9. Electrodynamics of Radiating Charges in a Gravitational Field

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The electrodynamics of a radiating charge and its electromagnetic field based upon the Lorentz–Abraham–Dirac (LAD) equation are discussed both with reference to an inertial reference frame and a uniformly accelerated reference frame. It is demonstrated that energy and momentum are conserved during runaway motion of a radiating charge and during free fall of a charge in a field of gravity. This does not mean that runaway motion is really happening. It may be an unphysical solution of the LAD equation of motion of a radiating charge due to the unrealistic point particle model of the charge upon which it is based. However it demonstrates the consistency of classical electrodynamics, including the LAD equation which is deduced from Maxwell's equations and the principle of energy-momentum conservation applied to a radiating charge and its electromagnetic field. The decisive role of the Schott energy in this connection is made clear and an answer is given to the question: What sort of energy is the Schott energy and where is it found? It is the part of the electromagnetic field energy which is proportional to (minus) the scalar product of the velocity and acceleration of a moving accelerated charged particle. In the case of the electromagnetic field of a point charge it is localized at the particle. This energy is negative if the acceleration is in the same direction as the velocity and positive if it is in the opposite direction. During runaway motion the Schott energy becomes more and more negative and in the case of a charged particle with finite extension, it is localized in a region with increasing extension surrounding the particle. The Schott energy provides the radiated energy of a freely falling charge. Also it is pointed out that a proton and a neutron fall with the same acceleration in a uniform gravitational field, although the proton radiates and the neutron does not. It is made clear that the question as to whether or not a charge radiates has a reference-dependent answer. An accelerated charge is not observed to radiate by an observer comoving with the charge, although an inertial observer finds that it radiates.

9.1 The Dynamics of a Charged Particle

The analysis of the energy-momentum balance of a radiating charge is usually based on the equation of motion of a point charge. The nonrelativistic version of the equation was discussed already more than 100 years ago by *H. A. Lorentz* [9.1]. The relativistic generalization of the equation was originally found by

M. Abraham [9.2] in 1905 and re-derived in 1909 by *M. von Laue* [9.3] who Lorentz transformed the nonrelativistic equation from the instantaneous rest frame of the charge to an arbitrary inertial frame. A new deduction of the Lorentz covariant equation of motion was given by *P. A. M. Dirac* in 1938 [9.4]. This equation is therefore called the Lorentz–Abraham–Dirac equation, or for short the LAD equation. A particularly interesting feature about Dirac's deduction is that it establishes a connection between Maxwell's equations and the equation of motion for a charged particle. It shows that the presence of the Abraham four-vector in the equation of motion (Sect. 9.1.2) comes from conservation of energy and momentum for a closed system consisting of a charge and its electromagnetic field.

9.1.1 The Nonrelativistic Equation of Motion of a Radiating Charge

In the nonrelativistic limit the equation of motion of a radiating charge, q, with mass m_0, acted upon by an external force, $\boldsymbol{f}_{\text{ext}}$, takes the form

$$m_0\ddot{\boldsymbol{r}} = \boldsymbol{f}_{\text{ext}} + m_0\tau_0\dddot{\boldsymbol{r}}\,, \quad \tau_0 = \frac{q^2}{6}\pi\varepsilon_0 m_0 c^3\,, \tag{9.1}$$

where the dot denotes differentiation with respect to the (Newtonian) time. If q and m_0 represent the charge and mass of an electron, respectively, τ_0 is of the same order of magnitude as the time taken by light to move a distance equal to the classical electron radius, i. e., $\tau_0 \approx 10^{-23}$ s. The general solution of the equation is

$$\ddot{\boldsymbol{r}}(T) = \mathrm{e}^{T/\tau_0} \times \left[\ddot{\boldsymbol{r}}(0) - \frac{1}{m\tau_0}\int_0^T \mathrm{e}^{-T'/\tau_0}\boldsymbol{f}_{\text{ext}}(T')\,\mathrm{d}T'\right]. \tag{9.2}$$

Hence the charge performs a runaway motion unless one chooses the initial condition

$$m\tau_0\ddot{\boldsymbol{r}}(0) = \int_0^\infty \mathrm{e}^{-T'/\tau_0}\boldsymbol{f}_{\text{ext}}(T')\,\mathrm{d}T'\,. \tag{9.3}$$

By combining (9.2) and (9.3) one obtains [9.5]

$$m\ddot{\boldsymbol{r}}(T) = \int_0^\infty \mathrm{e}^{-s}\boldsymbol{f}_{\text{ext}}(T+\tau_0 s)\,\mathrm{d}s\,. \tag{9.4}$$

This equation shows that the acceleration of the charge at a point of time T is determined by the future force, weighted by a decreasing exponential factor with value 1 at the time T, and a time constant τ_0, i. e., there is *pre-acceleration*.

In his discussion of (9.1) *Lorentz* [9.1] writes:

> *In many cases the new force represented by the second term in (9.1) may be termed a* resistance *to the motion. This is seen if we calculate the work of the force during an interval of time extending from $T = T_1$ to $T = T_2$. The result is*
>
> $$\int_{T_1}^{T_2} \dot{\boldsymbol{a}}\cdot\boldsymbol{v}\,\mathrm{d}T = [\boldsymbol{a}\cdot\boldsymbol{v}]_{T_1}^{T_2} - \int_{T_1}^{T_2} \boldsymbol{a}^2\,\mathrm{d}T\,. \tag{9.5}$$
>
> *Here the first term disappears if, in the case of periodic motion, the integration is extended to a full period, and also if at the instants T_1 and T_2 either the velocity or the acceleration is zero. Whenever the above formula reduces to the last term, the work of the force is seen to be negative, so that the name of resistance is then justly applied.*

P. Yi [9.6] gives the following interpretation:

> *The total energy of the system may be split into three pieces: the kinetic energy of the charged particle, the radiation energy, and the electromagnetic energy of the Coulomb field. In effect, the last acts as a sort of energy reservoir that mediates the energy transfer from the first to the second and* in the special case of uniform acceleration provides all the radiation energy without extracting any from the charged particle.

9.1.2 The Relativistic Equation of Motion of a Radiating Charge

The original relativistic equation of motion of a particle with rest mass m_0 and charge q (the LAD equation) may be written as [9.7]

$$F^{\mu}_{\text{ext}} + \Gamma^{\mu} = m_0\dot{U}^{\mu}\,, \tag{9.6}$$

where

$$\Gamma^{\mu} \equiv m_0\tau_0\left(\dot{A}^{\mu} - A^{\alpha}A_{\alpha}U^{\mu}\right), \tag{9.7}$$

and the dot denotes differentiation with respect to the proper time of the particle. Here F^{μ}_{ext} is the external force acting upon the particle, U^{μ} is its four-velocity and A^μ its four-acceleration. (Capital letters shall be used for four-vector components referring to an inertial frame, and units are used so that $c = 1$.)

The vector Γ^μ is called the *Abraham four-force* and is given by

$$\Gamma^\mu = \gamma\,(\boldsymbol{v}\cdot\boldsymbol{\Gamma}, \boldsymbol{\Gamma})\,, \tag{9.8}$$

where $\boldsymbol{\Gamma}$ is the three-dimensional force called the *field reaction force* [9.8], v is the ordinary velocity of the particle, and $\gamma = (1-v^2)^{-1/2}$. In an inertial reference frame the Abraham four-force may be written as

$$\Gamma^\mu = m_0\tau_0\gamma(\boldsymbol{v}\cdot\dot{\boldsymbol{g}}, \dot{\boldsymbol{g}})\,, \tag{9.9}$$

where $g = (A_\alpha A^\alpha)^{\frac{1}{2}}$ is the proper acceleration of the charged particle in the inertial frame. Hence

$$\boldsymbol{\Gamma} = m_0\tau_0\dot{\boldsymbol{g}}\,. \tag{9.10}$$

In flat spacetime there exist global inertial frames. However, in curved spacetime there are only *local* inertial frames. They are freely falling. Then $\boldsymbol{g}$ is the acceleration of the particle in a freely falling frame in which the particle is instantaneously at rest. In such a frame a freely falling particle has no acceleration. Hence, a particle falling freely in a gravitational field has vanishing four-acceleration. From (9.7) and (9.10) is seen that for such a particle the Abraham four-force vanishes. This is also valid for a charged particle emitting radiation while it falls. This case shall be treated in more detail in Sect. 9.5.

According to the Lorentz covariant Larmor formula, valid with reference to inertial systems, the energy radiated by the particle per unit time is (using the sign convention that the signature of the metric is $+2$),

$$P_{\mathrm{L}} = m_0\tau_0 A^\alpha A_\alpha = m_0\tau_0 g^2\,. \tag{9.11}$$

The radiated momentum per unit proper time is

$$P_{\mathrm{R}}^\mu = P_{\mathrm{L}} U^\mu\,. \tag{9.12}$$

From the equation of motion (9.6) we obtain the energy equation

$$\begin{aligned}\boldsymbol{v}\cdot\boldsymbol{F}_{\mathrm{ext}} &= \gamma^{-1}\left(m_0\dot{U}^0 - \Gamma^0\right)\\ &= m_0\gamma^3\boldsymbol{v}\cdot\boldsymbol{a} - \boldsymbol{v}\cdot\boldsymbol{\Gamma}\\ &= \frac{\mathrm{d}E_{\mathrm{K}}}{\mathrm{d}T} - \boldsymbol{v}\cdot\boldsymbol{\Gamma}\,,\end{aligned} \tag{9.13}$$

where $E_{\mathrm{K}} = (\gamma-1)m_0c^2$ is the kinetic energy of the particle and T is the coordinate time in the inertial frame. Note that the energy supplied by the external force is equal to the change of the kinetic energy of the charge when the Abraham four-force vanishes. Hence, it is tempting to conclude from the Abraham–Lorentz theory, i. e., from (9.10) and (9.13), that a charge having constant acceleration does not radiate. This is, however, not the case. The power due to the field reaction force is

$$\begin{aligned}\boldsymbol{v}\cdot\boldsymbol{\Gamma} &= m_0\tau_0\frac{\mathrm{d}}{\mathrm{d}T}(\gamma^4\boldsymbol{v}\cdot\boldsymbol{a}) - P_{\mathrm{L}}\\ &= -\frac{\mathrm{d}E_{\mathrm{S}}}{\mathrm{d}T} - \frac{\mathrm{d}E_{\mathrm{R}}}{\mathrm{d}T}\,,\end{aligned} \tag{9.14}$$

where E_{R} is the energy of the radiation field and E_{S} is the Schott energy defined by

$$E_{\mathrm{S}} \equiv -m_0\tau_0\gamma^4\boldsymbol{v}\cdot\boldsymbol{a} = -m_0\tau_0 A^0\,. \tag{9.15}$$

(This energy was called *acceleration energy* by *Schott* [9.9] but is now usually called *Schott energy*.) Hence, in the case of constant acceleration, when the Abraham four-force vanishes, the charge radiates in accordance with Larmor's formula, (9.11), and the rate of radiated energy is equal to minus the rate of change of the Schott energy. The energy equation may now be written as

$$\frac{\mathrm{d}W_{\mathrm{ext}}}{\mathrm{d}T} = \boldsymbol{v}\cdot\boldsymbol{F}_{\mathrm{ext}} = \frac{\mathrm{d}}{\mathrm{d}T}(R_{\mathrm{K}} + E_{\mathrm{S}} + E_{\mathrm{R}})\,, \tag{9.16}$$

where W_{ext} is the work on the particle due to the external force.

