Chapter 3
Normal Contact Problems with Axially-Symmetric Bodies Without Adhesion

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3.1 Mapping of Three-Dimensional Contact Problems onto One Dimension: The Basic Idea

The *method of dimensionality reduction* is based on the observation that certain types of three-dimensional contacts can be *exactly* mapped to one-dimensional linearly elastic foundations. Even one of the simplest contact problems offers us a taste of this method: If a flat cylindrical indenter is pressed into the surface of an elastic half-space (Fig. 3.1a), then the normal stiffness of the contact is *proportional to its diameter D*:

$$k_z = DE^*, \tag{3.1}$$

where E^* is the effective Young's modulus and is calculated from

$$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}, \tag{3.2}$$

using the Young's moduli of the contacting bodies E_1 and E_2 as well as their shear moduli ν_1 and ν_2.[1] The proportionality of the stiffness to the diameter can also be reproduced quite trivially by a *one-dimensional* linearly elastic foundation.

The linearly elastic foundation (Fig. 3.1b) is a series of independent, identical springs that are fixed to a rigid substrate separated from one another by a distance of Δx. In order to represent continua, the "discretization step" Δx must, of course, be sufficiently small, which we always silently imply. The number of springs that are in contact with the indenter is equal to $D/\Delta x$. If we denote the stiffness of a single spring as Δk_z, then the total stiffness of the contact is

$$k_z = \Delta k_z \frac{D}{\Delta x}. \tag{3.3}$$

[1] This result can be found in any book dealing with contact mechanics (see, for example [1]).

V.L. Popov and M. Heß, *Method of Dimensionality Reduction in Contact Mechanics and Friction*, DOI 10.1007/978-3-642-53876-6_3

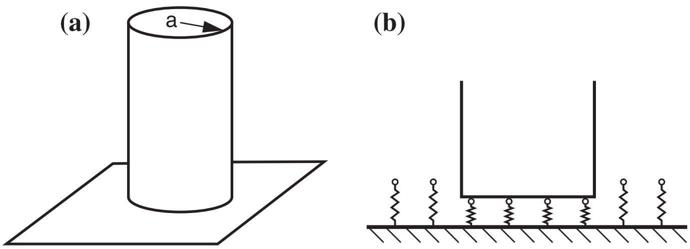

Fig. 3.1 (**a**) Contact between a flat, cylindrical indenter and an elastic half-space and (**b**) the one-dimensional model

In order for Eq. (3.1) to also be valid for the indentation into a linearly elastic foundation, the stiffness per unit length must be chosen to be equal to effective modulus E^*:

$$\frac{\Delta k_z}{\Delta x} = E^*. \tag{3.4}$$

According to this, the stiffness of every individual spring is

$$\Delta k_z = E^* \Delta x. \tag{3.5}$$

The proportionality of the stiffness to the diameter of the indenter is then met rather trivially in the case of an elastic foundation. In the following, it will be shown that the defined elastic foundation is also suitable for the mapping of a large number of other contact problems.

3.2 The Rules of Geike and Popov and the Rules of Heß for Normal Contact Problems

The relationship between normal force, the indention depth, and the contact radius can be reproduced exactly for a broad range of profiles by the reduced contact problem of a one-dimensional linearly elastic foundation. Thereby, the surface profile must merely be modified according to a few simple rules.

Let us first consider the contact between an elastic sphere with the radius R and an elastic half-space (the Herzian contact problem, Fig. 3.2a).[2] As early as 2005, Popov pointed out in a lecture[3] that also for a sphere (or a parabolic indenter) the relationship between normal force, the indentation depth, and the contact radius

[2] Strictly speaking, a parabolic profile with the radius of curvature R is considered.

[3] German–Russian Workshop "Numerical simulation methods in tribology: possibilities and limitations", Berlin University of Technology, March 14–17, 2005. Published in [2].

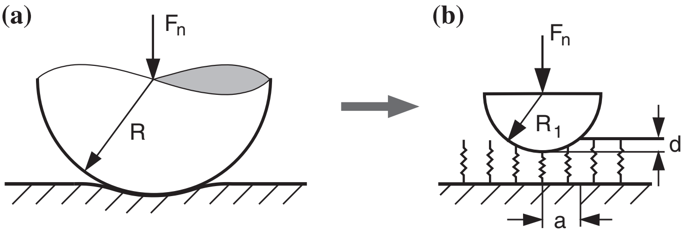

Fig. 3.2 (**a**) Contact between a sphere and an elastic half-space and (**b**) the one-dimensional model

can be *exactly* described by a one-dimensional model (Fig. 3.2b), provided that the radius is scaled by a factor of 1/2. At this point, we will describe the solution for a sphere in detail. In the following chapters, however, we will dispense with the details of the calculation due to their simplicity and only state the results.

