Probabilistic Approaches to the Rough Set Theory and Their Applications in Decision-Making

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Abstract. Rough sets were presented by Professor Zdzislaw Pawlak in a seminal paper published in 1982. Rough Sets Theory (RST) has evolved into a methodology for dealing with different types of problems, such as the uncertainty produced by inconsistencies in data. RST is the best tool for modeling uncertainty when it shows up as inconsistency, according to several analyses. This is the main reason for which the RST has been included in the family of Soft Computing techniques. The classical RST is defined by using an equivalence relation as an indiscernibility relation. This is very restrictive in different domains, so several extensions of the theory have been formulated. One of these alternatives is based on a probabilistic approach, where several variants have been proposed such as the Variable Precision Rough Sets model, Rough Bayesian model, and Parameterized Rough Set model. Here is presented an analysis about the evolution of the RST in order to enrich the applicability to solve real problems by means of the probabilistic approaches of rough sets and its application to knowledge discovering and decision making, two main activities in Business Intelligence.

1 Introduction

The aim of the information systems is to model the real world, but the uncertainty pervades our understanding of the real world; for this reason, it is necessary to consider and properly handle the uncertainty to implement computational systems and solve real problems [22]. The applications in Business Informatics are not the exceptions.

Logical experts and philosophers have considered the problems of vagueness and uncertainty for many years. More recently, computer scientists, particularly

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researchers related with Artificial Intelligence have worked out novel approaches on this research field, being the Fuzzy Sets Theory [40] one of the most outstanding and representative approaches.

Bonnisone and Tong [6] classify knowledge faultiness in: *uncertainty*, *imprecision* and *incomplete information*. This latter term is used to indicate the lack of a value, whereas "*imprecision*" denotes the existence of a value which cannot be measured with the suitable accuracy. Finally, the term "*uncertainty*" stands for the fact that an agent has formulated a subjective opinion on a veracity of a fact which is not known for sure. Bosc and Prade [8] coin two new terms: *vagueness* and *inconsistency*. The vagueness is a new category modeled by means of fuzzy sets but that essentially falls under the classification of imprecision, just like the aforementioned concept. The "inconsistency" describes a situation in which there are two or more conflicting values to be assigned to a variable [6] and [22].

This is the background where in Soft Computing is developed as an emergent approach of Computer Science whose goal is the remarkable ability of the human mind to reason and learn in an uncertain environment [7] and [13]. The essence of Soft Computing is to consider the pervading imprecision of the real world in the computational systems. For that reason, the ruling principle of Soft Computing is exploiting the tolerance, imprecision, uncertainty and partial truths in order to get flexibility, robustness, low solution costs and a better harmony with reality. Soft Computing is not a single methodology; on the contrary, it is an umbrella of approaches whose key members are fuzzy logic, neurocomputing and probabilistic reasoning, in addition to genetic algorithms, chaotic systems, belief networks and some elements of the learning theory (Lotfi A. Zadeh in [13]). The common denominator of these technologies is that they are not based in the classical reasoning and modeling approaches usually relying on Boolean logic, analytical models, hard classifications and determinist search. In a similar way to Zadeh, more recently Verdegay, Yager and Bonissone [29] provided their own definition of Soft Computing, in which the metaheuristics are emphasized.

It is quite common that a given combination of observations is associated with two or more different outcomes. According to [11], RST is the best tool for handling uncertainty when this is provoked by inconsistency. Li et al [16] argue that fuzzy logic, neural networks, and probabilistic reasoning were the initial components of Soft Computing, and that subsequently other components were added, such as the rough sets. In [2] and [3], the authors present an analysis of the relationship between rough sets and other components of Soft Computing.

The RST provides ways to directly model the uncertainty caused by inconsistencies in the information, and it also to takes into account the granularity of information. However, the classical approach based on a relation of inseparability is an equivalence relation; it is very strict to model real problems. In many real word applications, the assumption of exact data is not fulfilled and some objects are misclassified or condition attribute values are corrupted.

