

# Remarks on Schur's Conjecture

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**Abstract.** Let  $P$  be a set of  $n > d$  points in  $\mathbb{R}^d$  for  $d \geq 2$ . It was conjectured by Schur that the maximum number of  $(d - 1)$ -dimensional regular simplices of edge length  $\text{diam}(P)$ , whose every vertex belongs to  $P$ , is  $n$ . We prove this statement under the condition that any two of the simplices share at least  $d - 2$  vertices. It is left as an open question to decide whether this condition is always satisfied. We also establish upper bounds on the number of all 2- and 3-dimensional simplices induced by a set  $P \subset \mathbb{R}^3$  of  $n$  points which satisfy the condition that the lengths of their sides belong to the set of  $k$  largest distances determined by  $P$ .

## 1 Introduction

The investigation of the distribution of distinct distances induced by a finite set of points in Euclidean space was initiated by Erdős in 1946. It has become a classical topic in discrete and computational geometry, with applications in combinatorial number theory, the theory of geometric algorithms, pattern recognition, etc. A typical problem in the area is Erdős' unit distance problem [2,11]: what is the maximum number of unit distance pairs among  $n$  points in  $\mathbb{R}^d$ ?

In the present paper, we concentrate on graphs of diameters. The *diameter graph*  $D(P)$  of a finite set of points  $P$  in  $\mathbb{R}^d$  is the graph whose vertex set is  $P$ , and two vertices are connected by an edge if and only if their distance is the diameter of  $P$ .

Throughout this paper,  $d$  will always denote an integer which is at least 2.

One of the basic properties of graphs of diameters was formulated by Erdős [2]: the maximum number of diameters among  $n$  points in the plane is  $n$ ., Erdős generously attributed the statement to Hopf and Pannwitz [4], who in fact proved a slightly different statement. In 3 dimensions, a similar result was conjectured by Vázsonyi and proved by Grünbaum [5], Heppes [6], and Straszewicz [12]: the maximum number of diameters generated by  $n > 3$  points in  $\mathbb{R}^3$  is  $2n - 2$ . In higher dimensions, the analogous problem turned out to have a different flavor: Lenz found some simple constructions with a quadratic number of diameters.

In [10], instead of counting the number of edges, Schur, Perles, Martini, and Kupitz initiated the investigation of the number of cliques in a graph of diameters. A *k-clique*, that is, a complete subgraph of  $k$  vertices, in the graph of diameters of  $P$  corresponds to a regular  $(k - 1)$ -dimensional simplex (or, in short,  $(k - 1)$ -*simplex*) of side length  $\text{diam}(P)$  generated by  $P$ .

**Theorem A (Schur et al.).** *Any finite subset  $P \subset \mathbb{R}^d$  contains the vertices of at most one regular  $d$ -simplex of edge length  $\text{diam}(P)$ .*

The main result in [10] is the following.

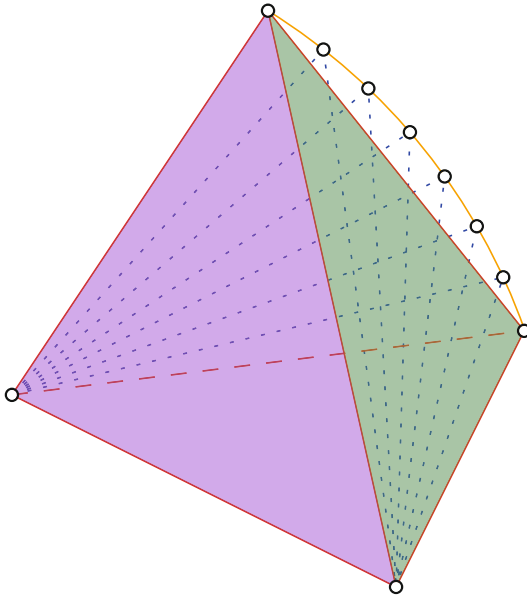
**Theorem B (Schur et al.).** *Any set  $P$  of  $n$  points in  $\mathbb{R}^3$  can generate at most  $n$  equilateral triangles of side length  $\text{diam}(P)$ .*

Theorem B can be regarded as another 3-dimensional generalization of the Hopf-Pannwitz result, according to which any set of  $n$  points in the plane has at most  $n$  diameters. It was conjectured by Z. Schur (see [10]) that this result can be extended to all dimensions  $d$ .

*Conjecture 1 (Schur).* The number of  $d$ -cliques in a graph of diameters on  $n$  points in  $\mathbb{R}^d$  is at most  $n$ .

