On the Computation of Choquet Optimal Solutions in Multicriteria Decision Contexts

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Abstract. We study in this paper the computation of Choquet optimal solutions in decision contexts involving multiple criteria or multiple agents. Choquet optimal solutions are solutions that optimize a Choquet integral, one of the most powerful tools in multicriteria decision making. We develop a new property that characterizes the Choquet optimal solutions. From this property, a general method to generate these solutions in the case of several criteria is proposed. We apply the method to different Pareto non-dominated sets coming from different knapsack instances with a number of criteria included between two and seven. We show that the method is effective for a number of criteria lower than five or for high size Pareto non-dominated sets. We also observe that the percentage of Choquet optimal solutions increase with the number of criteria.

Keywords: Choquet integral, Multicriteria decision making, Multiagent optimization, Fuzzy measure, Multiobjective optimization.

1 Introduction

The Choquet integral [1] is one of the most powerful tools in multicriteria decision making [2, 3]. A Choquet integral can be seen as an integral on a non-additive measure (or capacity or fuzzy measure). It presents extremely wide expressive capabilities and can model many specific aggregation operators, including, but not limited to, the weighted sum, the minimum, the maximum, all the statistic quantiles, the ordered weighted averaging operator [4], the weighted ordered weighted averaging operator [5], etc.

However, this high expressiveness capability has a price: while the definition of a simple weighted sum operator with p criteria requires $p-1$ parameters, the definition of the Choquet integral with p criteria requires setting of 2^p-2 values, which can be a problem even for low values of p.

Many approaches have been studied to identify the parameters of the Choquet integral [6]. Generally, questions are asked to the decision maker and the information obtained is represented as linear constraints over the set of parameters. An optimization problem is then solved in order to find a set of parameters which minimizes the error according to the information given by the decision maker.

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The approach considered in this paper is quite different: we will not try to identify the parameters of the Choquet integral but we will compute the solutions that are potentially optimal for at least one parameter set of the Choquet integral. Therefore, the parameters of the Choquet integral will not have to be determined. Instead, a set of solutions of smaller size comparing to the set of Pareto optimal solutions (which can be very huge in the case of multiobjective or multiagent problems) will be presented to the decision-maker. Each solution proposed will have interesting properties since they optimize at least one Choquet integral. Also, by computing all the Choquet optimal solutions, all the solutions that optimize one of the operators that the Choquet integral can model (weighted sum, ordered weighted averaging operator, etc.) will be generated.

We will present in the paper a new property that characterizes the Choquet optimal solutions. From this property, a general method to generate the Choquet optimal solutions is proposed. The method can be applied in different decision contexts involving multiple criteria or agents. The first application is in multicriteria decision making [7]: different alternatives are proposed to a decision maker and each alternative is evaluated according to a set of p criteria. No alternative Pareto dominates another and therefore no alternative can be a priori rejected. However, if we plan to use the Choquet integral in order to select the best alternative according to the preferences of the decision maker, we can first generate the solutions that are potentially optimal for at least one Choquet integral. This can be done in the absence of the decision maker. At the end, a smaller set comparing to the Pareto optimal set is proposed.

Another context in which the method can be applied is in decision contexts involving multiple agents, like multiagent knapsack problems [8], paper assignment problems [9], marriage problems in social networks [10], etc. In these problems, each agent has its own cost function and the aim is to generate a solution which is fair according to all the agents. Since the Choquet integral can model fairness operators like the max-min operator, the ordered weighted averaging and the weighted ordered weighted averaging operators, we can first compute all the potentially Choquet optimal solutions of these multiagent problems in order to generate a first set of candidate solutions.

The last application of the method is in multiobjective combinatorial optimization (MOCO) problems, which model situations where a decision-maker has to optimize several objectives simultaneously. These situations often come from a problem with a combinatorial number of solutions, for example spanning tree, shortest path, knapsack, traveling salesman tour, etc. [11]. To solve a MOCO problem, three different approaches are usually followed. In the *a posteriori* approach, all the Pareto optimal solutions are first generated. Once this has been done, the decision-maker is free to choose among all solutions the one that corresponds the best to his/her preferences. Another possibility, called the *a priori* approach, is to first ask the decision-maker what are his/her preferences among all the objectives and to compute an aggregation function [3] with specified parameters. The aggregation function is then optimized and at the end, only one solution is generally proposed to the decision-maker. A last possibility is to *interact* with the decision-maker along the process of generation of the solutions [12]. In this *interactive* approach, we ask the decision-maker to establish his/her preferences among different solutions, in order to guide the search, and to finally obtain a solution that suits him/her.

