Chapter 3
The Veneziano Model

All through 1969 people were adding legs to the Veneziano amplitude, or chopping it in half.

Claud Lovelace

3.1 Duality and the Beta Function Amplitude

In this chapter we look at the birth of dual models (or, in full, dual resonance models) and the beginnings of dual *theory* (of which the hadronic string theory is an example, providing an interpretation of its oscillator formalism). Given that these were found to admit an interpretation as a string system, they are usually believed to constitute the simultaneous birth of string theory. This is somewhat inaccurate since the very earliest work on dual models, as with the work on duality that preceded it, had no explicit connection whatsoever to string models. They were an attempt to incorporate the FESR duality together with the other S-matrix principles in a single model describing hadrons. There was, at best, some indirect evidence of non-locality from the high spins that (after the fact) might have been seen as a result of coming from an underlying string system. There was also the Regge behaviour that could also be reinterpreted in string theoretic terms after the fact. Indeed, one might as well mark the notion of the Regge trajectory (with its peculiar regularities) as the birth of strings if one is allowing *post hoc* string interpretations. I would urge that we should understand the birth of string theory (hadronic string theory, or 'old' string theory, that is) as taking place with the (multiple independent) discoveries of the idea that a possible system *responsible* for 'generating' the Veneziano formula is a family of harmonic oscillators, and of a very specific type like the string of a guitar (this we discuss in the next chapter). To claim otherwise is clearly to project the later interpretation onto the earlier work. Having said that, Veneziano's formula clearly

D. Rickles, *A Brief History of String Theory*, The Frontiers Collection,
DOI: 10.1007/978-3-642-45128-7_3, © Springer-Verlag Berlin Heidelberg 2014

paved the way for such string interpretations, which were of course interpretations of the structure it revealed.

The crucial step connecting duality to the other desirable properties of the scattering amplitude (such as Regge behaviour and crossing) was, then, taken by Gabriele Veneziano.[1] He observed that the Euler Beta function was able to model these features in a nice condensed (and, indeed, closed) form. Veneziano's model performed two feats: (1) it captured the empirical linear Regge trajectories relating M^2 and J; (2) it incorporated the mathematical properties of scattering amplitudes expected of strongly interacting systems (including the DHS duality identifying t and s processes) but, on account of the narrow-resonance approximation used, it did *not* satisfy unitarity. It was therefore (*almost*) a complete solution of the bootstrap—a model satisfying the conditions imposed on the S-matrix without employing quantum field theory. This was the first example of a dual resonance model. Veneziano presented his idea in July 1968, to a seminar group at which Sergio Fubini was in attendance. Encouraged by Fubini's response, Veneziano published soon after ([46, p. 214]—see Fig. 3.1).[2]

Veneziano was able to construct a representation of the 4-meson process $\pi\pi \rightarrow \pi\omega$, written in closed form, using products of Gamma functions:

$$A(s,t,u) = \frac{\Gamma[1-\alpha(s)]\Gamma[1-\alpha(t)]}{\Gamma[-\alpha(s)-\alpha(t)]} + \frac{\Gamma[1-\alpha(s)]\Gamma[1-\alpha(u)]}{\Gamma[-\alpha(s)-\alpha(u)]} + \frac{\Gamma[1-\alpha(t)]\Gamma[1-\alpha(u)]}{\Gamma[-\alpha(t)-\alpha(u)]} \tag{3.1}$$

Each summand will have a singularity (i.e. a pole) at negative integer values $(0, -1, -2, ...)$ of the argument.[3] These singularities point to locations of particles on Regge trajectories (i.e. poles in s or t). Hence, the Γ-function singularities reproduce the spectrum of particles lying on linearly rising Regge trajectories of ever

---

[1] As Christoph Schmid points out, achieving this was not an obvious possibility at the time: "[l]et me remind you that many people published 'proofs' that duality was impossible ... until Veneziano (1968) published his beautiful model. Since one example is stronger than a thousand 'proofs' to the contrary, people had to accept the fact that duality was possible" [42, p. 125]. The discovery also seems to have opened the floodgates, for some, as regards the possibility of saying something profound about hadronic scattering amplitudes (*behind* the various approximations). As David Fairlie writes, "to everyone's complete surprise Gabriele Veneziano came up with his famous compact form for a dual scattering amplitude, which encompassed contributions from many towers of resonances, and I felt that this was for me!" [20, p. 283]. It is, of course, often the case that the impact of some result is all the more impressive when its prior probability is very low. We see the same 'high impact' phenomenon following Michael Green and John Schwarz's anomaly cancellation result (for specific string theories) which had also been assigned a vanishingly small prior probability by the community of physicists working on it before their discovery.

[2] As David Olive recalls, Veneziano presented his discovery "in the ballroom of the Hofburg ... during the Vienna Conference on High Energy Physics (28 August–5 September 1968)" [38, p. 346].

[3] The Euler beta function is related to the Gamma function a follows: $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. Hence, we can also write Eq. 3.1 simply as $A(s,t,u) = A(s,t) + A(s,u) + A(t,u)$. This expression encodes the three possible permutations of the four scattered particles' labels (that are neither cyclic nor

IL NUOVO CIMENTO Vol. LVII A, N. 1 1° Settembre 1968

Construction of a Crossing-Simmetric, Regge-Behaved Amplitude for Linearly Rising Trajectories.

G. Veneziano (*)
CERN - Geneva

(ricevuto il 29 Luglio 1968)

Crossing has been the first ingredient used to make Regge theory a predictive concept in high-energy physics. However, a complete and satisfactory way of imposing crossing and crossed-channel unitarity is still lacking. We can look at the recent investigations on the properties of Reggeization at $t=0$ as giving a first encouraging set of results along this line of thinking (1). A technically different approach, based on superconvergence, has been also recently investigated (2), and the possibility of a self-consistent determination of the physical parameters, through the use of sum rules, has been stressed.

In this note we propose a quite simple expression for the relativistic scattering amplitude, that obeys the requirements of Regge asymptotics and crossing symmetry in the case of linearly rising trajectories. Its explicit form is suggested by the work of ref. (3) and contains only a few free parameters (**).

Our expression contains automatically Regge poles in families of parallel trajectories (at all t) with residue in definite ratios. It furthermore satisfies the conditions of superconvergence (4) and exhibits in a nice fashion the duality between Regge poles and resonances in the scattering amplitude.

(*) On leave of absence from the Weizmann Institute of Science, Rehovoth. Address after 1 September 1968: Department of Physics, M.I.T., Cambridge, Mass.

(1) For a general review of these problems see L. Bertocchi: *Proc. of the Heidelberg International Conference on Elementary Particles* (Amsterdam, 1967).

(2) Such an approach was proposed independently by M. Ademollo, H. R. Rubinstein, G. Veneziano and M. A. Virasoro: *Phys. Rev. Lett.*, **19**, 1402 (1967) and *Phys. Lett.*, **27**, B 99 (1968), and by S. Mandelstam: *Phys. Rev.*, **166**, 1539 (1968). Further developments and a number of references to related works can be found in ref. (3).

(3) M. Ademollo, H. R. Rubinstein, G. Veneziano and M. A. Virasoro: Weizmann Institute preprint (1968), submitted to *Phys. Rev.*

(**) We shall mostly work here in the approximation of real, linear trajectories and consequently of narrow resonances. We briefly discuss the effects of a nonzero imaginary part in the trajectory function which, in any case, we demand to have a linearly rising real part.