The fact that this bound is tight for any  $n > d$  can be shown by the following simple construction given in [10]. Let  $p_0, p_1, \dots, p_d$  be the vertices of a regular  $d$ -simplex inscribed in the unit sphere. The edge length of the simplex is  $\lambda_d = \sqrt{2(1 + 1/d)}$ . Denote by  $c$  the center of the  $(d - 2)$ -simplex  $p_0 p_1 \dots p_{d-2}$ . Consider the circle centered at  $c$  and passing through  $p_{d-1}$  and  $p_d$ , and let  $p_{d+1}, p_{d+2}, \dots, p_{n-1}$  be arbitrary points on the short arc between  $p_{d-1}$  and  $p_d$  of the circle. It is not difficult to see that the set  $P = \{p_0, p_1, \dots, p_{n-1}\}$  has diameter  $\lambda_d$  and determines exactly  $n$  regular  $(d - 1)$ -simplices of edge length  $\lambda_d$ . Figure 1 illustrates the case  $d = 3$  of this construction.

In a recent manuscript Kupavskii proved Conjecture 1 for  $d = 4$ .



**Fig. 1.** Construction for  $d = 3$

We can prove Schur's conjecture for point sets satisfying a special condition.

**Theorem 1.** *The number of  $d$ -cliques in a graph of diameters on  $n$  vertices in  $\mathbb{R}^d$  is at most  $n$ , provided that any two  $d$ -cliques share at least  $d - 2$  vertices.*

We do not have any example violating the additional condition and we believe that, in fact, it holds for all graphs of diameters. However, we were unable to prove that it is true in general.

*Problem 1.* Is it true that any two unit regular  $(d - 1)$ -simplices in  $\mathbb{R}^d$  must share at least  $d - 2$  vertices, provided the diameter of their union is one?

This is vacuously true for  $d = 2$ . For  $d = 3$  it follows, e.g., from Dolnikov's theorem [1,14] (a direct proof is given in [10]), and it is open for  $d \geq 4$ . We cannot even verify that two simplices must share at least *one* vertex (for  $d \geq 4$ ), so this step would already be a breakthrough. We propose the following still weaker conjecture.

*Conjecture 2.* Given two unit regular  $(d - 1)$ -simplices in  $\mathbb{R}^d$  with  $d \geq 3$ , we can choose a vertex  $u$  of one simplex and a vertex  $v$  of the other one, so that  $|uv| \geq 1$ .

This is only known to be true for  $d = 3$ . Obviously, a positive answer to Problem 1 would imply Conjecture 2. It seems that regularity of the simplices is not a crucial condition in Conjecture 2, and the following stronger version may be true.

*Conjecture 3.* Let  $a_1 \dots a_d$  and  $b_1 \dots b_d$  be two  $(d - 1)$ -simplices in  $\mathbb{R}^d$  with  $d \geq 3$ , such that all their edges have length at least  $\alpha$ . Then there exist  $i, j \in \{1, \dots, d\}$  such that  $|a_i b_j| \geq \alpha$ .

In other words, given  $d$  red and  $d$  blue points, we can find a red-blue distance that is at least as large as the smallest monochromatic distance. We can ask another more general question, which is probably very hard.

*Problem 2.* For given  $d$ , characterize all pairs  $k, \ell$  of integers such that for any set of  $k$  red and  $\ell$  blue points we can choose a red point  $r$  and a blue point  $b$  such that  $|rb|$  is at least as large as the smallest distance between two points of the same color.

From an easy packing argument one can see that there is a good choice of  $r$  and  $b$ , whenever at least one of the numbers  $k$  and  $\ell$  is large enough. The following theorem is a first step towards Problem 2.

**Theorem 2.** *For any set of  $2k$  points  $a_1, \dots, a_k, b_1, \dots, b_k$  in  $\mathbb{R}^d$  the following inequality holds:*

$$\max\{|a_i b_j| : 1 \leq i, j \leq k\} \geq \min\{|a_i a_j|, |b_i b_j| : 1 \leq i < j \leq k\},$$

*provided that  $k \geq c \cdot \sqrt{d} \cdot 2^{\frac{3d}{2}}$  with a large enough absolute constant  $c$ .*

Some generalizations of Theorems A and B to graphs of the  $k$ -th largest distances were established in [9]. In this paper we show how these theorems can be extended to *non-regular* triangles in  $\mathbb{R}^3$  whose all sides are large (i.e., among the  $k$  largest distances). For a given finite set  $P \subset \mathbb{R}^3$ , we let  $d_1 > d_2 > \dots$  be all distinct inter-point distances generated by point pairs in  $P$ , so that by  $d_k$  we denote the  $k$ -th largest distance generated by  $P$ .

**Theorem 3.** *For any  $k \in \mathbb{N}$  there is a constant  $c_k$  such that the following holds: any set  $P$  of  $n$  points in  $\mathbb{R}^3$  can generate at most  $c_k n$  triangles whose all sides have length at least  $d_k$ .*

This can be viewed as a 3-dimensional analogue of the well-known observation by Vesztergombi: the number of pairs at distance  $d_k$  among  $n$  points in the plane is at most  $2kn$  (see [15]). The analogous statement for large non-regular  $(d-1)$ -simplices in  $\mathbb{R}^d$  probably holds for  $d \geq 4$  as well, but this is open.