The one-dimensional substitution profile should have the radius of curvature of R_1 and is given by the equation

$$\tilde{z} = g(x) = \frac{x^2}{2R_1}. \tag{3.6}$$

If this profile is pressed into the elastic foundation to a depth of d, then we obtain the vertical displacement of the foundation at the point x:

$$u_z(x) = d - g(x) = d - \frac{x^2}{2R_1}. \tag{3.7}$$

The semi-span of the contact area (the "contact radius") a is given by requiring that $u_z(a) = 0$ and is

$$a = \sqrt{2R_1 d}. \tag{3.8}$$

The contribution of a single spring with a coordinate x to the normal force is

$$f_N = \Delta k_z \cdot u_z(x) = E^* \left(d - \frac{x^2}{2R_1} \right) \Delta x. \tag{3.9}$$

The total normal force is obtained by integration over the contact area:

$$F_N = \int_{-a}^{a} E^* \left(d - \frac{x^2}{2R_1} \right) dx = \int_{-\sqrt{2R_1 d}}^{\sqrt{2R_1 d}} E^* \left(d - \frac{x^2}{2R_1} \right) dx = \frac{4\sqrt{2}E^*}{3} \sqrt{R_1 d^3}. \tag{3.10}$$

If we now choose the radius of the "two-dimensional sphere" according to

$$R_1 = R/2, \tag{3.11}$$

("rule of Popov"), then we obtain the *exact* Herzian relationships for the contact radius and the normal force:

$$a = \sqrt{Rd}, \tag{3.12}$$

$$F_N(d) = \frac{4}{3}E^*\sqrt{Rd^3}. \tag{3.13}$$

In other words, the rule (3.11) means that the cross-section of the original three-dimensional profile (in our case, the sphere with the radius R) is stretched by a factor of 2 in the vertical direction.

In his dissertation from 2011, Heß [3] showed that a similar *exact* mapping is possible for an *arbitrary* axially-symmetric profile. In this chapter, we will apply the mapping rules determined by Heß without providing proof of their validity. A detailed derivation of these rules is provided in Chap. 17.

The focus of the following investigation is the contact between axially-symmetric bodies and an elastic half-space. Let the axis of symmetry be z and the surface of the elastic half-space be given by $z = 0$. We parameterize the surface of the half-space using the Cartesian coordinates x and y. Now, we consider an axially-symmetric body with the profile

$$\tilde{z} = f_n(r) = c_n r^n, \tag{3.14}$$

where $r = \sqrt{x^2 + y^2}$, C_n is a constant, and n represents an arbitrary positive number (not necessarily an integer). We now define a one-dimensional profile according to[4]

$$\tilde{z} = g_n(x) = \tilde{c}_n |x|^n. \tag{3.15}$$

As shown in Chap. 17, the contact between the three-dimensional profile (3.14) and the elastic half-space is equivalent to that of the two-dimensional profile (3.15) and the linearly elastic foundation (3.4) if the following *rule of Heß* is applied:

$$\tilde{c}_n = \kappa_n c_n, \quad \kappa_n = \frac{\sqrt{\pi}}{2} \frac{n\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2} + \frac{1}{2})}, \tag{3.16}$$

where $\Gamma(n)$ is the gamma function:

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} \mathrm{d}t. \tag{3.17}$$

[4] Let it be pointed out here that, as in the introductory examples, a one-dimensional profile is generally denoted with $g(x)$ and a three-dimensional profile with $f(r)$. Both are defined as being positive from the tip of the indenter upwards, which is additionally introduced as the coordinate $\tilde{z}$ (see Fig. 3.4).

The exact equivalence between the three-dimensional and one-dimensional problem is valid for the relationships between the normal force, the contact radius, and the indentation depth. In Table 3.1, the values of the scaling factor κ_n are presented for various values of n and in Fig. 3.3 for $0 < n \leq 5$, they are shown graphically.

Here, the values for a conical and a parabolic indenter are pointed out. The corresponding scaling factors are $\kappa_1 = \frac{1}{2}\pi$ and $\kappa_2 = 2$. The latter is, of course, consistent with the *rule of Popov*, which requires dividing the radius of curvature by 2.

The fact that it is possible to exactly map a three-dimensional contact problem to a one-dimensional linearly elastic foundation not only for profiles of the form (3.14), but rather *for arbitrary superpositions* of such forms is extremely important. We now consider a superposition of multiple profiles:

$$f(r) = \sum_{n=1}^{\infty} f_n(r) = \sum_{n=1}^{\infty} c_n r^n. \tag{3.18}$$

In this case, the rule of Heß is applied as follows: From the profile (3.18), a one-dimensional profile is generated

$$f(r) = \sum_{n=1}^{\infty} c_n r^n \Rightarrow g(x) = \sum_{n=1}^{\infty} \tilde{c}_n |x|^n. \tag{3.19}$$

In Chap. 17, it is shown that by indenting the profile (3.19) into a linearly elastic foundation with a stiffness according to (3.4), the relationships between the normal force, contact radius, and the indentation depth remain the same as those in the three-dimensional case.

Table 3.1 Scaling factor κ_n for various exponents of the form function

n	0.5	1	2	3	4	5	6	7	8	9	10
κ_n	1.311	1.571	2	2.356	2.667	2.945	3.2	3.436	3.657	3.866	4.063

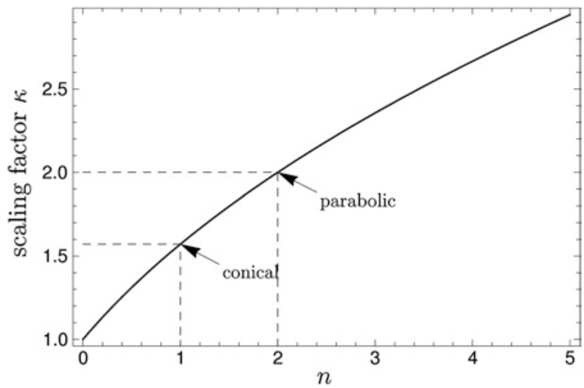

Fig. 3.3 Dependence of the scaling factor κ on the exponent n

The ability to map contacts between three-dimensional, axially-symmetric bodies of the form (3.14) to one-dimensional systems results from simple general scaling arguments and it is informative to discuss these briefly at this point. From dimensional analysis and self-affinity[5] of the profile (3.14), it arises that the contact radius and the indentation depth are related by the same exponential power n as $\tilde{z}$ and R:

$$d = \kappa_n c_n a^n, \tag{3.20}$$

where κ_n is a dimensionless constant. By pressing the one-dimensional profile (3.15) into the linearly elastic foundation, the indentation depth is trivially determined according to

$$d = \tilde{c}_n a^n. \tag{3.21}$$

By choosing a suitable $\tilde{c}_n = \kappa_n c_n$, one can always guarantee that the relationship between the indentation depth and the contact radius is correct in both cases. Furthermore, the differential contact stiffness is given in both the one-dimensional case as well as the three-dimensional case by

$$\frac{\partial F_N}{\partial d} = 2aE^* \tag{3.22}$$

(proof is given by Pharr et al. [4] or Popov [5]). By integrating this equation and taking (3.21) into consideration, the following relationship is obtained for both the one-dimensional and three-dimensional case:

$$F_N = \int \mathrm{d}F_N = 2E^* \int a\mathrm{d}(d) = 2E^* \int a\tilde{c}_n n a^{n-1}\mathrm{d}a = 2E^* \tilde{c}_n \frac{n}{n+1} a^{n+1}. \tag{3.23}$$

Inarguably, the force as a function of indentation depth must be the same in both cases:

$$F_N = \frac{2n}{n+1} E^* \tilde{c}_n^{-1/n} d^{\frac{n+1}{n}}. \tag{3.24}$$

If we constrain ourselves to the force–displacement relationship, then the ability to map three-dimensional systems to one-dimensional systems becomes even more general and is possible for arbitrary self-affine surfaces, regardless if they are axially-symmetric or not: The exponential dependence (3.24) is only contingent on the self-affinity and is valid for arbitrary surfaces with given Hurst exponents. Obviously, the correct coefficient can always be found by stretching the profile by the appropriate factor if the exponent in the force–displacement relationship is correct. As we will see in Chap. 10, this is also valid for self-affine, fractally rough surfaces. This paves the way for the fast calculation of contacts with rough surfaces and is, therefore, especially interesting.

Also, the superposition rule (3.19) has a simple physical meaning and requires only that the medium exhibits a linear behavior. Let us consider the two profiles

[5] For self-affinity, the following property is understood: If the profile (3.14) is stretched in the horizontal direction by the factor C and simultaneously in the vertical direction by a factor C^n, then one obtains the original profile. The exponent n is known as the *Hurst exponent*.

$f_1(r)$ and $f_2(r)$ being pressed into an elastic half-space. The first profile requires the indentation force $F_1(a)$ in order to obtain the contact radius a. The second profile, on the other hand, requires the force $F_2(a)$ in order to reach the *same contact radius a*. We denote the corresponding indentation depths with $d_1(a)$ and $d_2(a)$. If we initially consider the indention of $f_1(r)$ and then *additionally* apply $f_2(r)$ to the *same contact area,* with the radius a, then it directly follows from the linearity of the medium that the necessary force is

$$F_N(a) = F_1(a) + F_2(a). \tag{3.25}$$

The indentation depth, thereby, is

$$d(a) = d_1(a) + d_2(a). \tag{3.26}$$

These are exactly the two properties that are necessary for the mapping of superimposed profiles according to Eq. (3.19). In order to prevent confusion, we would like to stress that the principle of superposition is not valid (or is not exact) if the areas of application of both profiles are not the same.

3.3 General Mapping of Axially-Symmetric Profiles

The previous considerations dealt with the simplest mapping rules which are valid for contact profiles in the form of power functions. By choosing an arbitrary, positive real exponent and using the principle of superposition due to linearity, a large number of axially-symmetric contacts are able to be exactly mapped. The equivalence between one-dimensional and three-dimensional systems, however, is in no way restricted to such systems, but is generally valid for *all* axially-symmetric contacts with a simply connected contact area. The calculation of an equivalent profile using the profile function of the three-dimensional contact is conducted using the following formula:

$$g(x) = |x| \int_0^{|x|} \frac{f'(r)}{\sqrt{x^2 - r^2}} \mathrm{d}r, \tag{3.27}$$

the validity of which will be proven in Chap. 17. The fact that in the case of the power function (3.14), this rule leads to the simple scaling relation (3.16) is also explained here. Except for the explicit application of the formula (3.27), nothing changes in the procedure of the reduction method in order to determine the relationships between contact radius, indentation depth, and normal force. In the following, we would like to explain the procedure step by step using an example. For this, we consider the indentation of the following piecewise-defined profile into an elastic half-space:

$$f(r) = \begin{cases} 0 & \text{for } 0 \le r < b \\ \frac{r^2 - b^2}{2R} & \text{for } b \le r \le a \end{cases}. \tag{3.28}$$

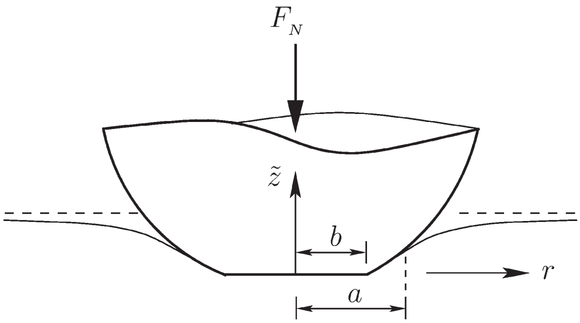

Fig. 3.4 Qualitative presentation of the indentation of a parabolic profile with a flattened tip into an elastic half-space