(4) For superconvergence we mean both the original sum rules proposed by V. de Alfaro, S. Fubini, G. Furlan and C. Rossetti: *Phys. Lett.*, **21**, 576 (1966), and the more recent generalized superconvergence (finite-energy) sum rules (see ref. (3) for detailed references). A unified treatment of all superconvergence sum rules has been given by S. Fubini: *Nuovo Cimento*, **52** A, 224 (1967).

Fig. 3.1 The first page of the chapter that is often seen as marking the origin of string theory (Photo credit: Springer, [49]) higher rotations. Notice also that $A(s,t) = A(t,s)$, so that the formula satisfies DHS duality. This was no mean feat and formed the basis of a mini-industry of research on dual models, leading later to dual theory, and from there to both aspects of string theory and (via the string picture) QCD.

It is simpler to see this duality by writing the amplitude just as a function of s and t, giving (where, again, $\alpha(s)$ is the Regge-Mandelstam trajectory[4]):

$$A(s,t) = \frac{\Gamma[-\alpha(s)]\Gamma[-\alpha(t)]}{\Gamma[-\alpha(s) - \alpha(t)]} = \mathrm{B}(-\alpha(s), -\alpha(t)) \tag{3.2}$$

Here, $\mathrm{B}(-\alpha(s), -\alpha(t))$ is the Euler Beta function and can be represented as an integral:

$$\mathrm{B}(-\alpha(s), -\alpha(t)) = \int_0^1 dx\, x^{-\alpha(s)-1}\,(1-x)^{-\alpha(t)-1} \tag{3.3}$$

The formula describes (for the specified linear trajectories $\alpha(s)$) an infinite set of (zero width) poles.[5] Once this model was out, the immediate challenge was to add unitarity; generalise it from four-particle to multi-particle amplitudes; add spin and isospin; and understand it from a more physical point of view (that is, understand *what* it is a model *of*). The first step crucial along these latter lines was the development of the harmonic oscillator representation of the generalised amplitude [24], which allowed for a demonstration of factorization (on which, see Sect. 3.4).

Though Veneziano's amplitude satisfied duality, it was valid only in an extreme approximation, namely the 'narrow resonance approximation' (in fact, with *infinitesimally* narrow resonances), with infinitely many poles—so: an infinite set of particles is found to be sufficient for both resonances and Regge poles (exchange particles). Recall that the width is related to stability, so that in the limit of zero-width a particle would never decay.[6] Further, the 'zero-width' approximation used implies that an infinite family of 1-hadron states make up the intermediate states *and* implies that an infinite family of such states will be exchanged, giving the strong forces. Duality requires that we don't add these together, but treat them as different approximations to the same physical process. The narrow resonance approximation is a very useful

---

(Footnote 3 continued)
anti-cyclic): (1234), (1243), and (1324)—cf. [18, p. 61]. Or, in plain words, these characterise the three perspectives from which one can view the scattering of the particles in the various channels.

[4] At this stage there was no known restriction on the value of the intercept, so it seemed it could be fixed to physically reasonable values. However, Virasoro would later show that consistency (specifically, being ghost-free) demanded a unit intercept, $\alpha(0) = 1$. Other models would require slightly different, but still fixed, intercept values.

[5] Rather surprisingly perhaps, Chew was not happy with the Veneziano model because of this approximation. He viewed it as conceding too much to the fundamentalist (read *arbitrary*) approach. According to Chew, the general S-matrix constraints ought to "fix particle widths as well as particle masses" [12, p. 26].

[6] The standard interpretation was to view the Veneziano amplitude as the first term in a Born approximation to a more complete version of the amplitude which would be 'generated' by adding loop corrections, hopefully thereby fixing the problem of unitarity.

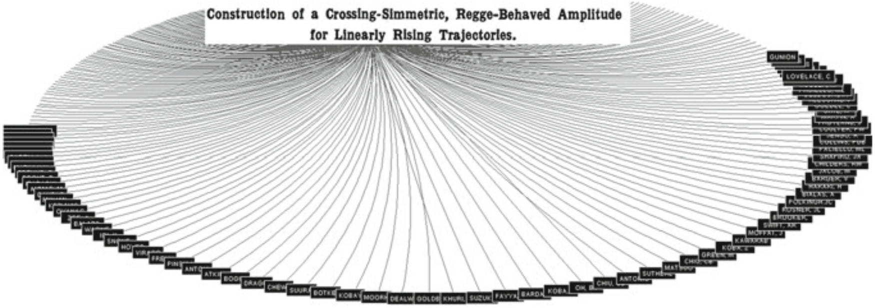


Fig. 3.2 The dramatic impact of Veneziano's paper. This network map shows the number of authors referring to the paper within a year of its publication, with each line representing a paper discussing the Veneziano model (Data generated using Thompson-Reuters, *Web of Knowledge*.)

tool for studying duality since one can get around the business (real though it is) of hadronic instability.

It was natural, therefore, to extend Veneziano's work to get around this shortcoming. The race to generalise, extend , and otherwise make sense of the Veneziano model was very dramatic, as can be seen in Fig. 3.2, which shows the number of papers citing Veneziano's paper within one year of its publication. There were a great many instances of independent simultaneous discoveries of the same results, making it particularly hard to pin down priority claims—and as a result I shall generally avoid this, except where priority is obvious.

It is worth pointing out that Veneziano's paper did not take with all strong interaction physicists; it was primarily deemed to be of importance to those already working on Regge theory and duality. For example, Gerardus 't Hooft's experience on first encountering it was as follows[7]:

> [W]e didn't understand it, I think. The understanding came much later, that this really was a string theory. So, Veneziano's paper was understood, but not as a string theory. It was some sort of abstract notion and ... it sounded like something very complicated, very difficult ... in particular how to show these theories of dispersion relations, are they unitary? And, all we knew was that the problem was complicated. People didn't give unique answers. Some people said, "yes," some people said, "no," some people said, "maybe." I mean, you had expressions which have pole singularities in the propagators. They have the right structure. They could obey dispersion relations. It's nearly right. So, in general the theory is nearly right, but there are still some things missing. And ... without the string interpretation these theories look very complicated. ... Later we realized ... that the particles are string-like. It became much more transparent. But then, I always thought there were fundamental difficulties with it, even in my days at CERN. Now and then I tried these theories, and often the theories were the strong interactions. But, later with QCD I realized these theories could be a good approximation ... to QCD. But, even as approximation[s] ... I failed to make sense of them. (Interview with the author, 10th February, 2010—AIP, p. 31; transcription courtesy of the American Institute of Physics)

[7] Note that 't Hooft's comments have some overlap with my reasons for being cautious about marking the birth of string theory with the construction of the Veneziano's model.

’t Hooft also recalls that there was no discussion of dual models at the Cargèse summer school on physics, in 1970, despite the fact that the likes of Jean-Loup Gervais were present—of course, ’t Hooft’s mind was focused firmly on field theory and gauge theory at the time, so he might well have been filtering out discussions that didn’t fit.