The corresponding result for (not necessarily regular) tetrahedra with large edges in  $\mathbb{R}^3$  is somewhat weaker in the sense that the bound depends not only on  $k$ , but also on the given tetrahedron. We will see in Section 4 that this kind of dependence is necessary.

**Theorem 4.** *For any tetrahedron  $T$  and any  $k$  there is a constant  $c(T, k)$  such that the following holds: any finite set  $P$  of points in  $\mathbb{R}^3$  spans at most  $c(T, k)$  tetrahedra congruent to  $T$ , provided that all edges of  $T$  have length at least  $d_k$ .*

If Conjecture 3 holds, then Theorem 4 can be generalized to higher dimensions. As for the planar case, it is an easy exercise to show that, for every  $k$ , there is a constant  $c_k$  such that any finite set of points in the plane spans at most  $c_k$  triangles, whose all sides have length at least  $d_k$ .

## 2 Proof of Theorem 1

We start with two lemmas that are borrowed from [13], where they are attributed to [8].

**Lemma 1 (Kupitz et al.).** *Let  $a, b, c, d$  be points on a 2-sphere of radius at least  $1/\sqrt{2}$  such that  $\text{diam}\{a, b, c, d\} = 1$  and  $|ab| = |cd| = 1$ . Then the short great circle arcs  $ab$  and  $cd$  must intersect.*

The maximum number of diameters in a finite set of points on a 2-sphere is the same as in the plane, as long as the radius of the sphere is large enough, compared to the diameter of the set.

**Lemma 2 (Kupitz et al.).** *Let  $S$  be a 2-sphere of radius at least  $1/\sqrt{2}$  in  $\mathbb{R}^3$ . If a set of  $n$  points on  $S$  has diameter 1, then the diameter occurs at most  $n$  times.*

Next, we establish Theorem 1, which says that Schur's conjecture (Conjecture 1) holds, provided that the given graph of diameters satisfies an additional condition: *any two  $d$ -cliques share at least  $d-2$  vertices.*

*Proof of Theorem 1.* Assume without loss of generality that the diameter of our set is equal to 1. We can also assume that every vertex belongs to at least two  $d$ -cliques, since otherwise we can proceed by induction. We start with several geometric observations.

Note that the vertices of a  $d$ -clique represent  $d$  affinely independent points, so their affine hull is  $(d - 1)$ -dimensional, i.e., a hyperplane. Therefore, the affine hull of the  $d$  vertices divides the space into two half-spaces.

We will use the expression *angle*  $uvw$  and notation  $\alpha(u, v, w)$  to refer to the following set of points:

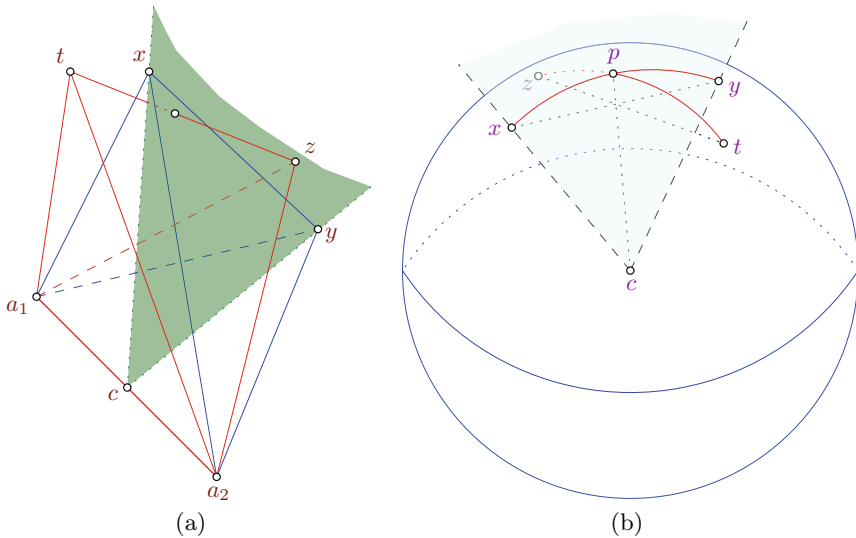
$$\alpha(u, v, w) = \{ \mu_1(u - v) + \mu_2(w - v) : \mu_1, \mu_2 \geq 0 \}.$$

**Lemma 3.** *If two  $d$ -cliques  $a_1 \dots a_{d-2}xy$  and  $a_1 \dots a_{d-2}zt$  share exactly  $d - 2$  vertices, then the open segment  $zt$  has exactly one common point with  $\text{aff}(a_1, \dots, a_{d-2}, x, y)$ , which lies inside  $\alpha(x, c, y)$ , where  $c = \frac{a_1 + \dots + a_{d-2}}{d-2}$  is the center of gravity of  $a_1 \dots a_{d-2}$ .*

