Chapter 2
Separation of the Elastic and Inertial Properties in Three-Dimensional Systems

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2.1 Introduction

For a wide class of "typical tribological systems," there are a number of properties that allow for the significant simplification of contact problems and, in this way, make fast calculations of multi-scalar systems possible. These simplified properties, which are used by the method of dimensionality reduction include

1. The ability to separate the elastic and inertial properties in three-dimensional systems
2. The close analogy between three-dimensional contacts and certain one-dimensional problems.

The first of these will be discussed in this chapter, while further chapters are dedicated to the second. The first property can be formulated into three statements:

(a) For sufficiently small velocities, deformations may be considered to be quasi-static.
(b) The potential energy, and therewith, the force–displacement ratio, is a *local* property which is only dependent on the configuration of the contact area and not on the form or size of the body as a whole.
(c) The kinetic energy, on the other hand, is a *global* property which is only dependent on the form and size of the body and not on the configuration of the micro-contacts.

These three listed statements are met in many macroscopic systems. In the following, we will consider them in detail individually.

V.L. Popov and M. Heß, *Method of Dimensionality Reduction in Contact Mechanics and Friction*, DOI 10.1007/978-3-642-53876-6_2

2.2 The Quasi-Static State

The separation of the elastic and inertial properties is only valid under the condition that the characteristic loading time T of a contact is much larger than the time that elastic waves in the continuum require to travel a distance on the order of magnitude of the diameter D of the contact area:

$$T > D/c, \tag{2.1}$$

where c is the speed of sound. For instance, if the characteristic time of changes in force in a wheel–rail contact is larger than the characteristic time of $T = 1\ \text{cm}/\left(5 \times 10^3\ \text{m/s}\right) = 2 \times 10^{-6}\ \text{s}$ (or the frequency is below 500 kHz), then they may be considered quasi-static. If this condition is met, then the deformation near the contact area is practically the same as in a static contact. This is, of course, the same for the contact forces.

If an even more stringent condition is met, namely,

$$T > R/c, \tag{2.2}$$

where R is the size of the entire system, then all particles in the continuum, with the exception of a small volume near the contact, move as a rigid body. In other words, the condition (2.2) means that the characteristic contact time is much larger than the period of the normal modes of the system. For a wheel–rail contact, this condition is met for frequencies below approximately 2 kHz.

If we continue with the example of a rolling wheel, then the characteristic contact time can be approximated as $T \approx D/v$, where v is the linear velocity (driving speed). Then, the quasi-static state condition simply means

$$v < c. \tag{2.3}$$

For a rough contact with a characteristic wavelength of λ, the characteristic time is $T \approx \lambda/v$, so that condition for the quasi-static state is much more restrictive: $\lambda/v > D/c$ or

$$v < c\frac{\lambda}{D}. \tag{2.4}$$

In most tribological systems, we are dealing with the movements of components whose relative velocities (e.g., a train at around 50 m/s) are orders of magnitude smaller than the speed of sound in these components (this is around 5×10^3 m/s for steel). Under these conditions, one can consider the problem to be *quasi-static* if one is interested in the wavelengths of the roughness that are roughly two orders of magnitude smaller than the diameter of the contact area.

2.3 Elastic Energy as a Local Property

Elastic interactions are local in the sense that they play a role only within a volume on the same order of magnitude as the diameter of the contact area and, therefore, are not dependent on the size or form of the body as a whole. Let us investigate

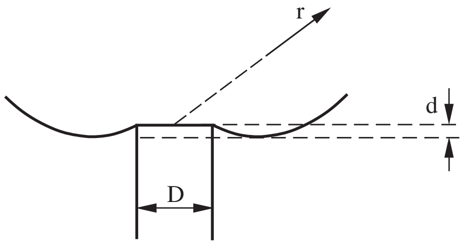

Fig. 2.1 Flat cylindrical indenter being pressed into an elastic body by a distance of *d*

this somewhat more closely by calculating the potential energy of a deformed contact area. We observe a cylindrical indenter that is pressed into a body by the distance d (Fig. 2.1).