3.2 Suzuki’s ‘Small Detour’

For me, the beta function amplitude was a small detour in my long career.
Mahiko Suzuki

Before we proceed to the near-industrial scale refinement job that followed the publication (and spread) of Veneziano’s discovery, I would first like to consider another figure that is often referred to as a kind of ‘co-discoverer’ of the link between the beta function and a dual amplitude: namely, Mahiko Suzuki, referred to in the first chapter. Suzuki’s story paints an interesting (and more complete) picture of the research landscape around the time of Veneziano’s own discovery.[8]

Suzuki received his PhD from the University of Tokyo in 1965, under the supervision of Hironari Miyazawa, who, curiously enough, himself proposed a fledgling version of supersymmetry for hadrons in 1966 (relating mesons and baryons)—though his version involved internal rather than space-time transformations. Miyazawa arranged for Suzuki to skip his final year of graduate school and join Gell-Mann’s Caltech group, as a Fulbright scholar, from 1965 to 1967—upgraded to a Richard Chase Tolman Research Fellowship in 2 years, thanks to Gell-Mann. Among those he shared an office with were David Horn and Christoph Schmid (the H and S from DHS-duality), and also Roger Dashen and Stephen Adler. Indeed, one of his early collaborations was on the FESR with Horn and Schmid. Though he left this collaboration early on, he readily admits that the FESR work was vital background for his own discovery of the beta function amplitude. Suzuki spent a year at the IAS in Princeton following his Caltech fellowship, and coincidentally Gell-Mann took his sabbatical there that year, cunningly co-arranged with Low, Goldberger, and Kroll, who also took their sabbaticals at the IAS that year. Also present as postdocs were Daniel Freedman and Jiunn-Ming Wang, who had just come up with their result about parallel Regge trajectories (see footnote 29). Suzuki came up with the idea that the beta function must be the scattering amplitude that incorporates duality during this visit, during the end of term break.

[8] Of course, it is the dissemination that (quite rightly) holds the weight in matters of scientific discovery, so I don’t mean to reduce Veneziano’s place in the history of dual theory and string theory with this discussion. My aim is to flesh out the background to the discovery and to present a piece of the history that has hitherto remained under wraps—my sincere thanks to Professor Suzuki for sharing his story with me.

Suzuki seems to have followed a similar path to Veneziano, namely going to the zero-width limit to achieve a simplification of the scattering problem and to make duality easier to satisfy in a transparent way. He describes his next steps as follows:

> I needed a gamma function of a Regge trajectory to incorporate a family of the Regge trajectory and its daughters in the intermediate state channel, and another gamma function for the Regge family in the force channel. After taking product (not sum) of these two gamma functions, I need the third function to make the high-energy (Regge) asymptotic behaviour in agreement with experiment. In the spring of 1968, I tried to realize this ideal limit of the hadron amplitudes. Once the problem was simplified so much, I had only to look for a right product of gamma functions (Private communication).

Thus the stage was set in such a way as to allow for a methodical search for an appropriate mathematical expression that supplied the required product. In this sense one can see very clearly that luck plays no role, and I expect that the same methodical procedure lay behind Veneziano's discovery. The book of formulas that Suzuki found the correct expression in was the 3-volume set *Sugaku Koshiki*,[9] by Moriguchi, Udagawa and Hitotsumatsu (Iwanami Shoten, 1956). The necessary function was on p. 2 of volume 3, in an entry entitled: "The asymptotic behavior of a ratio of gamma functions". Though the asymptotic behavior of the *Beta* function was not covered, Suzuki had no problem transforming it into that of the beta function.[10]

On this discovery, that ignited so much subsequent work after the publication of Veneziano's paper, he writes that "it was a small (I thought so, then) exciting discovery for me" (private communication). Hence, this reveals an interesting parallel discovery of the beta function-dual amplitude connection, replete with what looked like the right scientific context (with Horn, Schmid, and Gell-Mann all in Suzuki's loop). Suzuki prepared a paper containing the result, and planned to submit it to *Physics Letters B*, once he had arrived at CERN, where he was due to stay as a visiting scientist for a few months after his Fulbright had expired. He didn't feel any pressure since, as he puts it, he "did not anticipate anybody could possibly come up with this esoteric amplitude". On his arrival at CERN Suzuki handed over his manuscript (handwritten) to the secretary (a Madame Fabergé) for typesetting ready for mailing to the journal. However, Suzuki was told that the paper had first to be approved by a senior physicist. Suzuki went to Leon van Hove's office and explained his beta function amplitude idea, after which van Hove agreed to read the manuscript, whereupon he placed the paper in a drawer. It turned out that Schmid was a postdoc at CERN at the same time, and so he was there to greet Suzuki on his first day. On explaining his work to Schmid (following the customary academic greeting of 'what are you working on?'), Schmid pointed out that a young postdoc by the name of Veneziano had written a preprint of work that sounded similar. Suzuki rushed off

[9] Meaning "mathematical formulas". These volumes traveled with Suzuki when he left Japan for life in Pasadena. They still grace his shelves in Berkeley. Following an expression of interest in them from David Horn, Suzuki had a set mailed over as a present to Horn in 1966.

[10] Though it seems he initially stuck to writing it as the ratio of gamma functions, as expressed in his book of formulas. It was in fact Ling-Lie Wang (now Chau, currently at UC Davis) that mentioned that this ratio was simply the beta function, while chatting about their current research topics in the IAS library reading room.

to get a copy of the preprint and discovered that it was virtually identical, save for the choice of properties of the scattered particles, $\pi\pi \rightarrow \omega\pi$ in Veneziano's case and $\pi^- p \rightarrow \pi^0 n$ in Suzuki's case (where he had omitted spins as an inessential complication).[11] Suzuki realised he'd been "scooped" and retracted his paper from van Hove's office.

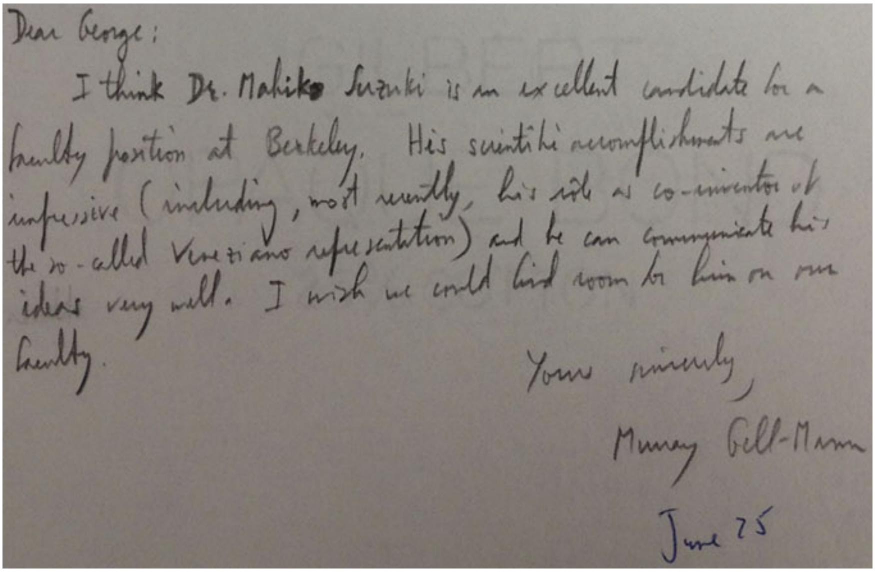
Thus goes the story of a parallel discovery of the beta function-dual amplitude. Had the timing been a little different, Suzuki might have been able to give a seminar on the result at the IAS, or at least discussed it further with people like Horn, Schmid, and Gell-Mann. Of course there are many such instances of multiple, parallel discoveries in science—not least the discovery of 'the Higgs mechanism'! Interestingly, Suzuki suggests that he and Veneziano were not alone in their search for an appropriate function. Chatting to Nambu during the late summer of 1968, while attending the biennial international high energy conference in Vienna, Suzuki discovered that Nambutoo was "casually combing for fun some mathematical books to search a function that satisfies the nuclear democracy or the duality" (private communication). Though Nambu didn't come to the beta function in his search, it highlights the fact that there was scientific convergence and, once again, discredits the notion that there was any kind of randomness involved in the discovery of the beta function amplitude.

