A Fuzzy Approach to Prospect Theory

Rafael Alejandro Espín Andrade, Erick González,
Eduardo Fernández, and Salvador Muñoz Gutiérrez

Abstract. The aim of this chapter is to revisit an experiment of Kahneman and Tversky to arrive at conclusions about Prospect theory and the ways of human thinking, but using a fuzzy approach, especially the compensatory one. New results shall be proved and others well-known shall be changed or confirmed. The study comprises the examination of logical predicates like those expressed by the following sentences: "if a scenario is probable then it is *convenient*", "there exist probable and *convenient* scenarios" and "all the scenarios are probable and *convenient*". According to the empirical results, the Reichenbach implication and the Geometric Mean are closest to the people's way of thinking.

1 Introduction

Prospect theory has been well accepted by Decision Theory community. This success is due to its right and simple answer to the question: actually how human beings make decisions under uncertainty? [8]. The expected utility theory, another classic, can't deal with situations where the subjectivity of persons is relevant and, hence, objectivity is not the only factor to be taken into account [6].

Prospect theory is a consequence of many experiments carried out by Kahneman and Tversky about the attitude of human being under uncertainty situations. They maintained the concept of *lottery*, used for computing expected utility functions, which consists of a set of premiums often representing money quantities,

Rafael A. Espín Andrade · Erick González · Salvador Muñoz Gutiérrez
"José Antonio Echeverría" Higher Technical Institute,
Havana, Cuba
e-mail: rafaelespin@yahoo.com,
erickgc@cemat.cujae.edu.cu,
salvador@ind.cujae.edu.cu

Eduardo Fernández
"Autonomous University of Sinaloa", Sinaloa, Mexico
e-mail: eddyf171051@gmail.com

R.A. Espín Andrade et al. (eds.), *Soft Computing for Business Intelligence*,
Studies in Computational Intelligence 537,
DOI: 10.1007/978-3-642-53737-0_3,

positive if they are gains or negative if they are losses, while being associated with the probability of occurrence, such that the probabilities of all the premiums sum one. They studied the shape, slope and other characteristics of a function, named *value function* that measures the risk attitude and preferences of persons. In this context, lotteries are called *prospects*.

On the other hand, Fuzzy logic is a multi-valued logic, with a wide range of applications [4]. Some of their essential properties are their facilities to model the "vagueness" proper to the natural language and the uncertainty. These properties are arguments to justify the relevance of searching for nexuses between Fuzzy logic and Prospect theory. Also, fuzzy logic has been a useful tool for modelling preferences.

The notion of t-norm and t-conorm doesn't seem to be adequate to solve problems in decision making; however, it is the most extended approach of all, even though empirical studies prove that some compensatory operators are closest to represent real human thinking than any t-norm or t-conorm system [10].

The insufficient study of compensatory operators in fuzzy literature [2], usually provokes that the concept of operator prevails over the concept of integrated operators' system. Maybe, the only exception in the literature is *Compensatory Fuzzy Logic* (CFL) [5]. The CFL consists of a set of axioms, some of them inspired in logic and others in Decision theory, which are grouped in a coherent way. It is a quartet of continuous operators (c, d, o, n) of, respectively, a conjunction operator, a disjunction operator, a fuzzy strict order operator and a negation operator.

The conjunction operator of the CFL could be defined with formulas of the quasi-arithmetic means and the disjunction operator could be their duals. CFL is a recommendable tool to be used in *Soft-computing*, which is the classification given by Zadeh [14] to all the branches of Artificial Intelligence opposites to *hard-computing*, such that a good or approximate solution is accepted, even if it is not optimal, and fuzzy logic is one of their bases.

CFL is designed to calculate using complex sentences expressed in natural language, and not the so usually exclusive employment of simple linguistic variables. The conception of this new tool is to reaffirm the Zadeh's idea to compute with words rather than with numbers [15]. This characteristic can be used to link CFL with Artificial Intelligence branches like *Knowledge Engineering*, the *Expert System*'s methodology [1].

The aim of this chapter is to revisit an experiment of Kahneman and Tversky [8] to arrive at conclusions about Prospect theory and the ways of human thinking, but using a fuzzy approach, especially the compensatory one. New results shall be proved and others well-known shall be changed or confirmed. The study comprises the examination of logical predicates like those expressed by the following sentences: "if a scenario is probable then it is *convenient*", "there exist probable and *convenient* scenarios" and "all the scenarios are probable and *convenient*".

In this chapter, a scenario is a premium, which is associated with a probability. An implication operator upon a set of five and a one-parameter family of compensatory systems will be selected for representing these predicates.

The chapter is structured as follows: next section, called *Preliminaries*, is divided in two parts, the first of them explains the basic concepts of Prospect theory and the second one exposes some notions about CFL, including the introduction of a compensatory one-parameter family. The third section describes the experiment of Kahneman and Tversky that shall be used in the chapter; some other notions like implication operators that will be useful are included. This section finishes with the description of a fuzzy approach to Prospect theory. The fourth section describes the analysis of the results.

2 Preliminaries

A *prospect* in Prospect theory, as a *lottery* in Utility theory, is represented by $L = (x_1, p_1, x_2, p_2, \dots, x_n, p_n)$, where p_i is the probability to obtain the potential outcome or premium x_i and $\sum_{i=1}^{n} p_i = 1$.

The detailed manner to measure the prospects can be found in [8], it is basically $V(L) = \sum_{i=1}^{n} \pi(p_i) v(x_i)$.

$\pi(p)$ is called the *weighting function* or *decision weight*, which maps over the probabilities and $v(x)$ is called the *value function*, which maps over the outcomes or premiums.

Let us note that probabilities aren't used directly in the final valorisation of the prospect, because they don't influence objectively the result, but subjectively, according to a function $\pi(p)$ defined by the decision maker. Usually, $\pi(p)$ is assumed by individual decision makers as non-linear weights, which are concave over certain interval [0, b] and convex over the interval [b, 1], where $0<b<1$.

The value function has the characteristics summarized below, according to empirical results:

1. There exists a *reference point* that is valued as indifferent by people. The other points are assumed like deviations from this point; therefore, people think in terms of gains and losses.
2. The function is concave over gains and convex over losses. That is to say, it is an s-shaped or sigmoidal function.
3. It is steeper for losses than for gains. This is because people experience losses more intensively than gains.

Hypothetical figures of a value function and a decision weight are represented in figures 1 and 2, respectively.

In brief, people are risk-averse for gains and risk-seeking for losses.

