Tight Bound on the Diameter of the Knödel Graph

Hayk Grigoryan and Hovhannes A. Harutyunyan

Department of Computer Science and Software Engineering, Concordia University, Montreal, Quebec, Canada H3G 1M8
h_grig@encs.concordia.ca, haruty@cse.concordia.ca

Abstract. The Knödel graph $W_{\Delta,n}$ is a regular graph of even order and degree Δ where $2 \leq \Delta \leq \lfloor \log_2 n \rfloor$. Despite being a highly symmetric and widely studied graph, the diameter of $W_{\Delta,n}$ is known only for $n = 2^{\Delta}$. In this paper we present a tight upper bound on the diameter of the Knödel graph for general case. We show that the presented bound differs from the diameter by at most 2 when $\Delta < \alpha \lfloor \log_2 n \rfloor$ for some $0 < \alpha < 1$ where $\alpha \to 1$ when $n \to \infty$. The proof is constructive and provides a near optimal diametral path for the Knödel graph $W_{\Delta,n}$.

Keywords: Knödel graph, diametral path, broadcasting, minimum broadcast graph.

1 Introduction

The Knödel graph $W_{\Delta,n}$ is a regular graph of even order and degree Δ where $2 \leq \Delta \leq \lfloor \log n \rfloor$ (all logarithms in this paper are base 2, unless otherwise specified). It was introduced by Knödel for $\Delta = \lfloor \log n \rfloor$ and was used in an optimal gossiping algorithm [17]. For smaller Δ, the Knödel graph is defined in [8].

The Knödel graph was widely studied as an interconnection network topology and proven to be having good properties in terms of broadcasting and gossiping. The Knödel graph $W_{\Delta,2^{\Delta}}$ is one of the three non-isomorphic infinite graph families known to be minimum broadcast and gossip graphs (graphs that have the smallest possible broadcast and gossip times and the minimum possible number of edges). The other two families are the well known hypercube [5] and the recursive circulant graph [18].

The Knödel graph $W_{\Delta-1,2^{\Delta}-2}$ is a minimum broadcast and gossip graph also for $n = 2^{\Delta} - 2 (\Delta \geq 2)$ [16],[3]. One of the advantages of the Knödel graph, as a network topology, is that it achieves the smallest diameter among known minimum broadcast and gossip graphs for $n = 2^{\Delta} (\Delta \geq 1)$. All the minimum broadcast graph families — k-dimensional hypercube, $C(4, 2^k)$-recursive circulant graph and $W_{k,2^k}$ Knödel graph — have the same degree k, but have diameters equal to k, $\lceil \frac{3k-1}{4} \rceil$ and $\lceil \frac{k+2}{2} \rceil$ respectively. A detailed description of some graph theoretic and communication properties of these three graph families and their comparison can be found in [6].

T. Lecroq and L. Mouchard (Eds.): IWOCA 2013, LNCS 8288, pp. 206–215, 2013.

As shown in [1], the edges of the Knödel graph can be grouped into dimensions which are similar to hypercube dimensions. This allows to use these dimensions in a similar manner as in hypercube for broadcasting and gossiping. Unlike the hypercube, which is defined only for $n = 2^k$, the Knödel graph is defined for any even number of vertices. Properties such as small diameter, vertex transitivity as a Cayley graph [15], high vertex and edge connectivity, dimensionality, embedding properties [6] make the Knödel graph a good candidate as a network topology and good architecture for parallel computing. $W_{\lfloor \log n \rfloor, n}$ guarantees the minimum time for broadcasting and gossiping. So, it is a broadcast and gossip graph [1],[7],[8]. Moreover, $W_{\lfloor \log n \rfloor, n}$ is used to construct sparse broadcast graphs of a bigger size by interconnecting several smaller copies or by adding and deleting vertices [13],[10],[9],[2],[4],[11],[16],[12].

Multiple definitions are known for the Knödel graph. We use the following definition from [8], which explicitly presents the Knödel graph as a bipartite graph.

