Modelling of Collective Animal Behavior Using Relations and Set Theory

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Abstract. In this paper we focus on developing the formal methods and techniques necessary to model and classify a collective animal behaviour. The benefits of using set theory are the possibility of a formal examination of the local problems and to organize individuals as elements of the considered classes, defined globally. In order to describe collective activity of animals, we proposed concepts of actions, behaviour and structures. To govern collective behaviour of animals we propose three key relations and mappings determined taxonomic order on them.

Keywords: collective animal bahaviour, relations.

1 Introduction

Group behaviour is a term coined in sociology, referring to the human activity, and thus an intelligent individuals living in groups. The observation of such behaviour, its description and modelling, cause many difficulties mainly due to the numerous and complex interactions between both, members of the group and its environment. Due to interaction the behaviour of the members of the group must be treated as a system of behaviours rather than the behaviour of separate and unrelated individuals.

At first we model a collective animal behaviour because such situations are much simpler than in humans.

There are a lot of authors who believe that many aspects of collective behaviour could be modelled mathematically and using mathematical abstractions drawn interesting and useful comparisons between diverse systems. An interesting and complete overview of the work in this area can be found in [13]. While examining group behaviour, we have to deal with a set of elements which are, for example, shoal of fish or a flock of birds. Hence, using the formal abstractions which provides a set theory seems most appropriate approach. In the literature on this subject [2, 7, 8, 10, 13] we often find a formulation of the relationships both between members of the group and its environment. It seems to be obvious that to describe them we should use the relations, but the literature on this subject there are the wide road of using functions [7–11, 15] , and does not find even narrow trail of relations and set theory. Deployment of relationships and

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use the set theory allows for standardization approach to modelling the collective behaviour in varied distributed systems. The benefits of using set theory are the possibility of a formal examination of the local problems and to organize individuals as elements of the considered classes, defined globally.

2 Basic Concepts and Notation

Let's consider a flock of birds in the air or a fish school in the water. Birds or fish form a set of elements, which we denote as X. Any element $x \in X$ has individual size but together they form different shapes and sizes which often far exceed the size and range of relationship of individual element. Let us to define a neighbourhood abstraction for set X, wherein an individual can identify a subset of other individuals around it by a variety of relationships and share state with them. Let N denotes neighbourhood, then

$$N \in Map(X, SubX) \tag{1}$$

where $SubX$ means a family of subsets of X.

Furthermore, if $N(x)$ indicates the neighbourhood of x then, using the neighbourhood relation (here denoted as η) we can define a collection of individuals which are neighbours of the given x as

$$N(x) = \{y \mid \ y \in X \ \wedge \ x\eta y\}, \tag{2}$$

and we can denote the set of neighbours of all nodes that belong to the set S as:

$$N(\mathrm{S}) = \{y \mid \ y \in X \ \wedge \ (\exists x \in \mathrm{S} \mid x\eta y)\}. \tag{3}$$

The neighbourhood relation is symmetric

$$(\forall x, y \in X)(x\,\eta\,y \Rightarrow y\,\eta\,x). \tag{4}$$

which implies, that if an individual x remains in a neighbourhood relation with y (i.e. x is in interaction with y) then the individual y is also in interaction with x. This results in neighbourhood integrity.

Watching the starlings' flock we can observe how complex configurations arise from repeated actions/interactions between the individual birds. Several authors [8, 10, 12, 13] proposed a models in which individual's activity follow a few simply rules (realize a simply actions) which change its state. As a result, we observe mesmerizing collective behaviour of flock.