For the displacement inside the elastic body at a large distance r from the indentation point, the following is valid:

$$u \approx \frac{D \cdot d}{r}. \tag{2.5}$$

The deformation can be estimated as $\varepsilon \approx \frac{\mathrm{d}u}{\mathrm{d}r} \approx -\frac{D \cdot d}{r^2}$ and the energy density, as $\mathcal{E} \approx \frac{1}{2}G\varepsilon^2 \approx \frac{1}{2}G\frac{D^2 \cdot d^2}{r^4}$. Through integration, the elastic energy is

$$U \simeq \int G\frac{D^2 \cdot d^2}{r^4}\pi r^2 \mathrm{d}r = \pi G D^2 \cdot d^2 \int \frac{\mathrm{d}r}{r^2}, \tag{2.6}$$

where G is shear modulus of the medium. This integral converges at the upper boundary (therefore, it can be set to infinity) and diverges at the lower limit. However, because the asymptote (2.5) is only valid for $r > D$, the elastic energy of the deformation within a volume with a linear dimension on the order of magnitude D dominates. In other words, the elastic energy is a local value that is only dependent on the configuration and deformation in the vicinity of the micro-contact. The size and form of the macroscopic body is irrelevant for the contact mechanics of this problem.

Incidentally, this property is not self-evident and would not, for example, be valid in a two-dimensional system. Instead of having Eq. (2.6), we would have the integral $\int \mathrm{d}r/r$ in the two-dimensional case, which diverges logarithmically on both boundaries. The elastic contact energy for the two-dimensional case is, therefore, dependent on the contact configuration as well as the size and form of the body as a whole.

2.4 Kinetic Energy as a Global Property

Exactly the opposite is true for the kinetic energy of the body. To illustrate this, let us consider a sphere landing on an indenter with a diameter of D (the contact radius remains the same) at a velocity of v (Fig. 2.2).

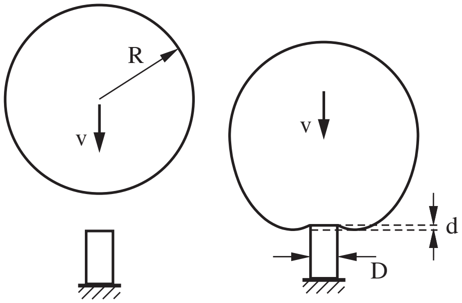

Fig. 2.2 Illustrating the kinetic energy of an elastic body landing on a rigid cylindrical indenter at a velocity of v

We assume that the condition (2.2) is met so that the elastic deformation in the entire body may be considered to be quasi-static. The center of gravity of the sphere x and the coordinate of the point of contact ξ are chosen as the generalized coordinates of the sphere. Accordingly, the indentation depth is equal to

$$d = x - \xi + R. \tag{2.7}$$

The potential energy of the sphere is a function of the indentation depth:

$$U = \frac{kd^2}{2} = \frac{k}{2}(x - \xi + R)^2, \tag{2.8}$$

where $k = E^*D$. E^* is here the effective Young modulus defined in the next Chapter [Eq. (3.2)]. The velocity field for a quasi-static indentation is obtained from (2.5) by differentiating the indentation depth with respect to time:

$$\dot{u} \approx \frac{D \cdot \dot{d}}{r} = \frac{D \cdot \left(\dot{x} - \dot{\xi}\right)}{r}. \tag{2.9}$$

The total kinetic energy is then composed of the kinetic energy of the movement of the center of mass and the kinetic energy of the deformation relative to the center of mass:

$$K = \frac{m\dot{x}^2}{2} + \frac{\rho}{2}\left(\dot{x} - \dot{\xi}\right)^2 \int \left(\frac{D}{r}\right)^2 \mathrm{d}V = \frac{m\dot{x}^2}{2} + \frac{m_1}{2}\left(\dot{x} - \dot{\xi}\right)^2, \tag{2.10}$$

with

$$m_1 \approx \rho D^2 \int \left(\frac{1}{r}\right)^2 2\pi r^2 \mathrm{d}r = 2\pi \rho D^2 R \approx m\left(\frac{D}{R}\right)^2. \tag{2.11}$$

A more accurate derivation leads to the result of $m_1 \approx 0.3\, m(D/R)^2$ for materials with $\nu = 1/3$ (see Problem 3 in this chapter). Note that this mass is on the same order of magnitude as the mass of a rod with the diameter D and the length R.