We propose here a new approach, between the *a posteriori* approach and the *a priori* approach, that consists in trying to find the set of solutions that are potentially optimal for at least one set of parameters of an aggregation function, and more specifically in this paper the Choquet integral.

Some papers already deal with the optimization of the Choquet integral of MOCO problems [13–15] but only when the Choquet integral is completely defined by the decision-maker. To our knowledge, the development of a method to generate the whole set of Choquet optimal solutions has not yet been studied, except the recent work of [16], where Lust and Rolland study the particular case of biobjective combinatorial optimization problems. They characterize the Choquet optimal solutions through a property and they define a method to generate all the Choquet optimal solutions. They apply the method to the biobjective knapsack problem and the biobjective minimum spanning tree.

We focus here on the general problem where the number of criteria can be more than two. We present a new property that characterizes the Choquet optimal set and develop a method based on this property to generate the Choquet optimal set, containing the solutions that are optimal solutions of Choquet integrals. We analyze the computational property of the method and we propose results for Pareto non-dominated sets. We will show that the Choquet integral becomes more expressive (can attain more Pareto optimal solutions) than the weighted sum, especially if the number of objectives increase.

The paper is placed in the context of MOCO problems and is organized as follows. In the next section, we first recall the definition of a MOCO problem and the Choquet integral. In section 3 we expose a property that characterizes the Choquet optimal set. In Section 4, we experiment the method on different instances of the multiobjective knapsack problem.

2 Aggregation Operators

In this section, we first introduce the formalism of a MOCO problem, and then present the weighted sum as it is the most popular aggregation operator. We then introduce the Choquet integral.

2.1 Multiobjective Combinatorial Optimization Problems

A multiobjective (linear) combinatorial optimization (MOCO) problem is generally defined as follows:

$$\begin{aligned} &\text{``}\max_x\text{''} f(x) = Cx = (f_1(x), f_2(x), \dots, f_p(x)) \\ &\text{subject to } Ax \leq b \\ &\qquad\qquad x \in \{0,1\}^n \end{aligned}$$

$$\begin{array}{ccl} x \in \{0,1\}^n & \longrightarrow & n \text{ variables}, i = 1,\dots,n \\ C \in \mathbb{R}^{p\times n} & \longrightarrow & p \text{ objective functions}, k = 1,\dots,p \\ A \in \mathbb{R}^{r\times n} \text{ and } b \in \mathbb{R}^{r\times 1} & \longrightarrow & r \text{ constraints}, j = 1,\dots,r \end{array}$$

A feasible solution x is a vector of n variables, having to satisfy the r constraints of the problem. Therefore, the feasible set in decision space is given by $\mathcal{X} = \{x \in \{0,1\}^n : Ax \leq b\}$. The image of the feasible set is given by $\mathcal{Y} = f(\mathcal{X}) = \{f(x) : x \in \mathcal{X}\} \subset \mathbb{R}^p$. An element of the set $\mathcal{Y}$ is called a cost-vector or a point.

Let us recall the concept of Pareto efficiency. We consider that all the objectives have to be maximized and we design by $\mathcal{P}$ the set of objectives $\{1,\dots,p\}$.

Definition 1. *The Pareto dominance relation (P-dominance for short) is defined, for all $y^1, y^2 \in \mathbb{R}^p$, by:*

$$y^1 \succ_P y^2 \iff [\forall k \in \mathcal{P}, y^1_k \geq y^2_k \text{ and } y^1 \neq y^2]$$

Definition 2. *The strict Pareto dominance relation (sP-dominance for short) is defined as follows:*

$$y^1 \succ_{sP} y^2 \iff [\forall k \in \mathcal{P}, y^1_k > y^2_k]$$

Within a feasible set $\mathcal{X}$, any element x^1 is said to be *P-dominated* when $f(x^2) \succ_P f(x^1)$ for some x^2 in $\mathcal{X}$, *P-optimal* (or *P-efficient*) if there is no $x^2 \in \mathcal{X}$ such that $f(x^2) \succ_P f(x^1)$ and *weakly P-optimal* if there is no $x^2 \in \mathcal{X}$ such that $f(x^2) \succ_{sP} f(x^1)$. The *P-optimal set* denoted by $\mathcal{X}_P$ contains all the P-optimal solutions. The image $f(x)$ in the objective space of a P-optimal solution x is called a P-non-dominated point. The image of the P-optimal set in $\mathcal{Y}$, equal to $f(\mathcal{X}_P)$, is called the Pareto front, and is denoted by $\mathcal{Y}_P$.

