Analytical Approximation Solution for Differential Equations with Piecewise Constant Arguments of Alternately Advanced and Retarded Type

Qi Wang[1,*], Cui Guo[2], and Ruixiang Zeng[1]

[1] School of Applied Mathematics, Guangdong University of Technology, Guangzhou, 510006, China
qiwang@gdut.edu.cn
[2] College of Science, Harbin Engineering University, Harbin, 510006, China

Abstract. In this work, the variational iteration method is used for analytic treatment of differential equations with piecewise constant arguments of alternately advanced and retarded type. In order to prove the precision of the results, some comparisons are also made between the exact solutions and the results of the numerical method and the variational iteration method. The obtained results reveal that the method is very effective and convenient for constructing differential equations with piecewise constant arguments.

Keywords: Variational iteration method, Piecewise constant arguments, Analytical approximation solution, Lagrange multiplier.

1 Introduction

Differential equations with piecewise constant arguments (EPCA) have received extensive investigations [1-5]. In EPCA, the derivatives of the unknown functions depend on not just the time t at which they are determined, but on constant values of the unknown functions in certain intervals of the time t before t. These equations have the structure of continuous dynamical systems in intervals of unit length. Continuity of a solution at a point joining any two consecutive intervals implies a recursion relation for the values of the solution at such points. Therefore, they combine the properties of differential equations and difference equations.

EPCA has been under intensive investigation for the last twenty years. The theory of EPCA was initiated in [6, 7] and developed by many authors. The general theory and basic results for EPCA have been thoroughly investigated in the book of Wiener [8]. For more detailed information about analytical solution and numerical treatment of EPCA, the reader is referred to [9-15] and the references therein.

In this paper, we will apply the analytical approximation technique: the variational iteration method to the following EPCA:

---

[*] Corresponding author.

Y. Yang, M. Ma, and B. Liu (Eds.): ICICA 2013, Part II, CCIS 392, pp. 242–251, 2013.

$$y'(t)+ay(t)+by\left(\left[t+\frac{1}{2}\right]\right)=0, \quad y(0)=y_0, \tag{1}$$

where $a,b,y_0 \in \mathbf{R}$ and $[\cdot]$ denotes the greatest integer function. Since the argument deviation $t-[t+1/2]$ is negative in $[n+1/2,n+1)$ and positive in $[n,n+1/2)$, (1) is said to be of alternately advanced and retarded type. The main purpose of this paper is to extend the variational iteration method to find the approximate solution of (1).

Many different methods have recently introduced to solve nonlinear problems, such as the homotopy analysis method (HAM) [16, 17], the variational iteration method (VIM) [18, 19], the Adomian's decomposition method (ADM) [20], and homotopy perturbation method (HPM) [21]. The VIM is strongly and simply capable of solving a large class of linear or nonlinear differential equations without linearization or small perturbation and also it reduces the size of calculations. The variational iteration method, which proposed by He [22-24], was successfully applied to autonomous ordinary and partial differential equations [25-27]. Recently, it has been used to solve effectively, easily and accurately a large class of linear and nonlinear differential equations.

2 Preliminaries

In this section, we will introduce some definitions and results which will be used later.

2.1 EPCA of Alternately Advanced and Retarded Type

Definition 1 (see [8]). A solution of (1) on $[0,\infty)$ is a function $y(t)$ that satisfies the conditions

(i) $y(t)$ is continuous on $[0,\infty)$;

(ii) The derivative $y'(t)$ exists at each point $t\in[0,\infty)$, with the possible exception of the point $t=n+1/2$, $n=0,1,\cdots$, where one-sided derivatives exist;

(iii) Eq. (1) is satisfied on $[0,1/2)$ and each interval $[n-1/2,n+1/2)$.