We now would like to illustrate the separation of the elastic and inertial properties of a contact and their accuracy using several concrete dynamic examples. The dynamic treatment of the system makes use of the Lagrange function, which

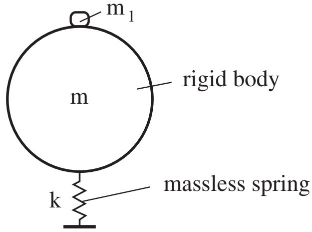

Fig. 2.3 Model for the impact of a sphere on a rigid cylindrical rod

is calculated as the difference between the kinetic energy (2.10) and the potential energy (2.8):

$$L = K - U = \frac{m\dot{x}^2}{2} + \frac{m_1}{2}\left(\dot{x} - \dot{\xi}\right)^2 - \frac{k}{2}(x - \xi + R)^2$$
$$= \frac{m\dot{x}^2}{2} + \frac{m_1}{2}\left(\dot{x} - \dot{\zeta}\right)^2 - \frac{k}{2}(x - \zeta)^2, \tag{2.12}$$

where we have introduced a new variable $\zeta = \xi - R$.

We consider three cases:

1. *Impact of the body with a stationary, rigid rod.* In this case, $\zeta = 0$ is valid for the entirety of the impact time and the Lagrange function takes the form

$$L = \frac{(m + m_1)\dot{x}^2}{2} - \frac{k}{2}x^2. \tag{2.13}$$

Therefore, the system is equivalent to a rigid body with the mass $(m + m_1)$ on a spring with the stiffness k, which is equal to the static contact stiffness (Fig. 2.3). The mass correction m_1 is on the order of magnitude of $m(D/R)^2$ and may be neglected for small contact diameters.

2. *"Base excitation."* We now assume that the coordinate of the contact area, and therefore, the coordinate ζ, is a given function of time: $\zeta = \zeta(t)$. The Lagrange function is then equal to

$$L = \frac{m\dot{x}^2}{2} + \frac{m_1}{2}\left(\dot{x} - \dot{\zeta}(t)\right)^2 - \frac{k}{2}(x - \zeta(t))^2 \tag{2.14}$$

and the Euler–Lagrange equation for the coordinate of the center of gravity is

$$(m + m_1)\ddot{x} + kx = k\zeta(t) + m_1\ddot{\zeta}(t) = \Delta F_N(t). \tag{2.15}$$

The acceleration term on the right-hand side of this equation is on the order of magnitude of $m_1\zeta/T^2$. Within the validity regime of the condition (2.2), we have the equation

$$\frac{m_1}{T^2}\zeta < \frac{m_1 c^2}{R^2}\zeta \approx \frac{mc^2D^2}{2R^4}\zeta = k\frac{mc^2D^2}{2kR^4}\zeta \approx k\frac{2\pi R^3\rho c^2 D^2}{EDR^4}\zeta = k\frac{2\pi D}{R}\zeta \ll k\zeta, \tag{2.16}$$

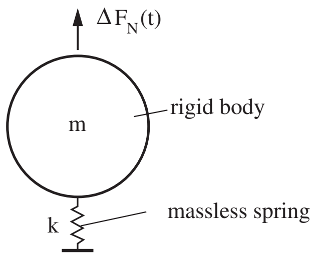

Fig. 2.4 Model of a non-stationary contact (e.g., a rolling sphere on a rough substrate)

where $c^2 = E/\rho$ has been substituted, E being Young's modulus. Therefore, the acceleration contribution can always be neglected with respect to the elastic contribution for small contact diameters. As in the first case, for sufficiently small contact diameter, the system can be modeled as a rigid body with the mass m bound to a spring (see Fig. 2.4).

3. *Freely oscillating surface.* If the body is held and a contact area with the diameter D indented and then instantaneously let free, then we obtain the Lagrange function by substituting $x = 0$ into (2.12):

$$L = \frac{m_1}{2}\dot{\zeta}^2 - \frac{k}{2}\zeta^2. \tag{2.17}$$

The movement of the surface would be an oscillation with the angular frequency ω_1:

$$\omega_1^2 = \frac{k}{m_1} \approx \frac{ED}{2\pi\rho D^2 R} \approx \frac{c^2}{2\pi DR}. \tag{2.18}$$

This frequency, however, is much larger than the natural frequency of the body $\omega_0^2 \approx c^2/R^2$. Therefore, the condition of validity for the Lagrange function (2.17) is not met: Resonance oscillations of a free surface *cannot* be dealt with using this approximation.