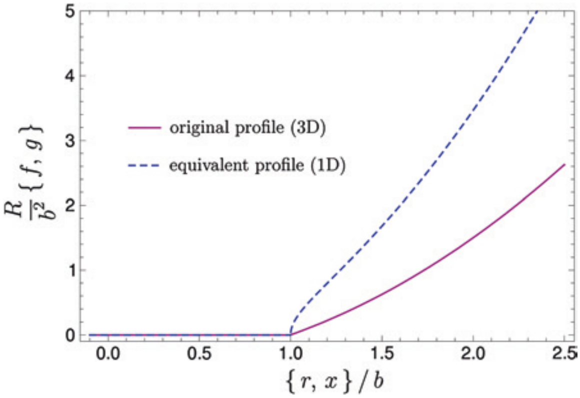

Fig. 3.5 Parabolic indenter with "worn" tip: comparison between original and equivalent profile

As can be gathered from Fig. 3.4, we can interpret the profile as an asperity which was originally parabolic, the tip of which, however, has been worn down through time.

The application of (3.27) requires the derivative of the original profile (3.28)

$$f'(r) = \begin{cases} 0 & \text{for } 0 \le r < b \\ \frac{r}{R} & \text{for } b \le r \le a \end{cases}, \tag{3.29}$$

which, after insertion into (3.27) and subsequent integration, leads to the equivalent one-dimensional profile[6]

$$g(x) = \begin{cases} 0 & \text{for } 0 \le |x| < b \\ \frac{|x|}{R}\sqrt{x^2 - b^2} & \text{for } b \le |x| \le a \end{cases}. \tag{3.30}$$

This profile is compared to the original in Fig. 3.5.

[6] Frequently, the one-dimensional profile is referred to in the following; this is to be understood, of course, as the profile in the one-dimensional model.

Naturally, the special case of $b = 0$ coincides with the mapping rule of Popov, of which one may be convinced by comparing (3.28) and (3.30) for this case.

For a known equivalent profile, we can now proceed to the solution of the contact problem using the aforementioned reduction process. In order to accomplish this, we must merely press the rigid profile described by (3.30) into the one-dimensional linearly elastic foundation, which results in a surface displacement of

$$u_z(x) = d - g(x) = d - \frac{|x|}{R}\sqrt{x^2 - b^2}. \tag{3.31}$$

The indentation depth, contact radius, and normal force must reveal the exact three-dimensional dependencies. The indentation depth as a function of contact radius results from requiring that the displacement at the edge of the contact approaches zero:

$$u_z(a) = 0 \quad \Rightarrow \quad d = g(a) = \frac{a}{R}\sqrt{a^2 - b^2}. \tag{3.32}$$

The normal force is the sum of the spring forces

$$F_N = E^* \int_{-a}^{a} \left[d - g(x)\right] \mathrm{d}x = 2E^* \int_{0}^{a} d \, \mathrm{d}x - \frac{2E^*}{R} \int_{b}^{a} x \sqrt{x^2 - b^2} \, \mathrm{d}x, \tag{3.33}$$

which provides

$$F_N(a) = \frac{2E^*}{3R}\left(2a^2 + b^2\right) \cdot \sqrt{a^2 - b^2}. \tag{3.34}$$

after integration and rearranging with the help of (3.32). The results (3.32) and (3.34) obtained by using the reduction method are exactly those derived by Ejike [6] for the three-dimensional problem. For the sake of completeness, let us state the relationship between normal force and indentation depth, which after solving (3.32) with respect to a and subsequently inserting this into (3.34), results in

$$F_N(d) = \frac{\sqrt{2}E^* b^3}{3R}\left(2 + \sqrt{1 + \left(\frac{2R}{b^2}d\right)^2}\right) \cdot \sqrt{-1 + \sqrt{1 + \left(\frac{2R}{b^2}d\right)^2}}. \tag{3.35}$$

Further contact problems that require the explicit application of formula (3.27) for the calculation of the equivalent profile can be found in the practice exercises at the end of this and the following two chapters.

3.4 The Mapping of Stress

In the one-dimensional contact problem with the linearly elastic foundation, the stresses are not able to be directly determined. Although the relationships between the force, displacement, and contact radius may be correctly obtained, it seems as if the contact-mechanical information dealing with the stress is lost. In reality, however, this is not the case. In the aforementioned dissertation by Heß [3], it was shown that the stress distribution for an arbitrary three-dimensional contact is able to be reproduced for a corresponding one-dimensional problem. The required derivations can be found in Chap. 17. In the present chapter, we will explain the rules for the calculation without the necessary evidence.

For the linearly elastic foundation, the spring forces $f_N(x)$ are directly given for every contact configuration. The distributed load $q(x)$ (or linear force density) is also able to be directly defined:

$$q(x) = \frac{f_N(x)}{\Delta x}. \tag{3.36}$$

Among others properties, it will be shown in Chap. 17 that the normal stress $\sigma_{zz}(r)$ in the contact area of a three-dimensional contact problem may be found from the distributed load $q(x)$ using the following integral transformation (the Abel transformation):

$$\sigma_{zz}(r) = \frac{1}{\pi} \int_r^\infty \frac{q'(x)}{\sqrt{x^2 - r^2}} \mathrm{d}x. \tag{3.37}$$

As an example of the application of this procedure, we once again consider the Hertzian contact problem. For the distributed load, it follows from (3.9) that

$$\begin{aligned} q(x) &= E^*\left(d - \tfrac{x^2}{2R_1}\right), && \text{for} \quad |x| < a = \sqrt{2R_1 d}. \\ q(x) &= 0, && \text{for} \quad |x| > a = \sqrt{2R_1 d} \end{aligned} \tag{3.38}$$

