On Properties of Forbidden Zones of Polygons and Polytopes*

Ross Berkowitz[1], Bahman Kalantari[1], Iraj Kalantari[2], and David Menendez[1]

[1] Rutgers, the State University of New Jersey
New Brunswick, NJ, USA
[2] Western Illinois University
Macomb, IL, USA

Abstract. Given a region R in a Euclidean space and a distinguished point $p \in R$, the *forbidden zone*, $F(R, p)$, is the union of all open balls with center in R having p as a common boundary point. The notion of forbidden zone, defined in [2], was shown to be instrumental in the characterization of *mollified zone diagrams*, a relaxation of *zone diagrams*, introduced by Asano, et al. [3], itself a variation of Voronoi diagrams. For a polygon P, we derive formulas for the area and circumference of $F(P, p)$ when p is fixed, and for minimum areas and circumferences when p ranges in P. These optimizations associate interesting new *centers* to P, even when a triangle. We give some extensions to polytopes and bounded convex sets. We generalize forbidden zones by allowing p to be replaced by an arbitrary subset, with attention to the case of finite sets. The corresponding optimization problems, even for two-point sites, and their characterizations result in many new and challenging open problems.

1 Introduction

The notion of the Voronoi diagram of a finite set of points in a Euclidean space is a rich concept with numerous applications. For a survey of results on Voronoi diagrams, see [4]. Voronoi diagrams have given rise to many variations. One of these, the *zone diagram* defined by Asano, et al. [3], is a new and rich variation of the Voronoi diagram for a given finite set of points in the Euclidean plane. The notion of a zone diagram and its existence was the main motivation behind defining mollified versions in [2], called *territory diagrams* and *maximal territory diagrams*. However, the study was also motivated by an intriguing relationship between approximations to Voronoi diagrams and certain regions of attraction in polynomial root-finding through iterations [5,6].

A mollified zone diagram can be viewed as a relaxation of a zone diagram in the sense that a zone diagram is a particular instance of the more general notion of maximal territory diagrams. The notion of a *forbidden zone* is intrinsic in the characterization of maximal territory diagrams in general and zone diagrams in

* This paper is an extended version of [1] and is dedicated to the memory of Sergio de Biasi.

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particular. In what follows in this section, we will briefly describe these notions in $\mathbb{R}^m$.

Given an n-tuple of point $P = \langle p_1, \ldots, p_n \rangle$, where $p_i \in \mathbb{R}^m$, and $\mathbf{R} = \langle R_1, \ldots, R_n \rangle$, where $R_i \subseteq \mathbb{R}^m$ with $p_i \in R_i$, we say $(P, \mathbf{R})$ is a *territory diagram* if for each $i = 1, \ldots, n$ we have

$$R_i \subseteq \operatorname{dom}\bigl(p_i, \bigcup_{j \neq i} R_j\bigr), \tag{1}$$

where for a given set $X \subseteq \mathbb{R}^m$ and a point p in $\mathbb{R}^m$, the *dominance region* of p with respect to X, $\operatorname{dom}(p, X)$, is defined as

$$\operatorname{dom}(p, X) \equiv \{z \in \mathbb{R}^m : d(z, p) \leq d(z, X)\}, \tag{2}$$

where $d(x, y) = \|x - y\|$ is the Euclidean distance between points x and y, and

$$d(z, X) = \inf_{x \in X} d(z, x) \tag{3}$$

In other words, each R_i must be contained in the set of all points that are closer to p_i than to all $R_j, j \neq i$. Given two territory diagrams $(P, \mathbf{R})$ and $(P, \mathbf{S})$ for the same tuple of sites P, we write $(P, \mathbf{R}) \preceq (P, \mathbf{S})$ if $R_i \subseteq S_i$ for all $i = 1, \ldots, n$. Additionally, we define $(P, \mathbf{R}) \prec (P, \mathbf{S})$ if $(P, \mathbf{R}) \preceq (P, \mathbf{S})$ but $\mathbf{R} \neq \mathbf{S}$.