Theorem 1 (see [8]). If $b\neq a/(e^{a/2}-1)$, then (1) *has on* $[0,\infty)$ *a unique solution*

$$y(t)=m(T(t))\lambda^{[t+1/2]}y_0,$$

where

$$m(t)=e^{at}+\left(e^{at}-1\right)a^{-1}b, T(t)=t-\left[t+\frac{1}{2}\right], \lambda=\frac{m(1/2)}{m(-1/2)}$$

Theorem 2 (see [8]). The zero solution of (1) *is asymptotically stable* for any given y_0 if and only if

$$-\frac{a\left(e^{a}+1\right)}{\left(e^{a/2}-1\right)^{2}}<b<-a, \text{ for } a>0,$$

$$b<-a \text{ or } b>-\frac{a\left(e^{a}+1\right)}{\left(e^{a/2}-1\right)^{2}}, \text{ for } a<0,$$

$$b<0 \text{ for } a=0.$$

2.2 Variational Iteration Method

The VIM is the general Lagrange method, in which an extremely accurate approximation at some special point can be obtained. Next, we will present the essential steps for using the VIM and the determination of the Lagrange multipliers.

Consider the following differential equation

$$Ly+Ny=g(t), \tag{2}$$

where L and N are linear and nonlinear operator, respectively, and $g(t)$ is the inhomogeneous term. According to VIM, we can write a correction functional as:

$$y_{n+1}(t)=y_n(t)+\int_0^t \lambda(s)[Ly_n(s)+N\tilde{y}_n(s)-g(s)]ds, \tag{3}$$

where λ is a general Lagrangian multiplier which can be identified optimally via integration by parts and the variational theory, and $\tilde{y}_n$ as a restricted variation which means $\delta\tilde{y}_n=0$. Having λ determined, an iteration formula, without restricted variation, should be used for the determination of the successive approximations $y_{n+1}(t)$ of the solution $y(t)$. The zero-th approximation y_0 can be any selective function. Consequently, the solution is given by

$$y(t)=\lim_{n\to\infty} y_n(t)\cdot \tag{4}$$

3 Applications

In this section, the VIM is successfully applied for solving a linear EPCA of alternately advanced and retarded type.

For (1), the correction functional reads

$$y_{n+1}(t)=y_n(t)+\int_0^t \lambda(s)\left(y_n^{'}(s)+ay_n(s)+b\tilde{y}_n\left(\left[s+\frac{1}{2}\right]\right)\right)ds. \tag{5}$$

Taking variational on both sides of (5), we have

$$\delta y_{n+1}(t)=\delta y_n(t)+\delta\int_0^t \lambda(s)\left(y_n^{'}(s)+ay_n(s)\right)ds$$

$$= (1+\lambda(t))\delta y_n(t) + \int_0^t \left(a\lambda(s) - \lambda'(s)\right)\delta y_n(s)ds ,$$

this yields the stationary conditions:

$$\begin{cases} 1+\lambda(t) = 0, \\ \lambda'(s) - a\lambda(s) = 0. \end{cases} \tag{6}$$

Thus

$$\lambda(s) = -e^{a(s-t)} , \tag{7}$$

so we obtain the following iteration formula

$$y_{n+1}(t) = y_n(t) - \int_0^t e^{a(s-t)} \left(y_n^{'}(s) + ay_n(s) + by_n\left(\left[s+\frac{1}{2}\right]\right)\right) ds , \tag{8}$$

and the following initial approximation is chosen

$$y_0(t) = y(0) = y_0,$$

so we have

$$y_0(t) = y_0,$$

$$y_1(t) = y_0(t) - \int_0^t e^{a(s-t)} \left(y_0^{'}(s) + ay_0(s) + by_0\left(\left[s+\frac{1}{2}\right]\right)\right) ds ;$$

$$y_2(t) = y_1(t) - \int_0^t e^{a(s-t)} \left(y_1^{'}(s) + ay_1(s) + by_1\left(\left[s+\frac{1}{2}\right]\right)\right) ds ;$$

$$y_3(t) = y_2(t) - \int_0^t e^{a(s-t)} \left(y_2^{'}(s) + ay_2(s) + by_2\left(\left[s+\frac{1}{2}\right]\right)\right) ds ;$$

