Characterizing Subset Spaces as Bi-topological Structures

Bernhard Heinemann

Faculty of Mathematics and Computer Science,
University of Hagen,
58084 Hagen, Germany
bernhard.heinemann@fernuni-hagen.de

Abstract. Subset spaces constitute a relatively new semantics for bi-modal logic. This semantics admits, in particular, a modern, computer science oriented view of the classic interpretation of the basic modalities in topological spaces à la McKinsey and Tarski. In this paper, we look at the relationship of both semantics from an opposite perspective as it were, by asking for a consideration of subset spaces in terms of topology and topological modal logic, respectively. Indeed, we shall finally obtain a corresponding characterization result. A third semantics of modal logic, namely the standard relational one, and the associated first-order structures, will play an important part in doing so as well.

Keywords: modal logic, topological semantics, subset spaces, knowledge and topological reasoning.

1 Introduction

Nowadays, successful applications of modal logic to computer science are abundant. We focus on a particular system from the realm of formal reasoning here, which may be seen as a cross-disciplinary framework for dealing with spatial as well as epistemic scenarios: the talk is of Moss and Parikh's bi-modal logic of subset spaces; see [12], [5], or Ch. 6 of [2].

We shall now indicate how the interrelation of the underlying ideas, *knowledge* and *spatiality,* is correspondingly revealed. The *epistemic state* of an agent under discussion, i.e., the set of all those states that cannot be distinguished by what the agent topically knows, can be viewed as a *neighborhood* U of the actual state x of the world. Formulas are then interpreted with respect to the resulting pairs x, U called *neighborhood situations.* Thus, both the set of all states and the set of all epistemic states constitute the relevant semantic domains as particular subset structures. The two modalities involved, K and $\Box$, quantify over all elements of U and 'downward' over all neighborhoods contained in U, respectively. This means that K captures the notion of knowledge as usual (see [7]), and $\Box$ reflects *effort to acquire knowledge* since gaining knowledge goes hand in hand with a shrinkage of the epistemic state. In fact, knowledge acquisition is this way reminiscent of a topological procedure. The appropriate logic for 'real' topological spaces, called

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topologic, was first determined by Georgatos in his thesis [8]. Meanwhile, a lot of work has been done on the development of a modal logical theory of subset spaces and, in particular, topological spaces on this basis; see [2] for a guide to the earlier literature. (To our knowledge, [10], [3], and [13], are the most recent papers in this field, with the last two forging links between subset spaces and *Dynamic Epistemic Logic (DEL);* see [6].)

The topological semantics of modal logic dates back to the late 1930s; see the respective notes in the paper [11]. In recent years, the research into logics based on this semantics has considerably been ramped up to satisfy requirements relating to spatial modelling and reasoning tasks in computer science; the handbook [5] contains lots of references regarding this as well (see, in particular, Ch. 5 and Ch. 10 there). The characteristic feature is the following interpretation of the modal box here: for every formula α, the validity domain of $\Box\alpha$ is defined to be the *interior* of the validity domain of α. With that, the well-known modal system $\mathsf{S4}$ has been proved to be the logic of the class of all topological spaces; see [11] again. – This is all that must be said about topological modal logic for the moment; more facts will be given in Section 3 below.

There is a translation from mono-modal to bi-modal formulas which conveys the already rather transparent connection between the two interpretations in topological spaces just mentioned. Its decisive clause reads $\Box\alpha \mapsto \Diamond\mathsf{K}\alpha$; see [5], Proposition 3.5. This translation even gives rise to an *embedding* of $\mathsf{S4}$ into *topologic;* see [5], Theorem 3.7. Thus, the elder, purely spatial formalism may be retrieved from a more comprehensive framework regarding epistemic issues, too.

Conversely, can subset spaces be identified in a purely topological way? – As it stands, this question is not raised precisely enough. So we must say that we are not looking for a somehow good-natured translation in the other direction here; this issue has already been discussed in [5], Sect. 3.2. Instead, our topic is the following. Subset spaces are closely related to certain bi-modal Kripke models having the same logic; see [5], Sect. 2.3. These structures of course are *bi-topological* since they validate, in particular, two modal logics containing $\mathsf{S4}$. Thus, our initial question is to be specified as follows: can a topological characterization of all those bi-topological structures that originate from a subset space be given and, should the situation arise, up to what extent in terms of topological modal logic? – The goal of this paper is to give an affirmative answer and a corresponding description, respectively.

The present paper grew out of a remark of Anil Nerode at LFCS 2013. It makes a contribution in several respects. First, it clarifies the interplay of the three semantics involved to a greater extent. Second, it facilitates an alternative view of subset spaces. In fact, it is generally very desirable (and common in mathematics) to have at hand different ways of seeing a subject, in order to be able to react on varying problems flexibly. Third, the crucial axiom schema of the logic of subset spaces, called the *Cross Axioms* in [5], is given a topological reading as a certain *cover property* here. And finally, a topological formulation of the properties defining subset spaces as first-order structures is supplied. All this makes this paper a theoretical one on a system being, on the other hand, of practical relevance to the reasoning process. The latter has been taken as a justification to submit the paper to LPAR.

We now proceed to the technical issues. In the following section, we first introduce the language for subset spaces, and we recapitulate the known relationship between subset spaces and Kripke models. Later on in this section, we review the logic arising from that language. Section 3 then deals with the basics of topological modal logic in more detail. In Section 4, the topological effect of the Cross Axioms is illuminated. The final technical section contains the characterization theorem announced above, before the paper is finished by some concluding remarks.

2 The Language and the Logic of Subset Spaces

In this section, we first fix the language for subset spaces, $\mathcal{L}$. After that, we link the semantics of $\mathcal{L}$ with the common relational semantics of modal logic. Finally, we recall some facts on the logic of subset spaces needed subsequently.

To begin with, we define the syntax of $\mathcal{L}$. Let $\mathsf{Prop} = \{p, q, \ldots\}$ be a denumerably infinite set of symbols called *proposition variables* (which should represent the basic facts about the states of the world). Then, the set Form of all $\mathcal{L}$-*formulas*[1] over Prop is defined by the rule $\alpha \ ::= \ \top \mid p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \mathsf{K}\alpha \mid \Box\alpha$. The *mono-modal fragment* MF of Form is obtained by disregarding the clause for K in this rule. Later on, the boolean connectives that are missing here are treated as abbreviations, as needed. The dual operators of K and $\Box$ are denoted by L and $\Diamond$, respectively; K is called the *knowledge operator* and $\Box$ the *effort operator*.