If the diameter of the contact is dependent on the indentation depth, then the corresponding potential energy of the contact $U(d)$ must be used in the Lagrange function:

$$L = \frac{m\dot{x}^2}{2} + \frac{m_1(t)}{2}\left(\dot{x} - \dot{\zeta}\right)^2 - U(x - \zeta), \tag{2.19}$$

where, as before,

$$m_1 \approx \frac{m}{2}\left(\frac{D(t)}{R}\right)^2. \tag{2.20}$$

As explained above, the second term in (2.19) can always be neglected as long as condition (2.2) is met. The model shown in Fig. 2.5 is the result.

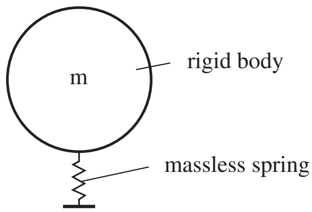

Fig. 2.5 Dynamic model for a non-linear (e.g., Hertzian) contact

Now, we will concentrate on the procedure for non-stationary force effects on a small contact area. As an example, we consider a rough sphere rolling on a rigid rough surface (although nominally flat) so that the potential energy is not only a function of indentation depth, but also an explicit function of time:

$$L = \frac{m\dot{x}^2}{2} + \frac{m_1(t)}{2}\left(\dot{x} - \dot{\zeta}\right)^2 - U(x - \zeta, t). \tag{2.21}$$

Due to the fact that the rolling takes place on a rigid surface, $\zeta = 0$ and the Lagrange function takes the form

$$L \approx \frac{m\dot{x}^2}{2} - U(x, t), \tag{2.22}$$

where we have neglected the mass correction. The corresponding Euler-Lagrange equation is then

$$m\ddot{x} = -\frac{\partial U}{\partial x} = F_N(x, t). \tag{2.23}$$

In this case, the system is equivalent to a rigid body on which the time-dependent contact forces act. If it is possible to divide the force into the part for the "smooth surface" and a stochastic part, according to the equation

$$F_N(x, t) = F_{N,0}(x) + \Delta F_N(t), \tag{2.24}$$

then the equation of motion takes the form

$$m\ddot{x} = F_{N,0}(x) + \Delta F_N(t). \tag{2.25}$$

This equation describes a rigid mass m coupled to the surface with a non-linear contact force $F_{N,0}(x)$ being acted upon by the exciting force $\Delta F_N(t)$. The corresponding model is the same as that in Fig. 2.4 with the exception that a non-linear spring is used here.

If the condition (2.2) is not met, but the condition (2.1) is still valid, then the body can no longer be treated as a rigid mass, such as in the case of high-frequency oscillations; however, the static equations can still be used to determine the contact forces. In this case, there is no simple model to describe the entire dynamics of the system, because the frequency is too high for the body to be assumed to be rigid. Therefore, the complete dynamic problem must be solved. The contact problem, however, remains quasi-static and provides a boundary condition for the elastic problem (Fig. 2.6). An example of such a dynamic case is presented in Problem 4.

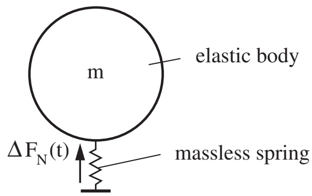

Fig. 2.6 Model for the case of high-frequency contact forces, for which condition (2.2) is not met, but the weaker condition (2.1) still holds true

2.5 Problems

Problem 1 Determine the contact time for an elastic sphere (radius R) impacting a rigid wall. (Hertz 1881, [1]).

Solution The approaching distance between the center of the sphere and the wall, starting at the moment of impact, is defined as x. The potential energy of the system is given by[1] $U = (8/15)E^* R^{1/2} d^{5/2}$, while the kinetic energy is equal to that of a rigid body. During the time of impact, the energy is conserved:

$$\frac{m}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{8}{15}E^* R^{1/2} x^{5/2} = \frac{m v_0^2}{2}, \tag{2.26}$$

where v_0 is the impact velocity. The minimum distance between the center of the sphere and the wall x_0 corresponds to the time at which the velocity $\mathrm{d}x/\mathrm{d}t$ disappears, and is equal to

$$x_0 = \left(\frac{15}{16}\frac{m v_0^2}{E^* R^{1/2}}\right)^{2/5}. \tag{2.27}$$

The length of impact τ (while x increases from 0 to x_0 and then decreases back to 0) is

$$\tau = \frac{2}{v_0}\int_0^{x_0} \frac{\mathrm{d}x}{\sqrt{1-(x/x_0)^{5/2}}} = \frac{2x_0}{v_0}\int_0^1 \frac{\mathrm{d}\xi}{\sqrt{1-\xi^{5/2}}} \approx \frac{2.94 x_0}{v_0}. \tag{2.28}$$

Problem 2 Solve Problem 1 assuming that the sphere is glued to a hard cylindrical foot, much like a golf tee, with a diameter of D so that the contact radius does not change.