2.2 Weighted Sum

The most popular aggregation operator is the weighted sum (WS), where non-negative importance weights $\lambda_i (i = 1,\dots,p)$ are allocated to the objectives.

Definition 3. *Given a vector $y \in \mathbb{R}^p$ and a weight set $\lambda \in \mathbb{R}^p$ (with $\lambda_i \geq 0$ and $\sum_{i=1}^p \lambda_i = 1$), the WS $f^{ws}_\lambda(y)$ of y is equal to:*

$$f^{ws}_\lambda(y) = \sum_{i=1}^{p} \lambda_i y_i$$

Note that there exist P-optimal solutions that do not optimize a WS, and they are generally called *non-supported* P-optimal solutions [11].

2.3 Choquet Integral

The Choquet integral has been introduced by Choquet [1] in 1953 and has been intensively studied, especially in the field of multicriteria decision analysis, by several authors (see [7, 2, 3] for a brief review).

We first define the notion of capacity, on which the Choquet integral is based.

Definition 4. *A capacity is a set function* $v\colon 2^{\mathcal{P}} \to [0,1]$ *such that:*

- $v(\emptyset) = 0$, $v(\mathcal{P}) = 1$ *(boundary conditions)*
- $\forall \mathcal{A}, \mathcal{B} \in 2^{\mathcal{P}}$ *such that* $\mathcal{A} \subseteq \mathcal{B}, v(\mathcal{A}) \leq v(\mathcal{B})$ *(monotonicity conditions)*

Therefore, for each subset of objectives $\mathcal{A} \subseteq \mathcal{P}$, $v(\mathcal{A})$ represents the importance of the coalition $\mathcal{A}$.

Definition 5. *The Choquet integral of a vector* $y \in \mathbb{R}^p$ *with respect to a capacity* v *is defined by:*

$$f_v^C(y) = \sum_{i=1}^{p} \big(v(Y_i^{\uparrow}) - v(Y_{i+1}^{\uparrow})\big) y_i^{\uparrow}$$
$$= \sum_{i=1}^{p} (y_i^{\uparrow} - y_{i-1}^{\uparrow}) v(Y_i^{\uparrow})$$

where $y^{\uparrow} = (y_1^{\uparrow}, \ldots, y_p^{\uparrow})$ *is a permutation of the components of* y *such that* $0 = y_0^{\uparrow} \leq y_1^{\uparrow} \leq \ldots \leq y_p^{\uparrow}$ *and* $Y_i^{\uparrow} = \{j \in \mathcal{P}, y_j \geq y_i^{\uparrow}\} = \{i^{\uparrow}, (i+1)^{\uparrow}, \ldots, p^{\uparrow}\}$ *for* $i \leq p$ *and* $Y_{(p+1)}^{\uparrow} = \emptyset$.

We can notice that the Choquet integral is an increasing function of its arguments.

We can also define the Choquet integral through the Möbius representation [17] of the capacity. Any set function $v\colon 2^{\mathcal{P}} \to [0,1]$ can be uniquely expressed in terms of its Möbius representation by:

$$v(\mathcal{A}) = \sum_{\mathcal{B} \subseteq \mathcal{A}} m_v(\mathcal{B}) \quad \forall \mathcal{A} \subseteq \mathcal{P}$$

where the set function $m_v : 2^{\mathcal{P}} \to \mathbb{R}$ is called the Möbius transform or Möbius representation of v and is given by

$$m_v(\mathcal{A}) = \sum_{\mathcal{B} \subseteq \mathcal{A}} (-1)^{(a-b)} v(\mathcal{B}) \quad \forall \mathcal{A} \subseteq \mathcal{P}$$

where a and b are the cardinals of $\mathcal{A}$ and $\mathcal{B}$.