Let $\boldsymbol{P}_{\mathrm{ext}}$ be the momentum delivered to the particle from the external force. Then $\mathrm{d}\boldsymbol{P}_{\mathrm{ext}}/\mathrm{d}T = \boldsymbol{F}_{\mathrm{ext}}$, and by means of (9.1), (9.2), and (9.7) we obtain

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{P}_{\mathrm{ext}}}{\mathrm{d}T} &= \boldsymbol{F}_{\mathrm{ext}} = m_0\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}T} - m_0\tau_0\left(\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}T} - g^2\boldsymbol{v}\right)\\ &= \frac{\mathrm{d}\boldsymbol{P}_M}{\mathrm{d}T} + \frac{\mathrm{d}\boldsymbol{P}_{\mathrm{S}}}{\mathrm{d}T} + \frac{\mathrm{d}\boldsymbol{P}_{\mathrm{R}}}{\mathrm{d}T}\,.\end{aligned} \tag{9.17}$$

Thus, according to (9.12) and (9.17) the four-momentum of the particle takes the form

$$P^\mu = P_M^\mu + P_{\mathrm{S}}^\mu\,, \tag{9.18}$$

where

$$P_M^\mu = m_0 U^\mu\,, \quad P_{\mathrm{S}}^\mu = -m_0\tau_0 A^\mu \tag{9.19}$$

are the mechanical four-momentum of the particle and the Schott four-momentum, respectively. In addition we have the four-momentum of the radiation field, which is not a state function of the particle.

9.1.3 Significance of the Schott Momentum

In order to demonstrate as clearly as possible the necessity of taking into account the Schott momentum in the dynamics of a charged particle we shall here consider circular motion with a constant speed [9.9]. Then $\boldsymbol{v} \cdot \boldsymbol{a} = 0$ and the Schott energy vanishes, but not the Schott momentum. It is

$$\boldsymbol{p}_{\mathrm{S}} = -m_0 \tau_0 \gamma^2 \boldsymbol{a} . \tag{9.20}$$

Since both the kinetic energy and the Schott energy are constant, (9.16) in this case reduces to

$$\frac{\mathrm{d}W_{\mathrm{ext}}}{\mathrm{d}T} = \boldsymbol{v} \cdot \boldsymbol{F}_{\mathrm{ext}} = \frac{\mathrm{d}E_{\mathrm{R}}}{\mathrm{d}T} = P_{\mathrm{L}} . \tag{9.21}$$

This equation shows that the radiated energy is provided by the tangential component of the external force.

Although the radiated energy per unit time is equal to the power due to the tangential component of the external force, the radiated momentum is not due only to this force. In order to see this most clearly we insert the expression for the centripetal acceleration into the expression (9.20) for the Schott momentum, which gives

$$\boldsymbol{p}_{\mathrm{S}} = -m_0 \tau_0 \gamma^2 \frac{v^2}{r} \boldsymbol{e}_r , \tag{9.22}$$

where $\boldsymbol{e}_r$ is the radial unit vector. The rate of change of the Schott momentum with respect to the inertial laboratory time is

$$\frac{\mathrm{d}\boldsymbol{p}_{\mathrm{S}}}{\mathrm{d}T} = m_0 \tau_0 \gamma^2 \frac{v^2}{r^2} \boldsymbol{v} . \tag{9.23}$$

Putting $\boldsymbol{F} = \boldsymbol{F}_{\parallel} + \boldsymbol{F}_{\perp}$ where $\boldsymbol{F}_{\parallel}$ and $\boldsymbol{F}_{\perp}$ are the components of $\boldsymbol{F}$ along and orthogonal to $\boldsymbol{v}$, we obtain

$$\boldsymbol{F}_{\perp} = \gamma m_0 \boldsymbol{a} , \tag{9.24}$$

and

$$\boldsymbol{F}_{\parallel} = \frac{\mathrm{d}\boldsymbol{p}_{\mathrm{S}}}{\mathrm{d}T} + P_{\mathrm{L}} \boldsymbol{v} . \tag{9.25}$$

For an uncharged particle the centripetal force $\boldsymbol{F}_{\perp}$ is the only force. But in the case of a charged particle a tangential force $\boldsymbol{F}_{\parallel}$ is necessary to keep the velocity constant. The radiated energy comes from the work performed by this force. The radiated momentum is partly due to $\boldsymbol{F}_{\parallel}$ and partly due to the change of the direction of the Schott momentum vector.

9.2 Schott Energy as Electromagnetic Field Energy

Already in 1915 *Schott* [9.10] argued that in the case of uniformly accelerated motion

> *the energy radiated by the electron is derived entirely from its acceleration energy; there is as it were internal compensation amongst the different parts of its radiation pressure, which causes its resultant effect to vanish.*

But what is the *acceleration energy*, now called *the Schott energy*? *Schott* [9.10] and later *Rohrlich* [9.8] noted that there is an important difference between the radiation rate and the rate of change of the Schott energy:

> *The radiation rate is always positive (or zero) and describes an* irreversible loss *of energy; the Schott energy changes in a* reversible *fashion, returning to the same value whenever the state of motion repeats itself.*

Rohrlich also wrote [9.11]:

> *If the Schott energy is expressed by the electromagnetic field, it would describe an energy content of the near field of the charged particle which can be changed* reversibly. *In periodic motion energy is borrowed, returned, and stored in the near-field during each period. Since the time of energy measurement is usually large compared to such a period only the average energy is of interest and that average of the Schott energy rate vanishes. Uniformly accelerated motion permits one to borrow energy from the near-field for large* macroscopic *time-intervals, and no averaging can be done because at no two points during the motion is the acceleration four-vector the same. Nobody has so far shown in detail just how the Schott energy occurs in the near-field, how it is stored, borrowed etc.*

A step toward answering this challenge was taken by *C. Teitelboim* [9.12]. He made a Lorentz invariant separation of the field tensor of the electromagnetic field of a point charge into two parts, $F^{\mu\nu} = F_{\mathrm{I}}^{\mu\nu} + F_{\mathrm{II}}^{\mu\nu}$, where $F_{\mathrm{I}}^{\mu\nu}$ is the velocity field and $F_{\mathrm{II}}^{\mu\nu}$ the acceleration field. Inserting these parts into the expression for the energy-momentum tensor of the electromagnetic field, Teitelboim found that the energy-momentum tensor contains terms of three types: a part $T_{\mathrm{I,I}}^{\mu\nu}$ independent of the acceleration, a part $T_{\mathrm{I,II}}^{\mu\nu}$ depending linearly upon the acceleration, and a part $T_{\mathrm{II,II}}^{\mu\nu}$ depending linearly upon the square of the acceleration of the charged particle producing the fields. Teitelboim then defined $T_{\mathrm{I}}^{\mu\nu} = T_{\mathrm{I,I}}^{\mu\nu} + T_{\mathrm{I,II}}^{\mu\nu}$ and $T_{\mathrm{II}}^{\mu\nu} = T_{\mathrm{II,II}}^{\mu\nu}$. The contribution of the interference between the fields I and II has been included in $T_{\mathrm{I}}^{\mu\nu}$, whereas the tensor $T_{\mathrm{II}}^{\mu\nu}$ is related only to the part of the field depending upon the square of the acceleration. Teitelboim showed that the energy-momentum associated with the field $F_{\mathrm{II}}^{\mu\nu}$ travels with the speed of light. The field fronts are spheres with centers at the emission points. The four-momentum associated with $T_{\mathrm{I}}^{\mu\nu}$ remains bound to the charge. Furthermore he calculated the four-momenta and their time derivatives associated with $T_{\mathrm{I}}^{\mu\nu}$ and $T_{\mathrm{II}}^{\mu\nu}$.

The results of Rohrlich and Teitelboim have been summarized by *P. Pearle* [9.13] in the following way:

The term Γ^{μ} in the Lorentz–Dirac equation, as given in (9.6), is called the Abraham force. Its first term, $m_0\tau_0\dot{A}^{\mu}$ is called the Schott term, and its second, $-m_0\tau_0 A^{\alpha}A_{\alpha}U^{\mu}$, the radiation reaction term. The zeroth component of the radiation reaction term is to be interpreted as the radiation rate. Indeed, the scalar product of this term with U_{μ} is the relativistic version of the Larmor formula. The spatial component of this term, proportional to $-\boldsymbol{v}$ like a viscous drag force, may similarly be interpreted as the radiation reaction force of the electron.

The physical meaning of the Schott term has been puzzled over for a long time. Its zero component represents a power which adds Schott acceleration energy *to the electron and its associated electromagnetic field. The work done by an external force not only goes into electromagnetic radiation and into increasing the electron's kinetic energy, but it causes an increase in the* Schott acceleration energy *as well. This change can be ascribed to a change in the* bound *electromagnetic energy in the electron's induction field, just as the last term of (9.14) can be ascribed to a change in the* free *electromagnetic energy in the electron's radiation field.*

What meaning should be given to the Schott term? Teitelboim [9.12] *has argued convincingly that when an electron accelerates, its near-field is modified so that a correct integration of the electromagnetic four-momentum of the electron includes not only the Coulomb four-momentum $(q^2/8\pi\varepsilon_0 r)U^{\mu}$, but an extra four-momentum $-m_0\tau_0 A^{\mu}$ of the bound electromagnetic field.*

It remained to obtain a more precise localization of the Schott field energy.

Rowe [9.15] modified Teitelboim's separation of the energy-momentum tensor of the electromagnetic field of a point charge, which he described by a delta-function, and introduced a separation into three symmetrical, divergence-free parts. In order to obtain a finite expression for the localization of the Schott energy as part of the energy of the electromagnetic field of a charged particle, *Eriksen* and *Grøn* [9.7] applied Rowe's separation to a charged particle with a finite radius and obtained the following result. The Schott energy is inside a spherical light front S touching the front end of a moving Lorentz contracted charged particle.