Solution The contact stiffness of a contact with a diameter of D is equal to $k = E^* D$ [2] (definition of material parameters see next Chapter) and the sphere can be considered to be a rigid mass as a first approximation. During the entire contact process, the differential equation

1 Definition of the material parameter E see next chapter.

$$m\ddot{x} + kx = 0 \tag{2.29}$$

is sufficient to describe the motion. The solution for the initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$ is $x = \frac{v_0}{\omega}\sin(\omega t)$, with $\omega^2 = k/m$. The length of time of the contact is equal to half of the period of the oscillation

$$\tau = \frac{\pi}{\omega} = \pi\sqrt{\frac{m}{E^* D}}. \tag{2.30}$$

Problem 3 Calculate the mass m_1 in Eq. (2.10) in the case of an elastic sphere.

Solution We will assume that the displacement relative to the center of mass in the entire volume of the sphere is the same as that in a half-space. If a round area with the diameter D is pressed into an elastic half-space by the distance u, then displacements result whose asymptotic forms (for $r \gg D$) appear as follows [3]:

$$u_x = \frac{1+\nu}{2\pi E}\left[\frac{xz}{r^3} - \frac{(1-2\nu)x}{r(r+z)}\right]F_z, \tag{2.31}$$

$$u_y = \frac{1+\nu}{2\pi E}\left[\frac{yz}{r^3} - \frac{(1-2\nu)y}{r(r+z)}\right]F_z, \tag{2.32}$$

$$u_z = \frac{1+\nu}{2\pi E}\left[\frac{2(1-\nu)}{r} + \frac{z^2}{r^3}\right]F_z \tag{2.33}$$

with

$$F_z = uE^* D. \tag{2.34}$$

Substitution of (2.34) into the equations for displacements results in

$$u_x = \frac{uD}{2\pi(1-\nu)}x\left[\frac{z}{r^3} - \frac{(1-2\nu)}{r(r+z)}\right], \tag{2.35}$$

$$u_y = \frac{uD}{2\pi(1-\nu)}y\left[\frac{z}{r^3} - \frac{(1-2\nu)}{r(r+z)}\right], \tag{2.36}$$

$$u_z = \frac{uD}{2\pi(1-\nu)}\left[\frac{2(1-\nu)}{r} + \frac{z^2}{r^3}\right]. \tag{2.37}$$

The corresponding velocities, under the assumption that the deformation is quasi-static, are equal to

$$\dot{u}_x = \frac{\dot{u}D}{2\pi(1-\nu)}x\left[\frac{z}{r^3} - \frac{(1-2\nu)}{r(r+z)}\right], \tag{2.38}$$

$$\dot{u}_y = \frac{\dot{u}D}{2\pi(1-\nu)} y\left[\frac{z}{r^3} - \frac{(1-2\nu)}{r(r+z)}\right], \tag{2.39}$$

$$\dot{u}_z = \frac{\dot{u}D}{2\pi(1-\nu)}\left[\frac{2(1-\nu)}{r} + \frac{z^2}{r^3}\right]. \tag{2.40}$$

With the spherical coordinates

$$x = r\cos\theta\cos\varphi, \tag{2.41}$$

$$y = r\cos\theta\sin\varphi, \tag{2.42}$$

$$z = r\sin\theta, \tag{2.43}$$

The equations (2.38)–(2.40) can be written as follows:

$$\dot{u}_x = \frac{\dot{u}D}{2\pi(1-\nu)}\frac{1}{r}\left\{\cos\theta\cos\varphi\left[\sin\theta - \frac{(1-2\nu)}{(1+\sin\theta)}\right]\right\}, \tag{2.44}$$

$$\dot{u}_y = \frac{\dot{u}D}{2\pi(1-\nu)}\frac{1}{r}\left\{\cos\theta\sin\varphi\left[\sin\theta - \frac{(1-2\nu)}{(1+\sin\theta)}\right]\right\}, \tag{2.45}$$

$$\dot{u}_z = \frac{\dot{u}D}{2\pi(1-\nu)}\frac{1}{r}\left[2(1-\nu) + \sin^2\theta\right]. \tag{2.46}$$