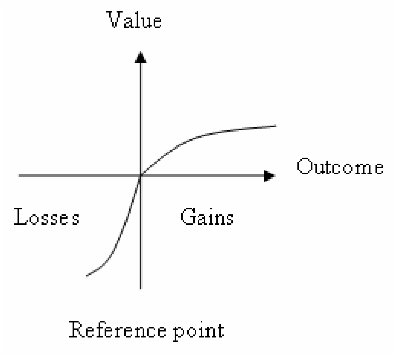

Fig. 1 A hypothetical Value Function

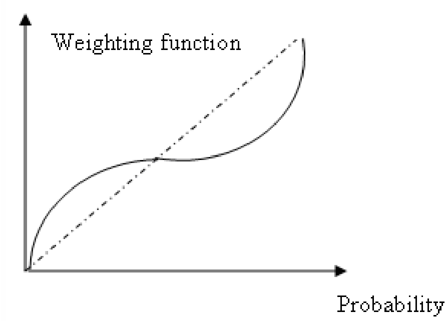

Fig. 2 A hypothetical weighting function

A *Compensatory Fuzzy Logic* (CFL) system is a quartet (*c*,*d*,*o*,*n*) of operators of conjunction, disjunction, fuzzy strict order and negation, respectively [5].

c and *d* map vectors of $[0,1]^n$ into [0,1], *o* is a mapping from $[0,1]^2$ into [0,1], and *n* is a unary operator of [0,1] into [0,1]. Some axiomatic must to be satisfied for the operators of conjunction and disjunction, like for example, Compensation Axiom¸ Symmetry Axiom and others [5].

A family of CFL systems may be obtained from the quasi-arithmetic means, with the following formula below [9]:

$$M_f(x_1, x_2, \dots, x_n) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right) \tag{1}$$

Where f(x) is a continuous and strictly monotonic function of one real variable.

In this chapter the one-parameter family with formula:

$$M_f(x_1, x_2, \dots, x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{1/p} \tag{2}$$

Where $p \in (-\infty, 0]$ satisfies the axiom of compensation, if the conjunction is defined as in (2). More details about the CFL and formulas of family (2) can be found in (Espin et al. 2011).

Therefore, conjunction is defined as follows:

$$c(x_1, x_2, \dots, x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{1/p} \tag{3}$$

The disjunction is defined as the dual of the conjunction, that is to say:

$$d(x_1, x_2, \dots, x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} (1 - x_i)^p\right)^{1/p} \tag{4}$$

The fuzzy negation is:

$$n(x) = 1 - x \tag{5}$$

The fuzzy strict order is:

$$o(x, y) = 0.5[c(x) - c(y)] + 0.5 \tag{6}$$

$p(\boldsymbol{x})$ is a formula of the propositional calculus in CFL.
This formula is valid in the CFL if it satisfies the condition (7), below:

$$f^{-1}\left(\frac{\int_{[0,1]^n} f(p(x))dx}{\int_{[0,1]^n} dx}\right) > \frac{1}{2} \tag{7}$$

3 The Experiments

This section begins with a useful resume of fuzzy implications.

In fuzzy literature the classification of implication operators is usually defined using other operators, like conjunction, disjunction and negation, but they are always based on t-norm and t-conorm paradigm. In this chapter, these concepts will be extended to any fuzzy system, including the compensatory ones. Here, when it would be necessary, the operators will preserve their exact definition, even if they don't correspond to any classification and taking into account that often the definition of an implication operator is associated with a specific t-norm and t-conorm.

The criteria for selecting implication operators for our purposes are the following:

1. The operator satisfies the truth-value table of the bivalent classical logic, when the truth-values calculus is restricted only to the set $\{0, 1\}$. Briefly, the truth-value of the formula $x \to y$ is 1 if $x = 0$ or $x = y = 1$, and is 0 if $x = 1$ and $y = 0$.
2. The operator must be a continuous function with regard to both arguments or it has a finite number of removable discontinuities.

The reason for imposing condition 1 is that this must be a natural extension of the mathematical logic. Whereas condition 2 guarantees the "sensitiveness" of the composed predicates, that is to say, any change in the simple predicates will be reflected in the final results of their corresponding composed predicates.

Some classifications definitions appeared in the literature are:

- S-implication [4]: $I_S(x, y) = d(n(x), y)$, where d and n are the disjunction and negation operators, respectively.
- R-implication [4]: $I_R(x, y) = sup\{z \in [0,1]: c(x, z) \le y\}$, where c is the conjunction operator.
- QM-implication [11], which is also known as QL-implication [4]: $I_{QL}(x, y) = d(n(x), c(x, y))$

- A-implication [12]: The operator satisfies a group of axioms, which implicitly associate it with the conjunction, disjunction and negation operators. For example, the Law of Importation $(x \wedge y \rightarrow z) \leftrightarrow (x \rightarrow (y \rightarrow z))$ is one of its axioms, where the symbol $\leftrightarrow$ is the logic equivalence.

The implication operators that have appeared in the literature satisfy the two conditions expressed above, and their classifications are:

- Reichenbach implication (S-implication): $x \rightarrow y = 1 - x + xy$
- Klir-Yuan implication (a variation of the above case without a classification): $x \rightarrow y = 1 - x + x^2 y$
- Natural implication (S-implication), see [5]: $x \rightarrow y = d(n(x), y)$
- Zadeh implication (QL-implication): $x \rightarrow y = d(n(x), c(x, y))$
- Yager implication (A-implication): $x \rightarrow y = y^x$

The formula of the equivalence is defined as: $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$. It is valid for any implication operator and any conjunction operator.

Other classifications can be found in [7].

This chapter shall revisit an experiment of Tversky and Kahneman appeared in [13]. The results are summarized in the table 1:

Table 1 Results of an experiment of Tversky and Kahneman

Premium 1	Premium 2	Probability 1	Probability 2	Equivalent
0	50	0.9	0.1	9
0	50	0.5	0.5	21
0	50	0.1	0.9	37
0	-50	0.9	0.1	-8
0	-50	0.5	0.5	-21
0	-50	0.1	0.9	-37
0	100	0.95	0.05	14
0	100	0.75	0.25	25
0	100	0.5	0.5	36
0	100	0.25	0.75	52
0	100	0.05	0.95	78
0	100	0.95	0.05	-8
0	100	0.75	0.25	-23.5
0	100	0.5	0.5	-42
0	100	0.25	0.75	-63
0	100	0.05	0.95	-84
0	200	0.99	0.01	10
0	200	0.9	0.1	20
0	200	0.5	0.5	76
0	200	0.1	0.9	131
0	200	0.01	0.99	188
0	-200	0.99	0.01	-3
0	-200	0.9	0.1	-23
0	-200	0.5	0.5	-89

Table 1 (*continued*)

Premium 1	Premium 2	Probability 1	Probability 2	Equivalent
0	-200	0.1	0.9	-155
0	-200	0.01	0.99	-190
0	400	0.99	0.01	12
0	400	0.01	0.99	377
0	-400	0.99	0.01	-14
0	-400	0.01	0.99	-380
50	100	0.9	0.1	59
50	100	0.5	0.5	71
50	100	0.1	0.9	83
-50	-100	0.9	0.1	-59
-50	-100	0.5	0.5	-71
-50	-100	0.1	0.9	-85
50	150	0.95	0.05	64
50	150	0.75	0.25	72.5
50	150	0.5	0.5	86
50	150	0.25	0.75	102
50	150	0.05	0.95	128
-50	-150	0.95	0.05	-60
-50	-150	0.75	0.25	-71
-50	-150	0.5	0.5	-92
-50	-150	0.25	0.75	-113
-50	-150	0.05	0.95	-132
100	200	0.95	0.05	118
100	200	0.75	0.25	130
100	200	0.5	0.5	141
100	200	0.25	0.75	162
100	200	0.05	0.95	178
-100	-200	0.95	0.05	-112
-100	-200	0.75	0.25	-121
-100	-200	0.5	0.5	-142
-100	-200	0.25	0.75	-158
-100	-200	0.05	0.95	-179

Columns 1, 2, 3 and 4 of table 1 represent prospects of two alternatives and the ultimate column summarizes equivalent values of their acceptance.