In order to describe collective activity of animals, we proposed concepts of actions, behaviour and structures. Action is considered as the property of each individual in group. The behaviour, on the other hand, is an external attribute, which can be considered either as an outcome of actions performed by the whole group or its subset. Action is a ternary relation which can be defined ($\times$ means Cartesian product) as follows:

$$Act : X \times State \rightarrow State. \tag{5}$$

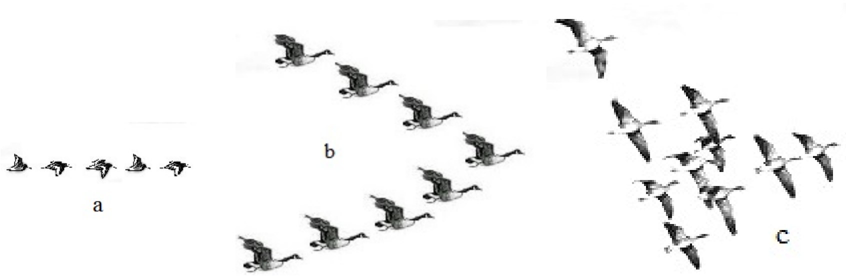

Fig. 1. Different formation of migrating birds: **(a)** line, **(b)** wedge and **(c)** delta

Let $Act(X)$ be a set of possible actions of elements of set X and $\mathcal{R}$ be an equivalence relation on X. We define behaviour *beh* as a set of actions which are equivalent in sense to realize a common goal *rel*

$$beh : [rel] = \{act \in Act \mid act\, \mathcal{R}\, rel\}. \tag{6}$$

Next, behaviours *beh* form a quotient set

$$Beh : Act(X)/\mathcal{R} = \{[rel] \subset Act(X) \mid act\, \mathcal{R}\, rel\}, \tag{7}$$

which consists of equivalent classes (6).

Similar approach has been applied to modelling the structure as a result of collective activity. Let $Stat(X)$ be a set of possible states of elements of set X and $\mathcal{S}$ be an equivalence relation on X. We define structure *struc* as a set of states of individuals which are equivalent in sense of collective activity *col*

$$struc : [col] = \{stat \in State \mid stat\, \mathcal{S}\, col\}. \tag{8}$$

Next, structures *struc* form a quotient set

$$Struc : State(X)/\mathcal{S} = \{[col] \subset State(X) \mid stat\, \mathcal{S}\, col\}, \tag{9}$$

which consists of equivalent classes (8).

As an examples of action *act* we can mention a change of direction of individual's movement as a result of repulsion, alignment or attraction for fish schools or bird flocks, and leaving a pheromone in the case of an ant reinforced its trail. Flocking starlings is one of the most spectacular examples of behaviour in all of nature. Predator - prey interactions like flash expansions of schooling fish forming vacuoles, bait balls cruising parabolas or vortices [8], and ant bridges building [9] are another good examples of patterns of behaviour *beh*. The canonical examples of structure *struct* are migrating birds formation in the sky [10]. Delta (skein) of ducks or V-shaped formation of geese [12] and finally line of oystercatchers flock as shown on Fig.1.

3 Relations, Mappings and Orders

The contribution of this paper is to introduce a novel, based on set theory, relational way of thinking about the modelling of collective animal behaviour. In the previous chapter we consider sets, now it's time to define the relations on it, which determine how simple behavioural rules of individuals can result in complex behavioural patterns reinforced by a number of set/group members (cardinality of set).

Our approach is solidified in three key relations called subordination (π), tolerance (ϑ) and collision ($\varkappa$). These relations allow us to think about collective group activity directly in terms of relationships between individuals and their group vicinity. Three relations mentioned above are defined as follows:

$$\textit{Subordination} \qquad \pi = \{< x, y >; x, y \in Act \mid x\,\pi\,y\} \tag{10}$$

The expression $x\,\pi\,y$ - means that action x is subordinated to the action y, in other words action y dominate over action x.

$$\textit{Tolerance} \qquad \vartheta = \{< x, y >; x, y \in Act \mid x\,\vartheta\,y\} \tag{11}$$

The expression $x\,\vartheta\,y$ - states that actions x and y tolerate each other,

$$\textit{Collision} \qquad \varkappa = \{< x, y >; x, y \in Act \mid x\,\varkappa\,y\} \tag{12}$$

The expression $x\varkappa\,y$- means that actions x and y are in collision one to another.