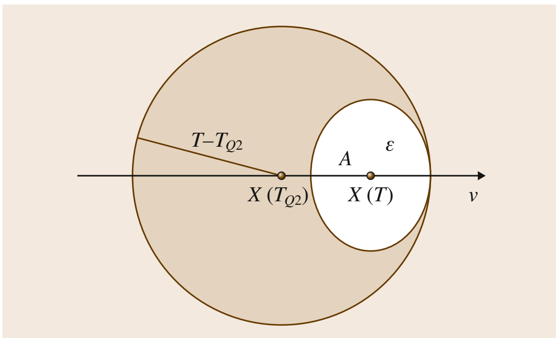

Fig. 9.1 A Lorentz contracted charged particle with proper radius ε moving to the right with velocity v. The field is observed at a point of time T, and at this moment the center of the particle is at the position $X(T)$. The circle is a field front produced at the retarded point of time T_{Q2} when the center of the particle was at the position $X(T_{Q2})$. The field front is chosen such that it just touches the front of the particle. The Schott energy is localized in the shaded region between the field front and the ellipsoid representing the surface of the particle. The velocity is chosen to be $v = 0.6$ (after [9.14], courtesy of the American Association of Physics Teachers)

From Fig. 9.1 one finds that at the point of time T the radius of the light front S that represents the boundary of the distribution of the Schott energy, is

$$T - T_{Q2} = \varepsilon \sqrt{\frac{1+v}{1-v}} , \tag{9.26}$$

where the field at the light front S is produced at the retarded point of time T_{Q2}, ε is the proper radius of the particle, and v is the absolute value of its velocity. Hence, unless the velocity of the charge is close to that of light, the Schott energy is localized just outside the surface of the charged particle. The radius of the light front S increases towards infinity for the field of a charge approaching the velocity of light, for example during runaway motion, which we shall now consider in Sect. 9.4.

In the deduction of the localization of the Schott energy we have not applied the equation of motion of the charge. We have only considered the field produced by the charge. Hence the deduction permits us to consider a charge with a finite proper radius. However, the LAD equation is deduced for a point charge. So relating the Schott energy to the conservation of energy for a radiating point charge and the field it produces, we should take the limit $\varepsilon \to 0$. In this limit it seems that the Schott energy is localized at the point charge. But it must be admitted that this limit seems rather unphysical, and our conclusion should rather be that this limit signals a breakdown of classical electrodynamics, or at least some sort of unsolved problem.

A slightly different perception of the Schott energy, still interpreted as electromagnetic field energy, has recently been given by *D. R. Rowland* [9.16]. He found that the Schott energy is the difference between the energy in the actual bound electromagnetic field of a charge and the energy in the bound field if the charge had moved with a constant velocity equal to its instantaneous velocity. Rowland's analysis further provides the following physical explanation of the existence of the Schott energy:

> *This difference arises because the bound fields of a charge cannot respond* rigidly *when the state of motion of a charge is changed by an external force. During uniform acceleration, the rate of change of this difference is just the negative of the rate at which radiation energy is created, and hence the power needed to accelerate a charged particle uniformly is just that which is required to accelerate a neutral particle with the same rest mass even though the charge is radiating.*

9.3 Pre-Acceleration and Schott Energy

The LAD equation has two strange consequences [9.17–25]: pre-acceleration, which is accelerated motion before a force acts; and run-away motion, which is accelerated motion of a charge after a force which acted upon it has ceased to act.

It has been claimed that during a period of pre-acceleration, before a charge is acted upon by an external force, the charge will not emit radiation [9.26]. In this section we will review a recent demonstration we have given where it was shown that a charge emits radiation during a period of pre-acceleration and that the radiation energy then comes from the Schott energy, which decreases during this period [9.27].

We shall consider a particle with charge Q and rest mass m_0 moving in an inertial frame and acted upon by an external force F of finite duration.

With P_{R}^{μ} as defined in (9.12) and P_{S}^{μ} in (9.19), the LAD equation can be written as

$$F^{\nu} = m_0 \dot{U}^{\nu} + \dot{P}_{\mathrm{S}}^{\nu} + \dot{P}_{\mathrm{R}}^{\nu} . \tag{9.27}$$

In the following we restrict ourselves to linear motion (along the x-axis). The equations can be simplified by introducing the rapidity of the particle,

$$\alpha = \operatorname{artanh} v . \tag{9.28}$$

Hence,

$$v = \tanh\alpha , \quad \gamma = \cosh\alpha , \quad \gamma v = \sinh\alpha ,$$
$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}\tau} = \frac{\dot{v}}{\cosh\alpha} = \frac{\dot{\alpha}}{\cosh^3\alpha} , \tag{9.29}$$

$$\begin{aligned} U^{\nu} &= (\cosh\alpha, \sinh\alpha, 0, 0) , \\ A^{\nu} &= \dot{U}^{\nu} = (\dot{\alpha}\sinh\alpha, \dot{\alpha}\cosh\alpha, 0, 0) , \end{aligned} \tag{9.30}$$

where $\dot{\alpha} = a_0$, i. e. $\dot{\alpha}$ is the acceleration in the inertial rest frame. The components of the mechanical, Schott, and radiation four-momenta may then be expressed as

$$m_0 U^{\nu} = m_0(\cosh\alpha, \sinh\alpha) , \tag{9.31}$$

$$P_S^\nu = -\frac{2}{3}Q^2\dot{\alpha}(\sinh\alpha, \cosh\alpha)\,, \qquad (9.32)$$

$$P_R^\nu = \frac{2}{3}Q^2 \int_{-\infty}^{\tau} \dot{\alpha}^2(\cosh\alpha, \sinh\alpha)\,d\tau\,. \qquad (9.33)$$

Differentiation gives

$$m_0\dot{U}^\nu = m_0\dot{\alpha}(\sinh\alpha, \cosh\alpha)\,, \qquad (9.34)$$

$$\begin{aligned}\dot{P}_S^\nu &= -\frac{2}{3}Q^2\ddot{\alpha}(\sinh\alpha, \cosh\alpha) \\ &\quad -\frac{2}{3}Q^2\dot{\alpha}^2(\cosh\alpha, \sinh\alpha)\,,\end{aligned} \qquad (9.35)$$

$$\dot{P}_R^\nu = \frac{2}{3}Q^2\dot{\alpha}^2(\cosh\alpha, \sinh\alpha) \qquad (9.36)$$

with the sum

$$\begin{aligned}m_0\dot{U}^\nu + \dot{P}_S^\nu + \dot{P}_R^\nu &= \left(m_0\dot{\alpha} - \frac{2}{3}Q^2\ddot{\alpha}\right) \\ &\quad \times (\sinh\alpha, \cosh\alpha)\,.\end{aligned} \qquad (9.37)$$

In terms of the rapidity the Minkowski force (9.27) reads as

$$F^\nu = (\gamma v F, \gamma F) = F(\sinh\alpha, \cosh\alpha)\,. \qquad (9.38)$$

The LAD equation for linear motion then takes the form [9.28]

$$F = m_0(\dot{\alpha} - \tau_0\ddot{\alpha})\,. \qquad (9.39)$$

Note that (9.39) transforms into the nonrelativistic equation of motion when α is replaced by v and proper time by laboratory time [9.29]. Equation (9.39) may be written as

$$\frac{d}{d\tau}\left(e^{-\tau/\tau_0}\dot{\alpha}\right) = -\frac{F}{m_0\tau_0}e^{-\tau/\tau_0}\,. \qquad (9.40)$$

Let τ_1 and τ_2 be two points of proper time with $\tau_1 < \tau_2$ and F a function of τ such that $F(\tau) = 0$ for $\tau < \tau_1$ and $\tau > \tau_2$. Then the general solution of (9.40) may be written as

$$e^{-\tau/\tau_0}\dot{\alpha}(\tau) = \int_{\tau}^{\tau_2} \frac{F(\tau')}{m_0\tau_0}e^{-\tau'/\tau_0}\,d\tau' + C_0\,, \qquad (9.41)$$

where C_0 is a constant. For $\tau > \tau_2$ the integral is zero, and

$$\begin{aligned}&\dot{\alpha}(\tau) = C_0 e^{\tau/\tau_0}\,, \\ \text{i.e.}\quad &\alpha(\tau) = C_0\tau_0 e^{\tau/\tau_0} + \text{const.}\end{aligned} \qquad (9.42)$$

When $C_0 \neq 0$ this is a *run away* solution. The rapidity increases without any boundary, and the velocity approaches the velocity of light when $\tau \to \infty$.

In this section we put $C_0 = 0$, which gives the following solution of (9.40),

$$\dot{\alpha}(\tau) = e^{\tau/\tau_0}\int_{\tau}^{\tau_2} \frac{F(\tau')}{m_0\tau_0}e^{-\tau'/\tau_0}\,d\tau'\,. \qquad (9.43)$$

The integral has the same value for all $\tau < \tau_1$ and is equal to zero for $\tau > \tau_2$. For convenience we introduce the notation

$$f(\tau) \equiv \int_{\tau}^{\tau_2} \frac{F(\tau')}{m_0\tau_0}e^{-\tau'/\tau_0}\,d\tau'\,. \qquad (9.44)$$

We put $\alpha(-\infty) = 0$ and obtain from (9.43) and (9.44), for

$$\begin{aligned}&\text{for}\quad \tau < \tau_1\,,\quad \dot{\alpha} = e^{\tau/\tau_0}f(\tau_1)\,, \\ &\alpha = \tau_0 e^{\tau/\tau_0}f(\tau_1),\end{aligned} \qquad (9.45)$$

$$\begin{aligned}&\text{for}\quad \tau_1 < \tau < \tau_2\,,\quad \dot{\alpha} = e^{\tau/\tau_0}f(\tau)\,, \\ &a = \tau_0 e^{\tau/\tau_0}f(\tau) + \frac{1}{m_0}\int_{\tau_1}^{\tau} F(\tau')\,d\tau',\end{aligned} \qquad (9.46)$$

$$\begin{aligned}&\text{for}\quad \tau > \tau_2\,,\quad \dot{\alpha} = 0\,, \\ &\alpha = \alpha(\tau_2) = \frac{1}{m_0}\int_{\tau_1}^{\tau_2} F(\tau')\,d\tau'\,.\end{aligned} \qquad (9.47)$$

Note that $\dot{\alpha} = 0$ for $\tau < \tau_1$ if $f(\tau_1) = 0$. That is, there is no pre-acceleration if $\int_{\tau_1}^{\tau_2} F(\tau')e^{-\tau'/\tau_0}\,d\tau' = 0$.

In order to discuss the energy and momentum of the particle and its field, we consider the formulation (9.27) of the LAD equation, which is a conservation equation of energy and momentum in differential form. Let τ_a and τ be two points of proper time with $\tau > \tau_a$. Then according to (9.27)

$$\int_{\tau_a}^{\tau} F^\nu\,d\tau = \Delta\left(m_0U^\nu\right) + \Delta P_S^\nu + \Delta P_R^\nu\,. \qquad (9.48)$$

For $\nu = 0$ the left-hand side is the work done by the external force and for $\nu = 1$ it is the delivered momentum. The Δ symbols refer to the increments from τ_a to τ.