Definition 1. *The Knödel graph on an even number of vertices n and of degree Δ were $2 \leq \Delta \leq \lfloor \log n \rfloor$ is defined as $W_{\Delta,n} = (V, E)$ where*

$$V = \{(i, j) \mid i = 1, 2\ j = 0, ..., n/2 - 1\},$$

$$E = \{((1, j), (2, (j + 2^k - 1) \bmod (n/2))) \mid \\ j = 1, ..., n/2\ k = 0, 1, ..., \Delta - 1\}.$$

We say that an edge $((1, j'), (2, j'')) \in E$ is r-dimensional if $j' = (j'' + 2^r - 1) \bmod (n/2)$ where $r = 0, 1, ..., \Delta - 1$. In this case, $(1, j')$ and $(2, j'')$ are called r-dimensional neighbors.

Fig. 1 illustrates $W_{3,14}$ and its 0, 1 and 2-dimensional edges. We can simplify the illustration of the Knödel graph by minimizing the number of intersecting edges. For this, we repeat few vertices and present the Knödel graph from Fig. 1 as illustrated in Fig. 2.

Despite being a highly symmetric and widely studied graph, the diameter of the Knödel graph $D(W_{\Delta,n})$ is known only for $n = 2^{\Delta}$. In [7], it was proved that $D(W_{\Delta,2^{\Delta}}) = \lceil \frac{\Delta+2}{2} \rceil$. The nontrivial proof of this result is algebraic and the actual diametral path is not presented. The problem of finding the shortest path between any pair of vertices in the Knödel graph $W_{\Delta,2^{\Delta}}$ is studied in [14], where an 2-approximation algorithm with the logarithmic time complexity is presented.

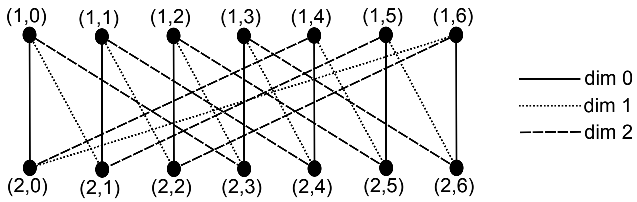

Fig. 1. The $W_{3,14}$ graph and its 0, 1 and 2-dimensional edges

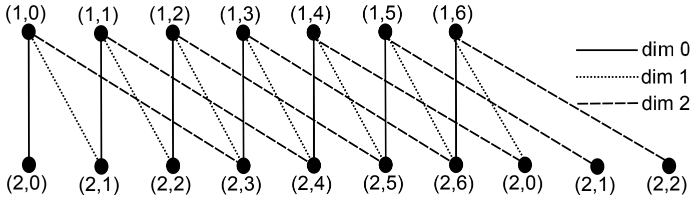

Fig. 2. The $W_{3,14}$ graph

Most properties of the Knödel graph are known only for $W_{\Delta,2^\Delta}$ and $W_{\Delta-1,2^\Delta-2}$. In this paper we present a tight upper bound on the diameter of the Knödel graph $D(W_{\Delta,n})$ for all even n and $2 \le \Delta \le \lfloor \log n \rfloor$. We show that the presented bound may differ from the actual diameter by at most 2 for almost all Δ. Our proof is constructive and provides a near optimal diametral path in $W_{\Delta,n}$.

Usually the partition in which a vertex occurs is not relevant, so we just use x to refer to either vertex $(1,x)$ or vertex $(2,x)$. The distance between vertices u and v is denoted by $dist(u,v)$. Using these notations and the vertex transitivity of the Knödel graph, we can state that $D(W_{\Delta,n}) = max\{dist(0,x) | 0 \le x < n/2\}$. In this paper, we actually give a tight upper bound on $dist(0,x)$ for all $0 \le x < n/2$.

2 Paths in the Knödel Graph

In this section we construct three different paths between two vertices in the Knödel graph $W_{\Delta,n}$. These paths have certain properties and are used in the next section to prove the upper bound on the diameter of $W_{\Delta,n}$.

