 19. In this case, the springs in the elastic formulation are replaced by spring–damper combinations, whose mathematical description can be obtained by replacing the material parameters in the Laplace domain.

7.6 The General Viscoelastic Material Law

In the previous sections, we have referred to the very simple special case of a linearly viscous material. The reason for this is the simplicity of the procedure and the clarity of the results. In this section, we show the results for the general case of an isotropic viscoelastic material. The exact derivations can be found in Chap. 19. The behavior of elastomers can be described by the relationships between deformation and stress with respect to compression

$$\sigma_{ii}(t) = \int_0^t K(t - t')\dot{\varepsilon}_{ii}(t')\mathrm{d}t' \tag{7.26}$$

and shear

$$s_{ik}(t) = 2\int_0^t G(t - t')\dot{e}_{ik}(t')\mathrm{d}t', \tag{7.27}$$

whereby we have denoted the shear component of the stress tensor with e_{ik}, in order to differentiate it from the previously introduced shear angle ε. The functions $K(t)$ and $G(t)$ are the corresponding relaxation functions. In Chap. 19, it will be shown how contacts with materials having this behavior can be solved using the reduction method. The springs in the linearly elastic foundation are replaced with elements having the characteristic

$$f_N(t) = 4\Delta x \int_0^t V(t-t')\dot{u}_z(t')\mathrm{d}t',\ V(t) := \mathcal{L}^{-1}\left\{\frac{G^*(s)(K^*(s)+G^*(s))}{K^*(s)+4G^*(s)}\right\}, \tag{7.28}$$

for which $\mathcal{L}^{-1}$ is the inverse Laplace transformation.[4] If the problem is limited to incompressible viscoelastic media (the compression modulus K is set to infinity), then the expression simplifies to

$$f_N(t) = 4\Delta x \int_0^t G(t-t')\dot{u}_z(t')\mathrm{d}t', \tag{7.29}$$

which agrees with Eq. (7.13).

7.7 Problems

Problem 1 The face (radius a) of a rigid, smooth cylindrical indenter is pressed into a linearly viscous half-space (viscosity η) with a constant force F_N (Fig. 7.7). Determine the indentation velocity and the indentation depth δ as a function of time with the help of the reduction method.

Solution The equivalent one-dimensional indenter is a rectangle with a width of $2a$ pressed into a chain of independent dampers. The distance between the dampers is Δx and the damping coefficient is $\Delta d_z = 4\eta\Delta x$. The external force is evenly distributed over the dampers so that every damper experiences a force of

$$f = \frac{\Delta x}{2a}F_N. \tag{7.30}$$

With this, all dampers are compressed with the velocity

$$\dot{\delta} = \frac{f}{\Delta d_z} = \frac{F_N}{8a\eta}. \tag{7.31}$$

The indentation depth is obtained by integrating with respect to time and is equal to

[4] The details of the notation are explained in detail in Chap.19.

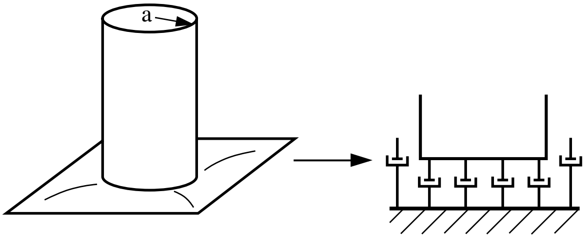


Fig. 7.7 Indentation of a cylindrical indenter into a viscous half-space

$$\delta(t) = \frac{F_N t}{8a\eta}. \tag{7.32}$$

Equation (7.31) can be obtained directly from the comparable elastic problem

$$F_N = 8Ga\delta \tag{7.33}$$

if the indentation depth and shear modulus are replaced by the velocity and shear viscosity in the result for the comparable elastic problem [1] (compare Chap. 17, Eq. 17.28):

$$F_N = 8\eta a\dot{\delta}. \tag{7.34}$$

It is easy to see that the equation is valid for an arbitrary-axially symmetric indenter as well a is considered to be the instantaneous value for the contact radius:

$$F_N(t) = 8\eta a(t)\dot{\delta}(t). \tag{7.35}$$

Problem 2 A rigid cone is pressed into a linearly viscous half-space (viscosity η) with a constant force F_N (Fig. 7.8). Determine the indentation depth as a function of time with the help of the reduction method.

Solution The surface of the cone can be described by the equation

$$f(r) = \tan\theta \cdot |r|. \tag{7.36}$$

The equivalent one-dimensional system (as described in Sect. 3.2) is

$$g(x) = \frac{\pi}{2}\tan\theta \cdot |x|. \tag{7.37}$$

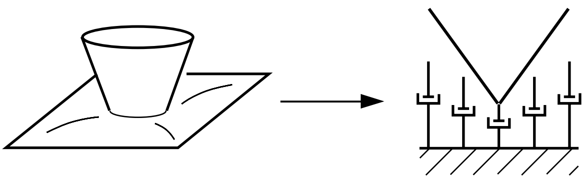


Fig. 7.8 Indentation of a cone into a viscous half-space

If the indenter is pressed to a depth of δ, then the vertical displacement of the foundation at point x is given by $u_z(x) = \delta - (\pi/2)\tan\theta \cdot |x|$. The contact radius is calculated by requiring that $u_z(a) = 0$, resulting in

$$a = \frac{2}{\pi}\frac{\delta}{\tan\theta}. \tag{7.38}$$

Equation (7.35) is also valid in this case. By inserting (7.38) into this equation, we obtain

$$F_N = \frac{16\eta}{\pi\tan\theta}\delta\dot{\delta}. \tag{7.39}$$

Separating the variables and integrating with the initial condition $\delta(0) = 0$ results in

$$F_N t = \frac{8\eta}{\pi\tan\theta}\delta^2. \tag{7.40}$$

The indentation depth as a function of time is then described by the equation

$$\delta(t) = \sqrt{\frac{\pi\tan\theta}{8}\frac{F_N}{\eta}t}. \tag{7.41}$$

Problem 3 A rigid axially-symmetric paraboloid is pressed into a half-space (viscosity η) with a constant force F_N (Fig. 7.9). Determine the indentation speed and indentation depth with the help of the reduction method.