On the other hand, the definition of lower approximation is also very strict; it is enough for two objects in a universe of one million objects that they were inseparable and belong to different classes for the system to result inconsistent, and consequently it will affect the lower approximation of those classes. The strict definition of the approximations has consequences in all the techniques that are built from rough sets. One of the most important aspects is the selection of features based on the concept of reduct (a reduct is generally defined as a minimal subset of attributes that can classify the same domain of objects as unambiguously as the original set of attributes).

The RST can be more flexible by using a weaker indiscernibility relation or relaxing the lower and upper definitions, the two basic concepts of the theory. The Pawlak's Rough Set Model may be extended by using an arbitrary binary relation instead of equivalent relations, [24]; by considering any binary relations on attribute values, instead of the trivial equality relation (=); an object *x* is related to another object *y*, based on an attribute *a*, if their values on *a* are related, with respect to a subset A of attributes, *x* is related to *y* if their values are related for every attribute in A. When all relations R_a are chosen to be =, the proposed definition is reduced to the definition in the Pawlak's Rough Set Model.

In this chapter the second alternativeis discussed, presenting an analysis of ways to make flexible the RST using a probabilistic approach. The main objective in this chapter is to provide an analysis and review of probabilistic approaches to rough sets. Several probabilistic extensions of the rough set model have been proposed to make the approach more applicable to real life data analysis problems, in which the information may be incomplete, with noises or uncertainties; important reviews about this subject are presented in [33] and [44]. This alternative approaches to the classical rough set theory that can be achieved by decreasing the classification precision in the knowledge obtained through rough set's analysis. Also, there is included the use of them to achieve decision-making troubles, which result to be of great interest in Business Informatics, especially in Business Intelligence.

Business Intelligence (BI) mainly refers to computer-based techniques used in identifying, extracting and analyzing business data. BI is a broad category of applications and technologies for gathering, storing, analyzing, and providing access to data to help enterprise users make better business decisions. Business Intelligence -understood broadly- can include the subset of competitive intelligence. Using data that has been stored, software applications are able to use this data to report past business information as well as predict future business information, including trends, threats, opportunities and patterns; to do this, BI includes techniques for different activities, among them decision making.

2 Rough Set Theory

The whole knowledge about the domain is contained in the set of objects, called Information System. A *decision system*$(U, A \cup \{d\}$, *where* $d \notin A)$ is obtained when a new attribute *d*, called decision attribute, is added for each object in U. A simple idea of rough sets is the following: objects having exactly the same values of condition attributes are indiscernible by using these attributes. This indiscernibility relation is the mathematical basis of RST. Such relation induces a partition of the universe U in equivalence classes; that is, a set of indiscernible objects according to the relation.

Any subset $X \subseteq U$ can be expressed exactly or approximately in terms of these sets by using two crisp sets called lower approximation and upper approximation. The lower approximation ($B_*(X)$) of a set of objects (concerning those attributes) is a collection of objects whose equivalence classes are fully contained into the set of objects we want to approximate; while the upper approximation ($B^*(X)$) of the same set of objects is a collection of objects whose equivalence classes are at least partially contained into the set of objects it is wanted to be approximated.

$$B_*(X) = \{x \in U \mid B(x) \subseteq X\} \tag{1}$$

$$B^*(X) = \{x \in U \mid B(x) \cap X \neq \emptyset\} \tag{2}$$

$$BN_B(X) = B^*(X) - B_*(X) \tag{3}$$

If the boundary region (BN_B) is empty ($BN_B(X) = \emptyset$ then *X* is *crisp* according to *B*; otherwise *X* is said to be *rough*. Objects members of the boundary region have a membership status that cannot be classified with certainty as members of the underlying concept. Using this approximations the positive, negative, and boundary regions of *X* can be defined: the positive region, $POS(X) = B_*(X)$, consists of all objects that are definitely contained in the set *X*, the negative region, $NEG(X) = U - B^*(X)$, consists of all objects that are definitely not contained in the set *X*, and the boundary region, defined by (3), consists of all objects that may be contained in *X*.

RST provides several measures to characterize a given set. These measures are very useful in order to evaluate the quality of the results computed via rough-set-based methods, for instance, the strength of the decision-making processes and the certainty of the discovered knowledge. Pawlak defines a measure to evaluate the quality of classification [21], this measure (γ_B) is used to calculate the degree of consistency of a decision system: If $\gamma_B(Y) = 1$, the decision system is consistent; otherwise it is inconsistent [28].