....

$$y(t) = \lim_{n\to\infty} y_n(t) \cdot \tag{9}$$

During the process of computation, we find that the greatest integer function $[\cdot]$ brings us much trouble. To overcome it, we introduce a method: consider the above iteration formula in a series of intervals:

$$[0,1/2),[1/2,3/2),[3/2,5/2),\cdots ;[(2n-1)/2,(2n+1)/2) , \ n=1,2,\cdots$$

Following this way, each integral in iteration formulas can be easily computed. Therefore, the following theorem is obtained.

Theorem 3. The VIM solution of (1) can be determined by (9) with the iteration (8).

Thus, the discussion is as follows.

When $t \in [0,1/2)$,

$y_{1,0}(t) = y_0$,

$$y_{1,1}(t)=y_{1,0}(t)-\int_0^t e^{a(s-t)}\left(y_{1,0}'(s)+ay_{1,0}(s)+by_{1,0}\left(\left[s+\frac{1}{2}\right]\right)\right)ds=-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-at},$$

$$y_{1,2}(t)=y_{1,1}(t)-\int_0^t e^{a(s-t)}\left(y_{1,1}'(s)+ay_{1,1}(s)+by_{1,1}\left(\left[s+\frac{1}{2}\right]\right)\right)ds=-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-at}.$$

Follow this way, we can obtain

$$y_{1,n}(t)=-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-at},\quad n\geq 1.$$

When $t\in[1/2,3/2)$,

$$y_{2,0}(t)=y_{1,1}(\frac{1}{2})=-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-at},$$

$$y_{2,0}(t)=y_{1,1}(\frac{1}{2})=-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-a/2},$$

$$y_{2,1}(t)=y_{2,0}(t)-\int_0^t e^{a(s-t)}\left(y_{2,0}'(s)+ay_{2,0}(s)+by_{2,0}\left(\left[s+\frac{1}{2}\right]\right)\right)ds$$

$$=-\frac{b}{a}(1-e^{-at})\left(-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-a/2}\right),$$

$$y_{2,2}(t)=y_{2,1}(t)-\int_0^t e^{a(s-t)}\left(y_{2,1}'(s)+ay_{2,1}(s)+by_{2,1}\left(\left[s+\frac{1}{2}\right]\right)\right)ds$$

$$=\left(\frac{b}{a}\right)^2(1-e^{-a})(1-e^{a/2-at})\left(-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-a/2}\right),$$

$$y_{2,3}(t)=y_{2,2}(t)-\int_0^t e^{a(s-t)}\left(y_{2,2}'(s)+ay_{2,2}(s)+by_{2,2}\left(\left[s+\frac{1}{2}\right]\right)\right)ds$$

$$=\left(\frac{b}{a}\right)^2(1-e^{-a})\left(-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-a/2}\right)(\frac{b}{a}e^{-a/2}-\frac{b}{a}-\frac{b}{a}e^{-a/2-at}+\frac{a+b}{a}e^{-at}-e^{a/2-at}),$$

....

When $t\in[3/2,5/2)$,

$$y_{3,0}(t)=y_{2,1}(\frac{3}{2})=-\frac{b}{a}\left(-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-a/2}\right)(1-e^{-3a/2}),$$

$$y_{3,1}(t)=y_{3,0}(t)-\int_0^t e^{a(s-t)}\left(y_{3,0}'(s)+ay_{3,0}(s)+by_{3,0}\left(\left[s+\frac{1}{2}\right]\right)\right)ds$$

$$=\frac{b}{a^2}\left(-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-a/2}\right)(1-e^{-3a/2})(b-(a+b)e^{-at}),$$