As seen from (9.45) the energies and momenta in the pre-acceleration period are given by (9.31)–(9.33)

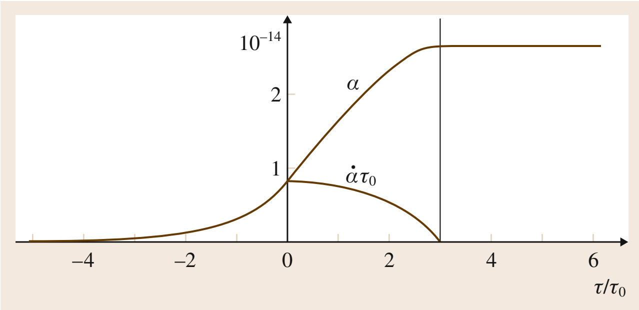

Fig. 9.2 α is the rapidity of an electron and τ_0 is the time taken by a light signal to travel a distance equal to two-thirds of the classical electron radius. A constant force acts from the proper time $\tau_1 = 0$ to the proper time $\tau_2 = 3\tau_0$. The electron, originally at rest, gets a motion (pre-acceleration) before the force acts (after [9.14], courtesy of the American Association of Physics Teachers)

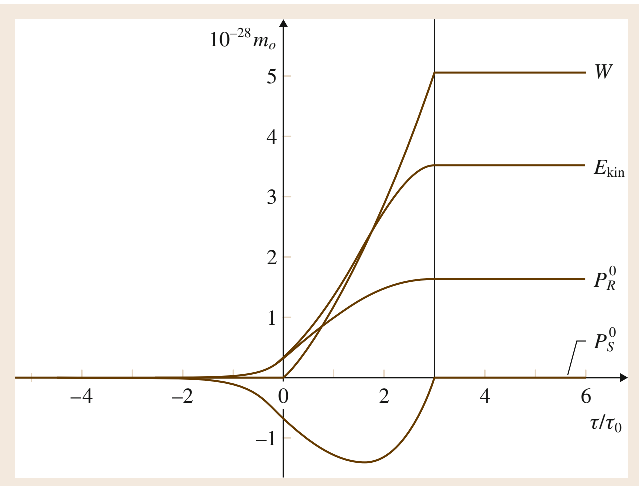

Fig. 9.3 The solution is the same as in Fig. 9.2. The graphs show the kinetic energy E_{kin}, the radiated energy P^0_{R}, the Schott energy P^0_{S}, and the external work W as a function of τ/τ_0. Note that $W = E_{\mathrm{kin}} + P^0_{\mathrm{R}} + P^0_{\mathrm{S}}$. In the pre-acceleration period $P^0_{\mathrm{R}} = E_{\mathrm{kin}}$ (after [9.14], courtesy of the American Association of Physics Teachers)

when we put $\dot{\alpha} = \alpha/\tau_0$. The integral in (9.43) is then solved by introducing $\mathrm{d}\tau = \tau_0\,\mathrm{d}\alpha/\alpha$. We put $\tau_a = -\infty$ and $\tau < \tau_1$. Due to the initial condition $\alpha(-\infty) = 0$ we obtain

$$\begin{aligned}\Delta\left(m_0 U^\nu\right) &= \left(E_{\mathrm{kin}}(\tau), P(\tau)\right)\\ &= m_0\left(\cosh\alpha - 1, \sinh\alpha\right),\end{aligned} \tag{9.49}$$

$$\Delta P^\nu_{\mathrm{S}} = P^\nu_{\mathrm{S}}(\tau) = m_0\left(-\alpha\sinh\alpha, -\alpha\cosh\alpha\right), \tag{9.50}$$

$$\begin{aligned}\Delta P^\nu_{\mathrm{R}} = P^\nu_{\mathrm{R}}(\tau) = m_0\,(&\alpha\sinh\alpha - \cosh\alpha + 1,\\ &\alpha\cosh\alpha - \sinh\alpha),\end{aligned} \tag{9.51}$$

where

$$\alpha = \tau_0 \mathrm{e}^{\tau/\tau_0} f(\tau_1). \tag{9.52}$$

This leads to

$$\Delta\left(m_0 U^\nu\right) + \Delta P^\nu_{\mathrm{S}} + \Delta P^\nu_{\mathrm{R}} = 0, \tag{9.53}$$

which says that the total increment of the energy and momentum of the system is zero, as it must be since the external force in the interval is zero.

A simple illustration of the above results is obtained by considering the special case where F is constant. We then put $g = F/m_0$, and the solution (9.45)–(9.47) takes the form

$$\begin{aligned}&\tau < \tau_1,\\ &\alpha = g\tau_0 \mathrm{e}^{(\tau-\tau_1)/\tau_0}\left(1 - \mathrm{e}^{-(\tau_2-\tau_1)/\tau_0}\right),\end{aligned} \tag{9.54}$$

$$\begin{aligned}&\tau_1 < \tau < \tau_2,\\ &\alpha = g\tau_0\left(1 - \mathrm{e}^{(\tau-\tau_2)/\tau_0}\right) + g(\tau - \tau_1),\end{aligned} \tag{9.55}$$

$$\tau > \tau_2, \quad \alpha = g(\tau_2 - \tau_1). \tag{9.56}$$

The rapidity α and its rate of change times τ_0 are shown graphically in Fig. 9.2. The corresponding curves for the work performed by the external force, $W = \int_{\tau_1}^{\tau} F^0\,\mathrm{d}\tau$, the kinetic energy of the particle, the radiation energy, and the Schott energy, as given in (9.49)–(9.51), are shown in Fig. 9.3.

In order to obtain some intuition about the quantities involved, we may refer to the figures, where we have put $\tau_2 - \tau_1 = 3\tau_0$, and the external force is due to the critical electrical field in air, $E = 2.4\times10^6\,\mathrm{V\,m^{-1}}$. Then for an electron, $g = 4.2\times10^{17}\,\mathrm{m\,s^{-2}}$ and $g\tau_0 = 2.6\times10^{-6}\,\mathrm{m\,s^{-1}}$. In ordinary units, where c is not taken to be 1, the factor $g\tau_0$ in (9.54), say, should be replaced by $g\tau_0/c = 0.88\times10^{-14}$. Hence, according to (9.54) the rapidity in the pre-acceleration period is of the order 10^{-14}, $\alpha(\tau_1) = 0.84\times10^{-14}$, $v(\tau_1) = c\tanh\alpha(\tau_1) = 2.5\times10^{-6}\,\mathrm{m\,s^{-1}}$. To lowest order in α (the next order is of the magnitude 10^{-42}) the expressions (9.49) and (9.50) for the changes of the kinetic energy and the Schott energy, and the emitted radiation energy in the pre-acceleration period reduce to

$$E_{\mathrm{kin}} = m_0(\cosh\alpha - 1) \approx \frac{1}{2} m_0\alpha^2, \tag{9.57}$$

$$P^0_{\mathrm{S}} = -m_0\alpha\sinh\alpha \approx -m_0\alpha^2, \tag{9.58}$$

$$P_R^0 = m_0\,(1+\alpha\sinh\alpha-\cosh\alpha) \approx \frac{1}{2}m_0\alpha^2\,, \tag{9.59}$$

where α is given by (9.54). These expressions show that radiated energy is approximately equal to the increase of kinetic energy.

9.4 Energy Conservation During Runaway Motion

Runaway acceleration seems to be in conflict with the conservation laws of energy and momentum. The momentum and the kinetic energy of the particle increase even when no force acts upon it. The charged particle even puts out energy in the form of radiation. Where do the energy and the momentum come from?

We shall here show that the source of energy and momentum in runaway motion is the so-called Schott energy and momentum [9.30]. During motion of a charge in which the velocity increases, the Schott energy has an increasingly negative value and there is an increasing Schott momentum directed oppositely to the direction of the motion of the charge.

We shall consider a charged particle performing runaway motion along the x-axis. Introducing the rapidity α of the particle its velocity and acceleration is expressed as in (9.29).

For $F=0$, i. e., for a free particle, the solutions of the LAD equation (9.40) are

1.
$$\dot\alpha = 0, \quad \text{i.e.} \quad \alpha = \text{const.}\,, \quad v = \text{const.}, \tag{9.60}$$
which is consistent with Newton's first law;
2.
$$\dot\alpha = k e^{\tau/\tau_0}\,, \quad k \neq 0\,, \quad \text{i.e.} \quad a \neq 0 \tag{9.61}$$
is the runaway solution.

As pointed out by *Dirac* [9.4] a particle in state 1 or 2 will remain in that state as long as no external force is acting. We shall here consider a particle which is at rest, i. e., in state 1., until it is acted upon by a force $F(\tau)$ pointing in the positive x-direction, i. e., we consider a solution to the LAD equation without pre-acceleration. The force acts from τ_1 to τ_2. For $\tau > \tau_2$ the particle is again free.

According to (9.43) $\dot\alpha$ is in the present case given by

$$\dot\alpha(\tau) = -\frac{e^{\tau/\tau_0}}{m_0\tau_0}\int_{-\infty}^{\tau} F(\tau')e^{-\tau'/\tau_0}\,d\tau'\,. \tag{9.62}$$

The integral vanishes for $\tau < \tau_1$, which gives $\dot\alpha = 0$ (and $\alpha = 0$). For $\tau > \tau_2$ the integral is independent of τ and we obtain the runaway motion (9.61). If the integral limit $-\infty$ in (9.62) is replaced by ∞, pre-acceleration is introduced, and the runaway motion disappears.

In the following, we examine (9.62) when the force F has constant value F_0 between τ_1 and τ_2, and is equal to zero outside this interval. The solution of the equation of motion is then

$$\tau < \tau_1\,, \quad \dot\alpha = 0\,, \quad \alpha = 0\,, \tag{9.63}$$

$$\tau_1 < \tau < \tau_2\,, \quad \dot\alpha = \frac{F_0}{m_0} - \frac{F_0}{m_0}e^{\frac{\tau-\tau_1}{\tau_0}}\,, \qquad \alpha = \frac{F_0}{m_0}(\tau-\tau_1) - \frac{F_0\tau_0}{m_0}\left(e^{\frac{\tau-\tau_1}{\tau_0}}-1\right), \tag{9.64}$$

$$\tau_2 < \tau\,, \quad \dot\alpha = -\frac{F_0}{m_0}\left(e^{-\frac{\tau_1}{\tau_0}} - e^{-\frac{\tau_2}{\tau_0}}\right)e^{\frac{\tau}{\tau_0}}\,, \qquad \alpha = \frac{F_0}{m_0}(\tau_2-\tau_1) - \frac{F_0\tau_0}{m_0}\left(e^{-\frac{\tau_1}{\tau_0}} - e^{-\frac{\tau_2}{\tau_0}}\right)e^{\frac{\tau}{\tau_0}}\,. \tag{9.65}$$

Equation (9.30) shows a strange aspect of the motion. The quantity $\dot\alpha$ contains two terms. The first expresses the relativistic version of Newton's second law, i. e., $F_0 = d(\gamma m_0 v)/dt$. However, the second term represents a runaway motion *oppositely directed* relative to the external force F_0, a highly unexpected mathematical result. According to (9.64) $\dot\alpha$ and α are oppositely directed relative to F_0 during the entire time interval $\tau_1 < \tau < \tau_2$.