A set of 2^p coefficients $m_v(\mathcal{A})$ ($\mathcal{A} \subseteq \mathcal{P}$) corresponds to a capacity if it satisfies the boundary and monotonicity conditions [18]:

1. $m_v(\emptyset) = 0, \; \sum_{\mathcal{A} \subseteq \mathcal{P}} m_v(\mathcal{A}) = 1$
2. $\sum_{\mathcal{B} \subseteq \mathcal{A},\; i \in \mathcal{B}} m_v(\mathcal{B}) \geq 0 \quad \forall \mathcal{A} \subseteq \mathcal{P}, i \in \mathcal{P}$

We can now write the Choquet integral with the use of Möbius coefficients. The Choquet integral of a vector $y \in \mathbb{R}^p$ with respect to a capacity v is defined as follows:

$$f_v^C(y) = \sum_{\mathcal{A} \subseteq \mathcal{P}} m_v(\mathcal{A}) \min_{i \in \mathcal{A}} y_i$$

3 Characterization of Choquet Optimal Solutions

We present in this section a characterization of the Choquet optimal solutions based on WS-optimal solutions, that is solutions that optimize a weighted sum. The set of Choquet optimal solutions of a MOCO problem with p objectives is called $\mathcal{X}_C$, and contains at least one solution $x \in \mathcal{X}$ optimal for each possible Choquet integral, that is $\forall v \in \mathcal{V}, \exists x_c \in \mathcal{X}_C \,|\, f_v^C(f(x_c)) \geq f_v^C(f(x)) \;\forall x \in \mathcal{X}$, where $\mathcal{V}$ represents the set of capacity functions defined over p objectives. Note that each Choquet optimal solution is at least weakly P-optimal [16].

In [16], Lust and Rolland studied the particular case of two objectives and they showed that $\mathcal{X}_C$ could be obtained by generating all WS-optimal solutions in each subspace of the objectives separated by the bisector ($f_1(x) \geq f_2(x)$ or $f_2(x) \geq f_1(x)$), and by adding a particular point M with $M_1 = M_2 = \max_{x\in\mathcal{X}} \min(f_1(x), f_2(x))$. We show here how this property can be generalized to more than two objectives.

We will work with the image of $\mathcal{X}_C$ in the objective space, $\mathcal{Y}_C$, equal to $f(\mathcal{X}_C)$. To each point $y_c \in \mathcal{Y}_C$ corresponds thus at least one solution x_c in $\mathcal{X}_C$.

Let σ be a permutation on $\mathcal{P}$. Let O_σ be the subset of points $y \in \mathbb{R}^p$ such that $y \in O_\sigma \iff y_{\sigma_1} \geq y_{\sigma_2} \geq \ldots \geq y_{\sigma_p}$.

Let p_{O_σ} be the following application:

$$p_{O_\sigma} : \mathbb{R}^p \rightarrow \mathbb{R}^p, (p_{O_\sigma}(y))_{\sigma_i} = (\min(y_{\sigma_1}, \ldots, y_{\sigma_i})), \forall i \in \mathcal{P}$$

For example, if $p = 3$, for the permutation (2,3,1), we have:

$$p_{O_\sigma}(y) = \big(\min(y_2, y_3, y_1), \min(y_2), \min(y_2, y_3)\big)$$

We denote by $\mathcal{P}_{O_\sigma}$ the set containing the points obtained by applying the application $p_{O_\sigma}(y)$ to all the points $y \in \mathcal{Y}$. As $(p_{O_\sigma}(y))_{\sigma_1} \geq (p_{O_\sigma}(y))_{\sigma_2} \geq \ldots \geq (p_{O_\sigma}(y))_{\sigma_p}$, we have $\mathcal{P}_{O_\sigma} \subseteq O_\sigma$.

3.1 Characterization Theorem

We propose a new characterization of the Choquet optimal set.

Theorem 1

$$\mathcal{Y}_C \cap O_\sigma = \mathcal{Y} \cap WS(\mathcal{P}_{O_\sigma})$$

where $WS(\mathcal{P}_{O_\sigma})$ designs the set of WS-optimal points of the set $\mathcal{P}_{O_\sigma}$.

This theorem characterizes the solutions which can be Choquet optimal in the set of feasible solutions as being, in each subspace of the objective space $\mathcal{Y}$ where $y_{\sigma_1} \geq y_{\sigma_2} \geq \ldots \geq y_{\sigma_p}$, the solutions that have an image corresponding to a WS-optimal point in the space composed of the original subspace plus the projection of all the other points following the application p_{O_σ}.