The data in table 1 will be interpreted with fuzzy models. Sigmoidal is the membership function that will be used, according to the recommendation appeared in [3].

The sigmoidal function formula is:

$$sigm(x, \alpha, y) = \frac{1}{1 + e^{-\alpha(x-y)}} \quad (8)$$

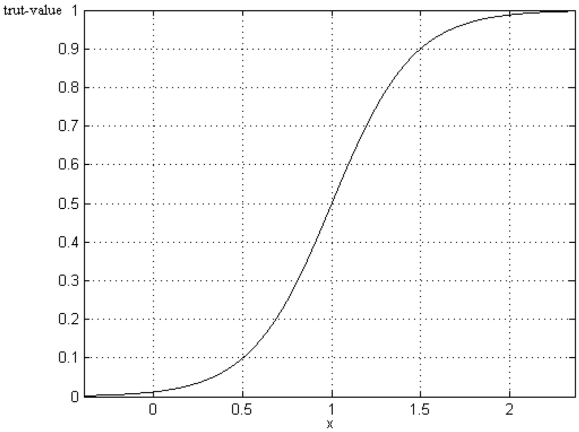

Fig. 3 A generic sigmoidal function with parameters $\gamma = 1$ and $\beta = 0.5$

Figure 3 is the graphic of a generic sigmoidal membership function, where $\gamma = 1$ and $\beta = 0.5$. $\alpha = 4.3944$ was calculated by the following formula below:

$$\alpha = \frac{\ln(0.9) - \ln(0.1)}{\gamma - \beta}$$

Let us note that $sigm(\gamma, \alpha, \gamma) = 0.5$, $sigm(\beta, \alpha, \gamma) = 0.1$ and it is s-shaped, different from function in figure 2 and equal to figure 1. $x = \gamma$ is an "indifferent" value and $x = \beta$ is "almost false" in formula (8).

Here a "scenario" is a premium associated with a probability and it will be classified with the term "convenient".

Three predicates will be calculated using fuzzy variables:

1. "If the scenario is probable then it is convenient".

"All the scenarios are probable and convenient". This statement measures the risk-aversion tendency by the decision makers.

"There exist probable and convenient scenarios". This statement measures the risk-seeking tendency by the decision makers.

It is converted in an optimization (maximization) problem which will be detailed below in order of apparition:

1. The first proposition is divided in the following two: "If all the scenarios are probable then they are convenient" and "If there are probable scenarios then they are convenient".

The maximization problems are respectively:

1.1. Maximize $P_{11}(x)$ such that $P_{11}(x)$ is:

$$\wedge_{i=1}^{56} ((u_p(p_i) \rightarrow u_x(x_i)) \wedge (u_p(1-p_i) \rightarrow u_x(y_i))) \leftrightarrow sigm(eq_i, \alpha_{eq}, \gamma_{eq}) \quad (9)$$

Where u_p and u_x are the sigmoidal functions sigm(p,α_p,γ_p) and sigm(x,α_x,γ_x), representing respectively the predicates "the scenario is probable" and "the scenario is convenient". sigm(eq_i,α_{eq},γ_{eq}) is the sigmoidal function of the equivalent values.

1.2. Besides, the second problem consists in maximizing $P_{12}(x)$ such that $P_{12}(x)$ is:

$$max \wedge_{i=1}^{56} ((u_p(p_i) \rightarrow u_x(x_i)) \vee (u_p(1-p_i) \rightarrow u_x(y_i))) \leftrightarrow sigm(eq_i, \alpha_{eq}, \gamma_{eq}) \quad (10)$$

2. The maximization problem is to find the maximum of $P_2(x)$ such that $P_2(x)$ is:

$$max \wedge_{i=1}^{56} ((u_p(p_i) \wedge u_x(x_i)) \wedge (u_p(1-p_i) \wedge u_x(y_i))) \leftrightarrow sigm(eq_i, \alpha_{eq}, \gamma_{eq}) \quad (11)$$

3. The maximization problem consists in maximizing $P_3(x)$ such that $P_3(x)$ is:

$$max \wedge_{i=1}^{56} ((u_p(p_i) \wedge u_x(x_i)) \vee (u_p(1-p_i) \wedge u_x(y_i))) \leftrightarrow sigm(eq_i, \alpha_{eq}, \gamma_{eq}) \quad (12)$$

Formulas 9-12 are aggregation operators for all the lotteries, the conjunctions $\wedge_{i=1}^{56}$ were defined on the set of the 56 lotteries, see table 1. u_p and u_x are modelled by using sigmoidal membership functions, the first of them represents the subjective perception of probability by people and the second one is the value function.

Because each lottery in table 1 consists in two scenarios with two probabilities, there are two evaluations for u_p and u_x in the lottery, first for p_i and $1 - p_i$, see third and fourth columns in table 1, and secondly for x_i and y_i, see the two first columns. The last column represents an equivalent valorisation of the lottery in the experiment. It is also modelled with a sigmoidal function which depends on two parameters, α_{eq} and γ_{eq}.

The search of the three sigmoidal functions u_p, u_x and $sigm(eq_i, \alpha_{eq}, \gamma_{eq})$ by each problem is reduced to the optimization on the space of the six parameters $\gamma_p, \alpha_x, \gamma_x, \alpha_{eq}$ and γ_{eq}, where the objective functions are those represented in formulas 9-12. Other unknown in formulas above are the implication operator → and hence, the equivalence ↔, therefore, the Reichenbach, Yager, Klir-Yuan, Natural and Zadeh are tested. CFL, depending on parameter p in formula (2) are tested too, and the search actually depends on eight parameters, if p and $\rightarrow$ are included.

The optimization problems, 1.1, 1.2, 2 and 3, are reduced to estimate the maximum truth-values of formulas 9-12 respectively, with a fixed $\rightarrow$ and varying the other seven parameters that were detailed in the paragraph above.

Every formula 9-12 is equivalent to a linguistic problem. For example, in formula 9, $((u_p(p_i) \rightarrow u_x(x_i)) \wedge (u_p(1-p_i) \rightarrow u_x(y_i)))$ means for the lottery i, "if the first scenario is probable then it is convenient and if the second scenario is probable then it is convenient". On the other hand, the logical equivalence $\leftrightarrow$ emulates the experimental equivalence summarized in the last column of table 1. This reasoning can be generalized to the other predicates which represent the other problems.

To sum up, each optimization problem depends on a CFL system. The one-parameter family of formulas 3, 4, 5, 6 will be one of the parameter to be estimated. Also, each problem derives in five cases, where the implication operator is applied from the five proposed in the beginning of the section.

Some heuristic restrictions of the alphas and gammas that will be applied are:

1. All the alphas are strictly equal to 0. This condition guarantees that sigmoidal is an increasing function and not a constant one, such as the case where it is equal 0.

The values of gammas are between the minimum and the maximum data in table 1, where they do not represent the equivalent values.

In case of the equivalent values of the last column in table 1, the gamma will be restricted between 0 and 76. As a result of the Prospect theory, it is well-known that people don't accept non-positive values with indifference; taking into account that gamma is the value which represents indifference (0.5). 76 is 20% of the absolute value of the maximum number in the last column in table 1, which has been selected heuristically.

