Soution of Convection-Diffusion Equations

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Abstract. Partial differential equations are an important part of mathematics in science and its numerical solution occupies an important position in the numerical analysis. Partial differential equations are closely related to human life and it has important research value. At present, people have studied its solutions in depths and achieved a lot of valuable results. The current solution is the finite element method and finite different method. The convection-diffusion equation is more closely related to human activities, especially complex physical processes. The behavior of many parameters in flow phenomena follows the convection-diffusion equation, such as momentum and heat. The convection-diffusion equation is also used to describe the diffusion process in environmental science, such as the pollutant transport in the atmosphere, oceans, lakes, rivers or groundwater. The research of the convection-diffusion equation is of great importance. Partial differential equation theory has important applications in the solution of the convection-diffusion equation. This chapter mainly talks about the application of the finite difference method in the solution of the convection-diffusion equation.

Keywords: Partial Differential Equations, Differential Format, Convection-Diffusion Equation, Finite Element Method.

1 Introduction

At present, the numerical solution of partial differential equations is mainly two categories: the finite difference method and the finite element method. The advantage of the finite element method is that the region boundaries are more flexible and the results are more accurate. But it also has disadvantages [1-3]. For example, it often requires us to solve the large banded sparse matrix and its computation and storage volume requirements are more difficult to achieve [4-7]. It is more difficult to achieve the implicit scheme and the workload in the preparation of the computer programs is larger [8]. As a result, these disadvantages hindered the further development and application of the finite element method [9, 10].

The finite difference method, as a traditional numerical method to solve partial differential equations, has achieved great success in the recent years [11, 12]. Researchers have achieved much good research result. The main idea of the finite difference method is to use a linear combination of discrete function value to substitute the derivative in order to achieve related difference format in differential equations [13-16]. By solving differential equations, we achieve the approximation of the solution of the differential equations. Our main purpose is to reduce the error and improve the accuracy by improving differential format [17, 18].

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2 The Finite Differential Method

Here is the initial value problem for hyperbolic equations and parabolic equations. Its solving region is $D_1 = \{(x,t) \mid -\infty < x < \infty, t \geq 0\}$.

We can draw two clusters of parallel lines in the upper half $x-t$ plane and have the upper half plane into a rectangular grid [19-22]. These lines can be called the grid lines and these intersections can be called the grid points or nodes. The lines paralleling the t axis are equidistant. We set the distance with $\Delta x > 0$ or h which is called space step. The lines paralleling x axis are not equidistant according to concrete problems. For simplicity, we assume that they are also equidistant. We set the distance with $\Delta t > 0$ sometimes τ . They are called time step. The two cluster grid lines can be written as such: $x = x_i = j\Delta x = jh, j = 0, \pm 1, \pm 2...$; $t = t_n = n\Delta t = n\tau, n = 0, 1, 2, \ldots$.

Mesh nodes (x_j, t_n) are sometimes abbreviated as (j, n).

Here is the initial boundary value problem for hyperbolic equations and parabolic equations. We assume the solving region is $D_1 = \{(x,t) \mid 0 < x < l, t \geq 0\}$.

The regional grid is constituted by straight lines paralleling t axis and x axis. They are $x = x_j, j = 0, 1, \ldots, J$, $t = t_n, n = 0, 1, 2, \ldots$.

In it, $x_i = j\Delta x = jh, \Delta x = h = \frac{l}{J}; t_n = n\Delta t = w\tau$.

Example3: Here is boundary value problem of elliptic equations. The solving region is a bounded domain D on the plane $x - y$ .Its boundary Γ is piecewise smooth curve. Take the steps which are along the x axis and the y axis and make two clusters of straight lines paralleling them.

$x = x_i = i\Delta x, i = 0, \pm 1, \pm 2, \ldots$, $y = y_i = j\Delta y, j = 0, \pm 1, \pm 2, \ldots$.

If the distance of two nodes along the x or y axis is only one step, the two nodes can be called two adjacent nodes. If a node's all four adjacent nodes belong to $D \cup \Gamma$, this node can be called an internal node. If a node's four adjacent does not belong to $D \cup \Gamma$, this node can be called boundary node.