Proof

In the following, we will denote O_σ as simply O for the sake of simplicity, and we will consider, without loss of generality, that the permutation σ is equal to $(1, 2, \ldots, p)$, that is $y \in O \Leftrightarrow y_1 \geq y_2 \geq \cdots \geq y_p$.

We know that $\mathcal{Y}_C \subseteq \mathcal{Y}$ and then $\mathcal{Y}_C \cap O \subseteq \mathcal{Y} \cap O$. We also know that $WS(\mathcal{P}_O) \subseteq O$ and then $\mathcal{Y}_C \cap WS(\mathcal{P}_O) \subseteq \mathcal{Y} \cap O$. Let y be in $\mathcal{Y}_C \cap O$.

- $[y \in \mathcal{Y}_C \cap O \Rightarrow y \in \mathcal{Y} \cap WS(\mathcal{P}_O)]$
  Let us write the Choquet integral of $y \in O$ related to a capacity v, with $Y_i = \{1, \ldots, i\}$ and $Y_0 = \emptyset$:

$$f_v^C(y) = \sum_{i=1}^{p} \big(v(Y_i) - v(Y_{i-1})\big) y_i = \sum_{i=1}^{p} \lambda_i y_i$$

  As v is monotonic for the inclusion, $\lambda_i = v(Y_i) - v(Y_{i-1})$ is always positive. We have also $\sum_{i=1}^{p} \lambda_i = \sum_{i=1}^{p} (v(Y_i) - v(Y_{i-1})) = v(\mathcal{P}) - v(\emptyset) = 1$.
  Let $z \in \mathcal{Y}$. As $y \in \mathcal{Y}_C$, we have $f_v^C(y) \geq f_v^C(z)$. We also have $\forall i \in \mathcal{P}$, $z_i \geq \min\{z_1, z_2, \ldots, z_i\} = (p_O(z))_i$, and as the Choquet integral is an increasing function of its arguments, we have $f_v^C(z) \geq f_v^C(p_O(z))$. And since $p_O(z) \in O$, we have $f_v^C(p_O(z)) = \sum_{i=1}^{p} \lambda_i p_O(z)_i$. Therefore we have $\forall z \in \mathcal{Y}$:

$$\sum_{i \in \mathcal{P}} \lambda_i y_i = f_v^C(y) \geq f_v^C(z) \geq f_v^C(p_O(z)) = \sum_{i \in \mathcal{P}} \lambda_i p_O(z)_i$$

  where $\lambda_i \geq 0 \ \forall i \in \mathcal{P}$ and $\sum_{i=1}^{p} \lambda_i = 1$. So $y \in WS(\mathcal{P}_O)$ and as $y \in \mathcal{Y}$, $y \in \mathcal{Y} \cap WS(\mathcal{P}_O)$.

- $[y \in WS(\mathcal{P}_O) \cap \mathcal{Y} \Rightarrow y \in \mathcal{Y}_C \cap O]$
  Let $y \in WS(\mathcal{P}_O) \cap \mathcal{Y}$. Then there are $\lambda_1, \ldots, \lambda_p \geq 0$ such that $\sum_{i=1}^{p} \lambda_i = 1$ and

$$\forall z \in \mathcal{Y}, \ \sum_{i \in \mathcal{P}} \lambda_i y_i \geq \sum_{i \in \mathcal{P}} \lambda_i p_O(z)_i$$

  By definition, $(p_O(z))_i = \min(z_1, \ldots, z_i), \forall i \in \mathcal{P}$.
  Let $\mathcal{A} \subseteq \mathcal{P}$. Let us define a set function m such that $m(\mathcal{A}) = \lambda_i$ if $\mathcal{A} = \{1, \ldots, i\}$ and $m(\mathcal{A}) = 0$ if not.