*Proof.* Since  $|a_i x| = |a_i y| = |a_i z| = |a_i t| = 1$  for all  $i = 1, \dots, d - 2$ , and

$$|cx| = |cy| = |cz| = |ct| = \sqrt{\frac{d-1}{2(d-2)}},$$

we know that the points  $x, y, z, t$  lie on a 2-sphere with center  $c$  and radius  $\geq 1/\sqrt{2}$  (Figures 2(a), 2(b)). Hence, we can apply Lemma 1 to the points  $x, y, z, t$  to conclude that the arcs  $xy$  and  $zt$  intersect at some point  $p$ . But then the segment  $cp$  is contained in  $\alpha(x, c, y)$  and it is intersected by the open segment  $zt$ . Therefore, the



**Fig. 2.** Proof of Theorem 1, Lemma 3

open segment  $zt$  intersects  $\text{aff}(a_1, \dots, a_{d-2}, x, y)$  at a point which lies in  $\alpha(x, c, y)$ , and in no other point, since otherwise the two  $d$ -cliques would lie in the same hyperplane and would necessarily coincide by Theorem A.  $\square$

**Lemma 4.** *There are no three  $d$ -cliques that share a  $(d - 1)$ -clique.*

*Proof.* Suppose the contrary: let  $a_1 \dots a_{d-1}x$ ,  $a_1 \dots a_{d-1}y$  and  $a_1 \dots a_{d-1}z$  be three  $d$ -cliques. Denote by  $c$  the center of gravity for  $a_1, \dots, a_{d-1}$ . Then the points  $x, y, z$  lie on the circle with center  $c$  and radius  $\sqrt{\frac{d}{2(d-1)}}$ , that is orthogonal to  $\text{aff}(a_1, \dots, a_{d-1})$ . Since the radius of the circle is at least  $1/\sqrt{2}$ , we have that  $\angle xcy, \angle ycz, \angle zcx \leq \frac{\pi}{2}$ . Hence, the points  $x, y, z$  lie on a half-circle and we can assume without loss of generality that  $y$  is between  $x$  and  $z$ . Note that  $|xy|, |yz| < 1$  and the points  $x$  and  $z$  lie on different sides of  $\text{aff}(a_1, \dots, a_{d-1}, y)$ . According to our initial assumption, there is at least one  $d$ -clique  $C$  containing  $y$  apart from  $a_1 \dots a_{d-1}y$ . Since  $C$  shares at least  $d - 2$  points with each of the cliques  $a_1 \dots a_{d-1}x$ ,  $a_1 \dots a_{d-1}y$  and  $a_1 \dots a_{d-1}z$  and, moreover,  $C$  cannot contain  $x$  or  $z$ , we conclude that  $C$  contains exactly  $d - 2$  of the points  $a_1, \dots, a_{d-1}$ . Without loss of generality, let  $C = ya_1 \dots a_{d-2}u$  and let  $u$  lie on the same side of  $\text{aff}(a_1, \dots, a_{d-1}, y)$  as  $x$ . Now, because of Lemma 3, the open segment  $a_{d-1}z$  contains a point from  $\alpha(u, c', y)$ , where  $c'$  is the center of gravity for  $a_1, \dots, a_{d-2}$ . However, the whole set  $\alpha(u, c', y)$  lies in the closed half-space that contains  $x$ , while the open segment  $a_{d-1}z$  lies entirely in the open half-space that contains  $z$ . This is a contradiction.  $\square$

It turns out that the above geometric observations provide enough information so that the proof can be finished more or less combinatorially. We distinguish two cases.

Case 1. *There is a  $(d + 1)$ -clique  $a_1 \dots a_{d+1}$ .*

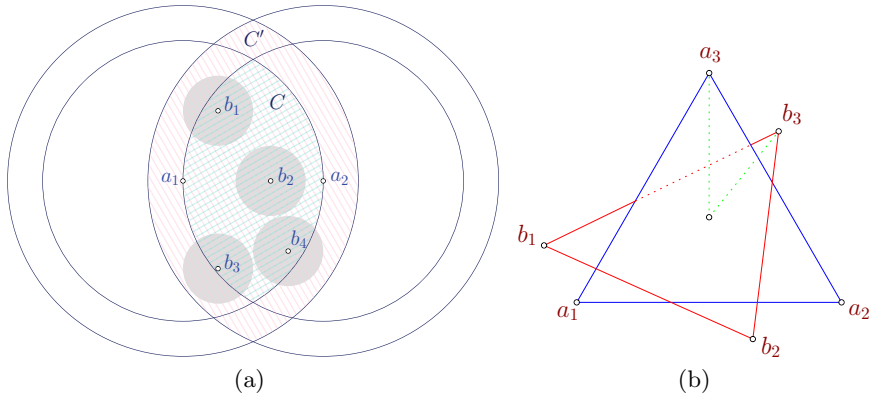
Suppose there is a  $d$ -clique  $C$  that contains a vertex  $x \notin \{a_1, \dots, a_{d+1}\}$ . By the assumption,  $C$  shares  $d - 2$  vertices with the clique  $a_1 \dots a_d$ , so we can assume that  $C$  contains  $a_1, \dots, a_{d-2}$ . But  $C$  also shares  $d - 2$  vertices with the clique  $a_2 \dots a_{d+1}$ , so we can also assume that  $C$  contains  $a_{d-1}$ . Therefore,  $C = a_1 \dots a_{d-1}x$ . Thus, we have three  $d$ -cliques containing  $a_1, \dots, a_{d-1}$ : namely,  $a_1 \dots a_d$ ,  $a_1 \dots a_{d-1}a_{d+1}$  and  $C$ . This is forbidden by Lemma 4. Hence we conclude that all  $d$ -cliques must be contained in  $a_1 \dots a_{d+1}$ , which gives us at most  $d + 1$  cliques, so in this case the statement is proven, since  $n \geq d + 1$ .