Before presenting our formal statements, let us get better understanding of the Knödel graph and the set of vertices which can be reached from vertex 0 using only 0 and $(\Delta-1)$-dimensional edges. Note that we can "move" in two different directions from vertex $0 = (1,0)$ or $0 = (2,0)$ of $W_{\Delta,n}$. Fig. 3 illustrates the discussed paths. We can choose the path $(1,0) \to (2, 2^{\Delta-1}-1) \to (1, 2^{\Delta-1}-1) \to (2, 2(2^{\Delta-1}-1)) \to \ldots$ or we can move in the opposite direction following the path $(1,0) \to (2,0) \to (1, n/2-(2^{\Delta-1}-1)) \to (2, n/2-(2^{\Delta-1}-1)) \to \ldots$. Every second edge in these paths is 0-dimensional. The $(\Delta-1)$-dimensional edges are used to move "forward" by $2^{\Delta-1}-1$ vertices, while the 0-dimensional edges are only to change the partition. These two paths will eventually intersect or overlap somewhere near vertex $\lceil n/4 \rceil$. Excluding vertex 0, we have only $n/2-1$ vertices in each partition. The $(\Delta-1)$-dimensional edges will split $W_{\Delta,n}$ into $\left\lceil \frac{n/2-1}{2^{\Delta-1}-1} \right\rceil$ segments, each having length $2^{\Delta-1}-1$, except the one containing vertex $\lceil n/4 \rceil$. We can perform only $\left\lfloor \frac{1}{2} \left\lceil \frac{n/2-1}{2^{\Delta-1}-1} \right\rceil \right\rfloor = \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor$ $(\Delta-1)$-dimensional passes in each of these two paths before they intersect. Therefore, we will never use more than $\left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor$ $(\Delta-1)$-dimensional passes to reach a vertex in $W_{\Delta,n}$.

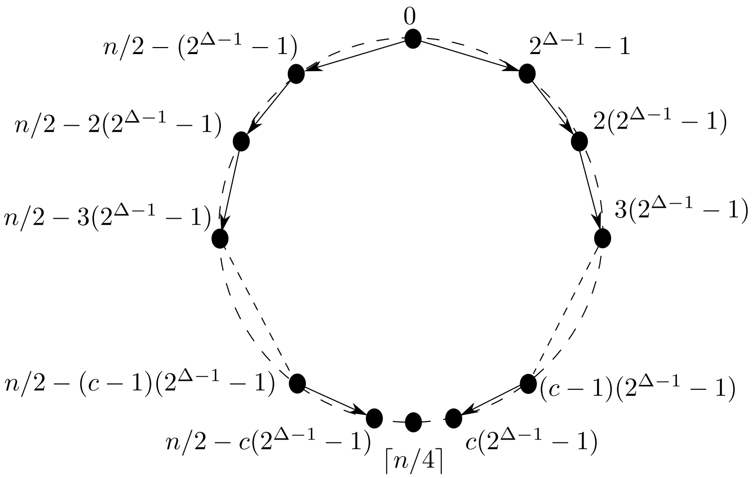

Fig. 3. Schematic illustration of the paths. $c = \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor$

Our first lemma constructs a path between vertex 0 and some vertex y which is relatively close to our destination vertex x. Vertex y will have a special form making such construction straightforward. Recall that x refers to $(1, x)$ or $(2, x)$, and y refers to $(1, y)$ or $(2, y)$.

Lemma 2. *For any vertex x of $W_{\Delta,n}$, by using at most $2c + 1$ edges where $c = \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor$, we can construct a path from vertex 0 to reach some vertex y such that $|x - y| \leq 2^{\Delta-1} - 1$.*

Proof. Our goal is to reach some vertex y of form $y = c(2^{\Delta-1} - 1)$ or $y = n/2 - c(2^{\Delta-1} - 1)$ such that $|x - y| \leq 2^{\Delta-1} - 1$. We use only 0 and $(\Delta - 1)$-dimensional edges and one of two paths described above and illustrated in Fig. 3. We consider two cases. In the first case we cover the values of x that can be reached by moving in “clockwise” direction from vertex 0. For the remaining values of x, we use the path from Fig. 3 moving to the opposite direction.

Case 1: $x < (c + 1)(2^{\Delta-1} - 1)$. By alternating between 0 and $(\Delta - 1)$-dimensional edges, we can reach a vertex y of form $y = c'(2^{\Delta-1} - 1)$ and closest to x from vertex $0 = (2, 0)$. We will need at most $2c' + 1$ edges for that. The path to reach $y = (1, y)$ will be $(2, 0) \to (1, 0) \to (2, 2^{\Delta-1} - 1) \to (1, 2^{\Delta-1} - 1) \to (2, 2(2^{\Delta-1} - 1)) \to ... \to (2, c'(2^{\Delta-1} - 1)) \to (1, c'(2^{\Delta-1} - 1)) = y$. It is clear that $c' \leq c = \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor$, hence the bound on the length of constructed path follows. From the form of y follows that $|x - y| \leq 2^{\Delta-1} - 1$. Fig. 4 shows the described path from $(2, 0)$ to $y = 6 = (1, 6)$.