$$y_{3,2}(t)=y_{3,1}(t)-\int_0^t e^{a(s-t)}\left(y_{3,1}'(s)+ay_{3,1}(s)+by_{3,1}\left(\left[s+\frac{1}{2}\right]\right)\right)ds$$

$$=\frac{b(a+b)}{a^2}\left(-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-a/2}\right)(e^{-at}-e^{-3a/2-at})(\frac{b}{a}(e^{a/2}-2e^{-a/2}+e^{at-2a})-1)$$

$$-\frac{b^2}{a^2}(1-e^{-3a/2})(e^{-at}+\frac{b}{a}-\frac{a+b}{a}e^{a/2-at})\left(-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-a/2}\right),$$

....

When $t\in[5/2,7/2)$

$$y_{4,0}(t)=y_{3,1}(\frac{5}{2})=\frac{b}{a^2}(-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-a/2})(1-e^{-3a/2})(b-(a+b)e^{-5a/2}),$$

$$y_{4,1}(t)=y_{4,0}(t)-\int_0^t e^{a(s-t)}(y_{4,0}^{'}(s)+ay_{4,0}+by_{4,0}([s+\frac{1}{2}]))ds$$

$$=\frac{b}{a^2}(-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-a/2})(1-e^{-3a/2})(b-(a+b)e^{-5a/2})(1-\frac{a+b}{a}(1-e^{-at})),$$

$$y_{4,2}(t)=y_{4,1}(t)-\int_0^t e^{a(s-t)}(y_{4,1}^{'}(s)+ay_{4,1}(s)+by_{4,1}([s+\frac{1}{2}]))ds$$

$$=\left\{1-\frac{a+b+1}{a}(1-e^{-at})+\frac{a+b}{a}(e^{-at}-e^{-2at})+a(a+b)e^{-at}-(a+b)[1-b(1-e^{3a})]t\right\}$$

$$\frac{b}{a^2}(-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-a/2})(1-e^{-3a/2})(b-(a+b)e^{-5a/2}),$$

....

When $t\in[7/2,9/2)$

$$y_{5,0}(t)=y_{4,1}(\frac{7}{2})=\frac{b}{a^2}(-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-a/2})(1-e^{-3a/2})(b-(a+b)e^{-5a/2})(1-\frac{a+b}{a}(1-e^{-7/2})),$$

$$y_{5,1}(t)=y_{5,0}(t)-\int_0^t e^{a(s-t)}(y_{5,0}^{'}(s)+ay_{5,0}(s)+by_{5,0}([s+\frac{1}{2}]))ds$$

$$=\frac{b}{a^2}(-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-a/2})(1-e^{-3a/2})(b-(a+b)e^{-5a/2})(1-\frac{a+b}{a}(1-e^{-7a/2}))(1-\frac{a+b}{a}(1-e^{-at})),$$

$$y_{5,2}(t)=y_{5,1}(t)-\int_0^t e^{a(s-t)}(y_{5,1}^{'}(s)+ay_{5,1}(s)+by_{5,1}([s+\frac{1}{2}]))ds$$

$$=\left\{1-(1-e^{-at})[1+\frac{b}{a}(1-\frac{a+b}{a}(1-e^{-4a})]+(a+b)te^{-at}\right\}\frac{b}{a^2}$$

$$(-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-a/2})(1-e^{-3a/2})(b-(a+b)e^{-5a/2})(1-\frac{a+b}{a}(1-e^{-7a/2})),$$

....