Then

$$\sum_{i\in\mathcal{P}} \lambda_i (p_O(z))_i = \sum_{i\in\mathcal{P}} \lambda_i \min(z_1,\ldots,z_i)$$
$$= \sum_{\mathcal{A}\subseteq\mathcal{P}} m(\mathcal{A}) \min_{i\in\mathcal{A}} z_i$$

Let us remind that the set function m corresponds to a capacity if:

1. $m(\emptyset) = 0, \sum_{\mathcal{A}\subseteq\mathcal{P}} m(\mathcal{A}) = 1$
2. $\sum_{\mathcal{B}\subseteq\mathcal{A},\ i\in\mathcal{B}} m(\mathcal{B}) \geq 0 \quad \forall \mathcal{A}\subseteq\mathcal{P}, i\in\mathcal{P}$

All these conditions are satisfied:

- $m(\emptyset) = 0$ by definition
- $\sum_{\mathcal{A}\subseteq\mathcal{P}} m(\mathcal{A}) = \sum_{i=1}^{p} \lambda_i = 1$
- all $m(\mathcal{B})$ are non-negative as $\lambda_i \geq 0$

So we have a set of Möbius coefficients such that $\forall z \in \mathcal{Y}$,

$$f_v^C(y) = \sum_{\mathcal{A}\subseteq\mathcal{P}} m(\mathcal{A}) \min_{i\in\mathcal{A}} y_i$$
$$= \sum_{i\in\mathcal{P}} \lambda_i y_i$$
$$\geq \sum_{i\in\mathcal{P}} \lambda_i p_O(z)_i$$
$$\geq \sum_{\mathcal{A}\subseteq\mathcal{P}} m(\mathcal{A}) \min_{i\in\mathcal{A}} z_i$$
$$\geq f_v^C(z)$$

□

4 Generation of Choquet Optimal Solutions

4.1 Algorithm for Generating $\mathcal{X}_C$

We present in this section an algorithm to generate the set $\mathcal{X}_C$ containing all the Choquet optimal solutions of a MOCO problem. The algorithm straightly follows from Theorem 1.

For all the permutations σ on $\mathcal{P}$, we have to:

1. Determine the projections with the application p_{O_σ}
2. Solve a WS problem

The projections are defined with the application

$$(p_{O_\sigma}(y))_{\sigma_i} = (\min(y_{\sigma_1}, \ldots, y_{\sigma_i})), \forall i \in \mathcal{P}$$

for each $y \in \mathcal{Y}$.

However, among these projections, only the P-non-dominated points are interesting (since if a point is P-dominated, its WS is inferior to the WS of at least another point). Therefore, to determine the projections, the following MOCO problem (called P_σ) has to be solved:

$$\text{``}\max\text{''} p(x) = \text{``}\max_{x \in \mathcal{X} \setminus \mathcal{X}_\sigma}\text{''} (f_{\sigma_1}(x), \min(f_{\sigma_1}(x), f_{\sigma_2}(x)), \ldots, \min(f_{\sigma_1}(x), f_{\sigma_2}(x), \ldots, f_{\sigma_p}(x)))$$

where $\mathcal{X}_\sigma$ is the set such that $x \in \mathcal{X}_\sigma \iff f_{\sigma_1}(x) \geq f_{\sigma_2}(x) \geq \ldots \geq f_{\sigma_p}(x)$.

Once the projections have been defined, a WS problem has to be solved, in $\mathcal{X}_\sigma$, and by adding the P-non-dominated points obtained from P_σ.

We give the main lines of the method in Algorithm 1.

Algorithm 1. `Generation of` $\mathcal{X}_C$

Parameters ↓: a MOCO problem
Parameters ↑: the set $\mathcal{X}_C$
Let σ be a permutation on $\mathcal{P}$, and Σ the set of permutations
Let $\mathcal{X}_\sigma$ be the set such that $x \in \mathcal{X}_\sigma \iff f_{\sigma_1}(x) \geq f_{\sigma_2}(x) \geq \ldots \geq f_{\sigma_p}(x)$
$\mathcal{X}_C \leftarrow \{\}$
for all $\sigma \in \Sigma$ **do**
  --| Determination of the projections:
  Solve the following MOCO problem, called P_σ, in $x \in \mathcal{X} \setminus \mathcal{X}_\sigma$:
  $\text{``}\max\text{''} p(x) = \text{``}\max_{x \in \mathcal{X} \setminus \mathcal{X}_\sigma}\text{''} (f_{\sigma_1}(x), \min(f_{\sigma_1}(x), f_{\sigma_2}(x)), \ldots, \min(f_{\sigma_1}(x), f_{\sigma_2}(x), \ldots, f_{\sigma_p}(x)))$
  Let $\mathcal{Y}_{P_\sigma}$ the Pareto non-dominated points obtained from solving (P_σ)
  Solve the WS problem (called WS_σ) $\max_{x \in \mathcal{X}_\sigma} f_\lambda^{ws}(f(x))$ with the additional points of $\mathcal{Y}_{P_\sigma}$.
  Let $\mathcal{X}_{ws_\sigma}$ the solutions obtained from (WS_σ)
  $\mathcal{X}_C \leftarrow \mathcal{X}_C \cup \mathcal{X}_{ws_\sigma}$
end for

4.2 Experiments

We present results for defined Pareto fronts, that is, a Pareto front is given, and the aim is to determine, among the P-non-dominated points, the Choquet optimal points.