The optimization will be based on the genetic algorithm coded in MATLAB.

4 Results

Tables 2-5 summarize the results for every optimization problem exposed above. Table 2, for example, may be read as following: The maximum truth-value of the objective function of formula (9) in the case of Reichenbach implication is 0.93791284, see second column and ultimate file. This is the biggest value by column in this table; hence, Reichenbach implication is the best of all implication operators for problem 1.1, which linguistically represents the predicate: "If all the scenarios are probable then they are convenient".

The values which maximize the problem 1.1 are: α_x=64, γ_x=1, α_p=11.0376854, γ_p=0.02615738, α_{eq}=45 and γ_{eq}=56. The last parameter estimated is $p = 0$, which corresponds to the Geometric Mean in formula (3).

Table 2 Estimated parameters for problem 1.1. "If all the scenarios are probable then they are convenient"

Estimated Parameters	Reichenbach	Yager	Klir-Yuan	Natural	Zadeh
α_x	64	0.88085938	64	57	128
γ_x	1	30.6367188	0	1	129
α_p	11.0376854	2.12890625	230	19.8595638	21.6115036
γ_p	0.02615738	1	0	0.10683823	0.4031105
α_{eq}	45	0.09375	65	32	74.8601074
γ_{eq}	56	60.7246094	1	57	73
P	0	0	0	0	0
Maximum truth-value	0.93791284	0.85109059	0.87150398	0.88978479	0.79259532

Every pair of parameters represents a sigmoidal membership function and hence, a fuzzy selection pattern by people. In case of the Table 2 they are plotted in figure 4.

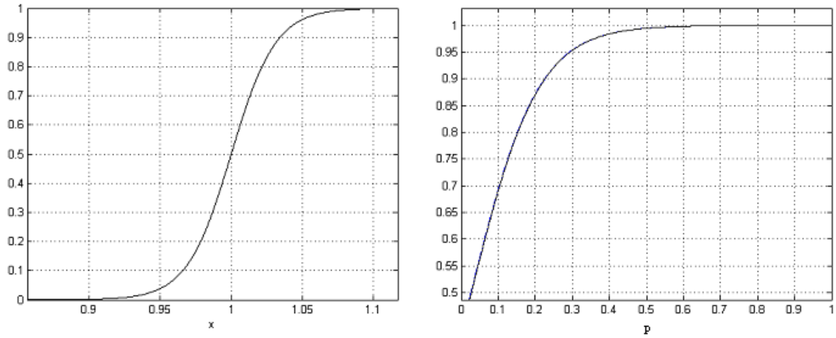

Fig. 4 Membership functions of the predicates: "The scenario is convenient" (left) and "the scenario is probable" (right), for the problem 1.1, with Reichenbach implication. See table 2

According to the meaning of each parameter, in problem 1.1, representing the predicate: "if the scenario is probable then it is convenient", people is indifferent when the function value is 1 and when the probability is 0.02615738, because $\gamma_x = 1$ and $\gamma_p = 0.02615738$, respectively.

Parameter $\alpha_x = 64$ corresponds to $\beta_x = 0.9657$, according to the formula of α, appeared above. Therefore, people consider "almost false" a value function equalling 0.9657.

A negative value of p suggests a "pessimistic" tendency in the people's behaviour. Let us note that p=0 or $p \approx 0$ for all the cases; therefore, people actually have a neutral's behaviour.

The reasoning above for the problem 1.1 can be extended to the other three problems, which their corresponding results are summarized in tables 3-5. Table 3, 4 and 5 summarize the values of the optimization problems with objective functions showed in equations (10), (11) and (12), respectively. Their corresponding figures are 5, 6 and 7.

Table 3 Estimated parameters for problem 1.2. "If there are probable scenarios then they are convenient"

Estimated parameters	Reichenbach	Yager	Klir-Yuan	Natural	Zadeh
α_x	32	0.0703125	64	31	128
γ_x	1.5	317.71875	1	1	1
α_p	230	7.5625	6.76686478	97	17.2717075
γ_p	0	1	0	1	0
α_{eq}	0.03500748	0	25	0	65
γ_{eq}	0	1	16	1	17
P	0	0	0	0	-1.9073E-06
Maximum truth-value	0.85970284	0.7147067	0.8335026	0.5411961	0.76028923

Table 4 Estimated parameters for problem 2. "All the scenarios are probable and convenient"

Estimated parameters	Reichenbach	Yager	Klir-Yuan	Natural	Zadeh
α_x	5.52869034	0.734375	6.02235603	7.0930481	12.0120811
γ_x	1	14.8125	1	1	1
α_p	230	230	230	230	230
γ_p	0	0	0	0	0
α_{eq}	62	0.09375	30	24	24
γ_{eq}	53	58.9453125	57	57	57
P	0	0	0	0	0
Maximum truth-value	0.90420119	0.81944258	0.89626517	0.83563393	0.87025163

Table 5 Estimated parameters for problem 3. "There exist probable and convenient scenarios"

Estimated parameters	Reichenbach	Yager	Klir-Yuan	Natural	Zadeh
α_x	97	0.3203	97	128	97
γ_x	1	18.6406	1	65	1
α_p	8.4261	6.7734	8.17059708	15.8297119	14.9041805
γ_p	0.354	0	0.24069786	0.58897972	0.82479858
α_{eq}	229.2813	0.0781	48	129	74.8599014
γ_{eq}	20.375	0	17	73	73
P	0	0	0	0	0
Maximum truth-value	0.9046	0.7803	0.87730427	0.83112339	0.76983507

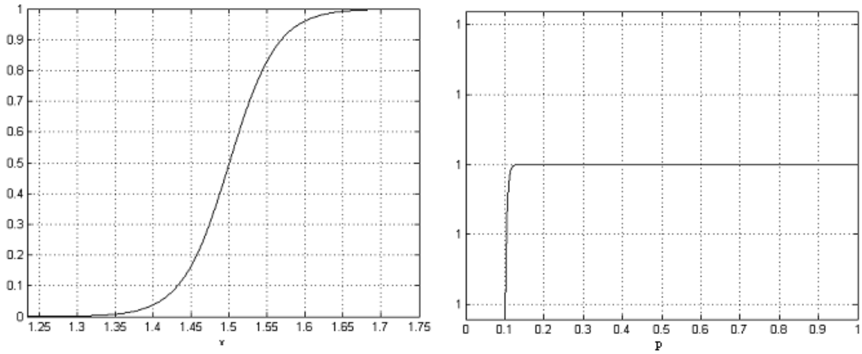

Fig. 5 Membership functions of the predicates: "The scenario is convenient" (left) and "the scenario is probable" (right), for the problem 1.2, with Reichenbach implication. See table 3.