It seems that Murray Gell-Mann had been aware of Suzuki's parallel discovery for he explicitly credits him as "co-inventor" in a reference for a position at Berkeley in 1969 (see Fig. 3.3). Suzuki recalls George Trilling, then Chairman of physics at Berkeley, introducing him as "co-discoverer of the Veneziano amplitude" during his department colloquium in the late summer of 1969.

3.3 Unitarity, Generalisations, and Extensions

The Veneziano model, though impressive, did not involve *all* the desirable properties of a good S-matrix bootstrap: it violated unitarity (i.e. the preservation of the probabilities summing to one [= unity], at each instant of time) and involved only four particles. Both a tree-level N-point amplitude and a treatment of loop amplitudes were needed to patch these problems and achieve a complete theory. Adding unitarity to the model would produce a representation of the world of an infinite number of resonances. This problem was solved while the model was still floating free of the string interpretation. As we see in the next section, which overlaps temporally with many of the issues of this section, ghost (negative probability) states were introduced along the way, and a framework also needed to be developed to remove these from the space of states, isolating a physical subspace.

[11] Not long after the publication of Veneziano's paper, both Claud Lovelace [33] and Joel Shapiro [43] independently constructed a Veneziano-like formula for the reaction involving $\pi + \pi \rightarrow \pi + \pi$ scattering.



Dear George:

I think Dr. Mahiko Suzuki is an excellent candidate for a faculty position at Berkeley. His scientific accomplishments are impressive (including, most recently, his role as co-inventor of the so-called Veneziano representation) and he can communicate his ideas very well. I wish we could find room for him on our faculty.

Yours sincerely,

Murray Gell-Mann

June 25

Fig. 3.3 Letter of reference for Mahiko Suzuki (addressed to George H. Trilling; dated 25th June, 1969), by Murray Gell-Mann, crediting Suzuki with independent discovery of the Beta function amplitude. *Image source* Gell-Mann papers, Caltech [Box 19, Folder 14]

Veneziano's original amplitude was for 4-particles, 2-in and 2-out, and only held in an extreme approximation of zero width resonances.[12] It was also an orbital model, lacking spinning particles, but this took a little longer to correct, as the search began for more realistic dual models.[13] First, in order to properly establish the Veneziano model, an N-point amplitude involving any number of loop contributions was required, as mentioned above. The first issue was quickly generalised to so-called 'production amplitudes,' involving more output particles than went in to the process. For example, Korkut Bardakçi and Henri Ruegg (both visiting CERN at the time) [3] first generalised the Veneziano model to five particle scattering, 2-in and 3-out. Miguel Virasoro [50] also found a 5-point amplitude. Soon after, Chan Hong-Mo [10] further extended this to a six-point function and from there to the N-point case—independent results of this kind were obtained again by Bardakçi and Ruegg [3], and also Goebel and Sakita [26] and Koba and Nielsen [32].

As Chan noted, in 1970, these many-particle generalisations of Veneziano's amplitude offer a better implementation of the bootstrap idea, since all particles can be treated as bound states of the other. It led some to think that, in Chan's words, it was

[12] This approximation basically means, in modern terms, that it is only carried out at the tree level, with 'external' particles, not to all orders.

[13] However, an early study of the problem of incorporating spin, in some detail, was that of Yasunori Miyata in Tokyo [36]. Later studies, as we see in a later chapter, correspond to what are now viewed as the first spinning string models of Pierre Ramond and Andrè Neveu and John Schwarz.

"more than just phenomenology" and might mark "the beginning of a new theory of strong interactions" [11, p. 379].

Within the context of these generalisations, David Fairlie and Keith Jones discovered the existence of a tachyonic ground state (i.e. for which $m^2 = -1$): "if one imposes the (unphysical) condition $\alpha(0) = 1$ demanding that the ground state is a tachyon (i.e. possesses a particle of negative mass squared), then the four- and five-point amplitudes can be expressed as integrals of a single integrand over the whole of the real line and the plane respectively" [21, p. 284].[14]

Ziro Koba[15] and Holger Nielsen established a highly influential framework for the N-particle amplitude in 1969—they too initially focused on the 5-point function. Nielsen had just finished his *Candidatus Scientiarum* degree at the end of 1968, and was able to briefly refer to the Veneziano model, after learning about it during a talk of Hector Rubinstein's that he'd seen at the Niels Bohr Institute (and considered to have been an highly influential episode).[16] The basic idea was to choose as variables points on a line in the projective plane ('Koba-Nielsen variables'). They presented their work at CERN shortly after developing the idea.

In the Koba-Nielsen framework (developed in [32]) the Beta function looks like:

$$B(\alpha(s), \alpha(t)) = \int_0^1 (1-x)^{-\alpha' t - \alpha_t - 1} x^{\alpha' s - \alpha_s - 1} \mathrm{d}x \tag{3.4}$$

The N-point amplitude takes the form (with subscripts labelling particles such that i refers to the ith particle, possessing momentum p_i):

$$A(s,t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^N z}{\mathrm{d}V_{abc}} \prod_1^{N-1} (z_i - z_{i+1})^{\alpha(0)-1} \prod_1^{N-1} \theta(z_i - z_{i+1}) \prod_{i<j} (z_i - z_j)^{2\alpha' p_i \cdot p_j} \tag{3.5}$$

[14] Fairlie compares the problem of the dual model tachyon to that weighing on Yang-Mills theory in the early days of its existence because of the zero mass particles described by the theory, which seemed clearly inadequate in accounting for spin-1 strongly interacting particles [20, p. 284]. As he notes, though his colleagues were sceptical of resolving the problem, the ground state tachyon was indeed eliminated thanks to a clever projecting out of the physically irrelevant sector of states by Gliozzi, Scherk, and Olive, in 1977 (see Sect. 7.3).

[15] The Japanese physicist Ziro Koba died on 28th September 1973. He had been a student of Tomonaga's. Apparently, he had once shaved his head as a self-punishment for making an error in a self-energy calculation he was carrying out for Tomonaga (see Madhusree Mukerjee's article on Nambu in *Scientific American*, February 1995, p. 38). He had neglected to include certain processes involving virtual pairs created via the Coulomb self-interaction of a vacuum electron—see *Progress in Theoretical Physics* **2**(2), pp. 216–217 (for the original calculation) and p. 217 for the retraction. Curiously, Koba had once shared an office with Yoichiro Nambu (who would later become the first to give the full string interpretation of the Veneziano formula) in Tokyo, just after the second world war.

[16] See http://theory.fi.infn.it/colomo/string-book/nielsen_note.txt. It seems that Hector Rubinstein was very effective in spreading the news of the Veneziano model. Leonard Susskind also credits Rubinstein with bringing the Veneziano amplitude to his attention while Rubinstein (then based in Israel) was visiting him in New York [48, p. 204].

Here $dV_{abc} = \frac{dz_a dz_b dz_c}{(z_b - z_a)(z_c - z_a)(z_a - z_c)}$ is a measure intended to formalise the conformal invariance of the formula, which is expressed in terms of invariance with respect to the modular (or Möbius: $SL(2, \mathbb{R})$) group $z' \rightarrow \frac{az+b}{cz+d}$ (with unit determinant: $ad - bc = 1$). The Möbius invariance of the amplitude ensures that duality is preserved. The conformal invariance was later interpreted via an analogy with electrostatics, by Fairlie and Nielsen (see Sect. 4.1). This approach was crucial in unpacking some of the deeper elements of the mathematical structure underpinning the dual models, especially that pertaining to string worldsheets.