In this classical rough set model, the lower and upper approximations are defined based on the two extreme cases (full inclusion or non-empty overlap) regarding the relationships between an equivalence class and a target set, this requirement limits unnecessarily the applications of rough sets in practical problems; in the next sections other approaches are analyzed, these are based on considering the degree of set overlap in the rough set formulation.

3 Rough Sets Based on Probabilistic Approaches

Based on the notion of rough membership functions, different approaches for the construction of a probabilistic rough set model have been developed. Yao et al. [38] put into groups the existing rough set models into two major classes, the algebraic and probabilistic rough set models, depending on whether statistical information is used; some of them are analyzed in this section. While non-probabilistic studies of rough sets focus on algebraic and qualitative properties of the theory, probabilistic approaches are more practical and capture quantitative properties of the theory [32]. Algebraic rough set approximations may be considered as qualitative approximations of a set, in this case the extent of overlap between a set and an equivalence class is not considered; by incorporating the overlap, probabilistic rough set approximations have been introduced. Probabilistic rough set approximations can be formulated based on the notions of rough membership functions and rough inclusion. The probabilistic approaches expand the positive and negative regions (defined from the lower and upper approximations) by providing probabilities that define boundary regions; since the boundary region introduces uncertainty into the discernibility of objects, the major challenge in data analysis by using rough sets is to minimize the size of this region, this is done by relaxing the definitions of the *POS* and *NEG* regions to include objects that would otherwise not have been previously included.

An attempt to use probabilistic information for approximations was suggested by Pawlak et al. [20]. Their model is based essentially on the majority rule. Yao and Wong [37] introduced a more general probabilistic approximation in the decision-theoretic rough set model (DTRSM).

$$B_*(X) - \alpha = \{x \in U \mid P(A|[x]) \geq \alpha\} \tag{4a}$$

$$B^*(X) - \beta = \{x \in U \mid P(A|[x]) > \beta\} \tag{4b}$$

Where $0 \leq \beta < \alpha \leq 1$. If α =1 and β =0, the classical lower and upper approximations are obtained. Based on Bayesian decision procedure, DTRSM provides systematic methods for deriving the required thresholds on probabilities for defining the three regions: positive region, boundary region and negative region. A review on decision-theoretic rough sets is presented in [9].

Liu et. al. [9] introduce three-way decision-theoretic rough sets and answer "why" and "how" to use DTRSM to solve practical problems. It divides the universe into three regions, which lead the generalized three-way decision rules. The probabilistic positive rules express that an object or object sets belong to one decision class when the threshold is more than α, which enable us to make decisions of acceptance; the probabilistic boundary rule express that an object or sets of objects belong to one decision class when the thresholds are between α and β, which lead to doubt about the decision; the probabilistic negative rules express that an object or sets of objects do not belong to one decision class when the threshold is less than β, which enable to make decisions of rejection. A great challenge for the probabilistic rough set models is the acquirement of a pair of thresholds. Unfortunately, the thresholds are usually given by expert's experience in most of the probabilistic rough sets.

3.1 Rough Sets Based on Rough Membership Function

By definition, elements in the same equivalent class have the same degree of membership. The rough membership may be interpreted as the probability of x belonging to X given that x belongs to an equivalence class, this interpretation leads to probabilistic rough sets; the probabilistic rough set models may be interpreted based on rough membership functions [38]. The rough membership function is defined by (5), this measure in the interval [0, 1].

$$\mu_X^B(x) = \frac{|X \cap B(x)|}{|B(x)|} \tag{5}$$

$B(x)$ denotes the equivalence class of object x according to the relation B. By definition, elements in the same equivalent class have the same degree of membership. This value may be interpreted analogously to conditional probability (as a frequency-based judgment of conditional probability). This interpretation leads to probabilistic rough sets [30] and [20]. Using the rough membership function, the lower and upper approximations are defined by (6) and (7).