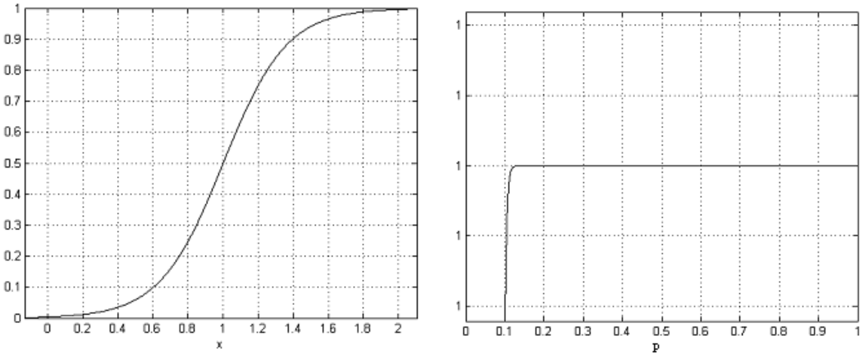

Fig. 6 Membership functions of the predicates: "The scenario is convenient" (left) and "the scenario is probable" (right), for the problem 2, with Reichenbach implication. See table 4.

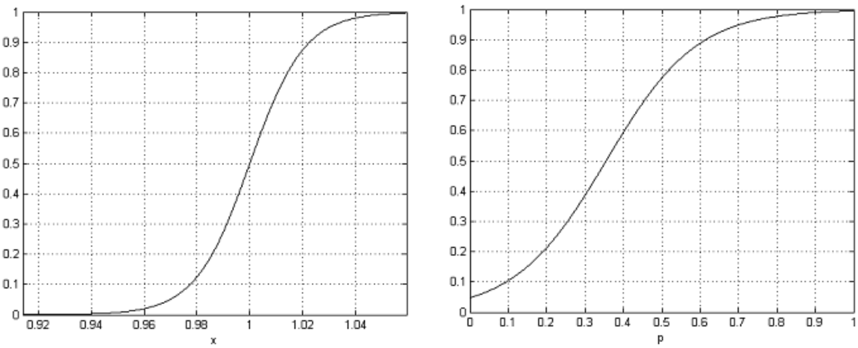

Fig. 7 Membership functions of the predicates: "The scenario is convenient" (left) and "the scenario is probable" (right), for the problem 3, with Reichenbach implication. See table 5.

These results allow arriving to some conclusions:

- All the predicates show better results with the Reichenbach implication.
- People measure preferences with Geometric Mean ($p = 0$).
- With Reichenbach implication, the values of indifference for the scenarios are equal or slightly bigger than 1.
- The probabilities are measured with small slopes and γ_p>0, for problems: "If all the scenarios are probable then they are convenient" and "There exist probable and convenient scenarios" (risk-seeking), see tables 2 and 5. Besides, the probabilities for: "If there are probable scenarios then they are convenient" and "There exist probable and convenient scenarios" (risk-aversion), have big slopes and the minimum of their values is 0.5 for the probability 0, see tables 3 and 4.

Other experiments made by authors, show that the shape of the membership function, like in figure 2, doesn't contribute to better results of the truth-values in the maximizations.

Tables below indicate the application of precedent results in the experiment summarized in table 1.

Table 6 Predicates "If all the scenarios are probable then they are convenient" and "If there are probable scenarios then they are convenient", respectively in columns 1 and 2 for Reichenbach, Yager and Klir-Yuan implications applied to the experiment of table 1. The next-to-last and last columns for every implication represent the conjunction and disjunction of the two predicates, respectively.

	Reichenbach				Yager				Klir-Yuan			
Lottery	∀	∃	∧	∨	∀	∃	∧	∨	∀	∃	∧	∨
1	0.01	1.00	0.09	1.00	0.00	0.86	0.05	0.62	0.71	0.53	0.84	1.00
2	0.07	1.00	0.27	1.00	0.03	0.63	0.14	0.40	0.71	0.82	0.84	1.00
3	0.55	1.00	0.74	1.00	0.18	0.84	0.39	0.64	0.71	0.96	0.84	1.00
4	0.00	0.00	0.00	0.00	0.00	0.83	0.00	0.59	0.00	0.19	0.00	0.16
5	0.01	0.00	0.00	0.00	0.00	0.59	0.00	0.36	0.00	0.03	0.00	0.16
6	0.00	0.00	0.00	0.00	0.00	0.84	0.00	0.60	0.00	0.19	0.00	0.16
7	0.01	1.00	0.08	1.00	0.00	0.89	0.04	0.67	0.71	0.51	0.84	0.97
8	0.02	1.00	0.14	1.00	0.01	0.78	0.07	0.53	0.71	0.64	0.84	1.00
9	0.07	1.00	0.27	1.00	0.03	0.66	0.14	0.43	0.71	0.82	0.84	1.00
10	0.28	1.00	0.53	1.00	0.10	0.75	0.28	0.52	0.71	0.93	0.84	1.00
11	0.66	1.00	0.81	1.00	0.21	0.87	0.42	0.68	0.71	0.97	0.84	1.00
12	0.01	1.00	0.08	1.00	0.00	0.89	0.04	0.67	0.71	0.51	0.84	0.97
13	0.02	1.00	0.14	1.00	0.01	0.78	0.07	0.53	0.71	0.64	0.84	1.00
14	0.07	1.00	0.27	1.00	0.03	0.66	0.14	0.43	0.71	0.82	0.84	1.00
15	0.28	1.00	0.53	1.00	0.10	0.75	0.28	0.52	0.71	0.93	0.84	1.00
16	0.66	1.00	0.81	1.00	0.21	0.87	0.42	0.68	0.71	0.97	0.84	1.00

Table 6 (*continued*)