A territory diagram $(P, \mathbf{R})$ is a *maximal territory diagram* if it is maximal with respect to the partial order $\prec$, i. e. if there exists no territory diagram $(P, \mathbf{S})$ for the same tuple of sites such that $(P, \mathbf{R}) \prec (P, \mathbf{S})$. The *forbidden zone* F_i with region R_i and a given site $p_i \in R_i$ is the set of all points that are closer to some point $y \in R_i$ than y is to p_i, that is:

$$F_i = \{z : d(z, y) < d(y, p_i) \text{ for some } y \in R_i\}. \tag{4}$$

In [2] it was shown that any maximal territory diagram $(P, \mathbf{R})$ satisfies

$$R_i = \operatorname{dom}\bigl(p_i, \bigcup_{j \neq i} R_j\bigr) - \bigcup_{j \neq i} F_j\ . \tag{5}$$

We thus see that forbidden zones arise in a natural way in mollified zone diagrams. However, one can note that they also arise in the study of Voronoi diagrams. Given a set of sites $P = \langle p_1, \ldots, p_n \rangle$, we note that the Voronoi cell of each particular site p_i, denoted by $V(p_i)$, is the largest set containing p_i for which the corresponding forbidden zone does not contain any other site.

In the following sections, we will formally define forbidden zones and state some of their basic properties. Next, we focus on the forbidden zones of various convex polygons. We develop formulas for the area, circumference, and particular regions within the forbidden zone when the site p is fixed in a convex polygon. Next, we consider optimization of each when p is allowed to range in the polygon. The optimization problems associate interesting *centers* to a polygon (even to a triangle), different from its classical sense of center. We extend our formulas for $p \notin P$. We also develop a formula for the area of the intersection of circles having a common boundary point. Aside from geometric interest, applications could be described. Finally, we extend some of the above results and optimizations to arbitrary polytopes and bounded convex sets.

2 Forbidden Zone of a Set with Respect to a Site

Definition 1. *The* forbidden zone $F(R,p)$ *for a region* $R \subseteq \mathbb{R}^m$ *and site* $p \in R$ *is the set of all points that are closer to some point* $y \in R$ *than* y *is to* p*, i.e.,*

$$F(R,p) = \{z : d(z,y) < d(y,p) \text{ for some } y \in R\}. \tag{6}$$

For an example, see Figure 1.

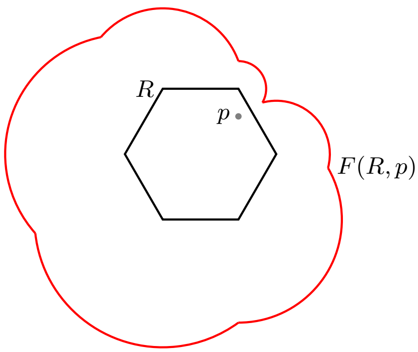


Fig. 1. The forbidden zone for a hexagonal region and a site

Theorem 1. *For a polytope* P *with vertices* v_i*,* $i = 1, \ldots, n$*, the forbidden zone is the union of the open balls centered at each* v_i *with radius* $d(p, v_i)$*. That is:*

$$F(P,p) = \bigcup_{i=1}^{n} \{z : d(z,v_i) < d(z,p)\}. \tag{7}$$

Proof. See [2]. □

The forbidden zone of a convex polygonal region and its site gives rise to several interesting problems in itself.

Calculating the *sum* of the areas of the related discs is simple, but the area of the union of the discs will be less than that sum due to overlap. However, we do not need to calculate the overlapping areas, because we can partition the forbidden zone into a set of non-overlapping triangles and regions.

Remark 1. In an recent, independent work on molecular structure, Kim, et al. [7], consider the more general problem of finding the area of several overlapping disks. Their approach differs from ours in emphasis: they begin with overlapping disks and derive an underlying polygon; we begin with a polygon and derive the overlapping disks based on a site. The structure of the forbidden zone allows us to simplify the area calculation and to optimize positions for the site based on different objectives.

2.1 The Forbidden Zone of a Convex Polygon

Consider a polygon P and a site $p \in P$. We will show how to divide the forbidden zone $F(P, p)$ into almost disjoint[1] triangles and sectors whose areas will sum up to the area of the forbidden zone.

For the following definitions and lemmas, we will assume that the polygon P has vertices v_i and angles θ_i, $i = 1, \ldots, n$.

We know from Theorem 1 that $F(P, p)$ is the union of open discs centered at each vertex v_i with radius $r_i = d(p, v_i)$. We will write C_i for the circle centered at v_i with radius r_i.