We now turn to the semantics of $\mathcal{L}$. For a start, we define the relevant domains. We let $\mathcal{P}(X)$ designate the powerset of a given set X.

Definition 1 (Semantic Domains).

1. *Let X be a non-empty set (of* states*) and $\mathcal{O} \subseteq \mathcal{P}(X)$ a set of subsets of X. Then, the pair $\mathcal{S} = (X, \mathcal{O})$ is called a* (subset) frame.
2. *Let $\mathcal{S} = (X, \mathcal{O})$ be a subset frame. Then the set*
$$\mathcal{N}_{\mathcal{S}} := \{(x, U) \mid x \in U \text{ and } U \in \mathcal{O}\}$$
is called the set of neighborhood situations *of $\mathcal{S}$.*
3. *Let $\mathcal{S} = (X, \mathcal{O})$ be a subset frame. An $\mathcal{S}$-*valuation *is a mapping $V : \mathsf{Prop} \to \mathcal{P}(X)$.*
4. *Let $\mathcal{S} = (X, \mathcal{O})$ be a subset frame and V an $\mathcal{S}$-valuation. Then, $\mathcal{M} := (X, \mathcal{O}, V)$ is called a* subset space (based on $\mathcal{S}$).

Note that neighborhood situations denominate the semantic atoms of our bimodal language. The first component of such a situation indicates the actual state of the world, while the second reflects the uncertainty of the agent in

[1] The prefix '$\mathcal{L}$' will be omitted provided there is no risk of confusion.

question about it. Furthermore, Definition 1.3 shows that values of proposition variables depend on states only. This is in accordance with the common practice in epistemic logic; cf. [7].

For a given subset space $\mathcal{M}$, we now define the relation of *satisfaction,* $\models_{\mathcal{M}}$, between neighborhood situations of the underlying frame and formulas from Form. Based on that, we define the notion of *validity* of $\mathcal{L}$-formulas in subset spaces and in subset frames. In the following, neighborhood situations are often written without parentheses.

Definition 2 (Satisfaction and Validity). *Let $\mathcal{S} = (X, \mathcal{O})$ be a subset frame.*

1. *Let $\mathcal{M} = (X, \mathcal{O}, V)$ be a subset space based on $\mathcal{S}$, and let $(x, U) \in \mathcal{N}_{\mathcal{S}}$ be a neighborhood situation. Then*

$$\begin{array}{lcl} x, U \models_{\mathcal{M}} \top & & \textit{is always true} \\ x, U \models_{\mathcal{M}} p & :\Longleftrightarrow & x \in V(p) \\ x, U \models_{\mathcal{M}} \neg\alpha & :\Longleftrightarrow & x, U \not\models_{\mathcal{M}} \alpha \\ x, U \models_{\mathcal{M}} \alpha \wedge \beta & :\Longleftrightarrow & x, U \models_{\mathcal{M}} \alpha \textit{ and } x, U \models_{\mathcal{M}} \beta \\ x, U \models_{\mathcal{M}} \mathsf{K}\alpha & :\Longleftrightarrow & \forall\, y \in U : y, U \models_{\mathcal{M}} \alpha \\ x, U \models_{\mathcal{M}} \Box\alpha & :\Longleftrightarrow & \forall\, U' \in \mathcal{O} : [x \in U' \subseteq U \Rightarrow x, U' \models_{\mathcal{M}} \alpha], \end{array}$$

   where $p \in \mathsf{Prop}$ and $\alpha, \beta \in \mathsf{Form}$. In case $x, U \models_{\mathcal{M}} \alpha$ is true we say that α holds in $\mathcal{M}$ at *the neighborhood situation x, U.*
2. *Let $\mathcal{M} = (X, \mathcal{O}, V)$ be a subset space based on $\mathcal{S}$. An $\mathcal{L}$-formula α is called* valid in $\mathcal{M}$ *iff it holds in $\mathcal{M}$ at every neighborhood situation of $\mathcal{S}$.*
3. *An $\mathcal{L}$-formula α is called* valid in $\mathcal{S}$ *iff it is valid in every subset space $\mathcal{M}$ based on $\mathcal{S}$; in this case, we write $\mathcal{S} \models \alpha$.*

Note that the idea of knowledge and effort described in the introduction is made precise by Item 1 of this definition. In particular, knowledge *is defined* as validity at all states that are indistinguishable to the agent; cf. [7].

Obviously, subset spaces are on the same level of language as are Kripke models in common modal logic (whereas subset frames correspond to Kripke frames).

Subset frames and spaces might be considered from a different perspective, as is known since [5] and reviewed in the following. Let a subset frame $\mathcal{S} = (X, \mathcal{O})$ and a subset space $\mathcal{M} = (X, \mathcal{O}, V)$ based on $\mathcal{S}$ be given. Take $W_{\mathcal{S}} := \mathcal{N}_{\mathcal{S}}$ as a set of worlds, and define two accessibility relations $R_{\mathcal{S}}^{\mathsf{K}}$ and $R_{\mathcal{S}}^{\Box}$ on $W_{\mathcal{S}}$ by

$$\begin{array}{l} (x, U)\, R_{\mathcal{S}}^{\mathsf{K}}\, (x', U') :\Longleftrightarrow\ U = U' \text{ and} \\ (x, U)\, R_{\mathcal{S}}^{\Box}\, (x', U') :\Longleftrightarrow\ (x = x' \text{ and } U' \subseteq U), \end{array}$$

for all $(x, U), (x', U') \in W_{\mathcal{S}}$. Moreover, let $V_{\mathcal{M}}(p) := \{(x, U) \in W_{\mathcal{S}} \mid x \in V(p)\}$, for every $p \in \mathsf{Prop}$. Then, bi-modal Kripke structures $S_{\mathcal{S}} := \big(W_{\mathcal{S}}, \{R_{\mathcal{S}}^{\mathsf{K}}, R_{\mathcal{S}}^{\Box}\}\big)$ and $M_{\mathcal{M}} := \big(W_{\mathcal{S}}, \{R_{\mathcal{S}}^{\mathsf{K}}, R_{\mathcal{S}}^{\Box}\}, V_{\mathcal{M}}\big)$ result in such a way that $M_{\mathcal{M}}$ is equivalent to $\mathcal{M}$ in the following sense.