Case 2: $x > n/2 - c(2^{\Delta-1} - 1)$. This case is similar to case 1 except in order to construct shorter path to y of form $y = n/2 - c'(2^{\Delta-1} - 1)$, we are moving from vertex $0 = (1, 0)$ in anticlockwise direction. The path for $y = (2, y)$ will

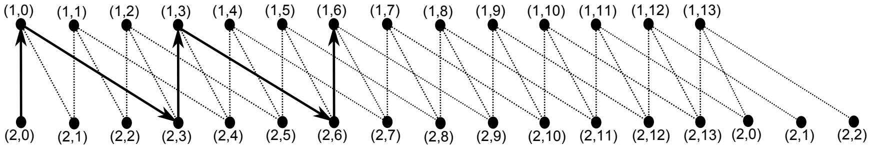

Fig. 4. A path between $(2, 0)$ and $(1, 6)$ vertices in $W_{3,28}$ graph

be $(1, 0) \to (2, 0) \to (1, n/2 - (2^{\Delta-1} - 1) \to (2, n/2 - (2^{\Delta-1} - 1)) \to ... \to (1, n/2 - (c' - 1)(2^{\Delta-1} - 1)) \to (2, n/2 - (c' - 1)(2^{\Delta-1} - 1)) = y$ and will have length at most $2c + 1$. Obviously we will have $|x - y| \le 2^{\Delta-1} - 1$ as well. □

The following lemma constructs a path between two vertices of $W_{\Delta,n}$ that are relatively close to each other. More precisely, when the difference of their labels is upper bounded by $2^{\Delta-1} - 1$. We construct a path between two vertices x_1 and x_2 which is not necessarily a shortest path between them. To reach the given vertex with label $x_2 > x_1$ from vertex labeled x_1, we first use a large dimensional edge to "jump over" vertex x_2 and reach some vertex $y \ge x_2$, such that $y - x_2$ is the smallest. After that, we start moving from y in backward direction till we reach x_2 from right. This backward steps are performed in a greedy way. At each step, we are using the largest dimensional edge to reach some new vertex y' such that $y' - x_2$ is minimal and y' is on the right side of x_2 i.e. $y' \ge x_2$.

Lemma 3 (Existence of a special path). *For any two vertices of* $W_{\Delta,n}$ *labeled* x_1 *and* x_2*, if* $|x_2 - x_1| \le 2^{\Delta-1} - 1$*, then there exists a special path between* x_1 *and* x_2 *of length at most* $2\Delta - 3$*. This path contains one "direct"* d*-dimensional edge where* $d \le \Delta - 1$*, some 0-dimensional edges and some edges having dimensions between* 1 *and* $d - 1$ *pointing in "backward" direction. The number of these backward edges is at most* $\Delta - 2$*.*

Proof. Without loss of generality, we assume that $x_1 = 0$ and $x_2 > x_1$. In order to construct the described path, we use an edge to get from vertex 0 to some vertex y closest to x_2 such that $y > x_2$ and y is directly connected to 0. This will be our "direct" d-dimensional edge. After reaching vertex y, we start to move in "backward" direction towards x_2. Once started moving in backward direction, the distance from y to x_2 which is upper bounded by $2^{\Delta-2}$, will be cut at least by half with each backward edge. Therefore we need at most $\Delta - 2$ backward edges. Combined with the 0-dimensional edges between these backward edges, this will give a path of length $2(\Delta - 2)$. By adding the initial edge, we get the $2\Delta - 3$ upper bound on the length of the constructed path.

Fig. 5 shows the described path between vertices $x_1 = (1, 0)$ and $x_2 = (2, 5)$. In the illustrated example $y = 7$, $d = 4$, the "direct" edge is $((1, 0), (2, 7))$ and the "backward" edges are $((2, 7), (1, 6))$ and $((2, 6), (1, 5))$.