When $t\in[9/2,11/2)$

$$y_{6,0}(t)=y_{5,1}(\frac{9}{2})=\frac{b}{a^2}(-\frac{b}{a}y_0+\frac{a+b}{a}y_0e^{-a/2})(1-e^{-3a/2})(b-(a+b)e^{-5a/2})$$

$$(1-\frac{a+b}{a}(1-e^{-7a/2}))(1-\frac{a+b}{a}(1-e^{-9a/2})),$$

$$y_{6,1}(t) = y_{6,0}(t) - \int_0^t e^{a(s-t)}(y_{6,0}^{'}(s) + ay_{6,0} + by_{6,0}([s+\frac{1}{2}]))ds$$
$$= \frac{b}{a^2}(-\frac{b}{a}y_0 + \frac{a+b}{a}y_0e^{-a/2})(1-e^{-3a/2})(b-(a+b)e^{-5a/2})(1-\frac{a+b}{a}(1-e^{-7a/2}))$$
$$(1-\frac{a+b}{a}(1-e^{-9a/2}))(1-\frac{a+b}{a}(1-e^{-at})),$$
$$y_{6,2}(t) = y_{6,1}(t) - \int_0^t e^{a(s-t)}(y_{6,1}^{'}(s) + ay_{6,1}(s) + by_{6,1}([s+\frac{1}{2}]))ds$$
$$= [1-\frac{a+b}{a}(1-e^{-at}) + \frac{b(a+b)}{a^2}(1-e^{-5a})(1-e^{-at})]\frac{b}{a^2}(-\frac{b}{a}y_0 + \frac{a+b}{a}y_0e^{-a/2})$$
$$(1-e^{-3a/2})(b-(a+b)e^{-5a/2})(1-\frac{a+b}{a}(1-e^{-7a/2}))(1-\frac{a+b}{a}(1-e^{-9a/2})),$$

....

In a word, in the interval $[(2n-1)/2,(2n+1)/2)$, $n=1,2,\cdots$ we have the following iteration formulas:

$$y_{n+1,m}(t) = y_{n+1,m-1}(t) - \int_0^t e^{a(s-t)}\left(y_{n+1,m-1}^{'}(s) + ay_{n+1,m-1}(s) + by_{n+1,m-1}\left(\left[s+\frac{1}{2}\right]\right)\right)ds$$
$$= y_{n+1,m-1}(t) - \int_0^t e^{a(s-t)}\left(y_{n+1,m-1}^{'}(s) + ay_{n+1,m-1}(s)\right)ds - \int_0^{\frac{1}{2}} e^{a(s-t)}by_{n+1,m-1}(0)ds$$
$$- \int_{\frac{1}{2}}^{\frac{3}{2}} e^{a(s-t)}by_{n+1,m-1}(1)ds - \cdots - \int_{\frac{2n-1}{2}}^{t} e^{a(s-t)}by_{n+1,m-1}(n)ds.$$

In view of (9), we can obtain the analytical approximation solution. Usually, the $m+1$ th approximation is used for numerical purposes.

4 Numerical Simulation

In this part, we will present some examples to test the effectiveness of VIM and the correctness of our conclusions. The software we use is Matlab R2012a. All the figures are produced on it.

Let $a=1, b=-2$ and $y_0=1$ in (1). In Fig. 1 we compare the 7th VIM solution with the numerical solution of the θ-method [14] using $h=0.05$ and $\theta=0.6$. The graphs of the true solution and 8th approximate solution are shown in Fig. 2. Moreover, we also plot the 9th approximate solution and the numerical solution of the θ-method with $h=0.04$ and $\theta=0.7$ in Fig.3, the 10th approximate solution and the numerical solution of the θ-method with $h=0.02$ and $\theta=0.85$ in Fig.4. From these figures we can see that the higher approximation has better property than the lower approximation. Therefore, the VIM is useful for seeking the approximation solution of EPCA.