Lemma 1. *The boundary of $F(P, p)$ consists of arcs from the circles C_i. Specifically, the boundary of the portion of C_i which is not overlapped by any other circle C_j, along with the points on the boundary where each C_i intersects with its neighbors C_{i-1} and C_{i+1}.*

We write q_i for the point on the boundary where C_i and C_{i+1} intersect. In the case where p lies on the line segment $\overline{v_i v_{i+1}}$, $q_i = p$; otherwise, this q_i is the *reflection* of p across $\overline{v_i v_{i+1}}$.

Next, we partition $F(P, p)$ by drawing lines connecting q_i to v_i and to v_{i+1}. We write S_i for the sector of C_i bounded by $\overline{q_i v_i}$ and $\overline{q_{i-1} v_i}$. We write T_i for the triangle formed by q_i, v_i, and v_{i+1}. (See Figure 2.)

Lemma 2. *The sectors S_i, triangles T_i, and polygon P almost partition $F(P, p)$ into non-overlapping regions.*

Now draw lines from p to each vertex v_i. These will almost partition P into triangles. We will write T'_i for the triangle formed by p, v_i, and v_{i+1}.

Lemma 3. *The triangles T_i and T'_i are congruent with the same area.*

Proof. It is sufficient to show that T_i and T'_i have corresponding sides of the same length. Since p and q_i both lie on the circles C_i and C_{i+1}, the corresponding sides $\overline{pv_i}$ and $\overline{pv_{i+1}}$ must have the same length as $\overline{q_i v_i}$ and $\overline{q_i v_{i+1}}$, respectively. The remaining side, $\overline{v_i v_{i+1}}$, is shared by the triangles. □

[1] By "almost disjoint" and "almost partition", we mean that the intersections are only the edges of a triangle, which have measure zero. Note also that by "triangle" and "polygon", we refer to the union of the interior and edges of the same.

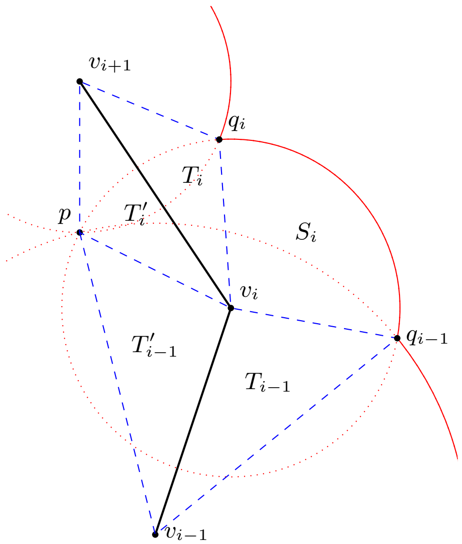


Fig. 2. Three adjacent vertices of a polygon, v_{i-1}, v_i, and v_{i+1}; the site p; the reflections of p across $\overline{v_{i-1}v_i}$, q_{i-1}, and $\overline{v_iv_{i+1}}$, q_i; the triangles T_{i-1} and T_i; their reflections inside P, T_{i-1}' and T_i'; and the sector S_i of C_i

Lemma 4. *The angle of the sector S_i is $2\pi - 2\theta_i$.*

Proof. Recall that the angle of P at v_i is θ_i. The line segment $\overline{pv_i}$ divides that angle into two parts, α and β, such that $\alpha + \beta = \theta_i$. These two parts are the angles of the triangles T_{i-1}' and T_i' at v_i. By Lemma 3, the angles of T_{i-1} and T_i at v_i must be α and β, respectively. These four triangles, along with the sector S_i, all share the vertex v_i and share both of their sides with their neighbors; so the sum of their angles must be 2π. Thus, the angle for S_i must be $2\pi - 2\alpha - 2\beta = 2\pi - 2\theta_i$. □

Now that these definitions are in place, we can calculate $|F(P,p)|$.

Theorem 2. *Given a convex polygon P with vertices v_i and interior angles θ_i, $i = 1, \ldots, n$, and a site $p \in P$,*

$$|F(P,p)| = 2|P| + \sum_{i=1}^{n} (\pi - \theta_i) r_i^2, \tag{8}$$

where $r_i = d(p, v_i)$.