17	0.00	1.00	0.07	1.00	0.00	0.93	0.03	0.74	0.68	0.50	0.73	0.74
18	0.01	1.00	0.09	1.00	0.00	0.90	0.05	0.69	0.71	0.53	0.84	1.00
19	0.07	1.00	0.27	1.00	0.03	0.74	0.15	0.50	0.71	0.82	0.84	1.00
20	0.55	1.00	0.74	1.00	0.18	0.85	0.39	0.65	0.71	0.96	0.84	1.00
21	0.74	1.00	0.86	1.00	0.23	0.89	0.45	0.71	0.71	0.97	0.84	1.00
22	0.00	0.05	0.01	0.03	0.00	0.86	0.00	0.62	0.21	0.28	0.26	0.27
23	0.00	0.00	0.00	0.00	0.00	0.80	0.00	0.55	0.00	0.19	0.00	0.16
24	0.01	0.00	0.00	0.00	0.00	0.53	0.00	0.32	0.00	0.03	0.00	0.16
25	0.00	0.00	0.00	0.00	0.00	0.84	0.00	0.60	0.00	0.19	0.00	0.16
26	0.00	0.05	0.01	0.03	0.00	0.89	0.00	0.67	0.00	0.28	0.00	0.16
27	0.00	1.00	0.07	1.00	0.00	1.00	0.04	0.96	0.68	0.50	0.73	0.74
28	0.74	1.00	0.86	1.00	0.23	1.00	0.48	0.94	0.71	0.97	0.84	1.00
29	0.00	0.05	0.01	0.03	0.00	0.83	0.00	0.59	0.21	0.28	0.26	0.27
30	0.00	0.05	0.01	0.03	0.00	0.89	0.00	0.67	0.00	0.28	0.00	0.16
31	1.00	1.00	1.00	1.00	1.00	0.87	0.93	1.00	1.00	0.98	1.00	1.00
32	1.00	1.00	1.00	1.00	1.00	0.69	0.83	1.00	1.00	0.97	1.00	1.00
33	1.00	1.00	1.00	1.00	1.00	0.86	0.93	1.00	1.00	0.98	1.00	1.00
34	0.00	0.00	0.00	0.00	0.00	0.82	0.00	0.58	0.00	0.19	0.00	0.00
35	0.01	0.00	0.00	0.00	0.00	0.54	0.00	0.32	0.00	0.03	0.00	0.00
36	0.00	0.00	0.00	0.00	0.00	0.83	0.00	0.59	0.00	0.19	0.00	0.00
37	1.00	1.00	1.00	1.00	1.00	0.91	0.95	1.00	1.00	0.98	1.00	1.00
38	1.00	1.00	1.00	1.00	1.00	0.81	0.90	1.00	1.00	0.97	1.00	1.00
39	1.00	1.00	1.00	1.00	1.00	0.72	0.85	1.00	1.00	0.97	1.00	1.00
40	1.00	1.00	1.00	1.00	1.00	0.78	0.88	1.00	1.00	0.97	1.00	1.00
41	1.00	1.00	1.00	1.00	1.00	0.88	0.94	1.00	1.00	0.98	1.00	1.00
42	0.00	0.00	0.00	0.00	0.00	0.84	0.00	0.60	0.00	0.24	0.00	0.00
43	0.01	0.00	0.00	0.00	0.00	0.68	0.00	0.43	0.00	0.08	0.00	0.00
44	0.01	0.00	0.00	0.00	0.00	0.52	0.00	0.31	0.00	0.03	0.00	0.00
45	0.01	0.00	0.00	0.00	0.00	0.71	0.00	0.46	0.00	0.08	0.00	0.00
46	0.00	0.00	0.00	0.00	0.00	0.86	0.00	0.63	0.00	0.24	0.00	0.00
47	1.00	1.00	1.00	1.00	1.00	0.92	0.96	1.00	1.00	0.98	1.00	1.00
48	1.00	1.00	1.00	1.00	1.00	0.84	0.92	1.00	1.00	0.97	1.00	1.00
49	1.00	1.00	1.00	1.00	1.00	0.78	0.88	1.00	1.00	0.97	1.00	1.00
50	1.00	1.00	1.00	1.00	1.00	0.82	0.90	1.00	1.00	0.97	1.00	1.00
51	1.00	1.00	1.00	1.00	1.00	0.89	0.95	1.00	1.00	0.98	1.00	1.00
52	0.00	0.00	0.00	0.00	0.00	0.84	0.00	0.59	0.00	0.24	0.00	0.00
53	0.01	0.00	0.00	0.00	0.00	0.66	0.00	0.42	0.00	0.08	0.00	0.00
54	0.01	0.00	0.00	0.00	0.00	0.48	0.00	0.28	0.00	0.03	0.00	0.00
55	0.01	0.00	0.00	0.00	0.00	0.69	0.00	0.44	0.00	0.08	0.00	0.00
56	0.00	0.00	0.00	0.00	0.00	0.85	0.00	0.61	0.00	0.24	0.00	0.00

Table 7 Predicates "If all the scenarios are probable then they are convenient" and "If there are probable scenarios then they are convenient" in columns 1 and 2 for Natural and Zadeh implications, respectively, applied to the experiment of table 1. The next-to-last and last columns for every implication represent the conjunction and disjunction of the two predicates, respectively.

	Natural				Zadeh			
Lottery	∀	∃	∧	∨	∀	∃	∧	∨
1	0.00	1.00	0.02	1.00	0.00	0.49	0.05	0.56
2	0.01	1.00	0.12	1.00	0.06	0.90	0.06	0.06
3	0.56	1.00	0.75	1.00	0.00	0.98	0.05	0.56
4	0.00	1.00	0.01	0.09	0.00	0.04	0.05	0.56
5	0.00	1.00	0.00	0.00	0.06	0.00	0.06	0.06
6	0.00	1.00	0.01	0.09	0.00	0.04	0.05	0.56
7	0.00	1.00	0.01	1.00	0.00	0.42	0.04	0.62
8	0.00	1.00	0.03	1.00	0.02	0.72	0.09	0.35
9	0.01	1.00	0.12	1.00	0.06	0.90	0.06	0.06
10	0.17	1.00	0.41	1.00	0.02	0.97	0.09	0.35
11	0.71	1.00	0.84	1.00	0.00	0.99	0.04	0.62
12	0.00	1.00	0.01	1.00	0.00	0.42	0.04	0.62
13	0.00	1.00	0.03	1.00	0.02	0.72	0.09	0.35
14	0.01	1.00	0.12	1.00	0.06	0.90	0.06	0.06
15	0.17	1.00	0.41	1.00	0.02	0.97	0.09	0.35
16	0.71	1.00	0.84	1.00	0.00	0.99	0.04	0.62
17	0.00	1.00	0.01	1.00	0.00	0.39	0.03	0.66
18	0.00	1.00	0.02	1.00	0.00	0.49	0.05	0.56
19	0.01	1.00	0.12	1.00	0.21	0.90	0.34	0.40
20	0.56	1.00	0.75	1.00	0.98	0.98	0.98	0.98
21	0.80	1.00	0.90	1.00	0.99	0.99	0.99	0.99
22	0.00	1.00	0.01	0.23	0.00	0.14	0.03	0.65
23	0.00	1.00	0.01	0.09	0.00	0.04	0.05	0.56
24	0.00	1.00	0.00	0.00	0.06	0.00	0.06	0.06
25	0.00	1.00	0.01	0.09	0.00	0.04	0.05	0.56
26	0.00	1.00	0.01	0.23	0.00	0.14	0.03	0.65
27	0.00	1.00	0.01	1.00	0.00	0.39	0.03	0.66
28	0.80	1.00	0.90	1.00	0.99	0.99	0.99	0.99
29	0.00	1.00	0.01	0.23	0.00	0.14	0.03	0.65
30	0.00	1.00	0.01	0.23	0.00	0.14	0.03	0.65
31	1.00	1.00	1.00	1.00	0.00	0.99	0.05	0.56
32	1.00	1.00	1.00	1.00	0.06	0.99	0.06	0.06
33	1.00	1.00	1.00	1.00	0.00	0.99	0.05	0.56
34	0.00	1.00	0.01	0.09	0.00	0.04	0.05	0.56

Table 7 (*continued*)

35	0.00	1.00	0.00	0.00	0.06	0.00	0.06	0.06
36	0.00	1.00	0.01	0.09	0.00	0.04	0.05	0.56
37	1.00	1.00	1.00	1.00	0.00	0.99	0.04	0.62
38	1.00	1.00	1.00	1.00	0.02	0.99	0.09	0.36
39	1.00	1.00	1.00	1.00	0.21	0.99	0.34	0.40
40	1.00	1.00	1.00	1.00	0.89	0.99	0.92	0.92
41	1.00	1.00	1.00	1.00	0.99	0.99	0.99	0.99
42	0.00	1.00	0.01	0.16	0.00	0.08	0.04	0.62
43	0.00	1.00	0.00	0.01	0.02	0.00	0.09	0.35
44	0.00	1.00	0.00	0.00	0.06	0.00	0.06	0.06
45	0.00	1.00	0.00	0.01	0.02	0.00	0.09	0.35
46	0.00	1.00	0.01	0.16	0.00	0.08	0.04	0.62
47	1.00	1.00	1.00	1.00	0.00	0.99	0.04	0.62
48	1.00	1.00	1.00	1.00	0.02	0.99	0.09	0.36
49	1.00	1.00	1.00	1.00	0.21	0.99	0.34	0.40
50	1.00	1.00	1.00	1.00	0.89	0.99	0.92	0.92
51	1.00	1.00	1.00	1.00	0.99	0.99	0.99	0.99
52	0.00	1.00	0.01	0.16	0.00	0.08	0.04	0.62
53	0.00	1.00	0.00	0.01	0.02	0.00	0.09	0.35
54	0.00	1.00	0.00	0.00	0.06	0.00	0.06	0.06
55	0.00	1.00	0.00	0.01	0.02	0.00	0.09	0.35
56	0.00	1.00	0.01	0.16	0.00	0.08	0.04	0.62