In 1969, Jack Paton and Chan Hong-Mo [39] further generalised the (already generalised, as above) Veneziano model by adding isospin factors (extending the reach of Veneziano models to non-neutral mesons). Though this later became the orthodox method of attaching quantum numbers to open string end-points, it should be understood that there is no question of the factors being associated with strings at this stage. Rather, the analysis is done using external lines (corresponding to the external particles) in a standard graph picture—though they do presciently mention in closing a similarity between their solution and quark pictures of the Harari-Rosner type.[17] The method is to assign an element of $SU(3)$ (that is, the 3×3 λ matrices of $SU(3)$) to the external lines (here denoted by π_a, K, and $\overline{K}$) of a scattering diagram (where τ_a is a Pauli matrix and a_i is the isospin label):

$$\pi_a \sim \begin{pmatrix} \tau_a & 0 \\ 0 & 0 \end{pmatrix}, \quad K \sim \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}, \quad \overline{K} \sim \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix} \tag{3.6}$$

The isopsin factors associated with some particular ordering (say, $1, 2, 3, \ldots, N$) are then given by the trace formula: $\mathrm{Tr}(\tau_{a_1} \tau_{a_2} \ldots \tau_{a_{\mathrm{N}}})$. Amplitudes are multiplied by such factors in order to implement isospin.

Virasoro [51] was also able to construct a novel dual amplitude, distinct from Veneziano's, but sharing analyticity, crossing-symmetry, Regge behaviour, and the other desirable properties. This took the form:

$$A(s, t, u) = \frac{\Gamma(\frac{1}{2} - \frac{1}{2}\alpha(s))\Gamma(\frac{1}{2} - \frac{1}{2}\alpha(t))\Gamma(\frac{1}{2} - \frac{1}{2}\alpha(u))}{\Gamma(1 - \frac{1}{2}\alpha(t) - \frac{1}{2}\alpha(u))\Gamma(1 - \frac{1}{2}\alpha(s) - \frac{1}{2}\alpha(u))\Gamma(1 - \frac{1}{2}\alpha(s) - \frac{1}{2}\alpha(t))} \tag{3.7}$$

This reduces to Veneziano's formula for intercept 2 ($\alpha(s) + \alpha(t) + \alpha(u) = 2$: to eliminate ghosts). Just as the Veneziano formula was taken up and extended and generalised in various ways, so too was Virasoro's version. Joel Shapiro constructed the N-point generalisation of Virasoro's original 4-point amplitude in 1970 [44], with the integral form:

[17] Rather interestingly, these factors would undergo successive transformations ("theoretical exapation" in the terminology of Chap. 7) as the understanding of dual models underwent its own transformation, first into string theory (amounting to an index known as the "Chan-Paton factor") characterising the endpoints of open strings) and then later as a result of developments leading to D-branes (amounting to an index characterising the surfaces the endpoints of open strings terminate on). (I should also point out, as a matter of historical accuracy, that Chan-Paton factors ought really to be called Paton-Chan factors, given that Paton was the lead author on the original paper.)

$$\int d^2z|z|^{k_1 \cdot k_2/2}|1-z|^{k_2 \cdot k_3/2} \tag{3.8}$$

Integrations go over the complex plane, and in contrast to the Veneziano amplitude (in Koba-Nielsen form), the integrand possesses $SL(2, \mathbb{C})$ invariance. Shapiro also generalised the electrostatic analogy to the Virasoro case, with the external particles living on the surface of the Argand plane.[18]

The next section deals with the factorization of the amplitude, which is an essential step in the study of radiative corrections (i.e. loop amplitudes). This latter task was initiated by Kikkawa, Sakita, and Virasoro [31]. This was a fairly radical (in the dual model context), intuitive attempt to restore unitarity in which the Veneziano model is considered to be the lowest-order term (i.e. a Born term) in a perturbation series, approximating a more complete unitary theory (with unitarity emerging via the loop expansion). The idea was to work by analogy with standard Feynman-diagram technology to get a "Feynman-like" theory, generating a perturbation series which, when summed, would give a unitary amplitude of resonances with non-vanishing widths.[19] This work depended on a thorough understanding of the factorization properties of tree diagrams which was achieved using an operator formalism developed by Sergio Fubini, David Gordon, and Gabriele Veneziano [24]. Using this formalism they were able to construct loop diagrams of any order by sewing together tree diagrams. The resulting loop amplitudes, constructed later, admitted a physical interpretation just in case Virasoro's unit intercept condition held and if a condition on the dimensionality

[18] Note that there exists a two-to-one correspondence between the operators in the original Veneziano model and in the Virasoro-Shapiro model (an important implication of this is a doubling of masses in the latter, as compared to the former). However, both the original Veneziano model and the Virasoro-Shapiro model are consistent only in $d = 26$ (a discovery that would be made in the year following these generalizations). The Virasoro-Shapiro model was later interpreted to be a closed-string analogue of the original Veneziano model.

[19] David Kaiser refers to superstring theory as a "sign of the S-matrix program's afterlife" which came about through "the transmogrification of Gabriele Veneziano's 1968 'duality' model" [29, p. 385]. He argues that Feynman diagrams ("paper tools") were at the heart of this transmogrification, claiming that "[t]oday's superstring theories owe their existence" to such tools. I would, however, say that the duality programme (initiated by DHS duality) initially marked a rather dramatic *failure* of Feynman diagrams, pointing to a need for diagrams ('duality diagrams') with very different representational characteristics. These map in an even more indirect way onto their target processes, as the Harari-Rosner diagrams make clear—functioning as equivalence classes of Feynman diagrams and thus superseding them. It took a little longer to interpret this equivalence class as emerging from the invariance properties of string worldsheets, and strictly speaking there was a discrete jump from pre- to post-duality programme Feynman diagrams. The Kikkawa, Sakita, Virasoro paper [31] was pivotal in the restablishment of Feynman diagram (or 'Feynman-like' diagrams, as they make clear) techniques, as were Holger Nielsen and David Fairle's 'fishnet diagrams' (discussed in Chap. 4). Subsequent usage of Feynman-like diagrams in superstring theory truly superseded their original role, since one eventually finds (thanks to modular invariance) that only one diagram is needed at each order of perturbation theory, which defeats their original purpose.

of spacetime held (namely, $d = 26$). The more general n-loop case was investigated by Kaku and Yu [30], Claud Lovelace[20] [34], and Alessandrini [1] (see also: [2]).

3.4 Factorization and the Beginnings of Dual Theory

In the context of a narrow resonance approximation, unitarity must be secured via a demonstration of factorizability. Physically, this procedure allows one to split the process up into incoming particles and outgoing particles.[21] In the case of the generalised N-point Veneziano amplitude, this lets one express the amplitude as a chain product of graph nodes and lines (or, more physically, vertices and propagators). The formal procedure corresponding to this involves isolating the residues at the poles of the amplitude (as functions of s). An amplitude factorizes just in case such a residue can be written as an expression for which each term must be the product of two factors, describing the number of incoming particles and outgoing particles respectively (along with their momenta). For the Veneziano amplitude, factorizability was proven independently by both Bardakçi and Mandelstam [5] and by Fubini and Veneziano [23]. Nambu too came up with a formulation in terms of infinitely many oscillators: [37]. The resulting system was found to be an infinite family of harmonic oscillators, $\alpha_n^\mu = \sqrt{n} a_n^\mu$ and $\alpha_{-n}^\mu = \sqrt{n} a_n^{\dagger\mu}$, allowing for an expression of the dual model spectrum.[22] The oscillators satisfied the following relations:

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0} \tag{3.9}$$

The realisation of [5, 23], and others that the Veneziano formula admitted a factorization in terms of an infinite set of harmonic oscillators then paved the way for a better understanding of the formula, offering a very clear path to a physical theory rather than an abstract model. The harmonic-oscillator formalism also opens up new computational and mathematical directions, of which the operator formalism for dual models was an instance.[23] The operators in this case were creation and annihilation operators for the oscillators. From this one can construct the dual model's spectrum as an infinity of states forming a Fock space, built up by the creation and annihilation

[20] Here, the loop corrections (known as 'M-loops') were conceptualised as integrals over a holed surface, with the number of handles (the genus) corresponding to the order in the perturbation series, much as in the modern string theoretic sense.