Proof. We know from Lemma 2 that

$$|F(P,p)| = |P| + \sum_{i=1}^{n} |T_i| + \sum_{i+1}^{n} |S_i|. \tag{9}$$

Furthermore, we know from Lemma 3 that $|T_i| = |T'_i|$, and therefore

$$\sum_{i=1}^{n} |T_i| = \sum_{i=1}^{n} |T'_i| = |P|. \tag{10}$$

We know that S_i is a sector of a disc of radius r_i, and from Lemma 4, we know its angle is $2\pi - 2\theta_i$. Therefore,

$$|S_i| = (\pi - \theta_i) r_i^2. \tag{11}$$

Substituting these values into (9) gives (8). □

We can also use this partition of $F(P, p)$ to determine its circumference, $\mathrm{Circ}(F(P, p))$.

Theorem 3. *Given a convex polygon P with vertices v_i and interior angles θ_i, $i = 1, \ldots, n$, and a site $p \in P$,*

$$Circ(F(P, p)) = 2 \sum_{i=1}^{n} (\pi - \theta_i) r_i. \tag{12}$$

Proof. From Lemma 1, we can see that the boundary of $F(P, p)$ consists of arcs, one from each circle C_i. By construction, these arcs must correspond to the sectors S_i, as the triangles T_i and T'_i can only intersect the boundary of $F(P, p)$ at a reflected point q_i.

From Lemma 4, we know the sector angle of S_i is $2\pi - 2\theta_i$, and therefore its arc must have length $(2\pi - 2\theta_i) r_i$. Simplifying and summing over all sectors gives (12). □

2.2 Moving the Site Outside the Region

The normal definition of a forbidden zone requires the site p to lie within the region R, but what if we relaxed that condition? We know from [2] that $F(S, q)$, $q \in S$, is equal to $F(conv(S), q)$, where $conv(S)$ is the convex hull of S. Therefore, for a region R and a site $p \notin R$, we can define $F(R, p)$ to be $F(conv(R \cup \{p\}), p)$.

If R is a polygon P, we can simply take the convex hull of p and the vertices v_i, getting a new convex polygon with vertices v'_i, and proceed from there.

3 Optimal Forbidden Zones for a Convex Polygon

Now that we know how to calculate the area and circumference of the forbidden zone of a convex polygon P and site $p \in P$, we can consider how $F(P, p)$ changes as we select different positions for p. In particular, we will consider how to choose p so as to minimize the area or circumference of $F(P, p)$, as well as a few other measures (see Figure 3 and Table 1)

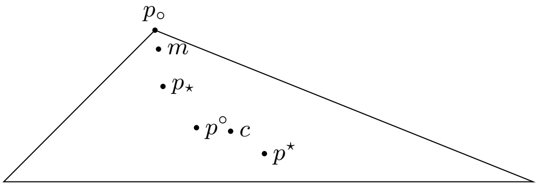
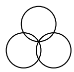


Fig. 3. Several "centers" of a triangle, including its center of mass (c), its Fermat point (m), and sites which minimize each of the area of the forbidden zone ($p^\star$), overlap of the forbidden zone ($p_\star$), the circumference of the forbidden zone (p°), and the "flower circumference" ($p_\circ$)

Table 1. Quantities Minimized by Various Polygon Centers

Centroid $\sum_{i=1}^{n} r_i^2$	Geometric Median $\sum_{i=1}^{n} r_i$	
Forbidden zone area $\sum_{i=1}^{n}(\pi - \theta_i) r_i^2$	Forbidden zone circumference $\sum_{i=1}^{n}(\pi - \theta_i) r_i$	
Flower area $\sum_{i=1}^{n} \theta_i r_i^2$	Flower circumference $\sum_{i=1}^{n} \theta_i r_i$	

3.1 Minimal Area

Imagine that we have a set of n radio transmitters, placed one at each of the vertices of a convex polygon P, and we want to set their strength so that every point in P will receive a signal from at least one transmitter. Assuming our transmitters broadcast in all directions up to some distance, r_i, and that the power required to transmit is proportional to r_i^2, then choosing $r_i = d(c, v_i)$, where c is the centroid

$$c = \frac{1}{n} \sum_{i=1}^{n} v_i \tag{13}$$

will produce a set of r_i's which minimizes the total power, $\sum_{i=1}^{n} r_i^2$. (Note that we are treating the vertices as vectors here.)