The derivative is $q'(x) = -E^* x/R_1$ within the contact area and zero outside of it. Insertion into (3.37) leads to

$$\sigma_{zz}(r) = -\frac{E^*}{\pi R_1} \int_r^\infty \frac{x\mathrm{d}x}{\sqrt{x^2 - r^2}} = -\frac{E^*}{\pi R_1} \int_r^a \frac{x\mathrm{d}x}{\sqrt{x^2 - r^2}} = -\frac{2}{\pi} E^* \left(\frac{d}{R}\right)^{1/2} \sqrt{1 - \left(\frac{r}{a}\right)^2}, \tag{3.39}$$

which corresponds exactly with the known Herzian solution.

Further examples to the calculation of the stress in axially-symmetric contacts according to Eq. (3.37) will be considered in the exercises at the end of this chapter.

3.5 The Mapping of Non-Axially-Symmetric Bodies

The equation for contact stiffness written in the form

$$k_z = 2E^* \beta \sqrt{\frac{A}{\pi}} \tag{3.40}$$

is also valid for non-circular cross-sections (A is the contact area). The constant β is always on the order of magnitude of 1 for "simple" profiles (see [7]):

$$\begin{array}{ll} \text{circular cross-section:} & \beta = 1.000 \\ \text{triangular cross-section:} & \beta = 1.034 \\ \text{square cross-section:} & \beta = 1.012 \end{array} \tag{3.41}$$

Equation (3.40) can be written in the form (3.1), if we define the effective diameter D as

$$D = 2\beta\sqrt{\frac{A}{\pi}}. \tag{3.42}$$

This rule allows for non-axially symmetric contacts to be mapped to a one-dimensional contact with a linearly elastic foundation.

3.6 Problems

Problem 1 Solve the problem of the contact between a cone and an elastic half-space (Fig. 3.6a) using the reduction method. Calculate the contact radius and the normal force as a function of the indentation depth.

Solution The form of the cone is described by the equation $f(r) = \tan\theta \cdot r$. The corresponding scaling factor has the value $\kappa_1 = \pi/2$, so that the one-dimensional profile is given by $g(x) = (\pi/2)\tan\theta \cdot |x|$. If the indenter is pressed to a depth of d, then the vertical displacement of the foundation at point x is given by $u_z(x) = d - (\pi/2)\tan\theta \cdot |x|$. We calculate the contact radius by demanding that $u_z(a) = 0$ and in this way, obtain the desired dependence on the indentation depth:

$$a = \frac{2}{\pi}\frac{d}{\tan\theta}. \tag{3.43}$$

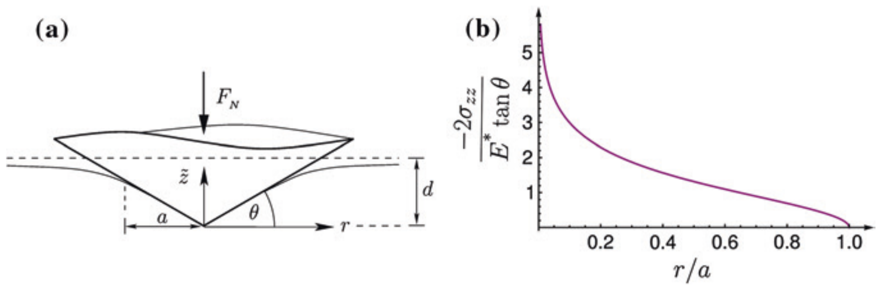

Fig. 3.6 (**a**) Contact between a rigid conical indenter and an elastic half-space. (**b**) Pressure distribution for the normal contact between a conical indenter and an elastic half-space

The normal force is obtained by "summing the spring forces":

$$F_N = 2E^* \int_0^a u_z(x)\mathrm{d}x = 2E^* \int_0^a (d - (\pi/2)\tan\theta \cdot x)\mathrm{d}x = \frac{2}{\pi} E^* \frac{d^2}{\tan\theta}. \quad (3.44)$$

Both results correspond *exactly*, of course, with those of the three-dimensional contact problem [8].

Problem 2 Let the profile $f(r) = C \cdot r^n$ be given for a rigid axially-symmetric indenter that is pressed into an elastic half-space. Determine the contact radius and the normal force in dependence on the indentation depth by using the reduction method.

Solution The equivalent one-dimensional profile is $g(x) = C\kappa_n |x|^n$. The contact radius is calculated from the condition $g(a) = d$ as

$$a = \left(\frac{d}{C\kappa_n}\right)^{1/n}. \quad (3.45)$$

The displacement field is determined by $u_z(x) = d - C\kappa_n |x|^n$ and for the normal force, we obtain

$$F_N = 2E^* \int_0^a u_z(x)\mathrm{d}x = 2E^* \int_0^a \left(d - C\kappa_n x^n\right)\mathrm{d}x = \frac{2n}{n+1} \frac{E^* d^{\frac{n+1}{n}}}{(C\kappa_n)^{1/n}}. \quad (3.46)$$

Once again, the results provide the *exact* dependencies of the three-dimensional problem (see Chap. 17).

Problem 3 Analyze the contact between a half-space and a superimposed profile of the form $f(r) = \frac{r^2}{2R} + |r| \tan\theta$ using the reduction method. Determine the contact radius and the normal force with respect to indentation depth.