Proposition 1. *For all $\alpha \in \mathsf{Form}$ and $(x, U) \in W_{\mathcal{S}}$, we have that $x, U \models_{\mathcal{M}} \alpha$ iff $M_{\mathcal{M}}, (x, U) \models \alpha$.*

Here (and later on as well), the symbol '$\models$' denotes the usual satisfaction relation of modal logic. – The proposition is easily proved by induction on α. We call $S_{\mathcal{S}}$ and $M_{\mathcal{M}}$ the Kripke structures *induced* by $\mathcal{S}$ and $\mathcal{M}$, respectively.

The question to what extent one can go the other way round, i.e., associate subset spaces to suitable Kripke structures so that the latter are the induced ones, will play an important part below. Some significant information on what 'suitable' means in this connection, is provided by looking at the *logic* of subset spaces (referred to as LSS later). Here is a sound and complete axiomatization (cf. [5], Sect. 2.2):

1. All instances of propositional tautologies
2. $\mathsf{K}(\alpha \to \beta) \to (\mathsf{K}\alpha \to \mathsf{K}\beta)$
3. $\mathsf{K}\alpha \to (\alpha \wedge \mathsf{KK}\alpha)$
4. $\mathsf{L}\alpha \to \mathsf{KL}\alpha$
5. $(p \to \Box p) \wedge (\Diamond p \to p)$
6. $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$
7. $\Box\alpha \to (\alpha \wedge \Box\Box\alpha)$
8. $\mathsf{K}\Box\alpha \to \Box\mathsf{K}\alpha$,

where $p \in \mathsf{Prop}$ and $\alpha, \beta \in \mathsf{Form}$; note that the last schema represents the aforementioned Cross Axioms. As a result, we obtain that LSS is sound and complete also with respect to the class of all *Kripke* models M such that

- the accessibility relation R of M belonging K is an equivalence (in other words, where K is an $\mathsf{S5}$-modality),
- the accessibility relation R' of M belonging to $\Box$ is reflexive and transitive (i.e., $\Box$ is $\mathsf{S4}$-like),
- the composite relation $R' \circ R$ is contained in $R \circ R'$ (this is usually called the *cross property*), and
- the valuation of M is constant along every R'-path, for all proposition variables.

The most interesting fact is the cross property here, formalizing the interplay between knowledge and effort. Thus, a bi-modal Kripke frame is called a *cross axiom frame,* iff its relations satisfy all these conditions apart from the last one; and a bi-modal Kripke model is called a *cross axiom model,* iff it is based on a cross axiom frame and the final requirement is satisfied, too. Now, it is easy to see that every induced Kripke frame is a cross axiom frame and every induced Kripke model is a cross axiom model. Hence we should find the candidates relating to the above question among these structures.

We are going to change from first-order to topological properties for now. However, we shall return to those later on.

3 Topological Modal Logic

The paper [1] as well as van Benthem and Bezhanishvili's chapter of the handbook [2] (that is, Ch. 5 there) contain all the facts from topological modal logic that are relevant for our purposes; these are freely quoted below.

First in this section, we revisit the topological semantics of modal logic. Let $\mathcal{T} = (X, \tau)$ be a topological space, V a $\mathcal{T}$-valuation in the sense of Definition 1, and $\mathcal{M} := (X, \tau, V)$. Then, the topological satisfaction relation $\models_t$ is defined canonically for $\top$, the proposition variables, and in the boolean cases, whereas the clause for the $\Box$-operator reads

$$\mathcal{M}, x \models_t \Box\alpha :\iff \exists U \in \tau : [x \in U \wedge \forall y \in U : \mathcal{M}, y \models_t \alpha],$$

for all $x \in X$ and every mono-modal formula $\alpha \in \mathsf{MF}$.[2]

We now connect $\models_t$ to the common relational semantics. As is known, a reflexive and transitive binary relation R on a set W is called a *quasi-order* on W. Quasi-ordered non-empty sets (W, R) are also called $\mathsf{S4}$-*frames* since the modal logic $\mathsf{S4}$ is sound and complete with respect to this class of structures. Given an $\mathsf{S4}$-frame (W, R), a subset $U \subseteq W$ is called *R-upward closed* iff $w \in U$ and $w\,R\,v$ imply $v \in W$, for all $w, v \in W$. (Correspondingly, *R-downward closed* sets are defined.) The set of all R-upward closed subsets of W is, in fact, a topology on W (with the R-downward closed sets being topologically closed ones). This topology, denoted by τ_R, is *Alexandroff,* i.e., the intersection of arbitrarily many open sets is again open. With that, we obtain the following correlation (which is easy to prove again).

Proposition 2. *Let $M = (W, R, V)$ be an $\mathsf{S4}$-model and $\mathcal{M}_M := (W, \tau_R, V)$ be based on the associated Alexandroff space. Then, for all $\alpha \in \mathsf{MF}$ and $w \in W$, we have that $M, w \models \alpha$ iff $\mathcal{M}_M, w \models_t \alpha$.*

And vice versa, starting from an Alexandroff space $\mathcal{T} = (X, \tau)$ yields an equivalent $\mathsf{S4}$-frame $S_{\mathcal{T}} := (X, R_\tau)$ by taking the *specialization order* R_τ of τ for the accessibility relation (i.e., $x\,R_\tau\,y :\iff x$ belongs to the closure $\overline{\{y\}}$ of $\{y\}$, for all $x, y \in X$); as for the equivalence just asserted, note that we have $\tau = \tau_{R_\tau}$ in this case (while $R = R_{\tau_R}$ is always true).

Restricting the just established one-to-one correspondence to spaces satisfying the separation axiom T_0 additionally (i.e., for any two distinct points there is an open neighborhood of either of them not containing the other one), yields a one-to-one correspondence between partially ordered sets and Alexandroff T_0-spaces; this is recorded for later purposes here.

The next topic to be treated is the topological impact of the modal system $\mathsf{S5}$. It is well-known that the accessibility relation of a Kripke frame validating this logic is an *equivalence.* The topological counterpart of the class of all such frames is given by the next proposition.

Proposition 3. *Let $\mathcal{T} = (X, \tau)$ be a topological space. Then, all $\mathsf{S5}$-sentences are (topologically) valid in $\mathcal{T}$ iff every τ-closed set is open.*

Here, we have used the obvious notion of topological frame validity. A proof of Proposition 3 only making recourse to the satisfaction relation $\models_t$ is given

[2] This formulation and the one given in the introduction are easily seen to be equivalent.

in [2], p. 253. However, one can also argue with the aid of the accessibility relation and the specialization order, respectively. For that, note that the R-upward closed sets are precisely the unions of equivalence classes whenever R is an equivalence relation. This implies that every τ_R-closed set is open in this case. On the other hand, the latter demand on τ entails that R_τ is symmetric and thus an equivalence, as can be seen easily.