In [4] we can find detailed study on properties of mentioned above relations. Here, we formulate them succinctly as:

$$\pi \cup \vartheta \cup \varkappa \subset Act \times Act \neq \emptyset \tag{13}$$

$$\iota \cup (\pi \cdot \pi) \subset \pi \tag{14}$$

where $\iota \subset Act \times Act$ is the identity on the set *Action*. Moreover,

$$\pi \cup \vartheta^{-1} \cup (\vartheta \cdot \pi) \subset \vartheta \tag{15}$$

where ϑ^{-1} is the converse of ϑ so,

$$\vartheta^{-1} = \{< x, y > \in X \times Y \mid y\,\vartheta\,x\} \tag{16}$$

Collision holds

$$\varkappa^{-1} \cup \{\pi \cdot \varkappa\} \subset \varkappa \subset \vartheta' \tag{17}$$

where ϑ' is the complement of ϑ so,

$$\vartheta' = \{< x, y > \in X \times Y \mid < x, y > \notin \vartheta\}. \tag{18}$$

The axiom (13) indicates that all these three relations are binary on nonempty set *Act*. The axiom (14) describes fundamental properties of subordination relation which is reflexive and transitive. Therefore it is also ordering relation on the set

Act. The axiom (8) states that subordination implies the tolerance. Hence we can obtain:

$$\{\forall x, y \in Act \mid x\,\pi\,y \Rightarrow x\,\vartheta\,y\} \tag{19}$$

and subordinated actions must tolerate all actions tolerated by dominants

$$\{\forall x, y, z \in Act \mid \{x\,\pi\,y \wedge y\,\vartheta\,z\} \Rightarrow x\,\vartheta\,z\}. \tag{20}$$

There are evident coincidences between relations (10)-(12) and proposed in [10, 13, 14] three rules which determine the individual animal movement:

a) adopt the same direction as your neighbours,
b) remain close to your neighbours,
c) avoid collisions with your neighbours.

Subordination (π) is an extension of alignment rules - a). Tolerance (ϑ) is an extension of cohesion (birds)/attraction (fish) rules - b). Finally, collision ($\varkappa$) is an extension of dispersion (birds)/ repulsion (fish) rules - c).

What is the advantage of employing relations to model animal collective behaviour? At the top of the list is topology. The most common mathematical models of animal collective behaviour employ metric distance model while using relation allows us to see distance as a property of topological space. A second factor is model plasticity. In traditional attempt we should determine the constant values of three radii and weighting factors of alignment, dispersion and repulsion forces. Relational attempt provides more sophisticated and powerful tools as: cardinality of each relation $\pi, \vartheta, \varkappa$ which can vary widely within different neighbourhoods; intensity quotients for each relation bounded with the way things are going in the group. This is the part of the story, but not all of it.

Relations also emphasize the importance of local decisions when we attempt to express the essence of distributed, large group of animal, behaviour patterns. Relational framework enables to firmly determine a relationship between sizes of neighbourhood and float border between local and global perceiving perspectives. To really crack the problem of collective animal behaviour, we need to figure out, how individuals faced with decisions and instructed by three simple rules of thumb, retain association between distinct sets of structure and behaviour patterns. The answer is - enforcement of order, as a result of countless conspecific and environmental factors.

Concerning the group of individuals (flock, herd, school), we define its subsets (2)-(3) and (10)-(12) (it's worth to remind here, that relation can be considered as a set). Since only subordination (10) is a transitive (see (14)), it is appropriate to employ the theory of set ordering [5].

Let X be a set of action

$$D_\pi \in Map(X, SubX) \tag{21}$$

where Map is used to mean a mapping of a set X into a $SubX$ (family of subsets of X). The (21) allows us to define four additional mappings of any elements x$\in X$:

$$A_\pi(x) = \{y \in X \mid x \in D_\pi(y)\}, \tag{22}$$

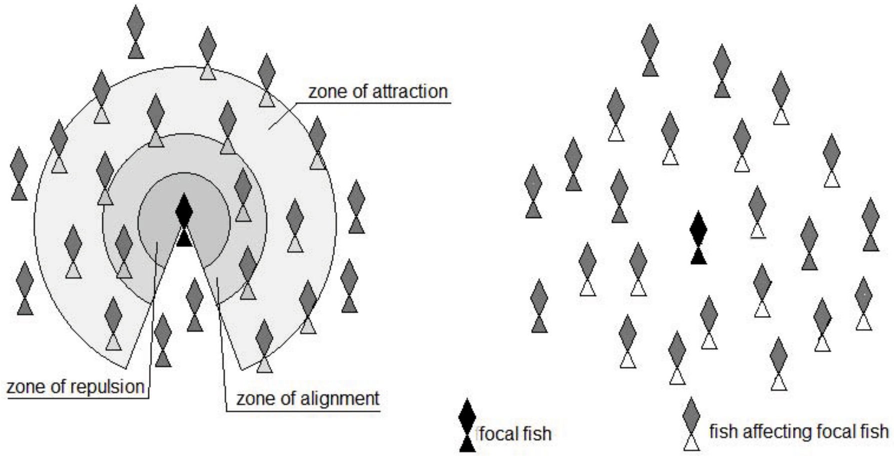

Fig. 2. Metric vs. topological distance in reference to fish school

$$M_\pi(x) = D_\pi(x) \cap A_\pi(x), \tag{23}$$

$$N_\pi(x) = D_\pi(x) \cup A_\pi(x), \tag{24}$$

$$C_\pi(x) = X \setminus N_\pi(x). \tag{25}$$

In fact, follows from (13)-(14), the mapping $D_\pi(x)$ (descendants) possess the following properties:

$$\cup D_\pi^{\vdash}(D_\pi(x)) \subset D_\pi(x), \qquad \textit{transivity}, \tag{26}$$

$$M_\pi(x) \subset \{x\}, \qquad \textit{antisymmetry}, \tag{27}$$

$$M_\pi(x) \neq \emptyset, \qquad \textit{weak reflexivity}, \tag{28}$$

$$N_\pi(x) = \emptyset, \qquad \textit{connectness}, \tag{29}$$

wehere, $D_\pi^{\vdash} \in Map\,(Sub(X), Sub\,(Sub(X))$ indicates an extension of D_π.
We conclude this section with the remark, that set of action (5) is weakly-ordered by any mapping $D_\star(x)$ iff the conditions (26), (28) are fulfilled. Further, if (27) holds, then set *Act* is partially ordered since:

$$M_\star(x) = \{x\}, \tag{30}$$

Finally, the set of actions *Act* is ordered totally when all four (26)-(29) conditions are fulfilled.

4 Conclusion

The novel mathematical tools for modelling of collective animal behaviour, based on relations and set theory, are the main purpose of this paper. As it has been argued in literature [2, 8, 10, 13], the behaviour of the large group of similar animals may be determined by simple behavioural rules of individuals. Realization of these rules can result in complex behavioural patterns and in the emergent properties, reinforced by a number of group members.

In both cases: of flock of starlings and school of fish, the results obtained by replacing a metric space by a topological space, allow better modelling of rapid changes of movement direction of a flock/school sub-groups. Adjusting the balance between subordination towards tolerance, by weakening (π) and strengthening (ϑ), and increasing cardinality/intensity quotients of (ϑ) relation, result in the very spectacular patterns of collective behaviour.

When considering emergence of structure in the flock of migrating birds or structure of shoaling fish, these phenomena are associated with mappings, which ordered subordination relation, since only this (π) relation is transitive. The V structure is the result of well-ordered and strength subordination (π). While we also strengthening tolerance (ϑ), the V structure becomes to be a delta. Going in opposite direction, i.e. completely eliminating (ϑ) but totally ordered subordination (π), we obtain a line of oystercatchers flock (see fig.1).

The existence of mapping (21) is the necessary, but not sufficient, condition for emergence of the structure. In case of flock of birds, the mapping results from aerodynamic rules and tends toward minimization of the energy consumption [1, 12]. A fish structure, commonly known as a bait ball, is a result of predators' activity and tends toward minimization of the surfaces exposed to attack. It is worth to notice, that mapping (21) works globally, since it is determined on X, and it is opposed to local (and individual) activity.

There is a plenty of empirical evidence that the proposed relational model approximates well many collective behaviours of animal group. But we also know that this model is not perfect and can fail catastrophically for some animal groups. There are two hopes for avoiding such situation: understanding better phenomena of group behaviour and improvement of methods and tools used for modelling of these processes. And the latter will still be the focus of our future work.

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