$$B_*(X) = \{x \in U \mid \mu_X^B(x) = 1\} \tag{6}$$

$$B^*(X) = \{x \in U \mid \mu_X^B(x) > 0\} \tag{7}$$

The former definitions of lower and upper approximations can be made more general by using an arbitrary precision threshold "τ", expression (8) and (9):

$$B_*\tau(X) = \{x \in U \mid \mu_X^B(x) \geq \tau\} \tag{8}$$

$$B^*\tau(X) = \{x \in U \mid \mu_X^B(x) > 1 - \tau\} \tag{9}$$

3.2 Variable Precision Rough Sets Model

The Variable Precision Rough Sets (VPRS) model is a generalized version of the conventional rough set approach which inherits all of its fundamental mathematical properties and aims at handling vague information. This model was introduced in [43]. The VPRS model defines the positive region as an area where, on the basis of available data, the rough membership of objects to the given set is certain to some degree.

The VPRS model allows for a controlled degree of misclassification. Any partially incorrect classification rule provides valuable trend information about future test cases if most of the available data which are applied to such a rule can be correctly classified. The target of this model is to loose the former definition of lower and upper approximations introduced in the classical rough set methodology.

This model deals with this type of information by introducing a precision parameterβ, the value of this parameter represents a bound on the conditional probability of a proportion of objects in a condition class, which are classified to the same decision class [27]. Ziarko in [43] considers the parameter β as an admissible level of classification error defined in the domain $\beta \in [0,0.5)$. Other alternative presented in [1] and [5] considered the parameter β to denote the proportion of correct classifications, in which case the appropriate range is $(0.5,1]$.

The concepts of β-lower approximation and β-upper approximations are defined as follows, where $D_0(Y/X)$ is an inclusion degree is defined by (12) and $\beta \in (0.5,1]$,

$$B_*^{\beta}(X) = \{x \in U : D_0(X/[x]_B) \geq \beta\} = \cup \{[x]_B : D_0(X/[x]_B) \geq \beta\} \tag{10}$$

$$\begin{aligned} B^{*\beta}(X) &= \{x \in U : D_0(X/[x]_B) > 1 - \beta\} \\ &= \cup \{[x]_B : D_0(X/[x]_B) > 1 - \beta\} \end{aligned} \tag{11}$$

Let U be a finite set, $F = \{X : X \subseteq U\}$ and $\subseteq$ a partial order relation on F. For all $X, Y \in F$, D_0 is computed as follows:

$$D_0(Y/X) = \begin{cases} \frac{|Y \cap X|}{|X|} & if\ X \neq \emptyset \\ 1 & \text{otherwise} \end{cases} \tag{12}$$

This means that an elementary class belongs to the lower approximation of X if and only if a 100% * β of its elements belong to X; in a similar way, an elementary class is excluded from the upper approximation of X if and only if a 100% * β of its elements does not belong to X. The grade of looseness allowed in our model is fixed in advance by properly setting the value of the parameter β. Due to the existence of β, the VPRS can resist data noise or remove data errors.

According to [27], the VPRS model lacks a feasible method to determine the value of the parameter β. Ziarko [43] proposed the β value to be specified by the user, Beynon proposed the allowable β value range to be an interval [5], and for the case of reduct calculation proposed two methods of selecting a β-reduct without determining a β value [4]. Other method to determine the precision parameter value in the context of reduct calculation is introduced in [27]. In a similar way, authors in [14] analyze the decision-theoretic rough set model from an optimization viewpoint.

3.3 Rough Bayesian Model

Sleezak [25] proposed an alternative parameterized rough set model, called Rough Bayesian model, in which the lower and upper approximations of X are defined as follows:

$$B_*(X, \varepsilon t) = \left\{ x \in U : \frac{|[x]_B \cap X|}{|X|} \geq \varepsilon t * \frac{|[x]_B \cap (U - X)|}{|(U - X)|} \right\} \quad (13)$$

$$B^*(X, \varepsilon q) = \left\{ x \in U : \frac{|[x]_B \cap X|}{|X|} \succ \varepsilon q * \frac{|[x]_B \cap (U - X)|}{|(U - X)|} \right\} \quad (14)$$

Where $\varepsilon t, \varepsilon q \in [1, +\infty]$, such that $\varepsilon t > \varepsilon q$.