Finally in this section, we deal with the question of topological completeness, which has been touched upon in the introduction already. Concerning $\mathsf{S4}$, several ways to establish this property can be found in the literature quoted so far.

We focus on our bi-modal setting now, in view of subsequent applications. It turns out that the most straightforward proceeding will do here, at least for the time being.

Definition 3 (Bi-topological Structures).

1. *Let X be a non-empty set and σ, τ topologies on X. Then, the tuple $\mathfrak{S} := (X, \sigma, \tau)$ is called a* bi-topological space.
2. *Let $\mathfrak{S} = (X, \sigma, \tau)$ be a bi-topological space and V an $\mathfrak{S}$-valuation. Then $\mathfrak{M} := (X, \sigma, \tau, V)$ is called a* bi-topological model.

Unless stated otherwise, formulas from Form will be interpreted in bi-topological structures by use of the bi-topological satisfaction relation $\models_t$ as from now;[3] the modality K should correspond to σ and the modality $\Box$ to τ in doing so.

Proposition 2 has an obvious bi-modal analogue which is formulated for the more special structures we are interested in here.[4]

Proposition 4. *Let $M = (W, \{R, R'\}, V)$ be a Kripke model such that R is an equivalence relation, R' a quasi-order, and, for all proposition variables, V is constant along every R'-path. Moreover, let $\mathfrak{M}_M := (W, \tau_R, \tau_{R'}, V)$. Then, for all $\alpha \in \mathsf{Form}$ and $w \in W$, we have that $M, w \models \alpha$ iff $\mathfrak{M}_M, w \models_t \alpha$.*

Let LS denote the bi-modal logic determined by the axiom schemata $1-7$ from above (and having *modus ponens* as well as the *necessitation rules for both modalities* as proof rules). Then, we obtain the following theorem.

Theorem 1. *The logic LS is sound and complete with respect to the class of all bi-topological models (X, σ, τ, V) satisfying the following requirements.*

1. *Every σ-closed set is open.*
2. *The topology τ is Alexandroff.*
3. *For every point $x \in X$, the valuation V is constant throughout the least τ-open set containing x.*

[3] Concerning notations in this regard, we do not distinguish between the common mono-modal case and the bi-modal one considered here; this should not lead to confusion.

[4] Some later auxiliary results too could have been stated in a more general form.

Proof. First note that, for every point $x \in X$, a least τ-open neighborhood of x really exists in case τ is Alexandroff. Now, the soundness of the axioms is clear from the above, except for Axiom 5. However, the validity of Axiom 5 can be established directly (i.e., by using the definition of $\models_t$).

As to completeness, note that the canonical model M_{LS} of the logic LS satisfies all the conditions that are stated for M in Proposition 4. Thus, it suffices to prove that $\mathfrak{M}_{M_{\mathsf{LS}}}$ meets the three requirements given in the theorem. Since the first and the second item are clear from the above again, an argument is needed for the third one only. For it, note that, in general, the least $\tau_{R'}$-open neighborhood of any point x is contained in the union of all R'-paths through x, provided that R' is a quasi-order. The path-constancy of the proposition variables, which is satisfied on the canonical model, therefore implies the validity of the third condition. This completes the proof of the theorem.

Can the preceding theorem be extended (in the correctly understood sense) to the logic LSS? – Among other things, this question will be discussed in the next section.

4 Topological Cross Axiom Spaces

It is not immediately clear how a topological counterpart of the cross property looks like. The 'naïve' LSS-analogue of Theorem 1 should, therefore, apply to the specialization orders of the topologies involved. We state the corresponding result at the beginning of this section. Afterwards, we show that a particular correspondence between topological concepts and the Cross Axioms appears nevertheless. – We need a certain converse of Proposition 4.

Proposition 5. *Let $\mathfrak{M} := (X, \sigma, \tau, V)$ be a bi-topological model, and let $M_{\mathfrak{M}} := (X, \{R_\sigma, R_\tau\}, V)$. Then, for all $\alpha \in$ Form and $x \in X$, we have that $\mathfrak{M}, x \models_t \alpha$ iff $M_{\mathfrak{M}}, x \models \alpha$.*

With that, the just announced theorem can be proved easily.

Theorem 2. *The logic LLS is sound and complete with respect to the class of all bi-topological models (X, σ, τ, V) satisfying the following requirements.*

1. *Every σ-closed set is open.*
2. *The topology τ is Alexandroff.*
3. *The specialization orders R_σ and R_τ satisfy the cross property (i.e., $R_\tau \circ R_\sigma \subseteq R_\sigma \circ R_\tau$).*
4. *For every point $x \in X$, the valuation V is constant throughout the least τ-open set containing x.*

Proof. Only the third item must be considered yet. First, note that the canonical relations of LLS (see [4], Definition 4.18) satisfy the cross property.[5] Hence

[5] A direct argument for this is given in [5], Proposition 2.2. Note that a simpler argument would do in case of a normal modal logic, since the Cross Axioms $\mathsf{K}\Box p \rightarrow \Box \mathsf{K} p$ with p a proposition variable are Sahlqvist formulas; see [4], Theorem 4.42.

completeness ensues in the same way as in the proof of Theorem 1. On the other hand, the soundness of LLS for the given class of structures follows from Proposition 5. This proves the theorem.

We now introduce a certain cover property for bi-topological spaces. Then we show that this property corresponds to the Cross Axioms in the same way as, for example, the transitivity of the accessibility relation associated with the operator $\Box$ corresponds to the formula schema $\Box\alpha \to \Box\Box\alpha$ (as related to the most basic modal logic).

Definition 4 (Cover Property). *Let $\mathfrak{S} = (X, \sigma, \tau)$ be a bi-topological space. Then, $\mathfrak{S}$ is said to satisfy the* cover property *iff, for all points $x \in X$, every τ-open cover $\mathcal{C}$ of any σ-open neighborhood of x contains a σ-open cover $\mathcal{C}'$ of some τ-open neighborhood of x (to the effect that $\bigcup\mathcal{C} \supseteq \bigcup\mathcal{C}'$).*

The desired correspondence between the cover property and the Cross Axioms is established by the following proposition (cf. Proposition 3).