A variety of finite difference methods for solving partial differential equations, using the series expansion method is the most commonly used method.

From the initial value of the convection equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, x \in R, t > 0 \tag{1}$$

$$u(x,0) = g(x), x \in R \tag{2}$$

And the diffusion equation initial value problem.

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}, x \in R, t > 0 \tag{3}$$

$$u(x,0)=g(x), x\in R \tag{4}$$

For discussion, assume that the $u(x,t)$ initial value problem of partial differential equations, the solution *Taylor* is sufficiently smooth progression commence there.

$$\begin{cases} \dfrac{u(x_j,t_{n+1})-u(x_j,t_n)}{\tau}=[\dfrac{\partial u}{\partial t}]_j^n+O(\tau) \\ \dfrac{u(x_j,t_{n+1})-u(x_j,t_{n-1})}{2\tau}=[\dfrac{\partial u}{\partial t}]_j^n+O(\tau^2) \\ \dfrac{u(x_{j+1},t_n)-u(x_j,t_n)}{h}=[\dfrac{\partial u}{\partial x}]_j^n+O(h) \\ \dfrac{u(x_j,t_n)-u(x_{j-1},t_n)}{h}=[\dfrac{\partial u}{\partial x}]_j^n+O(h) \\ \dfrac{u(x_{j+1},t_n)-u(x_{j-1},t_n)}{2h}=[\dfrac{\partial u}{\partial x}]_j^n+O(h^2) \end{cases} \tag{5}$$

$$\frac{u(x_{j+1},t_n)-2u(x_j,t_n)+u(x_{j-1},t_n)}{h^2}=[\frac{\partial^2 u}{\partial x^2}]_j^n+O(h^2) \tag{6}$$

Use type 1 and type 3 in (1.5):

$$\frac{u(x_j.t_{n+1})-u(x_j.t_n)}{\tau}+a\frac{u(x_{j+1}.t_n)-u(x_j.t_n)}{h}=[\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}]_j^n+o(\tau+h)$$

If $u(x,t)$ you meet the smooth solution of partial differential equations (1.1).

$[\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}]_j^n=0$ This can be seen, the partial differential equation (x_j,t_n) can be approximated in the Department with the following equation instead.

$$\frac{u_j^{n+1}-u_j^n}{\tau}+a\frac{u_{j+1}^n-u_j^n}{h}=0\ ,\ j=0,\pm1,\pm2,...,n=0,1,2,... \tag{7}$$

Where u_j^n is an $u(x_j,t_n)$ approximation? Finite difference equation (1.7) called the approximation of differential equations or difference equations can be rewritten into the form of easy calculation (1): $u_j^{n+1}=u_j^n-a\lambda(u_{j+1}^n-u_j^n)$.

Among them, $\lambda=\frac{\tau}{h}$ is known as grid.

Differential equation (1.7) coupled with the discrete form of the initial conditions (1.2).

$$u_j^0=\varphi_j\ ,\ j=0,\pm1,... \tag{8}$$

Time layer can advance, calculate the value of the layers, the differential equation (1.7) and (1.8) together constitute a differential format, advancing the first time layer to layer the first time, the formula (1.7) provides a direct calculation u_j^{n+1} of the expression of point by point, saying (1.7) the explicit form. The (5) and the type one and four style, you can get another differential equation (1.1) approximation of differential equations.

$$\frac{u_j^{n+1}+u_j^n}{\tau}+a\frac{u_j^n-u_{j-1}^n}{h}=0 \tag{9}$$

The first type of the Central and the fifth type (1.5), you can get another differential equation (1.1) approximation of differential equations.

$$\frac{u_j^{n+1}-u_j^n}{\tau}+a\frac{u_{j+1}^n-u_{j-1}^n}{2h}=0 \tag{10}$$

$$u_j^{n+1}=u_j^n-\frac{a\lambda}{2}(u_{j+1}^n-u_{j-1}^n) \tag{11}$$

Equation (1.10) is called the central difference scheme, (1.7) and (1.9) called eccentric differential format. Diffusion equation differential format using the same method can be constructed approximation.