However, total power might not be the appropriate measure to minimize. For example, we might want to minimize the total area where signal can be received (assume we are under some constraint to minimize broadcasts outside P). In this case, we want to choose $p \in P$ to minimize the total area of the forbidden zone.

To do this, we will reformulate that area in terms of the coordinates of the site p. Let P be a convex polygon with vertices $v_i = (x_i, y_i)$. The area of the forbidden zone with respect to a site $p = (x, y)$ is:

$$|F(P,p)| = 2|P| + \sum_{i=1}^{n} (\pi - \theta_i)[(x_i - x)^2 + (y_i - y)^2]. \tag{14}$$

Note that $|P|$ does not depend on the choice of p, so its contribution to the area can be ignored for this pursuit.

Since P is convex, $\theta_i < \pi$, meaning $|F(P,p)|$ is a convex function in p. Its minimizer is a solution of partial derivatives with respect to x, y set to zero. This gives,

$$\sum_{i=1}^{n} (\pi - \theta_i)(x_i - x) = 0, \quad \sum_{i=1}^{n} (\pi - \theta_i)(y_i - y) = 0. \tag{15}$$

Solving for x, y gives

$$x = \frac{\sum_{i=1}^{n} (\pi - \theta_i) x_i}{\sum_{i=1}^{n} (\pi - \theta_i)}, \quad y = \frac{\sum_{i=1}^{n} (\pi - \theta_i) y_i}{\sum_{i=1}^{n} (\pi - \theta_i)}. \tag{16}$$

We can write these as

$$x = \sum_{i=1}^{n} \alpha_i x_i, \quad y = \sum_{i=1}^{n} \alpha_i y_i, \tag{17}$$

where

$$\alpha_i = \frac{(\pi - \theta_i)}{\sum_{j=1}^{n} (\pi - \theta_j)} = \frac{\pi - \theta_i}{2\pi}. \tag{18}$$

The last equality holds because the sum of the angles of P is $(n-2)\pi$. So we have,

$$p^\star = \sum_{i=1}^{n} \alpha_i v_i. \tag{19}$$

Note that $\alpha_i > 0$ for all i and $\sum_{i=1}^{n} \alpha_i = 1$. Hence $p^\star$ is a convex combination of v_i's. We call it the *forbidden zone center* of the polygon. In contrast to the centroid c, $p^\star$ takes into account the overlap of the discs and minimizes the area of their union, rather than their sum.

3.2 Minimal Overlap

As we saw, the centroid c minimizes the area of the discs and the forbidden zone center $p^\star$ minimizes the area of the union of the discs. We may also ask which site minimizes the difference between these areas, which we will call the overlap or *flower* (because it often resembles a flower, as in Figure 4).

Extending our radio example, we might want to minimize the overlap between the broadcast ranges if the transmitters are made directional to avoid

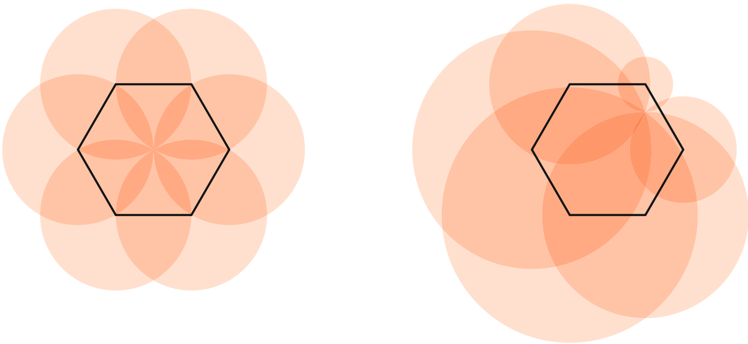


Fig. 4. Two hexagonal regions, with the discs for each vertex shaded to show their overlap, or "flower." Note that overlaps of four and even five discs occur when the site is off-center.

broadcasting outside P. Assuming the power needed is proportional to the area of the broadcast region, the minimal power requirement is one where the broadcast regions overlap the least.