Table 8 Predicate "All the scenarios are probable and convenient"

Lottery	Reichenbach	Yager	Klir-Yuan	Natural	Zadeh
1	0.17806275	0.58009323	0.28618914	0	0.00225796
2	0.65314345	0.8747217	0.76511623	0	0.04529298
3	0.9293946	0.96685524	0.9522479	0	0.63724983
4	0	0.02553434	0	0	0
5	0	0.02513789	0	0	0
6	0	0.02076962	0	0	0
7	0.14424611	0.52671956	0.23647781	0.0070434	0.00155506
8	0.32336441	0.72274194	0.47123815	0.03470226	0.00692078
9	0.65314345	0.87480273	0.76511623	0.25384054	0.04529298
10	0.8684096	0.94566937	0.9122294	0.80782289	0.29075092
11	0.94273215	0.97224114	0.96104794	0.95945049	0.73654208
12	0.14424611	0.52671956	0.23647781	0.0070434	0.00155506
13	0.32336441	0.72274194	0.47123815	0.03470226	0.00692078
14	0.65314345	0.87480273	0.76511623	0.25384054	0.04529298

Table 8 (*continued*)

15	0.8684096	0.94566937	0.9122294	0.80782289	0.29075092
16	0.94273215	0.97224114	0.96104794	0.95945049	0.73654208
17	0.12166376	0.48342306	0.20192342	0.00512731	0.001154
18	0.17806275	0.58011323	0.28618914	0.0104797	0.00225796
19	0.65314345	0.87480273	0.76511623	0.25384054	0.04529298
20	0.9293946	0.96717048	0.9522479	0.93985357	0.63724983
21	0.95157961	0.97572583	0.96690981	0.97043746	0.79980276
22	0	0.02554093	0	0	0
23	0	0.02552766	0	0	0
24	0	0.02512983	0	0	0
25	0	0.02076139	0	0	0
26	0	0.01830749	0	0	0
27	0.12166376	0.48342306	0.20192342	0.00512731	0.001154
28	0.95157961	0.97572583	0.96690981	0.97043746	0.79980276
29	0	0.02554093	0	0	0
30	0	0.01830749	0	0	0
31	0.94196679	0.98541558	0.96591403	0.0104797	0.63806891
32	0.87969053	0.98349646	0.94482962	0.25384054	0.0885345
33	0.94196679	0.98555361	0.96591403	0.93985357	0.63806891
34	0	8.4041E-06	0	0	0
35	0	8.2753E-06	0	0	0
36	0	6.8524E-06	0	0	0
37	0.95099281	0.98606545	0.97025924	0.0070434	0.73695177
38	0.91096126	0.98411595	0.95359026	0.03470226	0.29565947
39	0.87969053	0.98349646	0.94482962	0.25384054	0.0885345
40	0.91096126	0.98416925	0.95359026	0.80782289	0.29565947
41	0.95099281	0.98624983	0.97025924	0.95945049	0.73695177
42	0	8.4046E-06	0	0	0
43	0	8.3853E-06	0	0	0
44	0	8.2726E-06	0	0	0
45	0	7.7305E-06	0	0	0
46	0	6.4272E-06	0	0	0
47	0.95099281	0.98625032	0.97025924	0.95973609	0.73695177
48	0.91096126	0.9841712	0.95359026	0.81449187	0.29565947
49	0.87969053	0.98350713	0.94482962	0.44324606	0.0885345
50	0.91096126	0.9841712	0.95359026	0.81449187	0.29565947
51	0.95099281	0.98625032	0.97025924	0.95973609	0.73695177
52	0	2.7983E-09	0	0	0
53	0	2.7919E-09	0	0	0
54	0	2.7544E-09	0	0	0
55	0	2.5739E-09	0	0	0
56	0	2.14E-09	0	0	0

Table 9 Predicate "There exist probable and convenient scenarios"

Lottery	Reichenbach	Yager	Klir-Yuan	Natural	Zadeh
1	0.25078396	0.06590855	0.22175231	0.16974302	0.04963685
2	0.25078396	0.06590855	0.22175231	0.16974302	0.04963685
3	0.25078396	0.06590855	0.22175231	0.16974302	0.04963685
4	6.1029E-32	4.4792E-07	9.9667E-35	8.9904E-41	1.52E-68
5	6.1029E-32	4.4792E-07	9.9667E-35	8.9904E-41	1.52E-68
6	6.1029E-32	4.4792E-07	9.9667E-35	8.9904E-41	1.52E-68
7	0.25078333	0.06590838	0.22175174	0.16974259	0.04963672
8	0.25078396	0.06590855	0.22175231	0.16974302	0.04963685
9	0.25078396	0.06590855	0.22175231	0.16974302	0.04963685
10	0.25078396	0.06590855	0.22175231	0.16974302	0.04963685
11	0.25078333	0.06590838	0.22175174	0.16974259	0.04963672
12	0.25078333	0.06590838	0.22175174	0.16974259	0.04963672
13	0.25078396	0.06590855	0.22175231	0.16974302	0.04963685
14	0.25078396	0.06590855	0.22175231	0.16974302	0.04963685
15	0.25078396	0.06590855	0.22175231	0.16974302	0.04963685
16	0.25078333	0.06590838	0.22175174	0.16974259	0.04963672
17	0.24486462	0.06435289	0.21651821	0.16573651	0.04846525
18	0.25078396	0.06590855	0.22175231	0.16974302	0.04963685
19	0.25078396	0.06590855	0.22175231	0.16974302	0.04963685
20	0.25078396	0.06590855	0.22175231	0.16974302	0.04963685
21	0.24486462	0.06435289	0.21651821	0.16573651	0.04846525
22	0	4.7947E-19	0	0	0
23	0	4.9106E-19	0	0	0
24	0	4.9106E-19	0	0	0
25	0	4.9106E-19	0	0	0
26	0	4.7947E-19	0	0	0
27	0.24486462	0.06435289	0.21651821	0.16573651	0.04846525
28	0.24486462	0.06435289	0.21651821	0.16573651	0.04846525
29	0	5.4201E-35	0	0	0
30	0	5.4201E-35	0	0	0
31	1	1	1	1	1
32	1	1	1	1	1
33	1	1	1	1	1
34	5.7408E-92	4.7624E-15	4.092E-100	0	0
35	5.7408E-92	4.7624E-15	4.092E-100	0	0
36	5.7408E-92	4.7624E-15	4.092E-100	0	0
37	0.99999747	0.99999747	0.99999747	0.99999747	0.99999747
38	1	1	1	1	1