$$\frac{u_j^{n+1}-u_j^n}{\tau}+a\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}=0\ ,\ j=0,\pm1,\pm2,\ldots,n=0,1,2,\ldots \tag{12}$$

Consider diffusion equation (3) by integrating this equation, first of all selected points region, and located in the plane *x*-*t*, the integral region:

$D=\{(x,t)\,|\,x_j-\frac{h}{2}\le x\le x_j+\frac{h}{2},t_n\le t\le t_{n+1}\}$, internalizes $\iint_D \frac{\partial u}{\partial t}dxdt=\iint_D a\frac{\partial^2 u}{\partial t^2}dxdt$

Direct quartered can be

$$\int_{x_j-\frac{h}{2}}^{x_j+\frac{h}{2}}[u(t_n+\tau,x)-u(t_n,x)]dx=\int_{t_n}^{t_{n+1}}[\frac{\partial u}{\partial x}(t,x_j+\frac{h}{2})-\frac{\partial u}{\partial x}(t,x_j-\frac{h}{2})]dt$$

Numerical integration

$$[u(t_n+\tau,x_j)-u(t_n,x_j)]h\approx a[\frac{\partial u}{\partial x}(t_n,x_j+\frac{h}{2})-\frac{\partial u}{\partial x}(t_n,x_j-\frac{h}{2})]\tau \tag{13}$$

The resulting $\frac{u_j^{n+1}-u_j^n}{\tau}+a\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}=0$. That is (1.11), the integral method is also called the finite volume method.

The previous structure of differential format u_j^{n+1} are explicit, in each time level t_{n+1} can be independently worth the time layer, but $\frac{u(x_i,t_n)-u(x_i,t_{n-1})}{\tau}=[\frac{\partial u}{\partial t}]_j^n+o(\tau)$ not always the case, if adopted and style (1.6), you can get another of the diffusion equation differential format (1.3).

$$\frac{u_j^n-u_j^n}{\tau}-a\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}=0 \tag{14}$$

The finite difference scheme contains more than one node in the new time level; this finite difference scheme called the implicit scheme, most of the implicit scheme is suitable for solving initial boundary value problem of differential equations or to satisfy the cycle conditions of initial value problem.
$\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2}, 0<x<t, t>0$, $u(x,0)=g(x), 0<x<t$, $u(0,t)=u(l,t)=0, t>0$, in it $a>0$.

Diffusion equation (14) approximation with differential format, the initial conditions (12) with discrete, the discrete boundary conditions $u_0^n=0, n>0$.

$$u_j^n=0, n>0 \tag{15}$$

In it $J=\frac{l}{h}$.

Order $U^n=(u_1^n,u_2^n,\ldots,u_{j-1}^n)^T$ r shall be written as such:

$$AU^n=U^{n-1} \tag{16}$$

In it $A=\begin{bmatrix} 1+2a\lambda & -a\lambda & & & \\ -a\lambda & 1+2a\lambda & -a\lambda & & \\ & \ddots & \ddots & \ddots & \\ & & -a\lambda & 1+2a\lambda & -a\lambda \\ & & & -a\lambda & 1+2a\lambda \end{bmatrix}$, A is strictly diagonally dominant.

As a result, (16) Solvability A is a tri-diagonal matrix, can be used to catch up method. It can be seen from above, using an explicit format and effort solving, implicit scheme for solving does not seem to benefit, but the future will see the implicit scheme can be a large time step, so there is a significant benefit.

3 Solutions of Convection Diffusion Equations

For its simple structure, it is most likely think of is a direct discrete time derivative forward difference quotient, the spatial derivatives using the central difference quotient to approximate the differential equations (1.17), the following differential format:

$$\frac{u_j^{n+1}-u_j^n}{\tau}+a\frac{u_{j+1}^n-u_{j-1}^n}{2h}=\varepsilon\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2} \tag{18}$$

This is the convection diffusion equation (1.17) center Explicit Difference Scheme. (1.18) can be rewritten as such:

$$u_j^{n+1}=u_j^n-\frac{1}{2}\lambda(u_{j+1}^n-u_{j-1}^n)+\mu(u_{j+1}^n-2u_j^n+u_{j-1}^n) \tag{19}$$

The center of the convection-diffusion equation (1.18) explicitly format the impact on the diffusion effect, reducing the diffusion effect, we can establish the explicit form of the correction center.