3.4 Parameterized Rough Set Model

In [10], a generalization of the original definition of rough sets and variable precision rough sets is introduced, this generalization is based on the concept of absolute and relative rough membership, it is called Parameterized Rough Set model; according to the authors, the classical rough set model, the VPRS model, and the Rough Bayesian model are special cases of this.

The generalized VPRS model proposed in [10] assumes that, in order to include an object x in the positive region of set *X*, it is not sufficient to have a minimum percentage of objects from *X* in $[x]_R$, but it is also necessary that the percentage of objects from *X* in $[x]$ R is sufficiently greater than the percentage of objects from *X* outside $[x]_R$. In other words, it is necessary that both, the absolute and the relative memberships of *x* in *X* are not smaller than the given thresholds *t* and a, respectively.

This model is defined as follows: Let α and β, $\alpha \geq \beta$, be two real values in the range of variation of each relative rough membership $c(x, X)$ and $0 \leq q \leq t \leq 1$. The parameterized lower and upper approximations of *X* in *U* with respect to relative rough membership $c(x, X)$ are defined, respectively, by (15) and (16):

$$B_*(X, t, \alpha) = \left\{ x \in U : \frac{|[x]_B \cap X|}{|[x]_B|} \geq t \text{ and } c(x, X) \geq \alpha \right\} \quad (15)$$

$$B^*(X, q, \beta) = \left\{ x \in U : \frac{|[x]_B \cap X|}{|[x]_B|} \succ q \text{ or } c(x, X) \succ \beta \right\} \quad (16)$$

Where $c(x, X)$ is a relative rough membership measure; in [10] several expressions for $c(x, X)$ are proposed, for example (17):

$$c(x, X) = \frac{|[x]_B \cap X|}{|[x]_B|} - \frac{|X|}{|U|} \quad (17)$$

The Parameterized Rough Set model is the most general since it involves both, absolute and relative rough memberships; moreover, it can be generalized further by considering more than one relative rough membership.

3.5 Applications in Decision-Making

Decision-Making is a chosen strategy in order to achieve some purposes. Decision theory considers how is the best way to make decisions in the light of uncertainty about data. The basic approach to make decisions with a rough set model is to analyze a dataset in order to acquire lower and upper approximations. Immediate decisions (Unambiguous) can be formulated from the positive and negative regions, while Delayed decisions (Ambiguous) are based on the boundary region.

According to [38] the probabilistic rough set models are justified based on the framework of the decision theory; the results given in that work suggest that both algebraic rough set and probabilistic rough set models can be viewed as a special case of the decision theoretic framework. Several decision rules are derived using the probabilistic approach based on the membership function. The VPRSM has been used in many areas to support decision making [12]; for instance, a multi-attribute decision making method based on the concept of extended dominance relation and variable precision rough sets in this paper is proposed in [18], other example is presented in [15]. The use of a probabilistic approach for Decision-Making in Incomplete Information is analyzed in [39].

The concept of three-way decisions plays an important role in many real world Decision-Making problems; usually the Decision-Making is based on available information and evidence, when the evidence is insufficient or weak, it might be impossible to make either a positive or a negative decision. Yao [34], [35] and [36] propose to formulate decision rules according to three categories of decisions; this kinds of rules are derived from the three regions. As it was explained before, rules generated by the three regions form three-way decision rules: the positive rules generated by the positive region make decisions of acceptance; the negative rules generated by the negative region make decisions of rejection; and the boundary rules generated by the boundary region make deferred or non-committed decisions. Using this three-way decision approach, a solution to multi-category decision-making is proposed in [42]; other application is presented in [17], from the idea of three-way decisions of a new discriminate analysis approach by combining decision-theoretic rough sets and binary logistic regression is proponed. A multi-view decision method based on decision-theoretic rough set model is proposed in [41], in which optimistic decision, pessimistic decision, and indifferent decision are provided according to the cost of misclassification.

In real life, many important decision problems are not determined by a single decision-maker but by a group of them. In group Decision-Making, the members usually make judgments on the same decision problem independently. Due to the difference among them, there could be great disagreements on the same decision problem. Therefore, how to effectively integrate the evaluation of the decision-maker is an interesting problem. In [31], a study about how to use the variable precision rough set model as a tool to support group decision-making in credit risk management is presented. This technique is able to remove errors or inconsistency in a set of decision.