[21] Of course, this corresponds to the "chopping it in half" part of the Claud Lovelace quotation opening this chapter.

[22] Note that in their model of rising Regge trajectories in 1968 (still pre-Veneziano's model), Chu et al. [13] had guessed at the existence of a possible harmonic-oscillator potential as the 'force' causing the trajectories to rise.

[23] Pierre Ramond describes the creation/annihilation operator formalism as "clearly the window into the structures behind the dual models" ([41, p. 362]).

operators.[24] Once these were given, the problem of factorization of the generalised Veneziano amplitude (now written using the operator formalism) was a somewhat trivial matter to show.

A 'twisting operator,' generating twisted propagators by switching external particle lines in duality diagrams, was constructed soon after the operator formalism was devised, in September 1969, by Caneschi, Schwimmer, and Veneziano [8]. According to Stefano Sciuto, Fubini believed that all that remained to establish the dual theory on firm foundations at this point was to find the expression of the vertex for emission of a general state.[25] As he recalls, what Fubini said was:

> Now we know the spectrum, we have the propagator and we have the vertex for the emission of the lowest lying states, we only miss the vertex for the emission of a generic state: if we were able to get it, we would have a **theory**, not only a model [46, p. 215].

Fubini and Veneziano later constructed 'untwisted' vertex operators $V(z; p)$ in order to represent scattering amplitudes at vertices, showing also that the amplitude factorised:

$$V(z; p) =: e^{ip\cdot Q} \quad (Q^\mu = x^\mu + i\sum_{n\neq 0}\alpha_n^\mu/n) \tag{3.10}$$

This expression allows one to represent the creation or emission of a state at an interaction point using creation operators. And, likewise, joining or absorption of a state, in terms of annihilation operators. Not long after, it would be realised that such states admitted an interpretation in terms of strings.

The operator formalism was without a doubt a pivotal point in the history of S-matrix theory, dual theory, and string theory. In many ways it severed the umbilical cord between dual theory and S-matrix theory, allowing more orthodox tools and concepts from quantum field theory (such as Fock space, with creation and annihilation operators, control over physical and unphysical sectors, and so on) to be adapted to dual models.[26] This made the properties of dual models especially transparent. This formalism itself, as we will see in the following chapter, played a key role in pointing to a string picture because of the nature of the oscillators. Strings were by

---

[24] The oscillators were Bose fields in the first dual models. In the next chapter we look at the attempt to generalise this to include a fermionic sector, and also a combination of a bosonic and fermionic sector of states.

[25] As John Schwarz writes, "it suggested that these formulas could be viewed as more than just an approximate phenomenological description of hadronic scattering. Rather, they could be regarded as the tree approximation to a full-fledged quantum theory." [45, p. 55]. A fact that came as a surprise to Schwarz, and many others.

[26] Elena Castellani quite rightly puts great stress on the "continuous influence exercised by quantum field theory" in the development of string theory [9, p. 71]. Quantum field theory had at its disposal very many powerful tools for dealing with problems faced by dual models, not least the elimination of the ghosts from the spectrum of states, which were eliminated using a gauge-symmetry device known from QED—though, as we will see, an infinite-dimensional symmetry is required in the case of dual models (a result that would be understood in the string picture as arising from the infinitely many ghosts corresponding to string's infinite tower of vibrational modes).

no means a strange choice of system given the state of play after the construction of the operator formalism.

However, the oscillator formalism also revealed a serious problem: ghosts were seen to be exchanged at poles of the amplitude. In 1969, Fubini and Veneziano [23] showed that the dual resonance model's spurious states—the problematic time component of one of the modes of oscillation (caused by the indefinite metric of the Lorentz group)—could be viewed as unphysical degrees of freedom, and eliminated via a gauge choice, thus restricting the Hilbert space to physical states.[27] But there were *infinitely* many ghosts to fix, and Fubini and Veneziano's method was not general enough to cover all cases. At the close of 1969, Miguel Virasoro [52] devised an infinite Ward-like class of gauge (or 'subsidiary') conditions that *could* serve to cancel the infinity of ghosts, via a one-to-one correspondence (one subsidiary condition per mode of oscillation), for the (physically unrealistic) unit intercept case, $\alpha(0) = 1$, which meant that the lowest lying particle (the leading trajectory) was a tachyon with a massless first excited state.[28] The tachyon was seen as the price one had to pay for ghost elimination.

Virasoro was able to construct an infinite-dimensional (gauge) algebra from the oscillators, with generators $L_m = \frac{1}{2}\Sigma_n : \alpha_{m-n} \cdot \alpha_n.$[29] In the context of a two dimensional field theory one can label a point of the field with complex coordinates which will have the effect of singling out two classes of symmetry generator: $L[\xi]$ (responsible for generating holomorphic diffeomorphisms, or motions) and $\overline{L}[\overline{\xi}]$ (responsible for generating anti-holomorphic motions). The Virasoro algebra[30] is then the infinite-dimensional Lie algebra with basis $\{L_n | n \in \mathbb{Z}\}$, obeying the following commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \qquad (3.11)$$

The central (or 'anomaly') term c here is simply a c-number term that commutes with all other operators—that is, c is in the subgroup of operators that maximally commute with other elements (including non-symmetries): $[c, L_n] = 0$, $\forall n \in \mathbb{Z}$. Noether's theorem leads to c being referred to as the central *charge*, since conserved

---

[27] This is analogous to the situation that arises with timelike photons in QED, in which the spurious states also decouple.

[28] Virasoro was, of course, well aware of the overly restrictive nature of the unit intercept case, but expected that his method could be generalised to more physically realistic cases. Fubini and Veneziano [22] later did this using a projective operator language, providing ghost-cancellation up to the third excited level. Brower and Thorn [6] still later extended this to the ninth excited level.

[29] Here he was building on earlier work of Gliozzi [25], who had constructed a similar set of operators: $L_n = -\frac{1}{2}\sum_{m=-\infty}^{\infty} : a_{-m} \cdot a_{m+n}, \quad n = -1, 0, 1$ (note that this is Mandelstam's condensed version of the Gliozzi operators: [35, p. 282]. However, Virasoro extended this to *all* values of n.