Case 2. *There is no  $(d + 1)$ -clique.*

We have two subcases.

Subcase 2.1 *There are two  $d$ -cliques that share  $d - 1$  vertices.*

Let the cliques be  $a_1 \dots a_{d-1}x$  and  $a_1 \dots a_{d-1}y$ . Observe that  $|xy| < 1$ , since we assume there is no  $(d + 1)$ -clique. If there are no more  $d$ -cliques except for those generated by  $a_1, \dots, a_{d-1}, x, y$ , we are done. So we can suppose that there are some more  $d$ -cliques. Any new  $d$ -clique shares  $d - 2$  points both with  $a_1 \dots a_{d-1}x$  and



**Fig. 3.** (a) Proof of Theorem 2; (b) construction for  $d = 4$ : two equilateral triangles in two orthogonal planes with a common center at the origin

with  $a_1 \dots a_{d-1}y$ . Hence, any new clique contains *exactly*  $d - 2$  of the vertices  $a_1, \dots, a_{d-1}$ . We say that a  $d$ -clique is of *type*  $k$  if it contains all the vertices  $a_1, \dots, a_{d-1}$  except for  $a_k$ . Now we will again branch out into different cases.

First, let us see what happens if all  $d$ -cliques have the same type, e.g., they all contain the points  $a_1, \dots, a_{d-2}$ . The remaining two vertices of any  $d$ -clique must lie on the 2-sphere with center  $\frac{a_1 + \dots + a_{d-2}}{d-2}$  and radius  $\sqrt{\frac{d-1}{2(d-2)}} > \frac{1}{\sqrt{2}}$ . Thus, the number of  $d$ -cliques is no more than the number of unit-diameters among  $n - (d - 2)$  points on a 2-sphere of radius  $> 1/\sqrt{2}$ , which is at most  $n - (d - 2)$ , by Lemma 2.

Therefore, we can assume that there are at least two  $d$ -cliques of different types. Any two cliques of different types share exactly  $d - 3$  vertices among  $a_1, \dots, a_{d-1}$ , so they must share at least one more vertex. Again, we consider different cases.

Suppose there are two  $d$ -cliques of different types that share a vertex  $v$  outside of  $\{a_1, \dots, a_{d-1}, x, y\}$ . Let the cliques be  $a_1 \dots a_{d-2}uv$  and  $a_2 \dots a_{d-1}vw$ . Clearly,  $a_1 \dots a_{d-1}v$  is also a  $d$ -clique, so we have three  $d$ -cliques sharing  $d - 1$  points  $a_1, \dots, a_{d-1}$ , which is impossible, according to Lemma 4.

The second possibility that remains is that any two cliques of different types contain  $x$  or  $y$ . This means that either all cliques (apart from the initial two) contain  $x$  or all of them contain  $y$ . Without loss of generality, let all new cliques contain  $x$ . Notice that there can be at most one clique of each type, for if  $C_1$  and  $C_2$  were  $d$ -cliques of the same type, say, type 1, there would be three  $d$ -cliques sharing  $d - 1$  points  $x, a_2, \dots, a_{d-1}$ , contrary to Lemma 4. Consequently, in this case we have at most  $d + 1$  cliques, and the total number of vertices is at least  $d + 2$ .

Subcase 2.2 *Any two  $d$ -cliques share at most  $d - 2$  vertices.*

Let  $a_1 \dots a_{d-2}xy$  and  $a_1 \dots a_{d-2}zt$  be two  $d$ -cliques. None of the points  $x$  and  $y$  forms a diameter with any of the points  $z$  and  $t$ , since it would produce two  $d$ -cliques that share  $d - 1$  vertices. If all other cliques contain  $a_1, \dots, a_{d-2}$ , we are done as above, so without loss of generality suppose that there is a  $d$ -clique  $a_1 \dots a_{d-3}xuv$ . Clearly,  $u, v$  are new points, i.e., different from  $a_1, \dots, a_{d-2}, x, y, z, t$ . But now  $a_1 \dots a_{d-3}xuv$  and  $a_1 \dots a_{d-2}zt$  have only  $d - 3$  points in common, contradicting the assumption.