Solution The equivalent one-dimensional profile is

$$g(x) = \kappa_2 \frac{x^2}{2R} + \kappa_1 |x| \tan\theta = \frac{x^2}{R} + \frac{\pi}{2} |x| \tan\theta. \quad (3.47)$$

The contact radius is determined using the condition

$$g(a) = \frac{a^2}{R} + \frac{\pi}{2} a \tan\theta = d, \quad (3.48)$$

so that the following relationship between the contact radius and displacement results:

$$a = \sqrt{\left(\frac{\pi}{4} R \tan\theta\right)^2 + Rd} - \frac{\pi}{4} R \tan\theta. \quad (3.49)$$

The one-dimensional displacement field is given by $u_z(x) = d - \frac{x^2}{R} - \frac{\pi}{2}|x| \tan\theta$, where we obtain the equation

$$F_N = 2E^* \int_0^a u_z(x)\mathrm{d}x = 2E^* \int_0^a \left(d - \frac{x^2}{R} - \frac{\pi}{2}|x| \tan\theta\right)\mathrm{d}x \tag{3.50}$$

for the normal force, which leads to the following equation after integration:

$$F_N = 2E^* \left(da - \frac{a^3}{3R} - \frac{\pi}{4}a^2 \tan\theta\right). \tag{3.51}$$

Insertion of (3.49) and simple rearrangement with respect to the desired relationship between normal force and indentation depth leads to

$$\begin{aligned} F_N = \frac{\pi^3 R^2 (\tan\theta)^3 E^*}{96} &\left(\sqrt{1 + \frac{16d}{\pi^2 R (\tan\theta)^2}} - 1\right) \\ &\left(1 + \frac{32d}{R\pi^2 (\tan\theta)^2} - \sqrt{1 + \frac{16d}{\pi^2 R (\tan\theta)^2}}\right). \end{aligned} \tag{3.52}$$

Problem 4 Calculate the stress distribution between a flat cylindrical indenter and an elastic half-space with the help of the Abel transformation.

Solution We begin by calculating the distributed load in the one-dimensional case. For a flat cylindrical indenter, the distributed load is constant and equal to

$$q(x) = \begin{cases} F_N/(2a), & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}. \tag{3.53}$$

We obtain the derivative

$$q'(x) = \frac{F_N}{2a}(\delta(x+a) - \delta(x-a)), \tag{3.54}$$

where $\delta(x)$ denotes the Dirac delta function. The integral (3.37) takes the form

$$\sigma_{zz}(r) = \frac{1}{\pi} \int_r^\infty \frac{q'(x)}{\sqrt{x^2 - r^2}}\mathrm{d}x = \frac{1}{\pi}\frac{F_N}{2a} \int_r^\infty \frac{(\delta(x+a) - \delta(x-a))}{\sqrt{x^2 - r^2}}\mathrm{d}x. \tag{3.55}$$

For the Dirac delta function equation $\int f(x)\delta(x-a)\mathrm{d}x = f(a)$ is valid if the integration area contains the point $x = a$ and is otherwise zero. Thus, the integration in (3.55) results in

$$\sigma_{zz}(r) = \frac{1}{\pi}\frac{F_N}{2a} = \begin{cases} -\frac{1}{\sqrt{a^2 - r^2}}, & \text{for } |r| < a \\ 0, & \text{for } |r| > a \end{cases}. \tag{3.56}$$

This is the *exact* stress distribution that exists in the three-dimensional contact between a rigid flat cylindrical indenter and an elastic half-space [1].

Problem 5 Calculate the stress distribution in a contact between a rigid cone and an elastic half-space with the help of the Abel transformation.

Solution We consider the equivalent one-dimensional model from Problem 1. The vertical displacement of the foundation at the point x is $u_z(x) = d - (\pi/2)\tan\theta \cdot |x|$, from which we obtain the distributed load $q(x) = E^* \cdot u_z(x) = E^*(d - (\pi/2)\tan\theta \cdot |x|)$. In order to calculate the normal stress, we insert the derivative $q'(x) = -(\pi/2)E^*\tan\theta \cdot \mathrm{sign}(x)$ into Eq. (3.37)

$$\sigma_{zz}(r) = \frac{1}{\pi}\int_r^\infty \frac{q'(x)}{\sqrt{x^2 - r^2}}\mathrm{d}x = -\frac{1}{2}E^*\tan\theta \int_r^a \frac{\mathrm{d}x}{\sqrt{x^2 - r^2}}. \tag{3.57}$$

Taking the integral results in

$$\sigma_{zz}(r) = \begin{cases} -\frac{1}{2}E^*\tan\theta \cdot \ln\left(\frac{a}{r} + \sqrt{\left(\frac{a}{r}\right)^2 - 1}\right), & \text{for } r < a \\ 0, & \text{for } r > a \end{cases}, \tag{3.58}$$

which is, of course, also in this case the *exact* three-dimensional stress distribution. This is shown graphically in Fig. 3.6b.

Problem 6 Determine the normal force and normal stress for the contact between a rigid cylindrical indenter and a concave, parabolic profile (see Fig. 3.7) with the help of the reduction method. Instead of using the indentation depth d, the displacement should be formulated based on the geometric values of d_o and h. It is assumed that a complete contact is present.