The reason we chose this particular path between x_1 to x_2 is that the backward passes can be performed in the path constructed by Lemma 2. This will be crucial in the proof of the main theorem. □

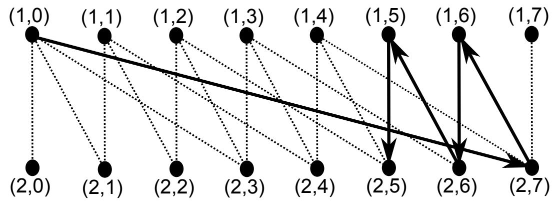

Fig. 5. A path between $(1, 0)$ and $(2, 5)$ vertices in a section of the Knödel graph of degree 5

Our last lemma deals with the problem of finding the shortest path in a particular section of the Knödel graph.

Lemma 4 (Shortest path approximation). *For any two vertices of* $W_{\Delta,n}$ *labeled* x_1 *and* x_2*, if* $|x_2 - x_1| \leq 2^d - 1$ *for some* $d \leq \Delta - 1$*, then there exist a path between* x_1 *and* x_2 *of length at most* $3\lceil d/4 \rceil + 4$*.*

Proof. Without loss of generality, we assume that $x_1 = 0$ and $x_2 > x_1$. Our goal is to construct a short path from vertex 0 to vertex $x_2 = x \leq 2^d - 1$. The proof is based on a recursive construction of a path between vertices 0 and x having length at most $3\lceil d/4 \rceil + 4$. The recursion will be on d.

The base case is when $d \leq 3$. This case is illustrated in Fig. 6, from which we observe that we can reach any vertex x where $0 \leq x \leq 2^d - 1 = 7$ with a path of length at most 4.

For $d > 3$, using at most three edges, we can cut the distance between 0 and x by a factor of 16. Fig. 7 presents a schematic illustration of this. We divide the initial interval of length $2^d - 1$ into eight smaller intervals $A_1, A_2, ..., A_8$, each having length at most $\lceil (2^d - 1)/8 \rceil$, where $A_i = [(i-1)m, im)$, $i = 1, ..., 8$ and $m = 2^{d-3}$.

It is not difficult to see that all these intervals, except A_6, have both their end vertices reachable from 0 by using at most three edges. For A_6, using at most 3 edges we can reach its middle vertex $11m/2 - 1$ and the end vertex $6m$. The paths, which use at most 3 edges, are illustrated in Fig. 7. This means that when $x \in A_i$ for all $1 \leq i \leq 8$, using at most three edges, we will be within distance $m/2$ from x. After relabeling the vertices, we will get the same problem of finding a path between vertices 0 and x, but the new x will be at least 16 times smaller.

It will take at most $\lceil \log_{16}(2^d - 1) \rceil$ recursive steps to reach the base case, and we will use at most three edges in each step. By combining this with at most 4 edges used for the base case, we will get that $dist(0, x) \leq 3\lceil \log_{16}(2^d - 1) \rceil + 4 \leq 3\lceil d/4 \rceil + 4$. □

We note that each recursive step in Lemma 4 involves only constant number of operations. Therefore the described path can be constructed by an algorithm of complexity $O(\log n)$.

Lemma 4 can be used to construct a short path between any two vertices of $W_{\Delta,n}$ for the case when $\Delta = \lfloor \log n \rfloor$. The length of the constructed path will

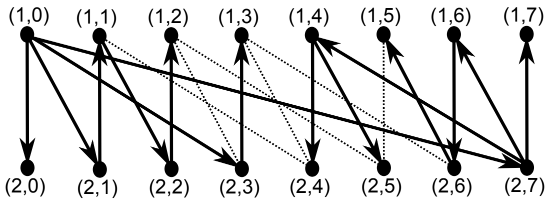

Fig. 6. Paths from vertex 0 to all other vertices $x \leq 7$ in a section of the Knödel graph

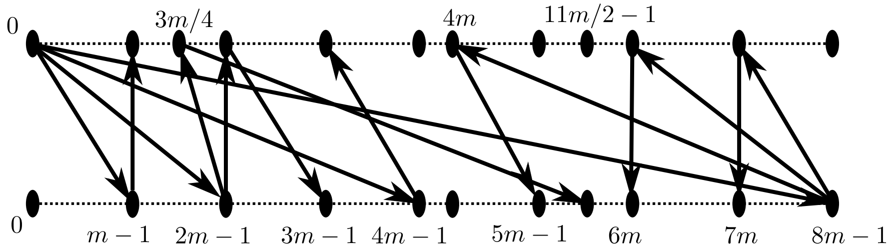

Fig. 7. Illustration of the recursive step. $m = 2^{d-3}$

be at most $3\lceil(\Delta-1)/4\rceil+4$. It follows that $D(W_{\Delta,n}) \leq 3\lceil(\Delta-1)/4\rceil+4$ for $\Delta = \lfloor \log n \rfloor$.