At the point of time $\tau = \tau_2$,

$$\dot\alpha(\tau_2) = \frac{F_0}{m_0}\left(1 - e^{\frac{\tau_2-\tau_1}{\tau_0}}\right), \tag{9.66}$$

$$\alpha\,(\tau_2) = \frac{F_0}{m_0}\left(\tau_2-\tau_1+\tau_0-\tau_0 e^{\frac{\tau_2-\tau_1}{\tau_0}}\right). \tag{9.67}$$

In order to simplify the expressions we let $\tau_2-\tau_1 \to 0$ and $F_0 \to \infty$ keeping the product $(\tau_2-\tau_1)\cdot F_0 \equiv P$

constant. We then find the limits

$$\dot{\alpha}(\tau_2) = -\frac{P}{m_0\tau_0}\,, \quad \text{i.e.} \quad a = -\frac{P}{m_0\tau_0}\,, \tag{9.68}$$

$$\alpha(\tau_2) = 0\,, \quad \text{i.e.} \quad v = 0\,. \tag{9.69}$$

In this limit the external force is expressed by a δ function

$$F(\tau) = \delta(\tau - \tau_1)P\,. \tag{9.70}$$

Putting $\tau_1 = 0$ we have the situation: for $\tau < 0$ the particle stays at rest. At $\tau = 0$ it is acted upon by the force

$$F = \delta(\tau)P\,, \tag{9.71}$$

giving the particle an acceleration oppositely directed relatively to the force and a vanishing initial velocity,

$$a(0) = a_0 = -\frac{P}{m_0\tau_0}\,, \quad v(0) = 0\,. \tag{9.72}$$

According to (9.63), (9.64) and (9.72) the motion is as follows,

$$\tau < 0\,, \quad \dot{\alpha} = 0\,, \quad \alpha = 0\,, \tag{9.73}$$

$$\tau > 0\,, \quad \dot{\alpha} = a_0 e^{\frac{\tau}{\tau_0}}\,, \quad \alpha = \tau_0 a_0 e^{\frac{\tau}{\tau_0}} - \tau_0 a_0\,. \tag{9.74}$$

The runaway motion for $\tau > 0$ is accelerated, and the velocity $v = \tanh\alpha$ approaches the velocity of light as an unobtainable limit (Fig. 9.4).

The problem is to explain how this is possible for a particle not acted upon by any external force. It must be possible to demonstrate that the energy and momentum of the particle and its electromagnetic field is conserved, and find the force causing the acceleration. Of essential importance in this connection is the Schott energy and the Schott momentum.

Noting that $\dot{\alpha}$ is the acceleration in the instantaneous inertial rest frame of the particle, we find the energies expressed by the rapidity utilizing, from (9.74), that $\dot{\alpha} = a_0 + \alpha/\tau_0$. The kinetic energy of the particle is

$$E_{\text{kin}} = m_0(\gamma - 1) = m_0(\cosh\alpha - 1)\,. \tag{9.75}$$

The radiation energy is

$$\begin{aligned} E_{\text{R}} &= m_0\tau_0 \int_0^\tau \dot{\alpha}^2 \cosh\alpha \,\mathrm{d}\tau \\ &= m_0(\alpha\sinh\alpha + a_0\tau_0\sinh\alpha - \cosh\alpha + 1)\,. \end{aligned} \tag{9.76}$$

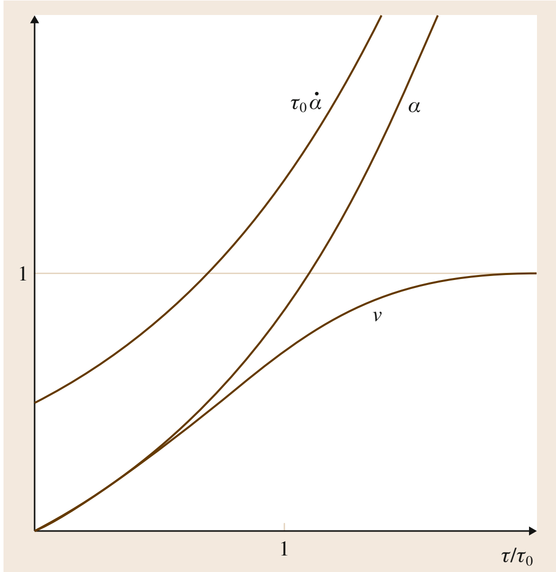

Fig. 9.4 The proper acceleration $\dot{\alpha}$, the velocity parameter α, and the velocity $v = \tanh\alpha$, as functions of the proper time for a particle performing runaway motion, starting from rest with positive acceleration. The quantity τ_0 is the time taken by a light signal to travel a distance equal to two-thirds of the particle's classical radius (after [9.14], courtesy of the American Association of Physics Teachers)

The Schott energy is

$$\begin{aligned} E_{\text{S}} &= -m_0\tau_0\gamma^4 va = -m_0\tau_0\dot{\alpha}\sinh\alpha \\ &= -m_0(\alpha + a_0\tau_0)\sinh\alpha\,. \end{aligned} \tag{9.77}$$

The sum of the energies is constant and equal to the initial value zero (Fig. 9.5).

Referring to Fig. 9.1 we see that during the runaway motion the sum of the increase of kinetic energy and radiation energy comes from tapping a reservoir of Schott energy which is initially localized very close to the charge. This field energy becomes more and more negative, and the radius of the spherical surface limiting the region with Schott energy increases rapidly. Using (9.26), (9.29), and (9.74) we find that it varies with the proper time of the particle as

$$T - T_{Q2} = \varepsilon e^{\alpha} = \varepsilon \exp\left[|a_0| \exp\left(\frac{\tau}{\tau_0}\right)\right], \tag{9.78}$$

where a_0 is given in (9.72).

Next we consider the momenta. The momentum of the particle is

$$P_{\text{kin}} = m_0\gamma v = m_0\sinh\alpha\,. \tag{9.79}$$

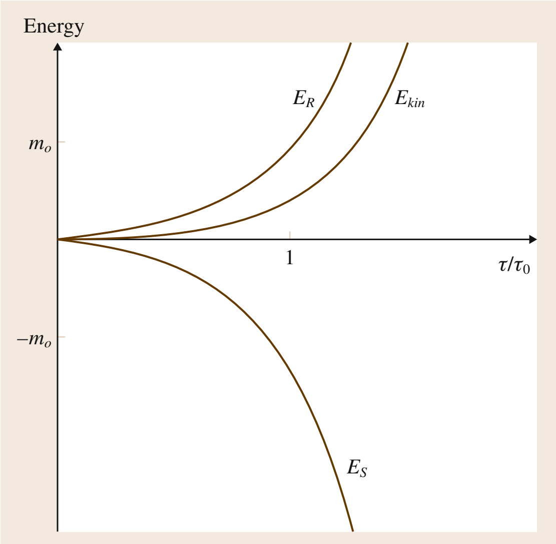

Fig. 9.5 The energies of a particle and its electromagnetic field while the particle performs runaway motion, as functions of τ/τ_0. Here E_{kin} is kinetic energy, E_{R} is radiated energy, and E_{S} is the Schott (or acceleration) energy (after [9.14], courtesy of the American Association of Physics Teachers)

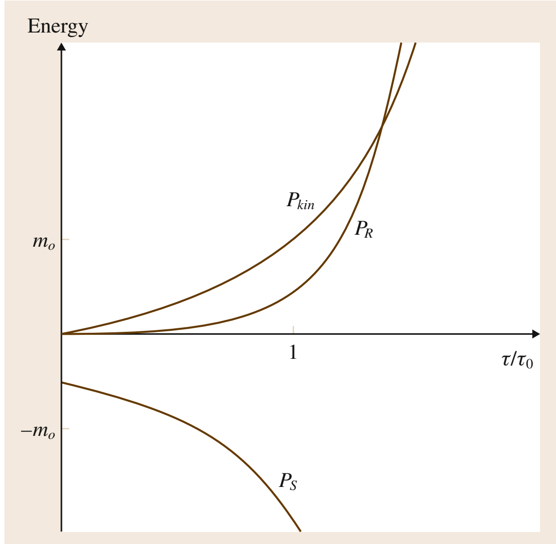

Fig. 9.6 The momentum of a particle and its electromagnetic field while the particle performs runaway motion, as functions of τ/τ_0. Here P_{kin} is kinetic momentum, P_{R} is radiated momentum, and P_{S} is Schott (or acceleration) momentum (after [9.14], courtesy of the American Association of Physics Teachers)

The momentum of the radiation is

$$\begin{aligned} P_{\mathrm{R}} &= m_0\tau_0 \int_0^{\tau} \dot{\alpha}^2 \sinh\alpha \,\mathrm{d}\tau \\ &= m_0(\alpha\cosh\alpha + a_0\tau_0\cosh\alpha - \sinh\alpha - a_0\tau_0)\ . \end{aligned} \tag{9.80}$$

The Schott momentum (acceleration momentum) is

$$\begin{aligned} P_{\mathrm{S}} &= -m_0\tau_0\gamma^4 a = -m_0\tau_0\dot{\alpha}\cosh\alpha \\ &= -m_0(\alpha + a_0\tau_0)\cosh\alpha\ . \end{aligned} \tag{9.81}$$

The sum of the momenta is constant and is equal to $-m_0 a_0\tau_0$, which is the initial Schott momentum (Fig. 9.6).

The forces which are responsible for the increase in the momentum of the particle (internal forces) are the following (for rectilinear motion in general). The radiation reaction force,

$$\Gamma_{\mathrm{R}} = -\frac{\mathrm{d}P_{\mathrm{R}}}{\mathrm{d}t} = -m_0\tau_0\dot{\alpha}^2\tanh\alpha\ , \tag{9.82}$$

and the acceleration reaction force,

$$\begin{aligned} \Gamma_{\mathrm{A}} &= -\frac{\mathrm{d}P_{\mathrm{S}}}{\mathrm{d}t} = -\frac{1}{\cosh\alpha}\dot{P}_{\mathrm{S}} \\ &= m_0\tau_0(\ddot{\alpha} + \dot{\alpha}^2\tanh\alpha)\ . \end{aligned} \tag{9.83}$$

The total field reaction force (also called the self-force) is

$$\Gamma = \Gamma_{\mathrm{R}} + \Gamma_{\mathrm{A}} = m_0\tau_0\ddot{\alpha}\ . \tag{9.84}$$

By means of (9.82)–(9.84) the forces are shown as functions of τ/τ_0 in Fig. 9.7.

Equation (9.82) shows that the radiation reaction force Γ_{R} is a force that retards the motion, acting like friction in a fluid. The *push* in the direction of the motion is provided by the acceleration reaction force, which is opposite to the change of Schott momentum per unit time. This force is opposite to the direction of the external force, i. e., it has the same direction as the runaway motion.