Solution First, we define the surface displacement within the contact area for the axially-symmetric contact. For this, we use the average displacement d_o in place of

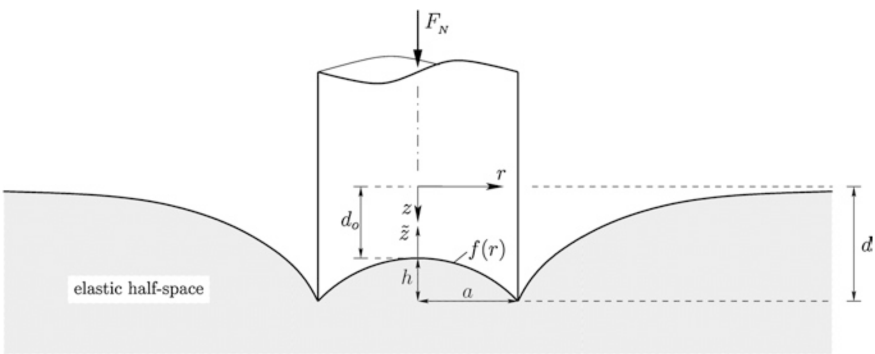

Fig. 3.7 Qualitative presentation of a (complete) indentation of a rigid cylindrical indenter with a concave, parabolic profile into an elastic half-space

the indentation depth d, so that $f(0) = 0$ is guaranteed for the concave profile in the same way as for the convex profile. Then, the following is valid:

$$u_z(r) = d_o - f(r) = d_o + \frac{h}{a^2} r^2. \tag{3.59}$$

The original profile contains a quadratic term, which we must simply multiply with the corresponding scaling factor in order to arrive at the geometry of the equivalent system:

$$g(x) = \kappa_2 f(|x|) = -2\frac{h}{a^2} x^2. \tag{3.60}$$

From the corresponding surface displacement in the one-dimensional model, we obtain a normal force of

$$F_N = 2E^* \int_0^a \left[d_o - g(x)\right] \mathrm{d}x = 2E^* \int_0^a \left(d_o + 2\frac{h}{a^2}x^2\right) \mathrm{d}x = 2E^* a\left(d_o + \frac{2}{3}h\right). \tag{3.61}$$

In order to calculate the normal stress in the original contact, we require the derivative of the distributed load $q'(x)$ in the reduced dimensions as well as the boundary condition $q(a)$:

$$q(x) = E^*\left(d_o + 2\frac{h}{a^2}x^2\right) \quad \Rightarrow \quad q'(x) = 4E^* \frac{h}{a^2} x \text{ and } q(a) = E^*(d_o + 2h). \tag{3.62}$$

Insertion of (3.62) into (3.37) leads to the desired normal stresses after integration and elementary rearrangement:

$$\sigma_{zz}(r) = -\frac{E^*}{\pi} \cdot \frac{d_o - 2h + 4h\left(\frac{r}{a}\right)^2}{\sqrt{a^2 - r^2}}. \tag{3.63}$$

Naturally, the results (3.61) and (3.63) correspond exactly to those of the three-dimensional, axially-symmetric contact, which is confirmed by comparison with the results given by Barber [9], if one takes into account the conversion $d_o = d - h$. Let it be once again insistently pointed out that a complete contact is assumed, which must satisfy the requirement of $\sigma_{zz}(0) \leq 0$. Then, from Eq. (3.63), the condition $d_o \geq 2h$ follows. Due to the fact that the reduction method in the form shown here is only suitable for the mapping of complete contacts (and not ring-shaped contact areas), this condition does not follow directly from the one-dimensional model. Furthermore, the exact mapping is only guaranteed for $F_N(d_o)$ and not for $F_N(d)$, because the maximum displacement (indentation depth) for concave profiles occurs on the boundary and not in the middle.

Problem 7 Formulate the method of dimensionality reduction for a transversally-isotropic medium.

Solution A transversally-isotropic medium is a medium that is isotropic in one plane. For crystalline bodies, this includes bodies in the hexagonal class of crystals. Also, a fiber composite with all fibers oriented in parallel is a transversally-isotropic medium. A linearly transversally-isotropic medium can be completely defined by 5 elastic moduli. If we denote the axis of symmetry to be "3," then the axes "1" and "2" are "equivalent" and can be chosen arbitrarily within the plane which they define. Hooke's law for such a medium is as follows:

$$\begin{aligned}
\sigma_{11} &= C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} \\
\sigma_{22} &= C_{12}\varepsilon_{11} + C_{11}\varepsilon_{22} + C_{13}\varepsilon_{33} \\
\sigma_{33} &= C_{13}(\varepsilon_{11} + \varepsilon_{22}) + C_{33}\varepsilon_{33} \\
\sigma_{12} &= (C_{11} - C_{12})\varepsilon_{12} \\
\sigma_{23} &= 2C_{44}\varepsilon_{23} \\
\sigma_{31} &= 2C_{44}\varepsilon_{31}\,.
\end{aligned} \tag{3.64}$$

The applicability of the method of dimensionality reduction is based solely on the fact that the differential stiffness of a medium is determined exclusively by the current contact area. For axially-symmetric profiles, it is given by the stiffness of the contact between a flat, rigid cylindrical indenter and the elastic half-space. Therefore, the rule for the application of the method of dimensional reduction to an arbitrary linearly elastic medium is as follows: First, the stiffness k_z of the contact with a flat cylindrical indenter with the diameter D and the equivalent one-dimensional system as a linearly elastic foundation with a stiffness per unit length of k_z/D are determined. This method can be applied to any medium for which a solution with a rigid cylinder is known.