3 Upper Bound on Diameter

In this section, using the lemmas from Section 2, we construct a path between vertices 0 and x for any vertex x in $W_{\Delta,n}$. The maximum length of such a path will be an upper bound on the diameter of $W_{\Delta,n}$.

Our first upper bound on $D(W_{\Delta,n})$ will trivially follow from Lemma 2 and Lemma 4.

Theorem 5 (Trivial). $D(W_{\Delta,n}) \leq 2\left\lfloor \frac{1}{2}\left\lceil \frac{n-2}{2^{\Delta}-2}\right\rceil\right\rfloor + 3\lceil(\Delta-1)/4\rceil+5.$

Proof. According to Lemma 2, for any vertex x in $W_{\Delta,n}$, we need at most $2\left\lfloor \frac{1}{2}\left\lceil \frac{n-2}{2^{\Delta}-2}\right\rceil\right\rfloor + 1$ edges to reach from vertex 0 to a vertex y of form $y = c(2^{\Delta-1}-1)$ or $y = n/2 - c(2^{\Delta-1}-1)$ within distance $2^{\Delta-1}-1$ from x i.e. $|x-y| \leq 2^{\Delta-1}-1$. Now we can apply Lemma 4 and claim that $dist(x,y) \leq 3\lceil d/4\rceil + 4$ where $d \leq \Delta - 1$. Thus, we have that $dist(0,x) \leq dist(0,y) + dist(y,x) \leq 2\left\lfloor \frac{1}{2}\left\lceil \frac{n-2}{2^{\Delta}-2}\right\rceil\right\rfloor + 3\lceil(\Delta-1)/4\rceil + 5$. □

Theorem 5 combines the paths described in Lemmas 2 and 4 in the most trivial way. With the slight modification of the path described in Lemma 2 and combining it with paths from Lemmas 3 and 4 we can significantly improve the presented upper bound on $D(W_{\Delta,n})$.

Theorem 6 (Main). *Let* $a = \left\lfloor \frac{1}{2}\left\lceil \frac{n-2}{2^{\Delta}-2}\right\rceil\right\rfloor$ *and* $b = \Delta - 2$ *(*$\Delta \geq 3$*). If* $a \geq b$ *then* $D(W_{\Delta,n}) \leq 2a + 3 = 2\left\lfloor \frac{1}{2}\left\lceil \frac{n-2}{2^{\Delta}-2}\right\rceil\right\rfloor + 3$*, otherwise* $D(W_{\Delta,n}) \leq 2a + 3\lceil(\Delta - 2 - a)/4\rceil + 7 \leq \frac{3}{4}\Delta + \frac{5}{4}a + \frac{17}{2}$.

Proof. Case 1: $a \geq b$. From Lemma 2, we recall that a is the maximum number of $(\Delta-1)$-dimensional edges necessary to reach a vertex of form $y = c(2^{\Delta-1} - 1)$ or $y = n/2 - c(2^{\Delta-1} - 1)$ closest to our destination vertex x. Recall that $b = \Delta - 2$ is the maximum number of "backward" edges used in the path from Lemma 3. We observe that when $a \geq b$, then all the "backward" passes can be performed by modifying the path described in Lemma 2 used to reach vertex y. We just need to replace some of the 0-dimensional edges from Lemma 2 used only for switching the graph partition with the corresponding "backward" passes from Lemma 3. As a result of this modification, instead of reaching y, with $2a + 1$ edges we will reach some vertex y' precisely at distance $2^{\Delta} - 1$ from x. By using one more $(\Delta-1)$-dimensional and one more 0-dimensional edge, we can perform the final pass and reach x with a path of length at most $2a + 3$.