Table 9 (*continued*)

39	1	1	1	1	1
40	1	1	1	1	1
41	0.99999747	0.99999747	0.99999747	0.99999747	0.99999747
42	0	4.9106E-19	0	0	0
43	0	4.9106E-19	0	0	0
44	0	4.9106E-19	0	0	0
45	0	4.9106E-19	0	0	0
46	0	4.9106E-19	0	0	0
47	0.99999747	0.99999747	0.99999747	0.99999747	0.99999747
48	1	1	1	1	1
49	1	1	1	1	1
50	1	1	1	1	1
51	0.99999747	0.99999747	0.99999747	0.99999747	0.99999747
52	0	5.2211E-27	0	0	0
53	0	5.2211E-27	0	0	0
54	0	5.2211E-27	0	0	0
55	0	5.2211E-27	0	0	0
56	0	5.2211E-27	0	0	0

Tables 6 and 7 summarize the calculus of the predicates: “If all the scenarios are probable then they are convenient” and “If there are probable scenarios then they are convenient”, corresponding to problem 1 by each of the 56 lotteries shown in table 1. The results were separated taking into account the five implication operators: Reichenbach, Yager and Klir-Yuan in table 6, Natural and Zadeh in table 7.

Every implication operator has associated four columns, the first of them, represented by symbol $\forall$, is the value of the predicate: “If all the scenarios are probable then they are convenient”. Sigmoidal functions of figure 4 were used.

The second column with symbol $\exists$ corresponds to results of the predicate: “If there are probable scenarios then they are convenient”, which uses the membership functions of figure 5. The third and fourth columns are conjunction (symbol $\wedge$) and disjunction (symbol $\vee$), respectively, of the two first columns.

The first column represents risk-aversion by people and the second one represents risk-seeking. The other two columns compute its aggregation using the conjunction and the disjunction.

For instance, with Reichenbach implication the first lottery in table 1 (0, 0.9; 50, 0.1), has truth-value 0.01 for the predicate “if all the scenarios are probable then they are convenient”, see second column and third row in table 6, and truth-value 1 for “If there are probable scenarios then they are convenient”. The conjunction and disjunction of these two truth-values may be found in the two next columns; they are 0.09 and 1, respectively. The computation is based on the sigmoidal functions obtained from the optimization problems 1.1 and 1.2, see tables 2 and 3.

Table 8 summarizes the results by every lottery in table 1 of the predicate: “All the scenarios are probable and convenient”, specifying the implication operator used in the calculation. Table 9 is structured as table 8, but here the predicate “There exist probable and convenient scenarios” is computed.

The membership functions appeared in table 4 and figure 6 were used to calculate the values of table 8. On the other hand, the elements of table 9 were calculated with the aid of the results in table 5 and figure 7.

5 Concluding Remarks

This chapter makes a fuzzy approach to Prospect theory by using the compensatory fuzzy logic. A one-parameter family of Compensatory Fuzzy Logic and five implication operators selected are used to obtain the maximization of four objective functions with the genetic algorithm coded in MATLAB. This approach is a revisit to a 1992 experiment of Kahneman and Tversky.

The family of CFL depends on a parameter p, equal to or less than 0 and they are based on the formula of the quasi-arithmetic mean. On the other hand, Reichenbach implication, Yager implication, Klir-Yuan implication, Natural implication and Zadeh implication are selected because they generalize the truth table of the bivalent logic, when they are restricted to values 0 or 1. Also, they are continuous or they have at most a finite number of removable discontinuities.

According to the empirical results, the Reichenbach implication and the Geometric Mean are closest to the people’s way of thinking. The sigmoidal membership functions of some predicates, like “the scenario is convenient” or “the scenario is probable” are found to be included in the composed predicates like “If the scenario is probable then it is convenient”, “All the scenarios are probable and convenient” or “There exist probable and convenient scenarios”.

1 or 1.5 are the values of indifference for the premiums, according to Reichenbach implication results. The membership function for probabilities changes for each predicate.

The sigmoidal functions were used for modelling every predicate, including those related with probabilities, even though its shape differs from the function illustrated in figure 2.

References

1. Buchanan, B.G., Shortliffe, E.H.: Knowledge Engineering. Rule-Based Expert Systems –The MYCIN Experiments of the Stanford Heuristic Programming Project, pp. 147–158. Addison-Wesley, Massachusetts (1984)
2. Detyniecki, M.: Mathematical aggregation operators and their application to video querying. Paris, University of Paris VI. Ph.D. Thesis (2001)
3. Dubois, D., Prade, H.: Criteria Aggregation and Ranking of alternatives in the framework of Fuzzy Sets Theory. In: Zimmermann, H.J., Gaines, B.R., Zadeh, L.A. (eds.) Fuzzy Set and Decision Analysis, pp. 209–240. North Holland, Amsterdam (1984)

4. Dubois, D. et al.: Fuzzy-Set Based Logic- An History -Oriented Presentation of their Main Developments Handbook of the History of Logic. In: Gabbay, D.M., Woods, J. (eds.), pp. 325–449. North-Holland, Elsevier BV (2007)
5. Espin, R., et al.: Un sistema lógico para el razonamiento y la toma de decisiones: la Lógica Difusa Compensatoria Basada en la Media Geométrica (A logic system for reasoning and decision making: Compensatory Fuzzy Logic on Geometric Mean). Revista Investigación Operacional 32, 230–245 (2011) (in Spanish)
6. French, S.: Decision Theory: An Introduction to the Mathematics of Rationality. Halsted Press, New York (1986)
7. Jayaram, B.: On the Law of Importation in Fuzzy Logic. IEEE Transactions on Fuzzy Systems 16, 130–144 (2008)
8. Kahneman, D., Tversky, A.: Prospect Theory: An Analysis of Decision under Risk. Econometrica 47, 263–291 (1979)
9. Mitrinovic, D.S., et al.: Classical and New Inequalities in Analysis (1993)
10. Dordrecht/Boston/London. Kluwer Academic Publishers
11. Mizumoto, M.: Pictorial Representations of Fuzzy Connectives: Part II. Cases of Compensatory Operators and Self-Dual Operators. Fuzzy Sets and Systems 32, 45–79 (1989)
12. Trillas, E., et al.: When QM-operators are implication functions and conditional fuzzy relations. International Journal of Intelligent Systems 15, 647–655 (2000)
13. Turksen, I.B., et al.: A new class of Fuzzy Implications: Axioms of Fuzzy Implication revisited. Fuzzy Sets and Systems 100, 267–272 (1998)
14. Tversky, A., Kahneman, D.: Advances in Prospect Theory: Cumulative Representation of Uncertainty. Journal of Risk and Uncertainty 5, 297–323 (1992)
15. Zadeh, L.: Some reflections on soft computing, granular computing and their roles in the conception, design and utilization of information/intelligent systems. Soft Computing 2, 23–25 (1998)
16. Zadeh, L.: From computing with numbers to computing with words –from manipulation of measurements to manipulation of perceptions. International Journal of Applied Mathematics and Computational Sciences 3, 307–324 (2002)