In group decision-making, the individual importance of the decision makers is introduced by using different weights for them; the analytical hierarchy process (AHP) technique is used to obtain members' weight, and aggregate group preference [23]. AHP is the multicriteria decision technique that can combine qualitative and quantitative factors for prioritizing, ranking and evaluating different decision alternatives. In [31], the VPRS and AHP are combined to obtain the weight of condition attribute sets decided by each decision maker, three VPRS-based models to obtain Integrated Risk Exposure (IRE) are discussed; the effectiveness of the methods is evaluated in an application in credit risk management, credit risk is represented by IRE.

One of the challenges a decision maker faces in using rough sets is to choose a suitable rough set model for data analysis; authors in [12] have observed that the availability of information regarding the analysis data is crucial for selecting a suitable rough set approach, and they present a list of decision types corresponding to the available information and user's needs.

3.6 An Example of Three-Way Decisions

Suppose a bank has to decide on the orders it receives from companies whether to grant a loan or not. The purpose of the credits can be to make investments, to cover unexpected expenses, to address problems of lack of financial capacity, etc. In order to do this, a criterion is established for helping the bank's management to decide on the granting of the credit.

The past experience of the bank can be used to set the criterion, on which credits have been effective or not. The available information is shown in Table 1, in which the applicant companies are described by a set of attributes; in this analysis is only considered the following information: the company's sector, business productivity, the company's production market, the company's finances state. Furthermore, it is known if the credit granted had a positive effect or not.

Table 1 Previous cases met by the bank

Company	Sector	Productivity	Market	Finances	Effectiveness
E1	Agricultural	low	limited	low	no
E2	Industry	high	wide	low	yes
E3	Industry	average	wide	average	no
E4	Agricultural	average	average	high	yes
E5	Industry	average	wide	average	yes
E6	Services	high	average	high	yes

From the information contained in Table 1, there may be formulated rules such as "three-way decisions", as follows:

R1: $Des([x]) \rightarrow_P Des(C), for\ [x] \subseteq POS(C)$, with all certainty the decision is C.

R2: $Des([x]) \rightarrow_B Des(C), for\ [x] \subseteq BND(C)$, uncertainty over the decision C.

R3: $Des([x]) \rightarrow_N Des(C), for\ [x] \subseteq NEG(C)$, with allcertainty the decision is NOT C.

After applying the definitions of RST the positive region of the decision Effectiveness class = "if" is:

POS(Effectiveness=”if”)= {E2, E4, E6}

BND (Effectiveness=”if”)= {E3, E5}

NEG (Effectiveness =”if”)= {E1}

Rule1 means that if the description of a company applying for a loan is inseparable from companies E2, E4, or E6, then it should be given the credit for sure. By rule 3, if it is inseparable of E1, with all certainty it must not be given the credit. According to the second rule, if it is inseparable from E3 or E5, there will be a doubt about granting the credit or not.

Obviously in a real situation where the available information is hundreds or thousands of cases rather than the 6 in Table 1, there can be inconsistencies on the information (such as between the cases of E3 and E5 in Table 1) but that because of the given amount of information available is not significant. In this case it would be necessary to use probabilistic approaches, and therefore the rules to use would be:

R1: $Des([x]) \rightarrow_P Des(C), for\ [x] \subseteq POS_{\alpha,\beta}(C)$

R2: $Des([x]) \rightarrow_B Des(C), for\ [x] \subseteq BND_{\alpha,\beta}(C)$

R3: $Des([x]) \rightarrow_N Des(C), for\ [x] \subseteq NEG_{\alpha,\beta}(C)$

Where

$$POS_{\alpha,\beta}(C) = \{x \in U : Pr(C|[x]) \geq \alpha\}$$

$$BND_{\alpha,\beta}(C) = \{x \in U : \beta < Pr(C|[x]) < \alpha\}$$

$$NEG_{\alpha,\beta}(C) = \{x \in U : Pr(C|[x]) \leq \beta\}$$

And

$Pr(C|[x]) = D0(C|[x]) = |C \cap [X]| \,/\, |[X]|$, from expression (12).

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