Proposition 6. *Let $\mathfrak{S} = (X, \sigma, \tau)$ be a bi-topological space. Then, all the Cross Axioms are (topologically) valid in $\mathfrak{S}$ iff $\mathfrak{S}$ satisfies the cover property.*

Proof. First, we prove that every Cross Axiom is valid in $\mathfrak{S}$ whenever $\mathfrak{S}$ satisfies the cover property. To this end, take any bi-topological model $\mathfrak{M} = (X, \sigma, \tau, V)$ based on $\mathfrak{S}$ and any point $x \in X$, and assume that $\mathfrak{M}, x \models_t \mathsf{K}\Box\alpha$ (with $\alpha \in \mathsf{Form}$). Then there exists a σ-open neighborhood U of x such that $\Box\alpha$ holds in $\mathfrak{M}$ throughout U. Thus, for all $y \in U$ there is a τ-open neighborhood U_y of y such that α holds in $\mathfrak{M}$ throughout U_y. Evidently, $\mathcal{C} := \{U_y \mid y \in U\}$ is a τ-open cover of the σ-open neighborhood U of x. According to the cover property, $\mathcal{C}$ contains a σ-open cover $\mathcal{C}'$ of some τ-open neighborhood U_x of x. Take any $z \in U_x$. Then there is a σ-open set $U' \in \mathcal{C}'$ containing z. We have $\mathfrak{M}, z \models_t \mathsf{K}\alpha$ because $U' \subseteq \bigcup\mathcal{C}' \subseteq \bigcup\mathcal{C}$. From that we obtain that $\mathfrak{M}, x \models_t \Box\mathsf{K}\alpha$, as z has been chosen arbitrarily. This shows that $\mathsf{K}\Box\alpha \to \Box\mathsf{K}\alpha$ is valid in $\mathfrak{S}$.

Second, suppose that the cover property is violated in $\mathfrak{S}$. Then there is a point $x \in X$ and a τ-open cover $\mathcal{C}$ of some σ-open neighborhood U of x such that no σ-open cover $\mathcal{C}'$ of any τ-open neighborhood U' of x is contained in $\mathcal{C}$. Define an $\mathfrak{S}$-valuation V as follows. Fix any $p \in \mathsf{Prop}$, let $V(p) := \bigcup\mathcal{C}$, and let V be arbitrary for the proposition variables different from p. Let $\mathfrak{M} := (X, \sigma, \tau, V)$. Then, $\mathfrak{M}, x \models_t \mathsf{K}\Box p$. On the other hand, for all τ-open neighborhoods U' of x there is a point $y \in U'$ such that every σ-open neighborhood U'' of y contains a point $z \notin \bigcup\mathcal{C}$, since otherwise we could construct a good-natured σ-open cover $\mathcal{C}'$ of some τ-open neighborhood of x. This implies that $\mathfrak{M}, x \models_t \Diamond\mathsf{L}\neg p$. It follows that some of the Cross Axioms are invalid in $\mathfrak{S}$.

Note that, in a sense, the just given argument is incompatible with the requirement on the constance of proposition variables as stated, e.g., in the fourth item of Theorem 2.

With a view to canonicity (and to the topological characterization result we have in mind), we now connect the cover property with the cross property (in the fashion of our reasoning after Proposition 3).

Proposition 7. *1. Let $S = (W, \{R, R'\})$ be a cross axiom frame, and let $\mathfrak{S}_S := (W, \tau_R, \tau_{R'})$. Then $\mathfrak{S}_S$ satisfies the cover property.*

2. Let $\mathfrak{S} = (X, \sigma, \tau)$ be a bi-topological space with σ and τ being Alexandroff. Suppose that $\mathfrak{S}$ satisfies the cover property. Then the associated Kripke frame $S_{\mathfrak{S}} := (X, \{R_\sigma, R_\tau\})$ satisfies the cross property.

Proof. 1. For every $u \in W$, let $R(u) := \{v \in W \mid u\,R\,v\}$ and $R'(u) := \{v \in W \mid u\,R'\,v\}$. Obviously, $R(u)$ and $R'(u)$ are the least τ_R-open and $\tau_{R'}$-open neighborhoods of u, respectively; moreover, $R(u)$ equals the R-equivalence class of u. Now, let $w \in W$ be any point, and let $\mathcal{C}$ be any $\tau_{R'}$-open cover of some τ_R-open neighborhood U_w of w. Take the $\tau_{R'}$-open neighborhood $R'(w)$ of w and define $\mathcal{C}' := \{R(v) \mid v \in R'(w)\}$. Then, $\mathcal{C}'$ clearly is a τ_R-open cover of $R'(w)$. We argue that $\bigcup \mathcal{C}' \subseteq \bigcup \mathcal{C}$. For this, take any $x \in \bigcup \mathcal{C}'$. Then, $x \in R(v)$ for some $v \in R'(w)$. Thus we have $w\,R'\,v\,R\,x$. Due to the cross property, it follows that $w\,R\,y\,R'\,x$, for some $y \in W$. We obtain $y \in R(w) \subseteq U_w$ because of the minimality of $R(w)$. And we get $x \in R'(y) \subseteq U$ for some $U \in \mathcal{C}$ because of the minimality of $R'(y)$ and the fact that $y \in U_w$. This shows that $x \in \bigcup \mathcal{C}$, as desired.

2. Let $x\,R_\tau\,y\,R_\sigma\,z$ be satisfied for any $x, y, z \in X$. We have $\sigma = \sigma_{R_\sigma}$ and $\tau = \tau_{R_\tau}$, since σ and τ are Alexandroff; this was mentioned right after Proposition 2 above. Thus, it makes sense to speak about the minimal τ-open cover $\mathcal{C}$ of the minimal σ-open neighborhood U_x of the point x on the one hand, on the other hand, we have $U_x = R_\sigma(x)$ and $\mathcal{C} = \{R_\tau(u) \mid u \in U_x\}$. According to the cover property, $\mathcal{C}$ contains a σ-open cover $\tilde{\mathcal{C}}$ of some τ-open neighborhood of x. For reasons of minimality, this means that $\mathcal{C}$ contains the cover $\mathcal{C}'$ of the minimal τ-open neighborhood of x defined in the first part of the proof (here with R_σ instead of R and R_τ instead of R' though) as well. From $x\,R_\tau\,y\,R_\sigma\,z$ we now infer $z \in \bigcup \mathcal{C}'$. Hence $z \in \bigcup \mathcal{C}$. This implies that there exists a point $v \in X$ such that $x\,R_\sigma\,v\,R_\tau\,z$, due to the choice of $\mathcal{C}$. Thus, the cross property is established.

As a consequence, we obtain the following characterization of bi-topological spaces arising from cross axiom frames.