The solution for the stiffness of a contact between a flat, cylindrical indenter and a transversally-isotropic medium (with an axis of symmetry parallel to the normal vector) can be directly taken from the work of Yu [10]. It is given by Eq. (3.1) with

$$E^* = \frac{2\left(\bar{C}_{13}^2 - C_{13}^2\right)}{\bar{C}_{13}(\nu_1 + \nu_2)}, \tag{3.65}$$

where the following relationships are introduced:

$$\nu_1 = \left[\frac{\left(\bar{C}_{13} - C_{13}\right)\left(2\bar{C}_{13} - I_0\right)}{4C_{33}C_{44}}\right]^{1/2} + \left[\frac{\left(\bar{C}_{13} + C_{13}\right)I_0}{4C_{33}C_{44}}\right]^{1/2} \tag{3.66}$$

$$\nu_2 = \left[\frac{\left(\bar{C}_{13} - C_{13}\right)\left(2\bar{C}_{13} - I_0\right)}{4C_{33}C_{44}}\right]^{1/2} - \left[\frac{\left(\bar{C}_{13} + C_{13}\right)I_0}{4C_{33}C_{44}}\right]^{1/2} \tag{3.67}$$

$$\bar{C}_{13} = (C_{11}C_{33})^{1/2} \tag{3.68}$$

$$I_0 = \bar{C}_{13} - C_{13} - 2C_{44}. \tag{3.69}$$

Insertion of (3.66)–(3.69) into (3.65) results in

$$E^* = \frac{2\sqrt{C_{44}}\left(C_{11}C_{33} - C_{13}^2\right)}{\sqrt{C_{11}}\sqrt{\left(\sqrt{C_{11}C_{33}} - C_{13}\right)\left(C_{13} + 2C_{44} + \sqrt{C_{11}C_{33}}\right)}}. \tag{3.70}$$

Problem 8 Determine the indentation depth and the normal force as a function of contact radius for the normal contact between a sphere of radius R and a linearly elastic half-space with the help of the reduction method. Contrary to the parabolic approximation of Hertz, here the exact spherical form should be taken into account and the equivalent profile should be calculated with the general Eq. (3.27).

Solution The exact profile of a sphere with a radius of R is given by the function

$$f(r) = R - \sqrt{R^2 - r^2}. \tag{3.71}$$

The first derivative of (3.71) is

$$f'(r) = \frac{r}{\sqrt{R^2 - r^2}}. \tag{3.72}$$

Inserting (3.72) into the general formula (3.27) leads to the equation

$$g(x) = x \int_0^x \frac{r}{\sqrt{R^2 - r^2} \cdot \sqrt{x^2 - r^2}} \mathrm{d}r = -x \int_{z(0)}^{0} \frac{\mathrm{d}z}{\sqrt{1 + z^2}}, \tag{3.73}$$

for which the elementary integral on the right results by using the substitution $z(r) = \dfrac{\sqrt{x^2 - r^2}}{\sqrt{R^2 - x^2}}$. The equivalent profile is

$$g(x) = x \cdot \operatorname{arsinh}\left(\frac{x}{\sqrt{R^2 - x^2}}\right) = \frac{1}{2} x \ln\left(\frac{R + x}{R - x}\right). \tag{3.74}$$

Figure 3.8 shows both of the “equivalent” profiles as well as their parabolic approximations. The dashed lines confirm the rule of Popov.

Simultaneously, the surface displacement of the linearly elastic foundation may be found with (3.74), which must tend to zero at the contact boundary and in this way, determines the indentation depth:

$$u_z(a) = 0 \quad \Rightarrow \quad d = g(a) = \frac{1}{2} a \ln\left(\frac{R + a}{R - a}\right). \tag{3.75}$$

The spring forces, which are proportional to the surface displacement, must be in equilibrium with the normal force

$$F_N = E^* \int_{-a}^{a} \left[d - g(x)\right] \mathrm{d}x = 2E^* da - E^* \int_0^a x \ln\left(\frac{R + x}{R - x}\right) \mathrm{d}x. \tag{3.76}$$

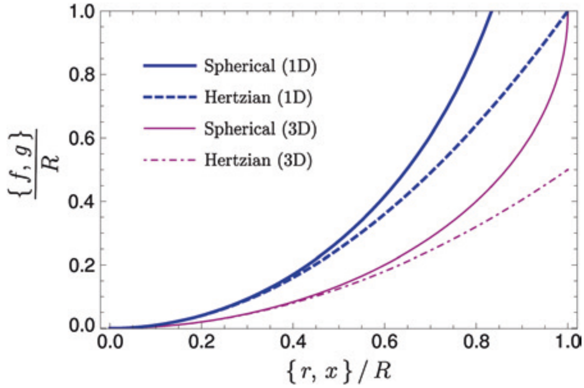

Fig. 3.8 Exact spherical form and parabolic approximation (Hertz) including the one-dimensional equivalents

A suitable partial integration initially provides

$$F_N = 2E^* a\left[d - \frac{R}{2} + \frac{R^2 - a^2}{4a}\ln\left(\frac{R+a}{R-a}\right)\right] \tag{3.77}$$

and after insertion of (3.75), the contact force is finally obtained as a function of contact radius:

$$F_N(a) = E^*\frac{R^2 + a^2}{2}\ln\left(\frac{R+a}{R-a}\right) - E^* Ra. \tag{3.78}$$

The indentation depth from (3.75) and the normal force from (3.78) correspond exactly to the three-dimensional contacts based on the solutions of Segedin [11], which are obtained using the Area-functions. Finally, let it be known that we could have just as well developed the spherical profile as a series. After multiplying the individual terms with the corresponding scaling factor, according to the rules of Heß, the equivalent profile (3.74) would have been given in the form of a power series. If the integral for the general formula (3.27) is not known, we have, in fact, no choice but to use this strategy.

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