[30] Note that the Virasoro algebra is the central extension of the 'Witt algebra' (over $\mathbb{C}$) defined by $[L_m, L_m] = (m-n)L_{m+n}$ (generators are $\{L_n : n \in \mathbb{Z}\}$)—see [53]. It acts as $[L_m, L_m]f = \{-z^{n+1}\frac{d}{dz}, -z^{m+1}\frac{d}{dz}\}f$. This is the Lie algebra associated with diffeomorphisms of the circle.

quantities are referred to as charges.[31] As Ramond notes, before this was understood there was "an intermediate period during which it was just written as 1/3 before it was realized that there were 26 (and not four) oscillators per mode" [40, p. 538]. The elements L_-, L_0, and L_+ also generate a Möbius algebra (a real subgroup of the projective algebra), satisfying:

$$[L_0, L_\pm] = \pm L_\pm\,, \quad [L_+, L_-] = -2L_0 \tag{3.12}$$

Using the operator formalism of [23], these operators (defining Virasoro's gauge conditions) can be written as (see [16, p. 93]):

$$L_0 = -\frac{p_0^2}{2} - \sum_{n=1}^{\infty} n a_n^\dagger a_n \tag{3.13}$$

$$L_+ = i p_0 a_1 - \sum_{n=1}^{\infty} \sqrt{n(n+1)} a_n^\dagger a_{n+1} \tag{3.14}$$

$$L_- = -i p_0 a_1^\dagger - \sum_{n=1}^{\infty} \sqrt{n(n+1)} a_n a_{n+1}^\dagger \tag{3.15}$$

Virasoro's algebra emerges precisely when the intercept is unity and a larger set of symmetries is induced, in which case the invariance group becomes infinite. Del Giudice and Di Vecchia [15] showed, shortly after Virasoro published his algebra, that physical (non-spurious) states (on the mass shell) must satisfy $L_l|\psi, \pi\rangle = 0$ and $[L_0 + 1]|\psi, \pi\rangle = 0$ (that is, physical states are orthogonal to spurious ones).[32] In 1972, Del Giudice and Di Vecchia, together with Fubini [17], constructed the space of physical (transverse, positive-norm) states (later called "DDF states") for the unit intercept case.[33] The DDF scheme involved the crucial result that the action of a vertex operator $V(z; p)$ on a physical state would spit out another physical state—a result that Brower would later build upon in [7].

Jumping ahead a little, the condition of unit intercept can be seen to follow from the definition of the appropriate physical vertex operators for the emission of ground

[31] The central charge c in the algebra is credited to Joseph H. Weis. Weis died, aged just 35, in a climbing accident in the French Alps (on the Grandes Jorasses) in August, 1978—he was killed with his climbing partner, and CERN physicist, Frank Sacherer. He had taken his PhD under Mandelstam at Berkeley, before taking up a postdoctoral position at MIT. He discovered the central charge during his study of 2D QCD. Though he never published it, he seems to have communicated his discovery to several people, and one can find him credited in, e.g., [6, p. 167], [22].

[32] These physical states were also shown to be eigenstates of the twist operator mentioned above [15, p. 587].

[33] Goddard and Thorn [27], and also Brower [7], later proved in 1972, that this space is complete when the number of transverse components of the oscillators is 24 (that is to say, the physical Hilbert space of states has dimension $\mathscr{T}^{24}$). There are no ghosts present when this condition, in addition to the unit intercept condition, is met. For $D > 26$ (where D is the spacetime dimension) ghosts appear, for $D < 26$ there are no ghosts.

state strings, $v_0(k)$. We can write this (following Clavelli, [14, p. 11]) as:

$$v_0(k) = \int d\tau : e^{i\sqrt{2\alpha'}k\cdot x(0,\tau)} \tag{3.16}$$

In order to constitute a physical state, this had better commute with the Virasoro constraints, $[L_N, v_0(k)] = 0$, which gives:

$$[L_N, v_0(k)] = N(1-\alpha' k^2)\int d\tau e^{-iN\tau} : e^{i\sqrt{2\alpha'}k\cdot x(0,\tau)} \tag{3.17}$$

One can achieve consistency here iff $1 - \alpha' k^2 = 0$. Since the intercept state is given by $\alpha(0) - \alpha' k^2 = 0$, it follows that $\alpha(0) = 1$ for the restriction to physical states. It also quickly follows from the unit intercept condition that the spin-1 state, $\alpha(0) - \alpha' k^2 = 1$, must have zero mass, $k^2 = 0$.

Finally, we should mention that the discovery of vertex operators in the context of the dual resonance model is curiously intertwined with their appearance in the theory of affine Lie algebras. At the root of the connection are the vertex operators developed in the course of the discovery of the operator formalism for the dual resonance models. The mathematical link here goes back to the fact that Regge trajectories involve an infinite number of resonances, so that symmetries based on these (the DHS duality symmetry) will involve infinite-dimensional groups. This physical situation led to the physicist's discovery of Kac-Moody algebras (later formulated as infinite-dimensional extensions of Lie algebras) within the context of strong interaction physics—see [19] for a historical discussion of the interplay between Kac-Moody algebras and physics.[34]

3.5 Summary

The Veneziano model, or rather the wider project of generalising it to multi-particle, multi-loop situations, was considered to be a genuinely possible route to a full theory of strong interaction physics. Accordingly, many physicist-hours were spent labouring on it, despite the fact that the framework was in many ways utterly detached from most areas of particle physics. The clear early problems with the Veneziano model (the lack of unitarity and the restriction to the 4-point scattering scenario) were resolved with remarkable speed and skill, well within two years, as was the problem of formulating the appropriate mathematical framework (replete with an understanding of the consistency conditions one must impose). The result was a general, elegant operator formalism for dual models that clearly pointed towards some underlying system responsible for generating the excitation spectrum.

---

[34] As Goddard and Olive note [28, p. 121], there is even an earlier precedent in the form of Tony Skyrme's construction of a fermionic field operator for the soliton in the sine-Gordon model [47].

The dual models were still facing problems on several fronts, including the lack of fermions in the spectrum. This would wait until the dual models had already undergone significant reinterpretation (including the beginnings of an interpretation as a theory of strings), though work had already begun prior to this interpretation and much of the initial work floated free of the string interpretation, based instead on the operator formulation, as we will see. The tachyon remained an issue, though it would be tamed to a certain extent when fermions were included.[35] Once the string picture begins to emerge, from 1969 onwards, we see a division into two classes of approach to dual models: a geometrical approach based on the string idea (and its associated worldsheet) and a more abstract approach based on the operator formalism. It would take some time for the string picture to fully take centre stage and develop computational prowess to rival the operator approach.

References

1. Alessandrini, V. (1971). A general approach to dual multiloop diagrams. *Il Nuovo Cimento A*, *2A*, 321–352.
2. Alessandrini, V., & Amati, D. (1971). Properties of dual multiloop amplitudes. *Il Nuovo Cimento A*, *4*(4), 793–844.
3. Bardakçi, K., & Ruegg, H. (1969). Meson resonance couplings in a 5-point Veneziano model. *Physics Letters*, *B28*, 671–675.
4. Bardakçi, K., & Ruegg, H. (1969). Reggeized resonance model for arbitrary production processes. *Physical Review*, *181*, 1884–1889.
5. Bardakçi, K., & Mandelstam, S. (1969). Analytic solution of the linear-trajectory bootstrap. *Physical Review*, *184*, 1640–1644.
6. Brower, R. C., & Thorn, C. B. (1971). Eliminating spurious states from the dual resonance model. *Nuclear Physics*, *B31*, 163–182.
7. Brower, R. C. (1972). Spectrum-generating algebra and no-ghost theorem for the dual model. *Physical Review D*, *6*, 1655–1662.
8. Caneschi, L., Schwimmer, A., & Veneziano, G. (1969). Twisted propagator in the operatorial duality formalism. *Physics Letters B*, *30*(5), 351–356.
9. Castellani, E. (2012). Early string theory as a challenging case study for philosophers. In A. Cappelli et al. (Eds.), *The birth of string theory* (pp. 63–78). Cambridge: Cambridge University Press.
10. Chan, H.-M., & Tsou, T.-S. (1969). Explicit construction of *n*-point function in generalized Veneziano model. *Physics Letters*, *B28*, 485–488.
11. Chan, H.-M. (1970). The generalized Veneziano model. *Proceedings of the Royal Society of London A.*, *318*, 379–399.
12. Chew, G. F. (1970). Hadron bootstrap: Triumph or frustration? *Physics Today*, *23*(10), 23–28.
13. Chu, S., Epstein, G., Kaus, P., Slansky, R. C., & Zacharia, F. (1968). Crossing-symmetric rising Regge trajectories. *Physical Review*, *175*(5), 2098–2105.
14. Clavelli, L. (1986). A historical overview of superstring theory. In L. Clavelli & A. Halperin (Eds.), *Lewes string theory workshop* (pp. 1–20). Singapore: World Scientific.
15. Del Giudice, E., & Di Vecchia, P. (1970). Characterization of the physical states in dual-resonance models. *Il Nuovo Cimento A10, 70*(4): 579–591.

