Nonlinear Control of a Gantry Crane

Bernd Kolar and Kurt Schlacher

Institute of Automatic Control and Control Systems Technology,
Johannes Kepler University Linz, Altenberger Strasse 69, 4040 Linz, Austria
{bernd.kolar,kurt.schlacher}@jku.at

Abstract. This paper treats the nonlinear control of a laboratory model of a gantry crane, where a trolley can be moved on a rail and the load is fixed at the end of a rope of variable length. The system is differentially flat, and the coordinates of the load, which also are the variables to be controlled, are a flat output. This fact allows us to determine a feedforward control law in a straightforward manner. Because of friction, the results achievable by a pure feedforward law are, as expected, not satisfying. This does not apply to the "pendulum subsystem", with the position of the trolley and the length of the rope as input, since it is almost free of friction. Therefore, a feedforward control for the "pendulum subsystem" is designed such that it shows an excellent tracking behavior. Finally, cascaded control is used for the guidance of the overall system.

Keywords: nonlinear control, underactuated mechanical system, differential flatness, feedforward control

1 Introduction

In this contribution, the design of a nonlinear controller for the laboratory model of a gantry crane is discussed. At this laboratory model, the motion of the load is restricted to a vertical plane. The task is to transfer the load from one rest position to another one. Since the crane is an underactuated mechanical system, the design of a controller is a lot more difficult compared to a fully actuated system. Moreover, the system is not input to state linearizable by static feedback (see [5]). However, the gantry crane is a differentially flat system and the coordinates of the load are a flat output (see e.g. [1], [2], [3], [4] and references therein). Since these coordinates are also the variables which are to be controlled, it is advantageous to use flatness based methods for the controller design.

The paper is structured as follows: Section 2 treats the modeling. After that, Section 3 deals with differential flatness. In Section 4, the design of reference trajectories for the load coordinates and a feedforward control are discussed. Finally, Section 5 deals with a feedforward control of the "pendulum subsystem" and measurement results are shown.

2 Modeling

In Fig. 1 the laboratory model is shown. The trolley can be moved on a rail and the load is fixed at the end of a rope of variable length. The length of the

R. Moreno-Díaz et al. (Eds.): EUROCAST 2013, Part II, LNCS 8112, pp. 289–296, 2013.

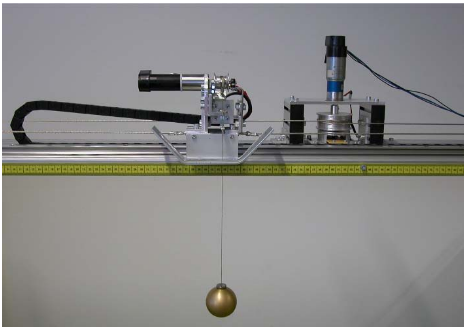


Fig. 1. Laboratory model

rope can be modified by coiling or uncoiling on a cylinder. The derivation of the mathematical model is based on the sketch shown in Fig. 2. The driving force F_{An} which acts on the trolley and the driving torque M_{An} which acts on the cylinder are the control inputs. The variable x_D denotes the position of the trolley, φ is the rotation angle of the cylinder and θ describes the pendulum angle. With R_T as the radius of the cylinder and L_0 as the length of the rope for $\varphi = 0$, the length of the rope is given by $L = L_0 + R_T\varphi$. The coordinates of the load are given by x_L and y_L. The parameters m_W and A_T are the mass of the trolley and the moment of inertia of the cylinder. The variable g represents the gravitational acceleration.

It is important to state that for modeling it was assumed that the rope is always stretched. This is true as long as the inequality $\ddot{y}_L < g$ is not violated. With this assumption, the gantry crane is a rigid multi-body system with holonomic constraints, which allows the usage of the Euler-Lagrange equations in standard form (see e.g. [6]) for the calculation of the equations of motion. They read as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right)^T - \left(\frac{\partial T}{\partial \mathbf{q}}\right)^T + \left(\frac{\partial V}{\partial \mathbf{q}}\right)^T = \mathbf{Q} \tag{1}$$

with the kinetic energy T, the potential V and the generalized forces $\mathbf{Q}$. The variable $\mathbf{q}$ represents the generalized coordinates. With the choice $\mathbf{q}^T = [x_D, \varphi, \theta]$ one obtains