There is a rather strange point here. We earlier identified the Schott energy as a field energy localized close to the charge [9.7]. Yet, in the present case the Schott

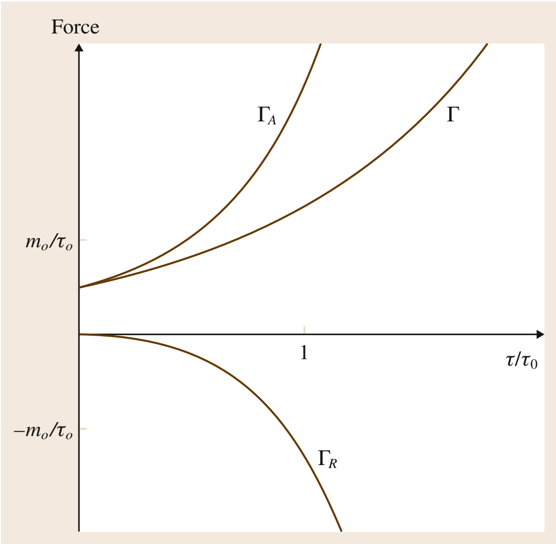

Fig. 9.7 The forces due to the electromagnetic field of a particle acting on the particle while it performs runaway motion, as functions of τ/τ_0. Here Γ_R is the radiation reaction force, Γ_A is the Schott (or acceleration) reaction force. Their sum is the field reaction force, $\Gamma = \Gamma_R + \Gamma_A$ (after [9.14], courtesy of the American Association of Physics Teachers)

momentum is oppositely directed to the motion of the charge. This is due to the fact that the Schott energy is negative. Hence even if the Schott momentum has a direction opposite to that of the velocity of the charge, it represents a motion of negative energy in the same direction as that of the charge.

In general the Schott energy is

$$E_S = -m_0 \tau_0 A^0 , \tag{9.85}$$

and the Schott momentum is

$$\boldsymbol{P}_S = -m_0 \tau_0 \boldsymbol{A} , \tag{9.86}$$

where $(A^0, \boldsymbol{A})$ is the four-acceleration of the particle. From the relation $A^0 = \boldsymbol{v} \cdot \boldsymbol{A}$ we obtain $E_S = \boldsymbol{v} \cdot \boldsymbol{P}_S$. It follows that for rectilinear motion $\boldsymbol{v}$ and $\boldsymbol{P}_S$ are oppositely directed when E_S is negative.

The Schott energy saves energy conservation for runaway motion of a radiating charge. Nevertheless, physicists doubt that runaway motion really exists. There is, however, no doubt that it is a solution of the LAD equation of motion of a charged particle. In this sense it is allowed, but maybe not everything that is allowed is obligatory. The physics equations seem to contain many possibilities that are not realized in our universe. Moreover, we seem to lack a criterion to eliminate those possibilities that do not exist physically. Hence one can only wonder why no runaways have ever been observed or why they could not be used as compact particle accelerators.

9.5 Schott Energy and Radiated Energy of a Freely Falling Charge

The Rindler coordinates (t, x, y, z) of a uniformly accelerated reference frame are given by the following transformation from the coordinates (T, X, Y, Z) of an inertial frame,

$$gt = \operatorname{artanh}\left(\frac{T}{X}\right) , \quad x = \sqrt{X^2 - T^2} , \tag{9.87}$$

with inverse transformation

$$T = x \sinh(gt) , \quad X = x \cosh(gt) . \tag{9.88}$$

Here g is a constant which shall be interpreted physically below.

Using Rindler coordinates, the line element takes the form [9.31]

$$\mathrm{d}s^2 = -g^2 x^2 \, \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 . \tag{9.89}$$

In the Rindler frame the nonvanishing Christoffel symbols are

$$\Gamma^x_{tt} = g^2 x , \quad \Gamma^t_{tx} = \Gamma^t_{xt} = \frac{1}{x} . \tag{9.90}$$

The Rindler coordinates are mathematically convenient, but not quite easy to interpret physically. An observer at rest in a uniformly accelerated reference frame in flat spacetime experiences a field of gravity. From the geodesic equation,

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0 , \tag{9.91}$$

which is also the equation of motion of a freely moving particle, follows that acceleration of a free particle instantaneously at rest is

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\Gamma^x_{tt} = -g^2 x \,. \tag{9.92}$$

Consider a fixed reference point $x = \text{constant}$ in the Rindler frame. It has velocity and acceleration

$$\begin{aligned} V &= \frac{\mathrm{d}X}{\mathrm{d}T} = \tanh(gt) \,, \\ A &= \frac{\mathrm{d}V}{\mathrm{d}T} = \frac{1}{x}\frac{1}{\cosh^3(gt)} \,, \end{aligned} \tag{9.93}$$

respectively, in the inertial frame. Hence, the acceleration of a reference point $x = \text{constant}$ at the point of time $t = 0$ is

$$A(0) = \frac{1}{x} \,. \tag{9.94}$$

This shows that the coordinate x of the Rindler frame has dimension 1 divided by acceleration and that it is equal to the inverse of the acceleration of the reference point that it represents at the point of time $t = 0$. The physical interpretation of the constant g then follows from (9.92). It represents the acceleration of gravity experienced in the Rindler frame at the reference point having acceleration equal to g relative to the inertial frame at the point of time $t = 0$.

The four-velocity and the four-acceleration of a particle moving along the x-axis are

$$\begin{aligned} u^\mu &= \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \gamma(1, v, 0, 0) \,, \\ \gamma &= (g^2x^2 - v^2)^{-\frac{1}{2}} \,, \end{aligned} \tag{9.95}$$

$$\begin{aligned} a^\mu &= \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta \\ &= \gamma^4 \left(a + g^2 x - \frac{2v^2}{x} \right) (v, g^2 x, 0, 0) \,, \end{aligned} \tag{9.96}$$

where $v = \mathrm{d}x/\mathrm{d}t$ and $a = \mathrm{d}v/\mathrm{d}t$.

As seen from the expression (9.7) for the Abraham four-force the field reaction force vanishes for a freely moving charge. Hence, such a charge falls with the same acceleration as a neutral particle. It has vanishing four-acceleration and follows a geodesic curve. This is valid in a uniformly accelerated reference frame in flat spacetime, but not in curved spacetime [9.32].

From the expression (9.96) it is seen that in the present case the equation of motion may be written as

$$x\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - 2\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + g^2x^2 = 0 \,. \tag{9.97}$$

The solution of this equation for a particle falling from $x = x_0$ at $t = 0$ is

$$x = \frac{x_0}{\cosh(gt)} \,. \tag{9.98}$$

Hence

$$\begin{aligned} v &= -gx_0 \frac{\sinh(gt)}{\cosh^2(gt)} \,, \\ \gamma &= \frac{\cosh^2(gt)}{gx_0} \,. \end{aligned} \tag{9.99}$$

In order to give a correct description of the radiation emitted by a charge valid in an accelerated frame of reference, one has to generalize the usual form of the Larmor formula for the radiated effect $\hat{P}$ valid in the orthonormal basis of an inertial frame,

$$\hat{P} = m_0 \tau_0 \hat{a}^2 \,, \tag{9.100}$$

where $\hat{a}$ is the acceleration of the charge in an inertial frame. This formula says that an accelerated charge radiates, which is a misleading statement. It sounds as if whether a charge radiates or not, is something invariant that all observers can agree upon. However, that is not the case. An accelerated observer permanently at rest relative to an accelerated charge would not say that it radiates. The covariant generalization of the formula is

$$P_\mathrm{L} = m_0 \tau_0 A_\mu A^\mu \,. \tag{9.101}$$

Freely falling charges have vanishing four-acceleration. Hence, this version of Larmor's formula seems to say that charges that are acted upon by nongravitational forces radiate. As mentioned above this is not generally the case. In fact, the formula above is not generally covariant. It is only Lorentz covariant because the components of the four-acceleration are presupposed to be given with reference to an inertial frame in this formula.

Saying that a charge radiates is not a reference-independent statement. This conclusion has been arrived at in different ways [9.33–38]. *M. Kretzschmar* and *W. Fugmann* [9.37, 38] generalized Larmor's formula (9.100) to a form which is valid not only in inertial reference frames, but also with respect to accelerated frames. A consequence of their formula is that a charge will be observed to emit radiation only if it accelerates relative to the observer. Whether it moves along a geodesic curve is not decisive. A freely falling charge, i. e., a charge at rest in an inertial frame may be observed to radiate, and a charge acted upon by non-geodesic forces may be observed not to radiate.

Hirayama [9.39] recently generalized the Lorentz covariant formula (9.101) to one which is also valid in a uniformly accelerated reference frame. He then introduced a new four-vector which may be called the *Rindler four-acceleration* of the charge. It is a four-vector representing the acceleration of the charge *relative to the Rindler frame*, and has components

$$\alpha^\mu = a^\mu - \frac{1}{gx^2(gx+v)}(v, g^2x^2, 0, 0) , \qquad (9.102)$$

where a^μ are the components in the Rindler frame of the four acceleration of the charge. For a freely falling charge $a^\mu = 0$. The generalized Larmor formula valid in a uniformly accelerated reference frame has the form

$$P = m_0\tau_0 g^2 x^2 \alpha_\mu \alpha^\mu , \qquad (9.103)$$

and has been thoroughly discussed by *Eriksen* and *Grøn* [9.31].

We shall now apply this formula to the charge falling freely from $x = x_0$ where it was instantaneously at rest. Then we need to calculate

$$\alpha_\mu \alpha^\mu = -g^2x^2(\alpha^t)^2 + (\alpha^x)^2 . \qquad (9.104)$$

Inserting the expressions (9.98) and (9.99) for x and v in (9.102), we obtain

$$\begin{aligned} \alpha^t &= \frac{1}{gx_0^2} e^{gt} \sinh(gt) \cosh^2(gt) , \\ \alpha^x &= -\frac{1}{x_0} e^{gt} \cosh^2(gt) , \end{aligned} \qquad (9.105)$$

which gives

$$\alpha_\mu \alpha^\mu = \frac{1}{x_0^2} e^{2gt} \cosh^2(gt) . \qquad (9.106)$$

Inserting this into (9.103) we find the power radiated by the freely falling charge

$$P = m_0\tau_0 g^2 e^{2gt} . \qquad (9.107)$$

The radiated energy is

$$E_R = \int_0^t P\,dt = \frac{m_0\tau_0}{2} g(e^{2gt} - 1) . \qquad (9.108)$$

One may wonder where this energy comes from. A proton and a neutron will perform identical motions during the fall, although the proton radiates energy and the neutron does not. The answer is: the radiated energy comes from the Schott energy. The Schott energy is given by

$$E_S = -m_0\tau_0 v \alpha^x . \qquad (9.109)$$

Inserting the expressions (9.99) and (9.105) for v and α^x, respectively, we obtain for the Schott energy as a function of time

$$E_S = -\frac{m_0\tau_0}{2} g(e^{2gt} - 1) . \qquad (9.110)$$

This shows that the radiation energy does indeed come from the Schott energy. Again the Schott part of the field energy inside the light front S in Fig. 9.1 becomes more and more negative during the motion, and the region filled with Schott energy which is inside the light front S and outside the particle, initially has a vanishing volume, but increases rapidly in size.