Define $O(P, p)$ as the difference between the sum of the areas of the discs centered at v_i with radius r_i and the area of the forbidden zone with site $p = (x, y)$.

$$O(P, p) = \left(\sum_{i=1}^{n} \pi r_i^2 \right) - |F(P, p)|$$
$$= \left(\sum_{i=1}^{n} \theta_i r_i^2 \right) - 2|P| \tag{20}$$

Since $|P|$ is independent of p, it is sufficient to minimize $\sum_{i=1}^{n} \theta_i r_i^2$. This gives

$$x = \frac{\sum_{i=1}^{n} \theta_i x_i}{\sum_{i=1}^{n} \theta_i}, \quad y = \frac{\sum_{i=1}^{n} \theta_i y_i}{\sum_{i=1}^{n} \theta_i}. \tag{21}$$

We can write these as

$$p_\star = \sum_{i=1}^{n} \alpha_i' v_i, \tag{22}$$

where

$$\alpha_i' = \frac{\theta_i}{\sum_{j=1}^{n} \theta_j} = \frac{\theta_i}{(n-2)\pi}. \tag{23}$$

Note that $O(P, p)$ is the difference between the sum of the areas of the discs and the area of the union of the discs. It is not necessarily the area of the overlapping region, as subregions where more than two discs intersect will be counted more than once.

3.3 Minimal Circumference

While $|F(P,p)|$ and $O(P,p)$ can be thought of as weighted centroids, since they minimize quantities of the form $\sum_{i=1}^{n} w_i r_i^2$, the circumference of the forbidden zone depends on r_i rather than r_i^2. This turns out to be an example of a *weighted geometric median* [8],

$$m(X) = \arg\min_{p \in \mathbb{R}^k} \sum_{x \in X} w_i d(p,x), \tag{24}$$

where the weights w_i are non-negative and sum to one.

In the circumference, the radii are weighed by the sector angle, $2\pi - 2\theta_i$, so we can produce a set of weights by dividing by their sum, as in (18). This gives an optimal solution

$$p^\circ \in \arg\min_{p \in P} \sum_{i=1}^{n} \frac{\pi - \theta_i}{2\pi} r_i. \tag{25}$$

In the case where P is a triangle, p° is a weighted Fermat point [9], and can be found using the methods discussed in [10]. More generally, p° can be found using the methods for finding weighed geometric medians given in [11].

3.4 Minimal Flower Circumference

Just as the flower's area was minimized by $p_\star$, we can find a minimizer of the "flower circumference":

$$p_\circ \in \arg\min_{p \in P} \sum_{i=1}^{n} \frac{\theta_i}{(n-2)\pi} r_i. \tag{26}$$

4 General Bounds on the Area of the Forbidden Zone

In this section, we find some general bounds on the area of the forbidden zones of arbitrary regions in the plane. We will assume throughout that our site p is contained in the region R. Otherwise it is not hard to check that choosing remote sites will yield arbitrarily large forbidden zones.

A few quick words on notation. For the remainder of this paper, given a set S, we will use $|S|$ to denote its Lebesgue measure. We will also use $B(x,r)$ to denote the ball of radius r about the point x with respect to the standard Euclidean norm.

Theorem 4. *For a convex region $R \subseteq \mathbb{R}^m$ with site $p \in \mathbb{R}^m$,*

$$2^m |R| \le |F(R,p)|. \tag{27}$$

Proof. If $|R| = 0$ or R is infinite, this is obvious.

Otherwise, we assume without loss of generality that our site p is the origin. For any $c \in \mathbb{R}$ and $x \in R$, it is clear that $d(cx,0) = cd(x,0)$. When $0 < c < 2$, we have $d(cx,x) = |c-1|d(x,0) < d(x,0)$ and therefore $cx \in F(R,0)$. This means that $cR \subseteq F(R,0)$ (see Figure 5). Since $|cR| = c^m|R|$, we conclude that $c^m|R| \le |F(R,0)|$. In the limit as c approaches 2, $2^m|R| \le |F(R,0)|$. □

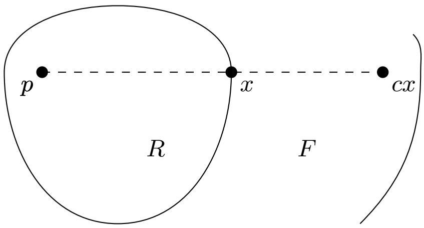


Fig. 5. Lower bound of forbidden zone in terms of volume

Corollary 1. *The volume of the forbidden zone of a ball is minimized when the site is the center of the ball.*