We have proved that  $n$  is an upper bound for the number of  $d$ -cliques. A construction from [10] showing that this bound can be achieved is given in Introduction. This completes the proof of Theorem 1. □

**Remark.** If the statement from Problem 1 is true, then Theorem 1 would confirm Schur's conjecture. The following weaker statement might be easier to prove: *There is a constant  $K(d)$  such that among any  $K(d)$  cliques in a graph of diameters, there are two cliques sharing a vertex.* If true, this would give a bound of the form  $k(d) \cdot n$  for Schur's conjecture. However, it appears that even this weaker form requires a new insight.

It is natural to extend Problem 1 to cliques that might have fewer than  $d$  vertices. In particular, is it true that a  $d$ -clique and a  $(d - 1)$ -clique in a graph of diameters in  $\mathbb{R}^d$  must share a vertex? For  $d = 2$  and  $d = 3$ , this is clearly false. It is also false in  $\mathbb{R}^4$ , as shown by the following construction (for  $k = 2$ ).

**Proposition 1.** *For every  $k \geq 2$ , there exist a unit regular  $(2k - 1)$ -simplex and a unit regular  $k$ -simplex in  $\mathbb{R}^{2k}$  that do not share a vertex, while the diameter of their union is 1.*

*Proof.* Consider a unit regular  $(2k - 1)$ -simplex  $\Delta = a_1 \dots a_{2k}$  in  $\mathbb{R}^{2k}$  and let  $u_1, \dots, u_k$  be the midpoints of the edges  $a_1a_2, a_3a_4, \dots, a_{2k-1}a_{2k}$ , respectively. Let the origin  $o = (0, \dots, 0)$  be the center of the simplex  $\Delta$  and let the simplex lie in the hyperplane  $x_{2k} = 0$ . For every  $n \geq 1$ , denote by  $r_n$  the circumradius of a unit regular  $n$ -simplex. We have that  $r_n = \sqrt{\frac{n}{2n+2}}$ . Denote by  $v_1, \dots, v_k$  the points such that  $|ov_i| = r_{k-1}$  and  $u_i$  lies on the segment  $ov_i$  for  $i = 1, 2, \dots, k$ . Then  $v_1v_2 \dots v_k$  is a unit regular  $(k - 1)$ -simplex with center  $o$ . Translate the points  $v_1, \dots, v_k$  by the vector  $(0, \dots, 0, \sqrt{\frac{3-2\sqrt{2}}{4k}})$  to get points  $w_1, \dots, w_k$ , and let  $w_{k+1} = (0, \dots, 0, \sqrt{\frac{3-2\sqrt{2}}{4k}} - \sqrt{\frac{k+1}{2k}})$ . Now it is not difficult to verify that  $\tilde{\Delta} = w_1 \dots w_k w_{k+1}$  is a unit regular  $k$ -simplex and that the pair of simplices  $\Delta$  and  $\tilde{\Delta}$  satisfies the needed conditions (we omit the straightforward calculation). □

The question whether a  $d$ -clique and a  $(d - 1)$ -clique in a graph of diameters in  $\mathbb{R}^d$  must share a vertex remains open for  $d \geq 5$ .



### 3 Proof of Theorem 2

*Proof of Theorem 2.* Suppose the contrary, i.e., that the maximum is strictly smaller than the minimum, while  $k \geq c \cdot \sqrt{d} \cdot 2^{\frac{3d}{2}}$  for a large enough  $c$ . Without loss of generality, we assume that

$$\min\{|a_i a_j|, |b_i b_j| : 1 \leq i < j \leq k\} = 1$$

and  $|a_1 a_2| = 1$ . Denote by  $C$  the intersection of two balls with centers  $a_1$  and  $a_2$  and radius 1 (Figure 3(a)). Then  $C$  contains all the points  $b_1, \dots, b_k$ . Since  $|b_i b_j| \geq 1$ , the balls centered at  $b_1, \dots, b_k$  with radii  $\frac{1}{2}$  do not overlap. Moreover, all these balls are contained in  $C'$ , which is the intersection of the balls with centers  $a_1$  and  $a_2$  and radius  $\frac{3}{2}$ . Let us estimate the volume of  $C'$ . Using the fact that the volume of a spherical cap of height  $h$  is

$$\frac{\pi^{\frac{d-1}{2}} r^d}{\Gamma\left(\frac{d+1}{2}\right)} \int_0^{\arccos \frac{r-h}{r}} \sin^d(t) dt,$$

where  $r$  is the radius of the sphere, we get

$$\begin{aligned} \text{Vol}(C') &= 2 \cdot \frac{\pi^{\frac{d-1}{2}} (3/2)^d}{\Gamma\left(\frac{d+1}{2}\right)} \int_0^{\arccos \frac{1}{3}} \sin^d(t) dt \\ &\leq 2 \cdot \frac{\pi^{\frac{d-1}{2}} (3/2)^d}{\Gamma\left(\frac{d+1}{2}\right)} \cdot \left(\frac{2\sqrt{2}}{3}\right)^d \cdot \arccos \frac{1}{3} = O\left(\frac{(2\pi)^{\frac{d}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}\right). \end{aligned}$$