The kinetic energy of the deformation is now calculated as

$$\begin{aligned}
K &\approx \frac{\rho}{2}2\pi\int_0^{\pi/2}\mathrm{d}\theta\int_0^{2R\sin\theta}\left(\dot{u}_x^2+\dot{u}_y^2+\dot{u}_z^2\right)r^2\cos\theta\,\mathrm{d}r\\
&= \frac{\pi\rho\dot{u}^2D^2R}{2\pi^2(1-\nu)^2}\int_0^{\pi/2}\left\{\cos^2\theta\left[\sin\theta - \frac{(1-2\nu)}{(1+\sin\theta)}\right]^2\right.\\
&\quad\left. + \left[2(1-\nu)+\sin^2\theta\right]^2\right\}\sin\theta\cos\theta\,\mathrm{d}\theta\\
&= \frac{\pi\rho\dot{u}^2D^2R}{2\pi^2(1-\nu)^2}\left(\frac{55}{12} - \frac{32}{3}\nu + 8\nu^2 - 2\ln 2\cdot(1-2\nu)^2\right)\\
&= \frac{1}{2}\cdot\frac{4\pi\rho R^3}{3}\frac{3\dot{u}^2D^2}{4\pi^2R^2}\cdot\frac{\frac{55}{12} - \frac{32}{3}\nu + 8\nu^2 - 2\ln 2\cdot(1-2\nu)^2}{(1-\nu)^2}.
\end{aligned} \tag{2.47}$$

From this, it follows that

$$m_1 = m\left(\frac{D}{R}\right)^2\frac{3}{4\pi^2}\cdot\frac{\frac{55}{12} - \frac{32}{3}\nu + 8\nu^2 - 2\ln 2\cdot(1-2\nu)^2}{(1-\nu)^2} = m\left(\frac{D}{R}\right)^2\delta(\nu). \tag{2.48}$$

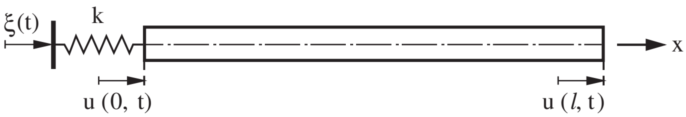

Fig. 2.7 Diagram for the contact described in Problem 4

For metallic materials ($\nu \approx 1/3$) and incompressible media ($\nu \approx 1/2$), $\delta(1/3) \approx 0.3$ and $\delta(1/2) \approx 0.38$, respectively.

Problem 4 A round rod with the diameter D_1 is excited in an area of constant diameter $D_2 \ll D_1$ by the harmonic oscillation $\xi = \xi_0 \cos \omega t$. Calculate the motion of the system.

Solution The system diagram is shown in Fig. 2.7, where $k = E^* D_2$.
The equation of motion for the elastic rod is

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}. \tag{2.49}$$

Here, $u(x,t)$ is the displacement of the point with the initial coordinate x and $c^2 = E/\rho$. The displacement of the "base point" we describe with ξ. The boundary condition at the left side of the rod is then

$$k[u(0,t) - \xi(t)] = AE \left.\frac{\partial u}{\partial x}\right|_{x=0}. \tag{2.50}$$

$A = \pi D_1^2/4$ is the cross section of the rod. The solution of Eq. (2.49), taking under consideration the unloaded end at the point $x = l$ and the boundary condition (2.50) with $\xi = \xi_0 \cos \omega t$ is

$$u(x,t) = \frac{k\xi_0 \cos \frac{\omega}{c}(x-l)}{k \cos \frac{\omega}{c} l - AE \frac{\omega}{c} \sin \frac{\omega}{c} l} \cos(\omega t). \tag{2.51}$$

If condition (2.2) is met, and therefore $\omega l/c \ll 1$, then the solution takes the form

$$u(x,t) = \frac{k\xi_0 \cos(\omega t)}{k - \frac{AlE\omega^2}{c^2}} = \frac{k\xi_0 \cos(\omega t)}{k - m\omega^2}, \tag{2.52}$$

where $m = \rho A l$ is the mass of the rod. In this limiting case, the displacement is not dependent of the coordinate x: The rod moves as a rigid structure with the mass m. The deviation from the approximation as a rigid mass, in this case, is of the second order of magnitude for $\omega l/c$.

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