[35] A complete understanding of the elimination of tachyons would take the best part of a decade to achieve, once supersymmetry was better understood.

16. Del Giudice, E., & Di Vecchia, P. (1971). Factorization and operator formalism in the generalized Virasoro model. *Il Nuovo Cimento A*, *5*(1), 90–102.
17. Del Giudice, E., Di Vecchia, P., & Fubini, S. (1972). General properties of the dual resonance model. *Annals of Physics*, *70*(2), 378–398.
18. Di Vecchia, P. (2008). *The birth of string theory*. In M. Gasperini & J. Maharana (Eds.), *String theory and fundamental interactions, Lecture Notes in Physics* (vol. 737, pp. 59–118). Berlin: Springer.
19. Dolen, L. (1995). The beacon of Kac-Moody symmetry for physics. *Notices of the American Mathematical Society, 42*, 1489–1495.
20. Fairlie, D. (2012). The analogue model for string amplitudes. In A. Capelli et al. (Eds.), *The birth of string theory* (pp. 283–293). Cambridge: Cambridge University Press.
21. Fairlie, D. B., & Jones, K. (1969). Integral representations for the complete four and five-point Veneziano amplitudes. *Nuclear Physics*, *B15*, 323–330.
22. Fubini, S., & Veneziano, G. (1970). Algebraic treatment of subsidiary conditions in dual resonance models. *Annals of Physics*, *63*, 12–27.
23. Fubini, S., & Veneziano, G. (1969). Level structure of dual resonance models. *Il Nuovo Cimento*, *A64*, 811–840.
24. Fubini, S., Gordon, D., & Veneziano, G. (1969). A general treatment of factorization in dual resonance models. *Physics Letters*, *B29*, 679–682.
25. Gliozzi, F. (1969). Ward-like identities and twisting operator in dual resonance models. *Lettere al Nuovo Cimento*, *2*(18), 846–850.
26. Goebel, C. J., & Sakita, B. (1969). Extension of the Veneziano form to n-particle amplitudes. *Physics Review Letters*, *22*, 257–260.
27. Goddard, P., & Thorn, C. B. (1972). Compatibility of the dual pomeron with unitarity and the absence of ghosts in dual resonance models. *Physics Letters*, *40B*, 235–238.
28. Goddard, P., & Olive, D. I. (1988). Kac-Moody and Virasoro algebras: A reprint volume for physicists. Singapore: World Scientific.
29. Kaiser, D. (2005). *Drawing theories apart: The dispersion of Feynman diagrams in postwar physics*. Chicago: University of Chicago Press.
30. Kaku, M., & Yu, L. (1970). The general multi-loop Veneziano amplitude. *Physics Letters B*, *33*(2), 166–170.
31. Kikkawa, K., Sakita, B., & Virasoro, M. A. (1969). Feynman-like diagrams compatible with duality. I. Planar diagrams. *Physical Review*, *184*, 1701–1713.
32. Koba, Z., & Nielsen, H. (1969). Reaction amplitude for N-Mesons: A generalization of the Veneziano-Bardakçi-Ruegg-Virasoro model. *Nuclear Physics, B*(10): 633–655.
33. Lovelace, C. (1968). A novel application of Regge trajectories. *Physics Letters B*, *28*, 264–268.
34. Lovelace, C. (1970). M-loop generalized Veneziano formula. *Physics Letters B*, *32*(8), 703–708.
35. Mandelstam, S. (1974). Dual-resonance models. *Physics Reports*, *13*(6), 259–353.
36. Miyata, Y. (1970). Pseudoscalar-vector meson scattering amplitude reduced from six-point Veneziano model. *Progress of Theoretical Physics*, *44*(6), 1661–1672.
37. Nambu, Y. (1970). Quark model and the factorization of the Veneziano amplitude. In R. Chand (Ed.), *Symmetries and quark models* (pp. 269–277). Singapore: World Scientific.
38. Olive, D. I. (2012). From dual fermion to superstring. In A. Cappelli et al. (Eds.), *The birth of string theory* (pp. 346–360). Cambridge: Cambridge University Press.
39. Paton, J. E., & Chan, H.-M. (1969). Generalized Veneziano model with isospin. *Nuclear Physics*, *B10*, 516–520.
40. Ramond, P. (1987). The early years of string theory: The dual resonance model. In R. Slansky & G. B. West (Eds.), *Theoretical advanced study institute lectures in elementary particle physics 1987: Proceedings* (pp. 501–571). Singapore: World Scientific.
41. Ramond, P. (2012). Dual model with Fermions: Memoirs of an early string theorist. In A. Capelli et al. (Eds.), *The birth of string theory* (pp. 361–372). Cambridge: Cambridge University Press.
42. Schmid, C. (1970). Duality and exchange degeneracy. In A. Zichichi (Ed.), *Subnuclear phenomena, Part 1* (pp. 108–143). New York: Academic Press.

43. Shapiro, J. A. (1968). Narrow-resonance model with regge behavior for $\pi\pi$ scattering. *Physical Review*, *179*(5), 1345–1353.
44. Shapiro, J. A. (1970). Electrostatic analogue for the Virasoro model. *Physics Letters B*, *33*, 361–362.
45. Schwarz, J. H. (2001). String theory origins of supersymmetry. *Nuclear Physics*, *B101*, 54–61.
46. Sciuto, S. (2012). The 3-Reggeon Vertex. In A. Capelli et al. (Eds.), *The birth of string theory* (pp. 214–217). Cambridge: Cambridge University Press.
47. Skyrme, T. H. R. (1961). Particle states of a quantized meson field. *Proceedings of the Royal Society of London A*, *A262*, 237–245.
48. Susskind, L. (2006). *The cosmic landscape*. New York: Back Bay Books.
49. Veneziano, G. (1968). Construction of a crossing-symmetric, Reggeon-behaved amplitude for linearly rising trajectories. *Il Nuovo Cimento*, *A57*, 190–197.
50. Virasoro, M. (1969). Generalization of Veneziano's formula for the five-point function. *Physical Review Letters*, *22*, 37–39.
51. Virasoro, M. (1969). Alternative constructions of crossing-symmetric amplitudes with Regge behavior. *Physical Review*, *177*(5), 2309–2311.
52. Virasoro, M. (1969). Subsidiary conditions and ghosts in dual-resonance models. *Physical Review D*, *1*(10), 2933–2936.
53. Wanglai, L., & Wilson, R. L. (1998). Central extensions of some lie algebras. *Proceedings of the American Mathematical Society*, *126*(9), 2569–2577.