But  $C'$  contains  $k$  non-overlapping balls of radius  $\frac{1}{2}$ , and, therefore,

$$k \cdot \frac{\pi^{\frac{d}{2}} 2^{-d}}{\Gamma\left(1 + \frac{d}{2}\right)} \leq O\left(\frac{(2\pi)^{\frac{d}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}\right).$$

Finally, taking into account the asymptotics  $\Gamma(x) \sim x^{x-\frac{1}{2}} e^{-x} \sqrt{2\pi}$ , we obtain  $k = O(\sqrt{d} \cdot 2^{3d/2})$ , with a contradiction, as long as  $c$  is large enough.  $\square$

**Remark.** On the other hand, we know that Theorem 2 does not hold with  $k \leq \lceil \frac{d+1}{2} \rceil$ . To see this, consider the following construction. Let  $a_1 \dots a_k$  be a regular  $(k-1)$ -dimensional simplex inscribed in the sphere

$$\{(x_1, \dots, x_d) : x_1^2 + \dots + x_{k-1}^2 = 1, x_k = \dots = x_d = 0\}$$

and let  $b_1 \dots b_k$  be a regular  $(k-1)$ -dimensional simplex inscribed in the sphere

$$\{(x_1, \dots, x_d) : x_k^2 + \dots + x_{2k-2}^2 = 1, x_1 = \dots = x_{k-1} = 0\}.$$

Then  $|a_i a_j| = |b_i b_j| = \sqrt{\frac{2k}{k-1}}$  for all  $i \neq j$ , while  $|a_i b_j| = \sqrt{2}$  (Figure 3(b)).

Thus, the smallest  $k(d)$  for which Theorem 2 holds is somewhere between  $d/2$  and  $c\sqrt{d} \cdot 2^{\frac{3d}{2}}$ . The gap is obviously quite large, and Conjecture 3 suggests the answer should be closer to the lower bound.

## 4 Proofs of Theorems 3 and 4

The proofs of Theorem 3 and Theorem 4 are both analogous to the proofs of the corresponding statements for regular simplices given in [9], with the only new ingredient being the next lemma.

**Lemma 5.** *Let  $a_1a_2a_3$  and  $b_1b_2b_3$  be two triangles in  $\mathbb{R}^3$  such that all their sides have length at least  $\alpha$ . Then there exist  $i, j \in \{1, 2, 3\}$  such that  $|a_ib_j| \geq \alpha$ .*

*Proof.* Suppose the contrary, i.e., that the two triangles are placed so that  $|a_ib_j| < \alpha$  for all  $i$  and  $j$ . Without loss of generality, let  $a_1a_2a_3$  lie in the plane  $x_3 = 0$ . By the pigeon hole there are two vertices of  $b_1b_2b_3$  that lie on the same side of  $x_3 = 0$ . Without loss of generality, let  $b_1$  and  $b_2$  lie in the half-space  $x_3 \geq 0$  and let  $b_1 = (0, 0, p)$  and  $b_2 = (0, q, r)$ , where  $p, q, r$  are non-negative and  $r \geq p$  (Figure 4). Translate the points  $b_1$  and  $b_2$  by the vector  $(0, 0, -p)$  to get new points  $c_1 = (0, 0, 0)$  and  $c_2 = (0, q, r - p)$ . Note that  $|c_1c_2| = |b_1b_2| \geq \alpha$  and  $|c_ia_j| \leq |b_ib_j| < \alpha$  for all  $i \in \{1, 2\}, j \in \{1, 2, 3\}$ . It follows that the points  $a_1, a_2, a_3$  must have non-negative second coordinates. Now we rotate the point  $c_2$  around  $c_1$  in the plane  $x_1 = 0$  until it hits the plane  $x_3 = 0$ . Thus, we replace  $c_2$  by  $c'_2 = (0, s, 0)$ , where  $s = \sqrt{q^2 + (r - p)^2}$ . Again,  $|c_1c'_2| = |c_1c_2| \geq \alpha$  and the distances between  $c'_2$  and  $a_j$  for  $j \in \{1, 2, 3\}$  are all smaller than  $\alpha$ . Indeed, letting  $a_j = (t, u, 0)$ , we have

$$|c'_2a_j| = \sqrt{t^2 + (u - s)^2} \leq \sqrt{t^2 + (q - u)^2 + (r - p)^2} = |c_2a_j| < \alpha,$$

where we used that  $u \geq 0$  and  $q \leq s$ .