Full and smooth solution set $u(x,t)$ to the convection-diffusion equation (1.18), the following equation can be:

$$\frac{\partial^2 u}{\partial t^2}=\varepsilon^2\frac{\partial^4 u}{\partial x^4}-2\varepsilon a\frac{\partial^3 u}{\partial x^3}+a^2\frac{\partial^2 u}{\partial x^2} \tag{20}$$

$$\frac{\partial^3 u}{\partial t^3}=\varepsilon^3\frac{\partial^6 u}{\partial x^6}-3\varepsilon^2 a\frac{\partial^5 u}{\partial x^5}+3\varepsilon a^2\frac{\partial^4 u}{\partial x^4}-a^3\frac{\partial^3 u}{\partial x^3} \tag{21}$$

Use the series *Taylor* to expand the type (1.18) and combination of (1.20) and (1.21):

$$\begin{aligned}
&\frac{u(x_j,t_{n+1})}{\tau}+a\frac{u(x_{j+1},t_n)-u(x_{j-1},t_n)}{2h}-\frac{u(x_{j+1},t_n)-2u(x_j,t_n)+u(x_{j-1},t_n)}{h^2}\\
&=\frac{\partial u}{\partial t}+\frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}+\frac{\tau^2}{6}\frac{\partial^3 u}{\partial t^3}+o(\tau^3)+a\frac{\partial u}{\partial t}+\frac{1}{6}ah^2\frac{\partial^3 u}{\partial t^3}+o(h^4)\\
&-\varepsilon\frac{\partial^2 u}{\partial x^2}-\frac{1}{12}\varepsilon h^2\frac{\partial^4 u}{\partial t^4}+o(h^4)\\
&=\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial t}-\varepsilon\frac{\partial^2 u}{\partial x^2}+\frac{\tau}{2}[\varepsilon^2\frac{\partial^4 u}{\partial x^4}-2\varepsilon a\frac{\partial^3 u}{\partial x^3}+a^2\frac{\partial^2 u}{\partial x^2}]+\\
&\frac{\tau^2}{6}[\varepsilon^3\frac{\partial^6 u}{\partial x^6}-3\varepsilon^2 a\frac{\partial^5 u}{\partial x^5}+3\varepsilon a^2\frac{\partial^4 u}{\partial x^4}-a^3\frac{\partial^6 u}{\partial x^3}]+\frac{a}{6}h^2\frac{\partial^3 u}{\partial x^3}-\frac{\varepsilon}{12}h^2\frac{\partial^4 u}{\partial x^4}+\\
&o(\tau^3+h^4)=\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}-(\varepsilon-\frac{a^2}{2}\tau)\frac{\partial^2 u}{\partial x^2}+\frac{1}{6}\varepsilon h(1-6\lambda-\lambda^2)\frac{\partial^3 u}{\partial t^3}+\dots
\end{aligned} \tag{22}$$

Upwind Difference Scheme to better reflect the case of convection-dominated, but this format is only first order accuracy, in order to better solve the various characteristics of the convection-diffusion equation and fully reflect the differential equations, we need to construct more accurate differential format index hybrid finite difference scheme.

For general convection-diffusion equation to calculate the analytical solution is impossible, in the entire solution region, so we can turn to consider the Can the analytical solution in the local area to solve the answer is yes. The basic idea of the hybrid finite analytic method is: first on the unit of local subdivision boundary conditions to obtain the analytical solution of this unit, and secondly, the use of this analytical deconstruction to create a finite difference scheme. Using the finite difference scheme for the hybrid finite analytic method called hybrid finite difference scheme.