$$T = \frac{1}{2}m_W\dot{x}_D^2 + \frac{1}{2}A_T\dot{\varphi}^2 + \frac{1}{2}\mathbf{v}_L^T m_L \mathbf{v}_L\,, \tag{2}$$

where

$$\mathbf{v}_L = \begin{bmatrix} \dot{x}_D - R_T\dot{\varphi}\sin(\theta) - (L_0 + R_T\varphi)\,\dot{\theta}\cos(\theta) \\ R_T\dot{\varphi}\cos(\theta) - (L_0 + R_T\varphi)\,\dot{\theta}\sin(\theta) \\ 0 \end{bmatrix} \tag{3}$$

is the velocity of the load. Furthermore one gets

$$V = -m_L g\,(L_0 + R_T\varphi)\cos(\theta) \tag{4}$$

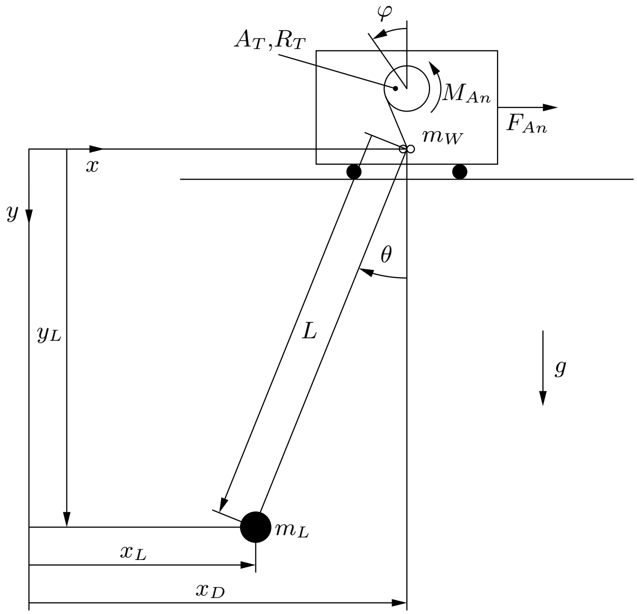


Fig. 2. Schematic for the mathematical modeling

and

$$\mathbf{Q}^T = [F_{An}, M_{An}, 0] \ . \tag{5}$$

Evaluating the Euler-Lagrange equations yields the equations of motion, which can be written in the form

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{Q} \tag{6}$$

with the mass matrix

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} m_W + m_L & -m_L R_T \sin(\theta) & -m_L \left(L_0 + R_T\varphi\right)\cos(\theta) \\ -m_L R_T \sin(\theta) & A_T + m_L R_T^2 & 0 \\ -m_L \left(L_0 + R_T\varphi\right)\cos(\theta) & 0 & m_L \left(L_0 + R_T\varphi\right)^2 \end{bmatrix} \tag{7}$$

and

$$\mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} m_L\dot{\theta}\left(\left(L_0 + R_T\varphi\right)\dot{\theta}\sin(\theta) - 2R_T\dot{\varphi}\cos(\theta)\right) \\ -m_L R_T\left(\left(L_0 + R_T\varphi\right)\dot{\theta}^2 + g\cos(\theta)\right) \\ m_L\left(L_0 + R_T\varphi\right)\left(2R_T\dot{\varphi}\dot{\theta} + g\sin(\theta)\right) \end{bmatrix} . \tag{8}$$

Since the mass matrix is positive definite, it can be inverted. Consequently, a state representation of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{9}$$

with

$$\mathbf{f}(\mathbf{x},\mathbf{u}) = \begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{M}(\mathbf{q})^{-1}\left(\mathbf{Q}-\mathbf{g}(\mathbf{q},\dot{\mathbf{q}})\right) \end{bmatrix}, \tag{10}$$

the state $\mathbf{x}^T = [\mathbf{q}, \dot{\mathbf{q}}]$ and the input $\mathbf{u}^T = [F_{An}, M_{An}]$ can be derived.

3 Differential Flatness

In the following we first give a definition of flat systems as it can be found for instance in [2], then we consider the gantry crane.