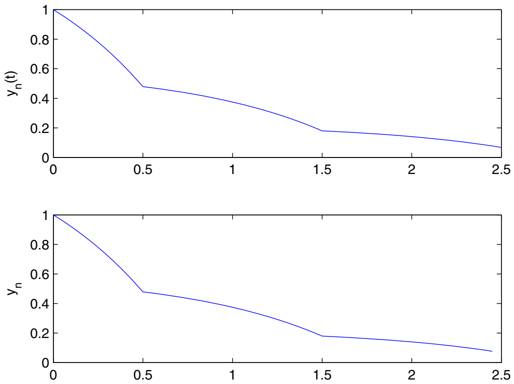

Fig. 1. A comparison between the 7th VIM solution (upper) and the numerical solution (lower)

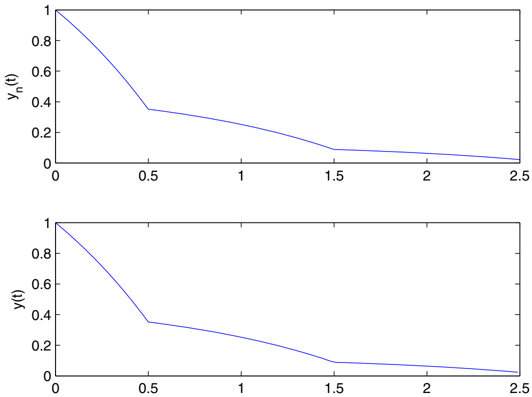

Fig. 2. A comparison between the 8th VIM solution (upper) and the true solution (lower)

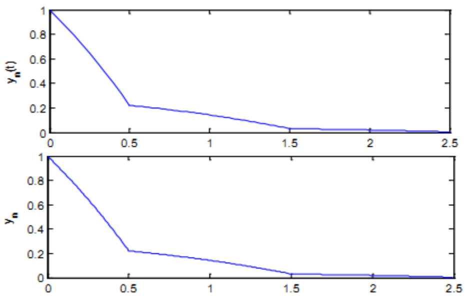

Fig. 3. A comparison between the 9th VIM solution (upper) and the true solution (lower)

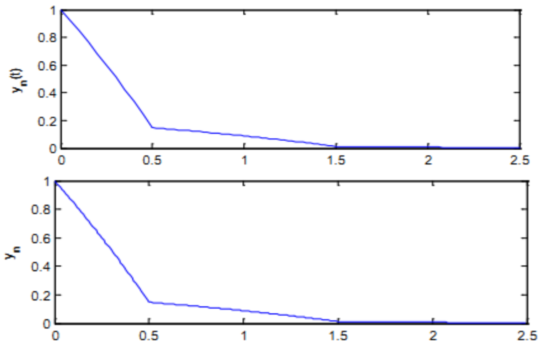

Fig. 4. A comparison between the 10th VIM solution (upper) and the true solution (lower)

5 Conclusions

The VIM has been successfully applied for solving linear EPCA of alternately advanced and retarded type. An illustrative example is solved exactly. The results reveal that the VIM is very effective and simple.

Acknowledgements. The authors are very grateful to the reviewers for their helpful suggestions. This work is supported by National Natural Science Foundation of China (No. 11201084).

References

1. Wiener, J., Debnath, L.: Partial Differential Equations with Piecewise Constant Delay. J. Comput. Int. J. Math. Math. Sci. 14, 485–496 (1991)
2. Nieto, J.J., Rodríguez-López, R.: Study of Solutions to Some Functional Differential Equations with Piecewise Constant Arguments. Abstr. Appl. Anal. 2012, 1–25 (2012)
3. Akhmet, M.U.: On the Reduction Principle for Differential Equations with Piecewise Constant Argument of Generalized Type. J. Math. Anal. Appl. 336, 646–663 (2007)
4. Seifert, G.: Second Order Neutral Delay Differential Equations with Piecewise Constant Time Dependence. J. Math. Anal. Appl. 281, 1–9 (2003)
5. Gurcan, F., Bozkurt, F.: Global Stability in a Population Model with Piecewise Constant Arguments. J. Math. Anal. Appl. 360, 334–342 (2009)
6. Shah, S.M., Wiener, J.: Advanced Differential Equations with Piecewise Constant Argument Deviations. Int. J. Math. Math. Sci. 6, 671–703 (1983)
7. Cooke, K.L., Wiener, J.: Retarded Differential Equations with Piecewise Constant Delays. J. Math. Anal. Appl. 99, 265–297 (1984)
8. Wiener, J.: Generalized Solutions of Functional Differential Equations. World Scientific, Singapore (1993)
9. Ivanov, A.F.: Global Dynamics of a Differential Equation with Piecewise Constant Argument. Nonlinear Anal. 71, e2384–e2389 (2009)
10. Akhmet, M.U., Yılmaz, E.: Impulsive Hopfield-Type Neural Network System with Piecewise Constant Argument. Nonlinear Anal.: RWA 11, 2584–2593 (2010)