First, the spatial variables x and time variables t equidistant mesh and analyzed in any one subdivision unit $(x,t)\in[x_{j-1},x_{j+1}]\times[t_{n-1},t_{n+1}]$. Equations (1.17) in this unit with freeze coefficient method $\frac{\partial u}{\partial t}$, and even if $\frac{\partial u}{\partial t}=k$, as a constant, this time into a second order k constant coefficient ordinary differential equation (1.17):

$$K+a\frac{du}{dx}=\varepsilon\frac{d^2u}{dx^2} \tag{23}$$

The analytical solution of the derived type (1.23) on $[x_{j-1},x_{j+1}]$ as follows:

$$u=c_1e^{\lambda x}+c_2-\frac{k}{a}x \tag{24}$$

In it, $\lambda=\frac{a}{\varepsilon}$. c_1,c_2 are the undetermined coefficients.

Differential equations satisfy the boundary conditions

$$\begin{cases} u(x_{j-1})=c_1e^{\lambda(j-1)}h+c_2-\frac{k}{a}(j-1)h \\ u(x_{j+1})=c_1e^{\lambda(j+1)}h+c_2-\frac{k}{a}(j+1)h \end{cases} \tag{25}$$

Solve it and achieve that:

$$\begin{cases} c_1=\dfrac{u(x_{j+1})-u(x_{j-1})+2h\frac{k}{a}}{e^{\lambda(j-1)h}(e^{2\lambda h}-1)} \\ c_2=u(x_{j-1})-\dfrac{u(x_{j+1})-u(x_{j-1})+2h\frac{k}{a}}{e^{2\lambda h}-1}+\frac{k}{a}(j+1)h \end{cases} \tag{26}$$

Sub-statute c_1, c_2 into (24):

$$u(x_j) = \frac{u(x_{j+1}) + e^{\lambda h} u(x_{j-1})}{1 + e^{\lambda h}} - k \frac{h}{a} \cdot \frac{e^{\lambda h} - 1}{e^{\lambda h} + 1} \tag{27}$$

De formats it and achieves it:

$$k = \frac{a\{(1 + e^{\lambda h}) u(x_j) - [u(x_{j+1}) + e^{\lambda h} u(x_{j-1})]\}}{h(1 - e^{\lambda h})} \tag{28}$$

Continue to deformity it and achieve it:

$$k = -a \frac{u(x_{j+1}) - u(x_{j-1})}{2h} - \frac{a\lambda}{2} \frac{1 + e^{\lambda h}}{1 - e^{\lambda h}} \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} \tag{29}$$

Time level $(n+1)$ analysis (24), if it is the highest possible accuracy, the difference quotient instead of k using the center, we can construct the following differential format:

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\tau} + a \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2h} = -\frac{ah}{2} \frac{1 + e^{\lambda h}}{1 - e^{\lambda h}} \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} \tag{30}$$

Format (1.30) combined with the initial conditions can only get on the even-numbered time value, the value can not be the odd time; this must be considered separately on the odd layer format.

At that time $n = 1$, (1.30) in the following format instead of

$$\frac{u_j^{n+1} - u_j^{n}}{2\tau} + a \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2h} = -\frac{ah}{2} \frac{1 + e^{\lambda h}}{1 - e^{\lambda h}} \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} \tag{31}$$

And combine the value of (1.30) and (1.31) all nodes in the entire grid can be obtained.

Building process can be seen from the format, the time derivative using the central difference quotient in the space on the local unit to solve the second order constant coefficient differential equations, boundary conditions, approximate analytical solution. This method combines the finite difference method and analysis, to be called the index hybrid finite analytic method, the format can be called the index hybrid finite analytic format (1.31).

We can verify that the truncation error $o(\tau^2 + h^2)$ in this format, and is absolutely stable.

4 Summary

Convection diffusion equations have been used in several of fields in social life and have important applications. Its solution is not only the practical application's needs, but also an important element in the theory of academic study. As a result, the research of the convection diffusion equations has great value. For this reason, in recent decades, the convection-diffusion equation theory has gained importance and rapid development and its solution is also changing every day. This paper first introduces the basic knowledge and solutions of the partial differential equations, and then use these to solve some simple problems, and in the end sublimate the above methods and then to solve some complex problems and made a number of high-precision solutions in order to make the result better and more reasonable.

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