3.1 Differential Flat Systems

Let us consider a nonlinear finite dimensional system of the form

$$S_i\left(\mathbf{z}, \dot{\mathbf{z}}, \ddot{\mathbf{z}}, \ldots, \mathbf{z}^{(\sigma_i)}\right) = 0, \quad i = 1, \ldots, q\,, \tag{11}$$

i.e. a system of ODEs in $\mathbf{z}^T = [z_1, \ldots, z_s]$. The quantities z_i are called the system variables and comprise the variables used in the modeling. This system is said to be differentially flat, if there exists a m-tuple $\mathbf{y}^T = [y_1, \ldots, y_m]$ of functions

$$y_i = \phi_i\left(\mathbf{z}, \dot{\mathbf{z}}, \ddot{\mathbf{z}}, \ldots, \mathbf{z}^{(\alpha_i)}\right), \quad i = 1, \ldots, m\,, \tag{12}$$

such that the following two conditions are satisfied:

1. There does not exist any differential equation of the form
$$R\left(\mathbf{y}, \dot{\mathbf{y}}, \ldots, \mathbf{y}^{(\beta)}\right) = 0\,. \tag{13}$$
In this case, $\mathbf{y}$ is said to be differentially independent.
2. All system variables can be (locally) expressed by $\mathbf{y}$ and its time derivatives, i.e.
$$z_i = \psi_i\left(\mathbf{y}, \dot{\mathbf{y}}, \ldots, \mathbf{y}^{(\gamma_i)}\right), \quad i = 1, \ldots, s\,. \tag{14}$$

In the case specified, $\mathbf{y}$ is called a flat output of the system.

3.2 Flatness based Analysis of the Gantry Crane

The mathematical model of the gantry crane consists of the equations of motion which we derived in Section 2 as well as the equations

$$x_L = x_D - (L_0 + R_T\varphi)\sin(\theta) \tag{15}$$

and

$$y_L = (L_0 + R_T\varphi)\cos(\theta)\,, \tag{16}$$

which describe the coordinates of the load as a function of the generalized coordinates. This system is of the form (11) and it can be shown that it is differentially flat (see [2]). A flat output is given by $\mathbf{y}^T = [x_L, y_L]$. Consequently, all system variables and in particular the control inputs can be expressed as functions of the coordinates of the load and their time derivatives. The parameterization of the control inputs is given by expressions of the form

$$F_{An} = f\left(\ddot{x}_L, x_L^{(3)}, x_L^{(4)}, y_L, \dot{y}_L, \ddot{y}_L, y_L^{(3)}, y_L^{(4)}\right) \tag{17}$$

$$M_{An} = f\left(\ddot{x}_L, x_L^{(3)}, x_L^{(4)}, y_L, \dot{y}_L, \ddot{y}_L, y_L^{(3)}, y_L^{(4)}\right) . \tag{18}$$

4 Trajectory Planning and Feedforward Control

First, this section deals with the design of the desired trajectories which should be tracked by the load. Then a feedforward control is discussed.

4.1 Reference Trajectories

The reference trajectories $x_{L,d}$ and $y_{L,d}$ for the load coordinates should convey the load from the initial rest position at the time $t = 0$ to the final rest position at the time $t = T_{end}$. Thus, the initial and the final values of the reference trajectories are determined by the initial and the final load position. The time derivatives of $x_{L,d}$ and $y_{L,d}$ up to a certain order have to be zero at $t = 0$ and $t = T_{end}$, because the system should be at rest then. Furthermore, in order to obtain continuous control inputs $F_{An,d}$ and $M_{An,d}$, the reference trajectories must be at least four times continuously differentiable. Often polynomials in t are used for the reference trajectories [1].

In the design of the reference trajectories, constraints have to be considered. First, $F_{An,d}$ and $M_{An,d}$ must not exceed the limitations of the control inputs. Second, the condition $\ddot{y}_{L,d} < g$ must hold, since it has been assumed that the rope is always stretched. Both conditions can be met by choosing a sufficiently great value for T_{end}.

4.2 Feedforward Control of the Gantry Crane

A feedforward control can be obtained easily by inserting the reference trajectories into the parameterization of the control inputs given by (17) and (18):

$$F_{An,d} = f\left(\ddot{x}_{L,d}, x_{L,d}^{(3)}, x_{L,d}^{(4)}, y_{L,d}, \dot{y}_{L,d}, \ddot{y}_{L,d}, y_{L,d}^{(3)}, y_{L,d}^{(4)}\right) \tag{19}$$

$$M_{An,d} = f\left(\ddot{x}_{L,d}, x_{L,d}^{(3)}, x_{L,d}^{(4)}, y_{L,d}, \dot{y}_{L,d}, \ddot{y}_{L,d}, y_{L,d}^{(3)}, y_{L,d}^{(4)}\right) . \tag{20}$$

If one implements this feedforward control at the laboratory model, the results are, as expected, rather poor. The reason is friction which affects the motion of the trolley and the rotation of the cylinder. Since the friction was not taken account of in the modeling, there is a considerable difference between the mathematical model and the behavior of the laboratory setup.