9.6 Noninvariance of Electromagnetic Radiation

F. Rohrlich was one of the first to note the noninvariance of electromagnetic radiation from a point charge against a transformation involving a relative acceleration between two reference frames [9.33]. He considered a uniformly accelerated charge in flat spacetime and concluded that a freely falling observer would see a supported charge in a uniformly accelerated reference frame radiating, and a supported observer would see a freely falling charge radiating. But a supported observer would not see a supported charge radiating, and a freely falling observer would not see a freely falling charge radiating. The same was later noted by *A. Kovetz* and *G. E. Tauber* [9.34], and an explanation for this was given by *D. G. Boulware* [9.36].

The nature of electromagnetic radiation is still a mystery. The wave–particle duality is something which seems to transcend our intuitive understanding. The waves of monochromatic light have infinite extension, but a photon is thought of as something having an exceedingly minute extension with a smallness only limited by the Heisenberg uncertainty relations.

Also thinking of electromagnetic radiation as a photon gas, and photons as a sort of object which you can detect with your apparatus, it seems exceedingly strange to claim that you can make the object vanish just by changing your state of motion. On the other hand that claim does not sound so impossible if you think of electromagnetic radiation as waves. The waves are a state of oscillation of electric and magnetic fields moving through space with the velocity of light. Maybe they can be transformed away?

That should indeed be possible. Think of a uniformly accelerated charge, radiating out an electromagnetic power. Transforming to the permanent rest frame of the charge the magnetic field vanishes. In this frame the charge does not radiate. Hence, saying that a charge radiates is not a reference-independent statement.

As mentioned in Sect. 9.5 *M. Kretzschmar* and *W. Fugmann* [9.37, 38] and *T. Hirayama* [9.39] deduced a generalized versions of Larmors's formula valid in uniformly accelerated reference frames. The significance of these formulae in connection with energy momentum conservation of a charged particle and its electromagnetic field has been thoroughly analyzed by *Eriksen* and *Grøn* [9.31].

The nonvanishing Christoffel symbols in the Rindler frame are given in (9.90), and the four-velocity and the four-acceleration of a particle moving along the x-axis have components given in (9.95) and (9.96). Transformation by means of (9.87) and (9.88) gives the following components of the four-velocity and four-acceleration in the inertial frame,

$$\begin{aligned} U^{\mu} &= (gxv^{t}, v^{x}, 0, 0)\,, \\ A^{\mu} &= (gxa^{t}, a^{x}, 0, 0)\,. \end{aligned} \tag{9.111}$$

Inserting $v = a = 0$ in (9.95) and (9.96) we find the four-velocity and four-acceleration of the reference particles in the Rindler frame

$$u^{\mu} = \left(\frac{1}{g}x, 0, 0, 0\right)\,, \quad g^{\mu} = \left(0, \frac{1}{x}, 0, 0\right)\,. \tag{9.112}$$

Using (9.96) we find that the proper acceleration $\hat{a}$ (relative to an instantaneous inertial rest frame) is given by $\hat{a}^2 = a_\mu a^\mu = A_\mu A^\mu$, i.e.,

$$\hat{a} = \gamma^3 gx\left(a + g^2 x - \frac{2v^2}{x}\right)\,. \tag{9.113}$$

For a particle instantaneously at rest at the point $x = x_1$ we obtain

$$\hat{a} = \frac{a}{g^2 x_1^2} + \frac{1}{x_1}\,, \tag{9.114}$$

where $1/x_1$ is the proper acceleration of the point $x = x_1$ in the Rindler frame. The difference $\hat{a} - 1/x_1$ will be denoted by $\tilde{a}$. We obtain

$$\tilde{a} = \frac{a}{g^2} x_1^2\,, \tag{9.115}$$

which may be interpreted as the acceleration of the particle relative to the Rindler system, measured by a standard clock carried by the particle.

According to the analysis of *Kretzschmar* and *Fugmann* [9.37, 38] the generalized Larmor formula as written out in a uniformly accelerated reference frame takes the form

$$P = \frac{2}{3} Q^2 (gx_1)^2\, \tilde{a}^2\,. \tag{9.116}$$

We shall now consider a freely falling charge in the Rindler frame. It is permanently at rest in the inertial comoving frame. Obviously it does not radiate as observed in this frame. But according to (9.116) it radiates as observed in the Rindler frame. In order to understand this from a field theoretic perspective in a similar way as was obtained with reference to an inertial frame in Sect. 9.2, we may again consider the Teitelboim partition of the field into a generalized Coulomb field I and a radiation field II. Calculating the flow of field energy of these types out of the Rindler section we arrived at in [9.31],

$$P_{\mathrm{I}} = \frac{2}{3} Q^2 g\,(v - gx_1)\left[\gamma^2 g\,(v - gx_1) + 2a^x\right], \tag{9.117}$$

$$P_{\mathrm{II}} = \frac{2}{3} Q^2 \left(\frac{a^x}{\gamma}\right)^2\,. \tag{9.118}$$

The emitted energy per emission time is

$$P = P_{\mathrm{I}} + P_{\mathrm{II}} = \frac{2}{3}\frac{Q^2}{\gamma^2}\left[a^x + \gamma^2 g\,(v - gx_1)\right]^2\,, \tag{9.119}$$

where

$$a^x = \gamma^4\left(a + g^2 x_1 - 2v^2/x_1\right) g^2 x_1^2\,. \tag{9.120}$$

We now apply these formulae to the special case of a freely falling charge in the Rindler frame. Then the four-acceleration vanishes, $a^x = 0$, which gives

$$P_\mathrm{I} = \frac{2}{3} Q^2 g^2 \gamma^2 (gx_1 - v)^2 \ , \quad P_\mathrm{II} = 0 \ . \tag{9.121}$$

In this case there is no emission of type II energy, only of type I.

This example shows the inadequacy of the Teitelboim separation with respect noninertial reference frames. In inertial frames radiation is associated with type II energy. However, as is seen from the present results, this is not the case in general. The separation in type I and II energy is based, respectively, on the vanishing and the nonvanishing of the four-acceleration of the charge, which means whether it is in free fall or not. The emission of radiation, on the other hand, depends upon the relative acceleration between the charge and the observer. Only in an inertial frame does the vanishing of the four-acceleration mean that the charge is not accelerated relative to the observer. It should also be noted that since there is a flux of type I energy out of the Rindler sector, type I energy is not a state function of the charge in the Rindler frame, as it is in an inertial frame.

Following *Hirayama* [9.39] we shall now present a separation of the electromagnetic field energy in two types, $\tilde{P}_\mathrm{I}$ and $\tilde{P}_\mathrm{II}$, making use of a modified acceleration called α. We write $\alpha^x = a^x - \Delta^x$, where Δ^x is a quantity independent of α^x, which is determined from the condition that there shall be no energy of the new type I emitted out of the Rindler system, $\tilde{P}_\mathrm{I} = 0$. Inserting $a^x = \alpha^x + \Delta^x$ into (9.121) and selecting the term of second order in α^x gives

$$\tilde{P}_\mathrm{II} = \frac{2}{3} Q^2 (\alpha^x/\gamma)^2 \ . \tag{9.122}$$

Since the total transport of energy out of the sector is independent of the partition used, we have $\tilde{P}_\mathrm{I} = P - \tilde{P}_\mathrm{II}$. Hence, by means of (9.122) we obtain,

$$\begin{aligned} \tilde{P}_\mathrm{I} = {} & \frac{2}{3} \frac{Q^2}{\gamma^2} \left(2\alpha^x + \Delta^x + \gamma^2 g v - \gamma^2 g^2 x_1\right) \\ & \times \left(\Delta^x + \gamma^2 g v - \gamma^2 g^2 x_1\right) . \end{aligned} \tag{9.123}$$

From the requirement that $\tilde{P}_\mathrm{I} = 0$ for all values of α^x follows

$$\Delta^x = \gamma^2 g (gx_1 - v) \ , \tag{9.124}$$

giving

$$\alpha^x = a^x - \gamma^2 g (gx_1 - v) \ , \tag{9.125}$$

and

$$\tilde{P}_\mathrm{I} = 0 \ , \quad \tilde{P}_\mathrm{II} = P \ . \tag{9.126}$$

Here α^x is just the x-component of *the acceleration of the charge relative to the Rindler frame* found by Hirayama using Killing vectors.

The covariant expression of the vector is

$$\begin{aligned} \alpha^\mu = {} & a^\mu - \left(g_\alpha g^\alpha\right)^{\frac{1}{2}} u^\mu - g^\mu \\ & - \left(g_\alpha g^\alpha\right)^{\frac{1}{2}} v_\beta u^\beta v^\mu - v_\alpha g^\alpha v^\mu \ . \end{aligned} \tag{9.127}$$

Using (9.95), (9.96) and (9.112), we have in our case

$$\begin{aligned} \left(g_\alpha g^\alpha\right)^{\frac{1}{2}} &= \frac{1}{x_1} \ , \\ v_\beta u^\beta &= -\gamma g x_1 \ , \\ v_\alpha g^\alpha &= \gamma \frac{v}{x_1} \ , \end{aligned} \tag{9.128}$$

and Hirayama's vector reads

$$\alpha^\mu = a^\mu - \frac{\gamma^2 (gx_1 - v)}{g x_1^2} \left(v, g^2 x_1^2, 0, 0\right) , \tag{9.129}$$

or, by means of (9.113),

$$\alpha^\mu = \left[\gamma^4 \left(a - \frac{v^2}{x_1}\right) + \frac{\gamma^2 v}{g x_1^2}\right] \left(v, g^2 x_1^2, 0, 0\right) . \tag{9.130}$$

It follows that $(\alpha^x/\gamma)^2 = g^2 x_1^2 \alpha_\mu \alpha^\mu$, which by means of (9.125) gives

$$P = \tilde{P}_\mathrm{II} = \frac{2}{3} Q^2 g^2 x_1^2 \alpha_\mu \alpha^\mu \ , \tag{9.131}$$

for the field energy produced per coordinate time which leaves the Rindler sector.

It is easily seen that the Hirayama separation is a proper generalization of the Teitelboim separation to accelerated frames, which reduces to the latter in inertial frames. To that end we describe the particle by the coordinate $\bar{x} = x_1 - 1/g$. Then the coordinate time for $\bar{x} = 0$ is equal to the proper time. Keeping $\bar{x}$ finite and letting $g \to 0$ we obtain the limits $x_1 \to \infty$, $gx_1 \to 1$, $ds^2 \to -dt^2 + dx^2$, $\gamma \to (1-v^2)^{-1/2}$. From (9.129) we then find that $\alpha^\mu \to a^\mu$, and from (9.131) that $P \to (2/3)\,Q^2 a_\mu a^\mu$.