Case 2: $a < b$. In this case we will be able to perform only some of the reverse passes from Lemma 3 by modifying the path from Lemma 2. More precisely, out of $b = \Delta - 2$ reverse passes, we will be able to perform only a of them in the modified path. We note that each reverse pass in Lemma 3 cuts the distance to x by half. This means that performing $b - a$ reverse passes in the path constructed by Lemma 2 of length $2a+3$, we will be within distance $2^{\Delta-2-a}$ from x compared to $2^{\Delta-2}$ without performing these reverse passes. Now we can use Lemma 4 with $d = \Delta - 2 - a$ and claim that we will be able to reach x by using at most $3\lceil(\Delta - 2 - a)/4\rceil + 4$ additional edges. Thus, $D(W_{\Delta,n}) \leq (2a + 3) + (3\lceil(\Delta - 2 - a)/4\rceil + 4) \leq \frac{3}{4}\Delta + \frac{5}{4}a + \frac{17}{2}$. □

4 Tightness of the Upper Bound

In this section we analyze the tightens of the upper bound on the diameter of the Knödel graph from Theorem 6. To do that we will first present a lower bound on the diameter of the Knödel graph.

Theorem 7 (Lower bound). $D(W_{\Delta,n}) \geq 2\left\lfloor \frac{1}{2}\left\lceil \frac{n-2}{2^{\Delta}-2}\right\rceil\right\rfloor + 1$.

Proof. First, note that in order to reach vertex $x = (1, c(2^{\Delta-1} - 1))$ where $c = \left\lfloor \frac{1}{2}\left\lceil \frac{n-2}{2^{\Delta}-2}\right\rceil\right\rfloor$ from vertex $(2, 0)$, we cannot construct a path shorter than the one described in Lemma 2 and illustrated in Fig. 3. This path contains exactly $c + 1$ 0-dimensional edges used for changing the graph partition and c $(\Delta - 1)$-dimensional edges used for moving towards x in the fastest possible way. Thus, the lower bound $D(W_{\Delta,n}) \geq 2c + 1 = 2\left\lfloor \frac{1}{2}\left\lceil \frac{n-2}{2^{\Delta}-2}\right\rceil\right\rfloor + 1$ follows. □

The following theorem shows that the presented upper bound is tight, in particular it is within additive factor 2, for almost all possible values of Δ.

Theorem 8 (Tightness). *For any $0 < \alpha < 1$, there exists some $N(\alpha)$ such that for all $n \geq N(\alpha)$ and $\Delta < \alpha \lfloor \log n \rfloor$, the $D(W_{\Delta,n}) \leq 2a + 3$ upper bound from Theorem 6 ($a = \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor$) differs from actual diameter by at most 2, i.e.* $2 \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor + 1 \leq D(W_{\Delta,n}) \leq 2 \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor + 3$.

Proof. From Theorem 7 it follows that the upper bound from Theorem 6 for the case when $a \geq b$ may differ from actual diameter by at most 2. Now, we find a sufficient condition for $a \geq b$ to be true. By observing that $a = \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor \geq \frac{n/2}{2^{\Delta}} - 1$ and $b = \Delta - 2 \leq \Delta - 1$ we get that if $\frac{n/2}{2^{\Delta}} - 1 \geq \Delta - 1$ then $a \geq b$ equality is true. After further simplification, we get the $2\Delta 2^{\Delta} \leq n$ sufficient condition for $a \geq b$ to be true.

It follows that for given n and Δ, where $2 \leq \Delta \leq \lfloor \log n \rfloor$ such that $\Delta 2^{\Delta+1} \leq n$, we have $2a + 1 \leq D(W_{\Delta,n}) \leq 2a + 3$. Finally, we observe that for any $0 < \alpha < 1$ and $\Delta < \alpha \lfloor \log n \rfloor$ the $\Delta 2^{\Delta+1} \leq n$ inequality is always true for sufficiently large n. □

Note that Theorem 6, in almost all cases, actually gives an approximation algorithm to find the diameter of $W_{\Delta,n}$ with an additive factor 2.

5 Summary

In this paper we obtained tight lower and upper bounds on the diameter of the Knödel graph $W_{\Delta,n}$ for all even n and $2 \leq \Delta \leq \lfloor \log n \rfloor$. We showed that the presented bound differs from actual diameter by at most 2 for almost all Δ. Our proofs are constructive and provide a near optimal diametral path in $W_{\Delta,n}$.