5 Feedforward Control of the "Pendulum Subsystem"

This section deals with the feedforward control of the so called "pendulum subsystem" of the gantry crane. This concept can be found for example in [1]. The equations

$$x_D = x_L + \frac{\ddot{x}_L y_L}{g - \ddot{y}_L} \tag{21}$$

and

$$\varphi = \frac{\sqrt{\left(\frac{\ddot{x}_L y_L}{g-\ddot{y}_L}\right)^2 + y_L^2} - L_0}{R_T} \tag{22}$$

represent the "pendulum subsystem". Naturally, the load coordinates also are a flat output with respect to this system. The tuple $\bar{\mathbf{u}}^T = [x_D, \varphi]$ can be interpreted as an input of the "pendulum subsystem" [1]. Since the friction which occurs in the pendulum motion is negligible, the real "pendulum subsystem" of the laboratory model is described almost exactly by (21) and (22). Therefore it can be expected that a feedforward control for the "pendulum subsystem" yields much better results than the feedforward control presented in the preceding section. The feedforward control $\bar{\mathbf{u}}_d^T = [x_{D,d}, \varphi_d]$ is given by

$$x_{D,d} = x_{L,d} + \frac{\ddot{x}_{L,d} y_{L,d}}{g - \ddot{y}_{L,d}} \tag{23}$$

$$\varphi_d = \frac{\sqrt{\left(\frac{\ddot{x}_{L,d} y_{L,d}}{g-\ddot{y}_{L,d}}\right)^2 + y_{L,d}^2} - L_0}{R_T}\,. \tag{24}$$

Since x_D and φ are no control inputs, they have to be used as reference variables for cascaded control circuits. In [1], the usage of PD-controllers is suggested. The control inputs are calculated from

$$F_{An} = F_{An,d} - k_{P,W}\Delta x_D - k_{D,W}\Delta\dot{x}_D \tag{25}$$

$$M_{An} = M_{An,d} - k_{P,T}\Delta\varphi - k_{D,T}\Delta\dot{\varphi} \tag{26}$$

with $\Delta x_D = x_D - x_{D,d}$, $\Delta\dot{x}_D = \dot{x}_D - \dot{x}_{D,d}$, $\Delta\varphi = \varphi - \varphi_d$ and $\Delta\dot{\varphi} = \dot{\varphi} - \dot{\varphi}_d$. The terms $F_{An,d}$ and $M_{An,d}$ correspond to a feedforward control of the gantry crane as discussed in the preceding section. The additional terms stabilize the input $\bar{\mathbf{u}}$ about the desired trajectory $\bar{\mathbf{u}}_d$. All variables which are required for the calculation of the control inputs F_{An} and M_{An} are either part of the state $\mathbf{x}^T = [\mathbf{q}, \dot{\mathbf{q}}]$ or can be calculated from the reference trajectories. Thus, equations of the form

$$F_{An} = f\left(\mathbf{x}, \mathbf{y}_d, \dot{\mathbf{y}}_d, \ddot{\mathbf{y}}_d, \mathbf{y}_d^{(3)}, \mathbf{y}_d^{(4)}\right) \tag{27}$$

$$M_{An} = f\left(\mathbf{x}, \mathbf{y}_d, \dot{\mathbf{y}}_d, \ddot{\mathbf{y}}_d, \mathbf{y}_d^{(3)}, \mathbf{y}_d^{(4)}\right) \tag{28}$$

can be derived. It has to be mentioned that neither the pendulum angle θ nor the associated angular velocity $\dot{\theta}$ occur in this control law.

Now the controller gains $k_{P,W}$, $k_{D,W}$, $k_{P,T}$ and $k_{D,T}$ have to be set properly. The calculation of the controller gains presented in [1] is based on the linearization of the crane about an equilibrium and the singular perturbation theory. In the following just the results are given.