Theorem 3. *Let $\mathfrak{S} = (X, \sigma, \tau)$ be a bi-topological space. Then there is a cross axiom frame $S = (W, \{R, R'\})$ such that $\sigma = \tau_R$ and $\tau = \tau_{R'}$ iff*

1. *every σ-closed set is open,*
2. *the topology τ is Alexandroff, and*
3. *$\mathfrak{S}$ satisfies the cover property.*

Proof. The necessity of the three conditions follows from both Proposition 7.1 and some of the results quoted in Section 3. Now, assume that these conditions are satisfied. Then σ is clearly Alexandroff. By Proposition 7.2, the frame $S_{\mathfrak{S}} = (X, \{R_\sigma, R_\tau\})$ satisfies the cross property. Moreover, R_σ is an equivalence and R_τ a quasi-order; see Section 3 again. Additionally, we have $\sigma = \tau_{R_\sigma}$ and $\tau = \tau_{R_\tau}$. This proves the theorem.

In the next section, we will prove a similar (but more complex) statement with regard to Kripke structures induced by subset spaces. This will be the main outcome of this paper.

By virtue of Theorem 3, a bi-topological space $\mathfrak{S} = (X, \sigma, \tau)$ is called a *topological cross axiom space* iff those three requirements are satisfied. And a bi-topological model $\mathfrak{M} = (X, \sigma, \tau, V)$ based on a topological cross axiom space is called a *topological cross axiom model* iff, for every point $x \in X$, the valuation V is constant throughout the least τ-open set containing x.

We conclude this section with a version of Theorem 2 having a purely topological reading.

Theorem 4. *The logic* LLS *is sound and complete with respect to the class of all topological cross axiom models.*

Proof. The soundness of LLS with respect to the given class of structures is clear from (the uncritical part of) Proposition 6. Concerning completeness, we must give reasons respecting just the cover property. We proceed as in the proof of Theorem 2 relating to this, and apply Proposition 7.1 additionally.

Theorem 3 and Theorem 4 comprise, in particular, all that we can achieve with regard to our characterization problem on the (modal-)logical side. However, more turns out to be possible on the topological one.

5 The Characterization Theorem

We shall now specify a couple of further requirements for bi-topological Alexandroff spaces to arise from induced cross axiom frames. This puts us in a position to state and prove the main result of this paper subsequently.

Definition 5 (Minimal Basis; Orthogonality).

1. *Let $\mathfrak{S} = (X, \sigma, \tau)$ be a bi-topological space such that σ and τ are Alexandroff. For any $x \in X$, let $R_\sigma(x) := \{y \in X \mid x\, R_\sigma\, y\}$ (as above), and let $R_\tau(x)$ be defined analogously. Then, the sets $\mathcal{B}_\sigma := \{R_\sigma(x) \mid x \in X\}$ and $\mathcal{B}_\tau := \{R_\tau(x) \mid x \in X\}$ are called the* minimal bases *of σ and τ, respectively.*
2. *Let $\mathfrak{S}$, x, and $R_\tau(x)$, be as above. Then we let $R_\tau^{-1}(x) := \{y \in X \mid y\, R_\tau\, x\}$ and $\overline{\mathcal{B}_\tau} := \{R_\tau^{-1}(x) \mid x \in X\}$. The latter set is called the* set of minimal τ-closed sets*.*
3. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ be two sets of subsets of X. These sets are said to be* orthogonal *iff any two members $A \in \mathcal{A}$ and $B \in \mathcal{B}$ intersect in at most one point.*

Note that $\mathcal{B}_\sigma$ and $\mathcal{B}_\tau$ are indeed bases of σ and τ, respectively. Moreover, note that, for every $x \in X$, the set $R_\tau^{-1}(x)$ is downward closed (see Section 3) and equals the closure $\overline{\{x\}}$ of $\{x\}$ actually; this justifies the naming in Definition 5.2. Finally, the condition stated in the third item reflects, at least in part, the geometric idea of orthogonality.

We obtain the following criterion resting on the just introduced notations.

Proposition 8. *Let $\mathfrak{S} = (X, \sigma, \tau)$ be a bi-topological space with σ and τ being Alexandroff.*

1. *The minimal bases $\mathcal{B}_\sigma$ and $\mathcal{B}_\tau$ are orthogonal iff, for all $x, y \in X$, there exists at most one R_τ-successor of x inside $R_\sigma(y)$.*
2. *The minimal base $\mathcal{B}_\sigma$ and the set $\overline{\mathcal{B}_\tau}$ of minimal τ-closed sets are orthogonal iff, for all $x, y \in X$, there exists at most one R_τ-predecessor of x inside $R_\sigma(y)$.*

Proof. 1. First, assume that $\mathcal{B}_\sigma$ and $\mathcal{B}_\tau$ are orthogonal. Let x, y be any points of X and suppose that two different R_τ-successors z_1, z_2 of x are contained in $R_\sigma(y)$. Then, however, a contradiction to the orthogonality of the minimal bases immediately results, since both $z_1, z_2 \in R_\tau(x)$. – The sufficiency of the condition can be seen easily as well.
2. This assertion can be proved in a similar manner.

For brevity, we say that a bi-topological space $\mathfrak{S}$ satisfies the *orthogonality properties* iff both conditions stated in Proposition 8 are met.

Our next requirement concerns a certain binary relation $\preccurlyeq_\mathfrak{S}$ on the minimal base $\mathcal{B}_\sigma$ of a bi-topological Alexandroff space $\mathfrak{S} = (X, \sigma, \tau)$. This relation should be a quasi-order and, in a sense, without a gap. The precise definitions follow right away.

Definition 6 ($\preccurlyeq_\mathfrak{S}$; Density Property). *Let $\mathfrak{S} = (X, \sigma, \tau)$ be a bi-topological space with σ and τ being Alexandroff.*

1. *For all $x, y \in X$, put $R_\sigma(x) \preccurlyeq_\mathfrak{S} R_\sigma(y) :\iff$ there are $x' \in R_\sigma(x)$ and $y' \in R_\sigma(y)$ such that $y' \in R_\tau(x')$.*
2. *The just defined relation $\preccurlyeq_\mathfrak{S}$ is said to satisfy the* density property *iff, whenever $R_\sigma(x) \preccurlyeq_\mathfrak{S} R_\sigma(y) \preccurlyeq_\mathfrak{S} R_\sigma(z)$, then, for any $x' \in R_\sigma(x)$ and $z' \in R_\sigma(z)$ such that $z' \in R_\tau(x')$, there exists $y' \in R_\sigma(y)$ satisfying $y' \in R_\tau(x')$ and $z' \in R_\tau(y')$.*

With that, we obtain the following result with the aid of standard arguments from the logic of subset spaces.