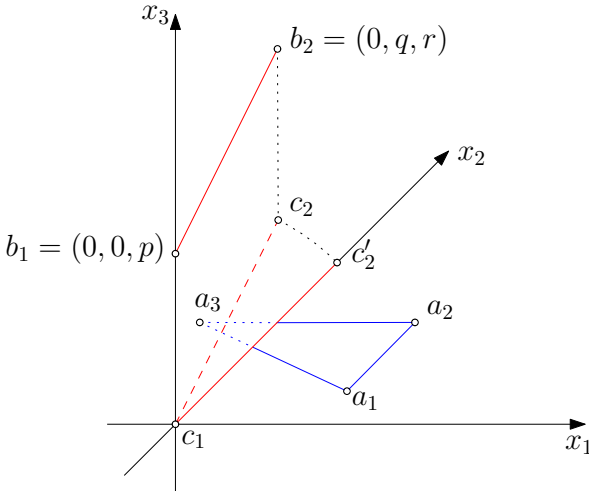
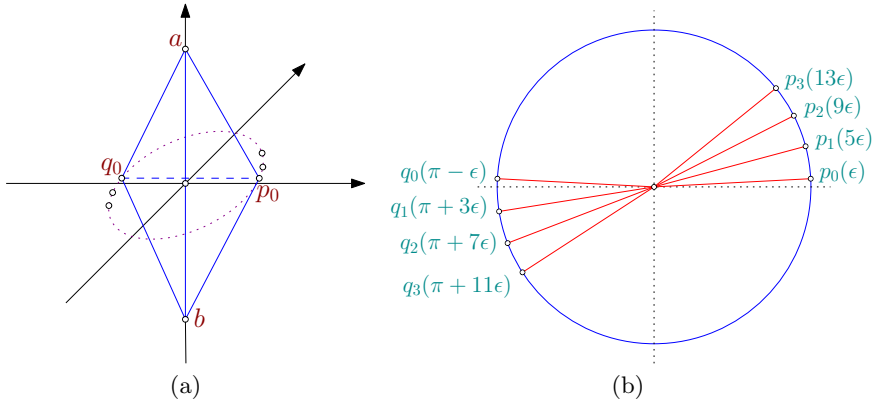


Fig. 4. Proof of Lemma 5

The points  $c_1, c'_2, a_1, a_2, a_3$  lie in the same plane and segment  $c_1c'_2$  can intersect at most two sides of triangle  $a_1a_2a_3$  at their interior points. So, without loss of generality, assume that  $c_1c'_2$  does not intersect  $a_1a_2$  at an interior point. Then either  $c_1, c'_2, a_1, a_2$  are in convex position or an extension of one of the segments  $c_1c'_2$  and  $a_1a_2$  intersects the other one. In either case one can easily show that one of the segments  $c_1a_1, c_1a_2, c'_2a_1, c'_2a_2$  has length at least  $\min\{|c_1c'_2|, |a_1a_2|\} \geq \alpha$ . Contradiction.  $\square$



**Fig. 5.** (a) Construction with many congruent large non-regular simplices; (b) points in the plane  $x_3 = 0$

**Remark.** Note that some dependence on  $T$  is necessary in Theorem 4, as shown by this simple construction. Take two points  $a = (0, 0, 1)$ ,  $b = (0, 0, -1)$ , and  $2n$  points in the plane  $x_3 = 0$  on the circle  $x_1^2 + x_2^2 = 1/4$  with polar coordinates as follows:

$$p_i = \left(\frac{1}{2}, (1 + 4i)\epsilon\right), \quad q_i = \left(\frac{1}{2}, \pi + (4i - 1)\epsilon\right),$$

for  $i = 0, 1, \dots, n - 1$  and small enough  $\epsilon > 0$  (Figures 5(a),5(b)). In this set of  $2n + 2$  points we have that

$$d_1 = |ab| = 2, d_2 = |ap_i| = \frac{\sqrt{5}}{2} \quad \text{and} \quad d_3 = |p_iq_i| = \sqrt{\frac{1}{2} + \frac{1}{2} \cos(2\epsilon)} < 1.$$

Recall that the distance between the points  $(r_1, \theta_1)$  and  $(r_2, \theta_2)$  in polar coordinates is equal

$$\sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)}.$$

Also, we can check that for all  $i, j$  we have

$$|p_iq_j| = \sqrt{\frac{1}{2} + \frac{1}{2} \cos((4(j - i) - 2)\epsilon)} \leq \sqrt{\frac{1}{2} + \frac{1}{2} \cos(2\epsilon)} = d_3,$$

since  $|4(j-i) - 2| \geq 2$ . It remains to notice that the chosen points span  $2n - 1$  tetrahedra with edge lengths  $d_1, d_2, d_2, d_2, d_2, d_3$ . Those are the tetrahedra  $abp_iq_j$  for all  $i, j \in \{0, 1, \dots, n-1\}$  such that  $j-i \in \{0, 1\}$ . Thus, for  $k = 3$  we can have an arbitrarily large number of tetrahedra whose all edges have lengths at least  $d_k$ .

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