11. Yuan, R.: On Favard's Theorems. J. Differential Equations 249, 1884–1916 (2010)
12. Lv, W.J., Yang, Z.W., Liu, M.Z.: Stability of Runge-Kutta Methods for the Alternately Advanced and Retarded Differential Equations with Piecewise Continuous Arguments. Comput. Math. Appl. 54, 326–335 (2007)
13. Liu, M.Z., Ma, S.F., Yang, Z.W.: Stability Analysis of Runge-Kutta Methods for Unbounded Retarded Differential Equations with Piecewise Continuous Arguments. Appl. Math. Comput. 191, 57–66 (2007)
14. Wang, Q., Zhu, Q.Y., Liu, M.Z.: Stability and Oscillations of Numerical Solutions of Differential Equations with Piecewise Continuous Arguments of Alternately Advanced and Retarded Type. J. Comput. Appl. Math. 235, 1542–1552 (2011)
15. Wang, Q., Zhu, Q.Y.: Stability Analysis of Runge-Kutta Methods for Differential Equations with Piecewise Continuous Arguments of Mixed Type. Int. J. Comput. Math. 88, 1052–1066 (2011)
16. Allan, F.M.: Construction of Analytic Solution to Chaotic Dynamical Systems Using the Homotopy Analysis Method. Chaos Solitons and Fractals 39, 1744–1752 (2009)
17. Liao, S.J.: Beyond Perturbation: Introduction to the Homotopy Analysis Method. CRC Press, Boca Raton (2004)
18. He, J.H., Wu, X.H.: Variational Iteration Method: New Development and Applications. Comput. Math. Appl. 54, 881–894 (2007)
19. He, J.H., Wu, G.C., Austin, F.: The Variational Iteration Method which Should be Followed. Nonlinear Sci. Lett. A 1, 1–30 (2010)
20. Al-Sawalha, M.M., Noorani, M.S.F., Hashim, I.: On Accuracy of Adomian Decomposition Method for Hyperchaotic Rössler System. Chaos Solitons and Fractals 40, 1801–1807 (2009)
21. He, J.H.: Homotopy Perturbation Method for Bifurcation of Nonlinear Problems. Int. J. Nonlinear Sci. Numer. Simu. 6, 207–208 (2005)
22. He, J.H.: Approximate Analytical Solution for Seepage Flow with Fractional Derivatives Inporous Media. Comput. Method Appl. Mech. Eng. 167, 57–68 (1998)
23. He, J.H.: Variational Iteration Method–A Kind of Non-linear Analytical Technique: Some Examples. Int. J. Nonlinear Mech. 34, 699–708 (1999)
24. He, J.H.: Variational Iteration Method for Autonomous Ordinary Differential Systems. Appl. Math. Comput. 114, 115–123 (2000)
25. He, J.H.: Variational Iteration Method-Some Recent Results and New Interpretations. J. Comput. Appl. Math. 207, 3–17 (2007)
26. Wazwaz, A.M.: The Variational Iteration Method for Analytic Treatment for Linear and Nonlinear ODEs. Appl. Math. Comput. 212, 120–134 (2009)
27. Ali, A.H.A., Raslan, K.R.: Variational Iteration Method for Solving Partial Differential Equations with Variable Coefficients. Chaos Solitons and Fractals 40, 1520–1529 (2009)