Proposition 9. *Let $\mathfrak{S} = (X, \sigma, \tau)$ be a topological cross axiom space. Then, the corresponding relation $\preccurlyeq_\mathfrak{S}$ is*

1. *a quasi-order in any case, and*
2. *even a partial order if, in addition, the relation R_τ is antisymmetric and the minimal bases $\mathcal{B}_\sigma$ and $\mathcal{B}_\tau$ are orthogonal.*

Proof. 1. The reflexivity of $\preccurlyeq_\mathfrak{S}$ is obvious. In order to establish the transitivity of this relation, we take advantage of the cross property, which is satisfied by the frame $S_\mathfrak{S} = (X, \{R_\sigma, R_\tau\})$ according to Proposition 7.2.
2. This follows from the definitions with the aid of the cross property again.[6]

[6] Note that this result can under certain conditions be obtained 'purely logically', by adding a particular axiom schema for tree-like structures; see [9], Proposition 3.5.

Finally, we introduce a condition that may appear somewhat odd to the reader at first glance. However, its significance will become clear from the construction in the proof of Theorem 5 below.

Definition 7 (Tamely Ramified). *Let $\mathfrak{S} = (X, \sigma, \tau)$ be a bi-topological space such that σ and τ are Alexandroff. Moreover, let $\mathcal{B}_\sigma$ and $\mathcal{B}_\tau$ be the minimal bases of σ and τ, respectively. Then we say that $\mathcal{B}_\sigma$ is* tamely ramified across *$\mathcal{B}_\tau$, iff the following is satisfied for any two $R_\sigma(x), R_\sigma(y) \in \mathcal{B}_\sigma$: if every point of $R_\sigma(y)$ is contained in the symmetric closure, taken with respect to R_τ, of $R_\sigma(x)$, then $R_\sigma(x) \preccurlyeq_{\mathfrak{S}} R_\sigma(y)$.*

It turns out that tame ramification in the sense of the previous definition is always factual for spaces that are derived from induced Kripke frames.

Proposition 10. *Let $\mathfrak{S} = (X, \sigma, \tau)$ be a bi-topological space and $\mathcal{S} = (X, \mathcal{O})$ a subset frame such that $\sigma = \tau_{R^{\mathsf{K}}_{\mathcal{S}}}$ and $\tau = \tau_{R^{\Box}_{\mathcal{S}}}$. Then $\mathcal{B}_\sigma$ is tamely ramified across $\mathcal{B}_\tau$.*

Proof. Due to the fact that $R^{\Box}_{\mathcal{S}}$ originates from the containment relation, the correctness of the assertion can be seen rather easily.

The preparatory work towards our main result has been completed by the last proposition. Thus, we are in a position to prove our final theorem now.

Theorem 5. *Let $\mathfrak{S} = (X, \sigma, \tau)$ be a bi-topological space. Then there is a subset frame $\mathcal{S} = (X, \mathcal{O})$ such that $\sigma = \tau_{R^{\mathsf{K}}_{\mathcal{S}}}$ and $\tau = \tau_{R^{\Box}_{\mathcal{S}}}$ iff*

1. *every σ-closed set is open,*
2. *the topology τ is Alexandroff and satisfies the separation property T_0,*
3. *$\mathfrak{S}$ satisfies the cover property,*
4. *$\mathfrak{S}$ satisfies the orthogonality properties,*
5. *the relation $\preccurlyeq_{\mathfrak{S}}$ is a quasi-order satisfying the density property,*
6. *the minimal basis $\mathcal{B}_\sigma$ of σ is tamely ramified across the minimal basis $\mathcal{B}_\tau$ of τ, and*
7. *every element of the minimal basis $\mathcal{B}_\sigma$ contains a τ-open point.*

Proof. The left-to-right direction is easy to prove. Let $\mathcal{S} = (X, \mathcal{O})$ be a subset frame such that $\sigma = \tau_{R^{\mathsf{K}}_{\mathcal{S}}}$ and $\tau = \tau_{R^{\Box}_{\mathcal{S}}}$. Items 1 and 2 then follow from topological modal logic; see Section 3. Item 3 is clear from Proposition 7.1, since $R^{\mathsf{K}}_{\mathcal{S}}$ and $R^{\Box}_{\mathcal{S}}$ satisfy the cross property; see the end of Section 2. Furthermore, one is quickly convinced that the first-order conditions corresponding to the orthogonality and the density properties are applicable to $R^{\mathsf{K}}_{\mathcal{S}}$ and $R^{\Box}_{\mathcal{S}}$. By Proposition 8, $\mathfrak{S}$ satisfies the orthogonality properties, and by Proposition 9.1, the relation $\preccurlyeq_{\mathfrak{S}}$ is a quasi-order which, in particular, satisfies the density property. Proposition 10 guarantees that the last but one item is satisfied. For the last one, note that the openness of $\{x\}$ exactly means that x has no R_τ-successor apart from x itself.

For the other direction, assume that the seven requirements are met by $\mathfrak{S}$. It suffices to show that $S_{\mathfrak{S}} = (X, \{R_\sigma, R_\tau\})$ is isomorphic to the Kripke frame

$S_{\mathcal{S}} = \big(W_{\mathcal{S}}, \{R_{\mathcal{S}}^{\mathsf{K}}, R_{\mathcal{S}}^{\Box}\}\big)$ induced by some subset frame $\mathcal{S} = (Y, \mathcal{O})$, for we will have $\sigma = \tau_{R_\sigma} = \tau_{R_{\mathcal{S}}^{\mathsf{K}}}$ and $\tau = \tau_{R_\tau} = \tau_{R_{\mathcal{S}}^{\Box}}$ in this case (after identifying isomorphic structures). – The following is clear from our previous statements and results.

(a) The relation R_σ is an equivalence (see Proposition 3 and the remark thereafter), and the relation R_τ is a partial order (see the remarks after Proposition 2).
(b) $S_{\mathcal{S}}$ satisfies the cross property (see Proposition 7.2).
(c) For all $x, y \in X$, there exists at most one R_τ-successor of x inside the equivalence class $R_\sigma(y)$ of y, and there exists at most one R_τ-predecessor of x inside $R_\sigma(y)$ (see Proposition 8).
(d) The relation $\preccurlyeq_{\mathfrak{S}}$ is a partial order (see Proposition 9.2).