Calculating the bound energy in the Rindler frame we find that the total energy of the charge and its field is [9.31]

$$\tilde{U} = \tilde{U}_{\mathrm{I}} + \tilde{U}_{\mathrm{II}} = -\frac{1}{2}Q^2 g + g^2 x_1^2 \gamma m_0 + \tilde{E}_{\mathrm{S}} + \tilde{E}_{\mathrm{R}} \,. \tag{9.132}$$

The first term on the right-hand side has no obvious physical interpretation. The second is the mechanical energy of the particle. The third term is the acceleration energy or the Schott energy, when the partition of the field is made according to the acceleration α^μ,

$$\tilde{E}_{\mathrm{S}} = -\frac{2}{3}Q^2 v \alpha^x \,. \tag{9.133}$$

The quantity $\tilde{E}_{\mathrm{S}}$ is analogous to the Schott energy

$$E_{\mathrm{S}} = -\frac{2}{3}Q^2 V A^X \,, \tag{9.134}$$

in an inertial system according to the Teitelboim partition. The fourth term is the radiation energy in the α^μ-partition,

$$\tilde{E}_{\mathrm{R}} = \frac{2}{3}Q^2 \int\limits_{-\infty}^{t_1} g^2 x^2 \alpha_\mu \alpha^\mu \, \mathrm{d}t \,. \tag{9.135}$$

By differentiation upon the proper time of the particle, i. e., $\mathrm{d}/\mathrm{d}\tau = \gamma(\mathrm{d}/\mathrm{d}t_1)$, we find the formula

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\gamma g^2 x_1^2\right) = a^t g^2 x_1^2 = -a_t \,, \tag{9.136}$$

by which the energy equation (9.132) becomes

$$\frac{\mathrm{d}\tilde{U}}{\mathrm{d}\tau} = -m_0 a_t + \tilde{\Gamma}_0 \,, \tag{9.137}$$

where

$$\begin{aligned}\tilde{\Gamma}_0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\tilde{E}_{\mathrm{S}} + \tilde{E}_{\mathrm{R}}\right) \\ &= \frac{2}{3}Q^2\left(\frac{\mathrm{d}\alpha_t}{\mathrm{d}\tau} - v_t \alpha_\nu \alpha^\nu\right) .\end{aligned} \tag{9.138}$$

The quantity $\tilde{\Gamma}_0$ is interpreted as a component of the Abraham vector in the Rindler frame. Let us compare it with the time component of the corresponding vector Γ^μ given by *Hirayama* [9.39]. From his (9.50) we obtain for the motion in the x-direction,

$$\Gamma_0 = \frac{2}{3}Q^2\left\{v^\nu \nabla_\nu \alpha_t - \frac{\alpha_t}{\gamma v x} - v_t \alpha_\nu \alpha^\nu\right\} . \tag{9.139}$$

Inserting

$$v^\nu \nabla_\nu \alpha_t = \frac{\mathrm{d}\alpha_t}{\mathrm{d}\tau} - \Gamma^\sigma_{t\beta} v^\beta \alpha_\sigma = \frac{\mathrm{d}\alpha_t}{\mathrm{d}\tau} + \frac{\alpha_t}{\gamma v x}, \tag{9.140}$$

we obtain

$$\Gamma_0 = \frac{2}{3}Q^2\left\{\frac{\mathrm{d}\alpha_t}{\mathrm{d}\tau} - v_t \alpha_\nu \alpha^\nu\right\} , \tag{9.141}$$

which is equal to $\tilde{\Gamma}_0$ as given by (9.141). Thus for the Abraham vector in the Rindler frame we have

$$\Gamma_0 = \tilde{\Gamma}_0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\tilde{E}_{\mathrm{S}} + \tilde{E}_{\mathrm{R}}\right) . \tag{9.142}$$

Note that when $\Gamma_0 = 0$, the radiation energy is supplied by the Schott energy. This is quite similar to the corresponding case in an inertial frame. From [9.6, eq. (3.2)] we then have $\Gamma_T = \mathrm{d}/\mathrm{d}\tau(E_{\mathrm{S}} + E_{\mathrm{R}})$, where $E_{\mathrm{S}} = (2/3)\,Q^2 A_T$ and $E_{\mathrm{R}} = \frac{2}{3}Q^2 \int_{-\infty}^{T} A_\mu A^\mu \,\mathrm{d}T$.

From the Lorentz invariance of Maxwell's equations it follows that the existence of electromagnetic radiation is Lorentz invariant. The quantum mechanical photon picture of radiation suggests that its existence is generally invariant. However, as we have shown in this section, the equations of classical electrodynamics imply that this is not the case. The existence of radiation from a charged particle is not invariant against a transformation involving reference frames that accelerate or rotate relative to each other. Even if a charge accelerates as observed in an inertial frame, it does not radiate as observed from its permanent rest frame.

D. R. Rowland [9.16] recently explained this in the case of a uniformly accelerated charge in the following way:

> *The electric field lines of the charge in the Rindler frame in which it is at rest lie along the geodesics for photons for that frame of reference. This means that relative to the Rindler frame, the photons emitted by the charge are purely longitudinal, not transverse, meaning that they are virtual rather than real (i. e. radiation) photons.*

9.7 Other Equations of Motion

The problems with the LAD equation of motion of a charged point particle, i. e., pre-acceleration and run-away solutions, have motivated several researchers to propose alternative equations of motion. *R. T. Hammond* [9.40] recently reviewed some proposals for constructing a new equation of motion of a radiating electron. The equation of motion is written in the form (9.6). In the case of the LAD equation the vector Γ^μ is given by (9.7). We now consider a charged particle in an external electromagnetic field as described with reference to an inertial frame. Then the external force is given by $F^\mu = qF^{\mu\sigma}U_\sigma$, where $F^{\mu\sigma}$ are the components of the electromagnetic field tensor.

Another equation that has been much used, is the so-called Landau–Lifschitz (LL) equation of motion [9.41], which is

$$\Gamma^\mu = \tau_0 \left(q\dot{F}^{\mu\sigma}U_\sigma + q^2\left(F^{\mu\gamma}F^\alpha_\gamma U_\alpha + F^{\nu\gamma}F^\alpha_\nu U_\alpha U^\mu\right)\right) . \tag{9.143}$$

In the nonrelativistic limit the LL equation takes the form

$$m_0\dot{\boldsymbol{v}} = \boldsymbol{f}_{\text{ext}} + \tau_0\dot{\boldsymbol{f}}_{\text{ext}} . \tag{9.144}$$

In the absence of an external force this reduces to Newton's first law, and there is now a runaway solution. Moreover, there is no pre-acceleration. Hammond pointed out, however, that in the deduction of this equation one utilizes a condition which in the case of a charge oscillating with frequency Ω takes the form (inserting the velocity of light in this formula),

$$\Omega << (f_{\text{ext}}/m_0c\tau_0)^{\frac{1}{2}} . \tag{9.145}$$

Hammond notes that for weak enough external forces this condition may never be satisfied. He comments further on this [9.40]:

> *Of course in this case the net force is extremely small, but for long times, such as charged particles in a galactic orbit, we see that we cannot even use the LL equation. Thus there is an entire range in which the LL equation seems to fail.*

G. W. Ford and *R. F. O'Connell* constructed a more general equation of motion for a radiating charged particle taking a possible electron structure into account [9.42–46] This equation is

$$\Gamma^\mu = \tau_0 q\left(\left(F^{\mu\sigma}U_\sigma\right)^{\cdot} + U^\mu U_\alpha\left(F^{\alpha\beta}U_\beta\right)^{\cdot}\right) . \tag{9.146}$$

In the nonrelativistic limit this is the same as the nonrelativistic version of the LL equation.

Another approach was developed by *A. D. Yaghjian* [9.47]. He modeled a particle by a shell and assumed that no forces act upon the shell until the time when the force is applied, and obtained

$$\Gamma^\mu = \tau_0\theta(\tau)(\ddot{U}^\mu + U^\mu\dot{U}_\alpha U^\alpha) , \tag{9.147}$$

where $\theta(\tau)$ is the step function. Due to the presence of the step function the equation of motion with this expression for Γ^μ avoids pre-acceleration.

Yet another approach is followed by *Hammond* [9.40, 48]. In [9.40] he considers the nonrelativistic case in one dimension and writes the equation of motion as

$$m_0\dot{v} = f_{\text{ext}} - f , \tag{9.148}$$

where f is the radiation reaction force. Then he assumes that the radiated effect is given by fv. Combining this with Larmor's formula (9.11) he finds

$$fv = m_0\tau_0(\dot{v})^2 . \tag{9.149}$$

Eliminating f from (9.148) and (9.149) he arrives at

$$m_0v\dot{v} = vf_{\text{ext}} - m_0\tau_0\dot{v}^2 . \tag{9.150}$$

Integration gives

$$\frac{1}{2}m_0v^2 = \int f_{\text{ext}}\,\mathrm{d}x - \int P_{\mathrm{L}}\,\mathrm{d}t . \tag{9.151}$$

Hence, the increase of kinetic energy equals the work done by the external force minus the energy radiated away. Hammond then says that (9.150) is free of the plagues of the LAD equation. However, that is not quite so. There is a reminiscence of the runaway solution. If there is no external force $f_{\text{ext}} = 0$, (9.150) reduces to $v = -\tau_0\dot{v}$ with general solution $v = v_0\mathrm{e}^{-t/\tau_0}$. Thus, there is an exponentially decaying *runaway solution*. It may be noted that a solution of (9.150) of the same form, $v = v_0\mathrm{e}^{-t/2\tau_0}$ is obtained if there is an external friction like force proportional to the velocity, $f_{\text{ext}} = -(m_0/4\tau_0)v$.

9.8 Conclusion

Seemingly there is a problem with energy conservation connected with the LAD equation of motion of a radiating charge in combination with the Larmor formula for the effect of the radiation emitted by an accelerated charged particle, although a general analysis implies energy conservation for a dynamics based upon these equations [9.49]. The equation of motion has runaway solutions in which a charge accelerates and emits radiation even when it is not acted upon by any exterior force. Where does the increase of kinetic energy and radiation energy come from?

In the present article it has been shown how the Schott energy provides both an increase of the kinetic energy of the particle and the energy it radiates. The Schott energy is the part of the electromagnetic field energy which is proportional to the acceleration of the charge, and for nonrelativistic motion of the charge it is localized close to the charge [9.7]. The Schott energy has the curious property that it can become increasingly negative, which makes it possible to use it as a sort of inexhaustible source of energy in the case of runaway motion.

Also the case of a freely falling charge in the gravitational field which exists in a uniformly accelerated reference frame in flat spacetime, is quite strange. The comoving frame of the charge is an inertial frame in which it is permanently at rest. Obviously the charged particle does not radiate in this frame. Nevertheless it radiates as observed in the accelerated frame [9.31]. Again one may wonder: Where does the radiated energy come from? Again the answer is: It comes from the Schott energy.

We have here demonstrated how this comes about by calculating the radiated energy and the Schott energy as functions of time for runaway motion and for freely falling motion in a gravitational field. This provides an interesting application of the LAD equation that may be useful in the teaching of the electrodynamics of radiating charges. It has been shown that it is necessary to take the Schott energy into account in order to avoid apparent energy paradoxes in the theory of radiating charges based on the LAD equation.

The necessity of taking the Schott energy into account for energy-momentum conservation may point to a problem with the LAD equation or the point particle model of a charge. Whereas there is a physical basis for the Schott energy in the electromagnetic field of a point charge, an energy that becomes negative without bound and supplies limitless radiation energy and kinetic energy of runaway solutions may be a sign of the breakdown of the LAD equation [9.40].

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