Solution The surface of the paraboloid is described by the equation

$$f(r) = \frac{r^2}{2R}. \tag{7.42}$$

The one-dimensional indenter is a parabola that (according to Sect. 3.2) is scaled by a factor of 2:

$$g(x) = \frac{x^2}{R}. \tag{7.43}$$

If the indenter is now pressed to a depth of δ, then the contact radius is

$$a = \sqrt{R\delta}. \tag{7.44}$$

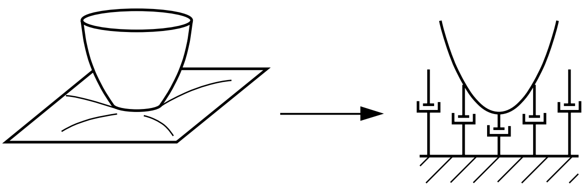


Fig. 7.9 Indentation of an axially-symmetric paraboloid into a viscous half-space

In this case, the force is also found using Eq. (7.35). Inserting (7.44) into (7.35) results in

$$F_N = 8\eta R^{1/2}\delta^{1/2}\dot{\delta}. \tag{7.45}$$

Integration with the initial condition $\delta(0) = 0$ results in

$$F_N t = \frac{16}{3}\eta R^{1/2}\delta^{3/2}. \tag{7.46}$$

The indentation depth as a function of time is then

$$\delta = \left(\frac{3F_N t}{16\eta R^{1/2}}\right)^{2/3}. \tag{7.47}$$

Differentiating with respect to time results in the indentation speed as a function of time:

$$\dot{\delta} = \frac{2}{3}\left(\frac{3F_N}{16\eta R^{1/2}}\right)^{2/3} t^{-1/3}. \tag{7.48}$$

This result is the exact solution to the corresponding three-dimensional problem and is also able to be derived without using the reduction method [7].

Problem 4 A rigid conical indenter is pressed into a viscoelastic (Kelvin body with the shear modulus G and the viscosity η) half-space with a constant force F_N. Find the dependence of the indentation depth on time.

Solution The equivalent one-dimensional indenter is given by Eq. (7.37) and the contact radius by Eq. (7.38) (it is not dependent on the rheological properties of the medium). To determine the force, we must now use the superposition of the elastic component (Eq. 3.44)

$$F_{N,el} = \frac{8G}{\pi}\frac{\delta^2}{\tan\theta} \tag{7.49}$$

and the viscous component (Eq. 7.39):

$$F_N = \frac{8G}{\pi}\frac{\delta^2}{\tan\theta} + \frac{16\eta}{\pi\tan\theta}\delta\dot{\delta}. \tag{7.50}$$

This equation can be written in the form

$$\frac{\pi\tan\theta\cdot F_N}{8G} = \delta^2 + 2\tau\delta\dot{\delta} = \delta^2 + \tau\frac{\mathrm{d}\left(\delta^2\right)}{\mathrm{d}t}, \tag{7.51}$$

where $\tau = \eta/G$ is the relaxation time of the medium. Integration with the initial condition $\delta(0) = 0$ results in

$$\delta^2(t) = \frac{\pi\tan\theta\cdot F_N}{8G}\left(1 - e^{-t/\tau}\right). \tag{7.52}$$

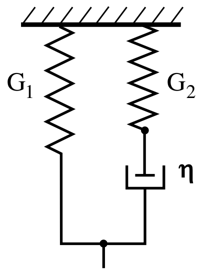


Fig. 7.10 Standard model for an elastomer consisting of a spring and a parallely attached Maxwell element

Problem 5 A rigid cylindrical indenter is pressed into an elastomer, which is described by the "standard model" [1] (Fig. 7.10). Find the dependence of the indentation depth on time.

Solution The standard model for an elastomer is shown in Fig. 7.10. It consists of a Maxwell element (a stiffness G_2 and damper η in series) and a stiffness G_1 attached in parallel.

The one-dimensional opposing side is a foundation of elements with a separation distance of Δx, the individual components of which can be characterized by the parameters $4G_1\Delta x$, $4G_2\Delta x$, and $4\eta\Delta x$. The equivalent one-dimensional indenter is a rectangle with the width $2a$. The normal force is

$$F_N = 8G_1 a u_z + 8G_2 a(u_z - u_1), \tag{7.53}$$

where u_1 satisfies the following equation:

$$u_z = u_1 + \tau \dot{u}_1 \tag{7.54}$$

and $\tau = \eta/G_2$. Solving the equation with the initial conditions $u_z(0) = 0$ and $u_1(0) = 0$ results in

$$u_1(t) = \frac{F_N}{8G_1 a}\left(1 - \exp\left(-\frac{G_1 t}{\tau(G_1 + G_2)}\right)\right), \tag{7.55}$$

$$u_z(t) = \frac{F_N}{8G_1 a}\left(1 - \frac{G_2}{G_1 + G_2}\exp\left(-\frac{G_1 t}{\tau(G_1 + G_2)}\right)\right). \tag{7.56}$$

In the special case of $G_2 \gg G_1$, we obtain the result for the Kelvin body:

$$u_z(t) = \frac{F_N}{8G_1 a}\left(1 - \exp\left(-\frac{G_1 t}{\eta}\right)\right). \tag{7.57}$$

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