The set Y will be obtained as a certain set of partial functions on $\mathcal{B}_\sigma$ shortly.[7] For this purpose, let $x, y \in X$ be given and suppose that $R_\sigma(x) \preccurlyeq_{\mathfrak{S}} R_\sigma(y)$. It can be concluded from (b) and (c) that the relation R_τ restricted to $R_\sigma(x)$ in the domain and $R_\sigma(y)$ in the range, is an injective and surjective partial function, say $f_{R_\sigma(x), R_\sigma(y)}$. This function is, in fact, *strictly* partial because of the last condition stated in the theorem. Now, let Y be the set of all partial functions $f : \mathcal{B}_\sigma \to X$ having a domain $\mathrm{dom}(f)$ that is maximal with respect to the following three conditions:

- $f(R_\sigma(x)) \in R_\sigma(x)$, for all $R_\sigma(x) \in \mathrm{dom}(f)$;
- $f(R_\sigma(y)) = f_{R_\sigma(x), R_\sigma(y)} \circ f(R_\sigma(x))$, for all $R_\sigma(x), R_\sigma(y) \in \mathrm{dom}(f)$ satisfying $R_\sigma(x) \preccurlyeq_{\mathfrak{S}} R_\sigma(y)$;
- the range of f is R_τ-*connected,* i.e., for all $x, y \in \mathrm{range}(f)$, $x \, R_\tau^s \, y$ is valid, where R_τ^s denotes the symmetric closure of R_τ.

Note that (d) and the density property imply the coherence of the second condition, whence the process of maximizing the domain is really possible.

For every $x \in X$, let $U_{R_\sigma(x)} := \{f \in Y \mid f(R_\sigma(x)) \text{ exists}\}$, and let $\mathcal{O} := \{U_{R_\sigma(x)} \mid x \in X\}$. Then, $S_{\mathfrak{S}} \cong S_{\mathcal{S}}$ is valid for the subset frame $\mathcal{S} := (Y, \mathcal{O})$. To see this, note that a one-to-one mapping h from the set $\mathcal{N}_{\mathcal{S}}$ of all neighborhood situations of $\mathcal{S}$ onto the set of all points of X is mediated by $f, U_{R_\sigma(x)} \mapsto f_{R_\sigma(x)}$, where $f_{R_\sigma(x)} := f(R_\sigma(x))$, in such a way that, for all $f, g \in Y$ and $x, y \in X$ with $f \in U_{R_\sigma(x)}$, we have that

$$g \in U_{R_\sigma(x)} \iff g_{R_\sigma(x)} \, R_\sigma \, f_{R_\sigma(x)}.$$

All this is rather easy to prove, and the claimed isomorphism is established with regard to the K-component thus. As to the $\Box$-part, we prove that, for all $f \in Y$ and $x, y \in X$ such that $f \in U_{R_\sigma(x)} \cap U_{R_\sigma(y)}$,

$$U_{R_\sigma(y)} \subseteq U_{R_\sigma(x)} \iff f_{R_\sigma(x)} \, R_\tau \, f_{R_\sigma(y)},$$

showing the compatibility of the containment relation $\subseteq$ with the accessibility relation R_τ. The right-to-left direction is more or less obvious. For the left-to-right direction, assume that $U_{R_\sigma(y)} \subseteq U_{R_\sigma(x)}$. This means that, for all $f \in Y$, if

[7] We once more note that $\mathcal{B}_\sigma$ equals the set of all R_σ-equivalence classes.

$f_{R_\sigma(y)}$ is defined, then $f_{R_\sigma(x)}$ is defined as well. According to the way the elements of Y have been obtained, we conclude that every point of $R_\sigma(y)$ is contained in the symmetric closure, taken with respect to R_τ, of $R_\sigma(x)$ from that. Now, the ramification condition applies, ensuring the existence of points $x' \in R_\sigma(x)$ and $y' \in R_\sigma(y)$ which materialize $R_\sigma(x) \preccurlyeq_{\mathfrak{S}} R_\sigma(y)$. Since $f \in U_{R_\sigma(x)} \cap U_{R_\sigma(y)}$, it follows that $f_{R_\sigma(x)} \, R_\tau \, f_{R_\sigma(y)}$ holds as well, whence the left-to-right direction is proved, too. Consequently, h is an isomorphism, as desired.

The proof of Theorem 5 lights up the relative proximity of the topological and the relational semantics of modal logic once again.

Finally in this section, we fix the analogue of Theorem 5 for bi-topological *models*. In fact, the following corollary is obtained as an immediate consequence of that theorem.

Corollary 1. *A bi-topological model $\mathfrak{M}$ is determined by a subset space in the sense of the preceding theorem, iff*

1. *the bi-topological space underlying $\mathfrak{M}$ satisfies all the conditions stated there, and*
2. *the valuation of $\mathfrak{M}$ meets the constancy property as formulated, e.g., in the third item of Theorem 1.*

6 Concluding Remarks

Investigations into multi-topological structures appear rather unfrequent in topological modal logic; see [2], Sect. 2 and Sect. 3 of Ch. 5, for some hints. The present paper adds a new facet to this field by working out a hitherto undiscovered connection between bi-modal logic and bi-topological spaces. We have, actually, given a bi-modally oriented characterization of bi-topological spaces arising from subset spaces here.

The second contribution of this paper is Theorem 4, stating the soundness and completeness of the logic of subset spaces, LLS, with respect to the class of all topological cross axiom spaces. In this connection, the question arises whether this theorem can also be proved in a 'more topological' way, i.e., by means of the approach to topological canonicity undertaken, e.g., in [1], Sect. 3.1.

Our new approach raises several issues that should be treated by future research. We only mention two of the questions coming up here, in particular, by confining ourselves to the framework of subset spaces. What is the bi-topological effect of those additional schemata that are relevant to the logic of *special classes* of subset spaces? And can notably topological spaces be characterized along the lines followed in this paper? – Here is a concrete starting point towards a possible answer. The *Weak Directedness Axioms* of common modal logic, $\Diamond\Box\alpha \rightarrow \Box\Diamond\alpha$, come along with the closure of the open sets under finite intersections in *topologic;* see [5]. In topological modal logic, we have a corresponding class of spaces: the *extremally disconnected* ones (where the closure of each open set is clopen by definition); see [2], Sect. 2.6 of Ch. 5. But we neither know up to now whether the latter property is sufficient for the closure under finite intersections (as related to the subset space semantics), nor how the *Union Axioms* of *topologic* can be captured within the new framework.

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