The parameters $k_{P,W}$ and $k_{D,W}$ are calculated from the equations

$$k_{P,W} = \frac{m_W \lambda_{W,1} \lambda_{W,2}}{\varepsilon_W^2} - \frac{m_L g}{L_s} \tag{29}$$

and

$$k_{D,W} = \frac{m_W \left(\lambda_{W,1} + \lambda_{W,2}\right)}{\varepsilon_W} \tag{30}$$

with $\varepsilon_W = 0.1$. Here the parameters $\lambda_{W,1}$ and $\lambda_{W,2}$ have to be chosen in the region of $\sqrt{g/L_s}$. The variable L_s is the pendulum length in the equilibrium which was used for the linearization. The parameters $k_{P,T}$ and $k_{D,T}$ must be set such that the eigenvalues $\lambda_{T,1}$ and $\lambda_{T,2}$ of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k_{P,T}}{A_T + m_L R_T^2} & -\frac{k_{D,T}}{A_T + m_L R_T^2} \end{bmatrix} \tag{31}$$

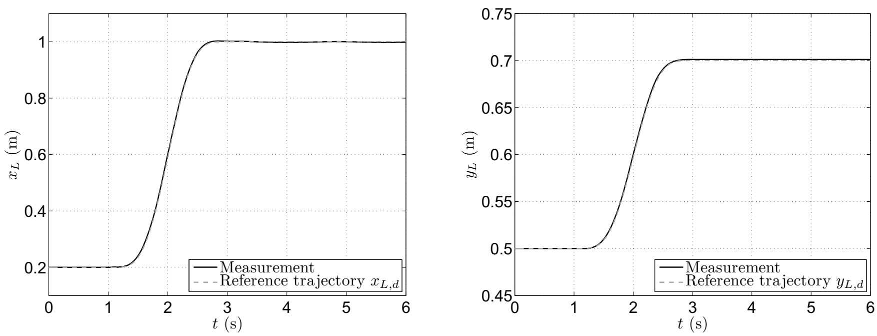
have negative real parts. The controller parameters can be calculated from the eigenvalues with

$$k_{P,T} = \left(A_T + m_L R_T^2\right) \lambda_{T,1} \lambda_{T,2} \tag{32}$$

and

$$k_{D,T} = -\left(A_T + m_L R_T^2\right)\left(\lambda_{T,1} + \lambda_{T,2}\right) . \tag{33}$$

In Fig. 3 measurement results are shown. One can see that the load tracks the reference trajectory closely. Since the used concept is still a feedforward control, the tracking behavior is only precise as long as the load is not affected by disturbances. The PD-controllers only ensure that $\bar{\mathbf{u}}$ tracks the desired trajectories $\bar{\mathbf{u}}_d$. If there is no disturbance, (21) and (22) coincide with the behavior of the real



(a) x-coordinate of the load (b) y-coordinate of the load

Fig. 3. Measurement results for the feedforward control of the "pendulum subsystem"

"pendulum subsystem" and therefore the load will track the reference trajectory. In case of a disturbance, there will be a deviation between the actual and the desired trajectory which is not counteracted by the controller.

6 Conclusion

In this contribution we have dealt with the nonlinear, flatness based control of a gantry crane. We derived a mathematical model and a feedforward control law. Then we discussed a feedforward control for the "pendulum subsystem" and provided measurement results to demonstrate the excellent tracking behavior which can be achieved with this approach.

Acknowledgments. Support of the present work from the Comet K2 Austrian Center of Competence in Mechatronics (ACCM) is gratefully acknowledged.

References

1. Rudolph, J.: Beiträge zur flachheitsbasierten Folgeregelung linearer und nichtlinearer Systeme endlicher und unendlicher Dimension. Shaker Verlag, Aachen (2003)
2. Rudolph, J.: Flatness Based Control of Distributed Parameter Systems. Shaker Verlag, Aachen (2003)
3. Delaleau, E., Rudolph, J.: Control of flat systems by quasi-static feedback of generalized states. Int. J. Control 71, 745–765 (1998)
4. Fliess, M., Lévine, J., Martin, P., Rouchon, P.: On differentially flat nonlinear systems. In: Fliess, M. (ed.) Nonlinear Control Systems Design, pp. 408–412. Pergamon Press (1992)
5. Isidori, A.: Nonlinear control systems, 3rd edn. Springer, London (1995)
6. Spong, M.W., Vidyasagar, M.: Robot Dynamics and Control. Wiley, Chichester (1999)