Recall that the only known results, regarding the diameter of the Knödel graph, were the exact value $D(W_{\Delta,2^{\Delta}}) = \left\lceil \frac{\Delta+2}{2} \right\rceil$ [7] and an 2-approximation algorithm with logarithmic time complexity for finding shortest path between any pair of vertices in $W_{\Delta,2^{\Delta}}$ [14]. Lemma 4 provides $D(W_{\Delta,2^{\Delta}}) \leq 3 \lceil (\Delta - 1)/4 \rceil + 4$. Comparing this with the exact expression above, we see that Lemma 4 provides an 3/2-approximation algorithm for the problem of finding a diametral path. This is much better than the 2-approximation algorithm presented in [14]. However, we note that [14] addresses more general problem of finding a shortest path in the Knödel graph and the 2-approximation ratio is for the shortest path between any two vertices, while our result is only for the diametral path.

Our future research will be focused on routing and broadcasting problems in the Knödel graph $W_{\Delta,n}$ for all $2 \leq \Delta \leq \lfloor \log n \rfloor$.

References

1. Bermond, J.C., Harutyunyan, H.A., Liestman, A.L., Perennes, S.: A note on the dimensionality of modified knödel graphs. Int. J. Found. Comput. Sci. 8(2), 109–116 (1997)

2. Bermond, J.C., Fraigniaud, P., Peters, J.G.: Antepenultimate broadcasting. Networks 26(3), 125–137 (1995)
3. Dinneen, M., Fellows, M., Faber, V.: Algebraic constructions of efficient broadcast networks. Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, 152–158 (1991)
4. Dinneen, M.J., Ventura, J.A., Wilson, M.C., Zakeri, G.: Construction of time-relaxed minimal broadcast networks. Parallel Processing Letters 9(1), 53–68 (1999)
5. Farley, A.M., Hedetniemi, S., Mitchell, S., Proskurowski, A.: Minimum broadcast graphs. Discrete Mathematics 25, 189–193 (1979)
6. Fertin, G., Raspaud, A.: A survey on knödel graphs. Discrete Applied Mathematics 137(2), 173–195 (2004)
7. Fertin, G., Raspaud, A., Schröder, H., Sýkora, O., Vrťo, I.: Diameter of the knödel graph. In: Brandes, U., Wagner, D. (eds.) WG 2000. LNCS, vol. 1928, pp. 149–160. Springer, Heidelberg (2000)
8. Fraigniaud, P., Peters, J.G.: Minimum linear gossip graphs and maximal linear (δ, k)-gossip graphs. Networks 38(3), 150–162 (2001)
9. Harutyunyan, H.A.: Minimum multiple message broadcast graphs. Networks 47(4), 218–224 (2006)
10. Harutyunyan, H.A.: An efficient vertex addition method for broadcast networks. Internet Mathematics 5(3), 211–225 (2008)
11. Harutyunyan, H.A., Liestman, A.L.: More broadcast graphs. Discrete Applied Mathematics 98(1-2), 81–102 (1999)
12. Harutyunyan, H.A., Liestman, A.L.: On the monotonicity of the broadcast function. Discrete Mathematics 262(1-3), 149–157 (2003)
13. Harutyunyan, H.A., Liestman, A.L.: Upper bounds on the broadcast function using minimum dominating sets. Discrete Mathematics 312(20), 2992–2996 (2012)
14. Harutyunyan, H.A., Morosan, C.D.: On the minimum path problem in knödel graphs. Networks 50(1), 86–91 (2007)
15. Heydemann, M.C., Marlin, N., Pérennes, S.: Complete rotations in cayley graphs. European Journal of Combinatorics 22(2), 179–196 (2001)
16. Khachatrian, L.H., Harutounian, O.S.: Construction of new classes of minimal broadcast networks. In: Conference on Coding Theory, Dilijan, Armenia, pp. 69–77 (1990)
17. Knödel, W.: New gossips and telephones. Discrete Mathematics 13(1), 95 (1975)
18. Park, J.H., Chwa, K.Y.: Recursive circulant: a new topology for multicomputer networks (extended abstract). In: International Symposium on Parallel Architectures, Algorithms and Networks, ISPAN 1994, pp. 73–80 (1994)