To generate Pareto fronts with different numbers of objectives, we have applied a heuristic to several multiobjective knapsack instances. We have used knapsack instances with random profits. The heuristic is an adaptation of the one presented in [19]. Note that the aim is not to generate the best possible approximation of the Pareto front of these instances, but only to generate a set of P-non-dominated points. The results are given in Table 1, for $p = 2, \ldots, 7$, and for 3000 points.

We respectively indicate the number of criteria, the number of WS-optimal points, the number of Choquet optimal points, the proportion of WS-optimal points under the total number of points, the proportion of Choquet optimal

Table 1. Random multiobjective knapsack instances (3000 points)

# Crit	# WS	# C	% WS	% C	% C not WS
2	123	128	4.10	4.27	3.91
3	184	240	6.13	8.00	23.33
4	240	380	8.00	12.67	36.84
5	282	485	9.40	16.17	41.86
6	408	676	13.6	22.53	39.64
7	528	1016	17.6	33.87	48.03

points under the total number of points and the proportion of Choquet optimal points that are not WS-optimal.

We see that if the number of Choquet optimal points and the number of Choquet optimal points that are not WS-optimal points are very small for $p = 2$, these number grows rapidly with the number of criteria: for $p = 7$, we have that 33.87% of the P-non-dominated points are Choquet optimal points, and 48.03% of them are not WS-optimal. We see thus that when the number of criteria increases, the Choquet integral allows to attain considerably more P-non-dominated points than the WS.

In table 2, we indicate the CPU times needed to generate the Choquet optimal points of these sets (on a Intel Core i7-3820 at 3.6GHz). We see that if the CPU times are reasonable for $p \leq 5$, they become rapidly high for $p = 6$ (more than 4 minutes) or $p = 7$ (more than 1 hour). We also compare the CPU times obtained with the Algorithm 1 with a method based on a linear program: for each point of the Pareto front, we check if there exists a capacity v such that the Choquet integral of this point is better that all the other points. We see that this method is more effective once $p \geq 6$.

Table 2. CPU Random multiobjective knapsack instances (3000 points)

# Crit	CPU(s) Algorithm 1	CPU(s) LP
2	19.91	46.32
3	12.26	32.22
4	19.89	44.76
5	33.43	66.80
6	291.58	88.78
7	4694.23	110.56

In table 3, we compare the CPU times obtained by both methods, for $p = 5$, according to the number of solutions (between 100 and 3000). We see that until the number of solutions is equal to 2000, the method based on the linear program is more effective. It is only for sets with at least 2000 solutions that the method based on the Algorithm 1 becomes faster, since it is only for high size sets that we can take the most of enumerating all the permutations.

Table 3. CPU Random multiobjective knapsack instances (5 criteria)

# Sol	CPU (s) LP	CPU (s) Algorithm 1
100	0.54	3.11
250	0.85	3.49
500	2.01	5.34
1000	6.78	9.95
2000	24.72	20.57
3000	73.65	33.43

5 Conclusion

We have introduced in this paper a new characterization of the Choquet optimal solutions in multicriteria decision contexts, and more specifically for multiobjective combinatorial optimization problems. We have also presented an algorithm to obtain these solutions based on this characterization. The experimentations showed that increasing the number of objectives increase the expressiveness of the Choquet integral comparing to the WS (more P-non-dominated points can be attained). This work about generating all Choquet-optimal solutions opens many new perspectives:

- Following [20], it will be interesting to study and to define what brings exactly and concretely (for a decision maker) the Choquet optimal solutions that are not WS optimal solutions, given that they are harder to compute.
- We have shown that it can be very time-consuming to apply the general method developed in this paper due to the increase of parameters of the Choquet integral. Therefore, dedicated methods to compute all the Choquet optimal solutions of specific problems could be studied.
- It will be interesting to study if we can adapt the characterization of the Choquet optimal solutions to more restrictive set of capacities, such that k-additive capacities [21].

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