Proof. First, we show that if we take our site to be the origin and our region to be $R = B(0, r)$, then $F = B(0, 2r)$. It is clear by the previous argument that $B(0, 2r) \subseteq F$. For the other direction, we again recall that $F = \bigcup_{x\in R} B(x, \|x\|)$. Therefore, we have that any point $z \in F$ must have the form $x+y$, where $x \in R$ and $\|y\| \leq \|x\| < r$, and so we have by the triangle inequality that

$$\|z\| = \|x + y\| \leq \|x\| + \|y\| \leq 2\|x\| < 2r. \tag{28}$$

So we must have that $F \subseteq B(0, 2r)$, completing the argument that $B(0, 2r) = F$. Therefore, $|F| = 2^m|B(0, r)| = 2^m|R|$, which is minimal by our bound proven above. □

Another problem is bounding the minimal possible and maximal possible areas of the forbidden zone of a fixed region R with respect to choosing a site $p \in R$. First we recall a definition and then we proceed to prove our result.

Definition 2. *The diameter of a set $S \subseteq \mathbb{R}^m$ is*

$$\delta(S) \equiv \sup_{x,y\in S} \|x - y\|. \tag{29}$$

Theorem 5. *If F is the forbidden zone of a region $R \subseteq \mathbb{R}^m$ with respect to any site $p \in R$, then*

$$\frac{\omega_m}{2^{m-1}}\delta^m(R)^m \leq |F| \leq \omega_m 2^m \delta^m(R), \tag{30}$$

where ω_m is the volume of the unit m-ball.

Proof. For the upper bound, we repeat the fact that $F = \bigcup_{x\in R} B(x, \|x - p\|)$ and note that because $p \in R$ for any $x \in R$ we have $\|x - p\| < \delta(R)$. So, for each $x \in R$, we have

$$B(x, \|x - p\|) \subseteq B(p, 2\|x - p\|) \subseteq B(p, 2\delta(R)). \tag{31}$$

(See Figure 6.) So we therefore have that $F \subseteq B(p, 2\delta(R))$ and as a consequence,

$$|F| \leq |B(p, 2\delta(R))| = 2^m\delta^m(R)\omega_m. \tag{32}$$

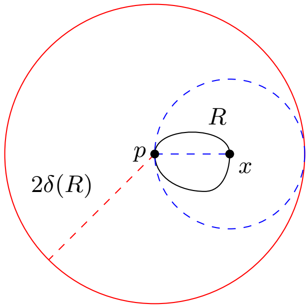


Fig. 6. Upper bound for the forbidden zone in terms of its diameter

For the lower bound, fix $\epsilon > 0$. By the definition of $\delta(R)$ there exist $x, y \in R$ such that $\|x - y\| \geq \delta(R) - \epsilon$. Because we know that the forbidden zone of R is the same as the forbidden zone of the convex hull of R we may assume that

$$\ell = \{tx + (1-t)y : 0 \leq t \leq 1\} \tag{33}$$

is contained in R. In particular we have $F(R, p) \supseteq F(\ell, p)$. So we have reduced this to the problem of computing the minimal forbidden zone of a line of length $\delta(R)$ (see Figure 7).

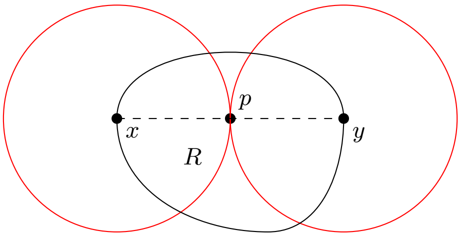


Fig. 7. Lower bound for the forbidden zone in terms of its diameter

To do this we note that for any site p we have, by the triangle inequality, that $\|x - y\| \leq \|x - p\| + \|y - p\|$. Let $B_x = B(x, \|x - p\|)$, $B_y = B(y, \|y - p\|)$ and $s \in (0, 1]$ such that $s(\|x - p\| + \|y - p\|) = \|x - y\|$. We have, by the reverse triangle inequality, that $sB_x \cap sB_y = \varnothing$.

$$\begin{aligned}
|F| &= \left| \bigcup_{z \in \ell} B(x, \|p - z\|) \right| \\
&\geq |sB_x \cup sB_y| \\
&\geq \min_{r \in [0, \|x-y\|]} |B(x, r)| + |B(y, \|x - y\| - r)|
\end{aligned}$$

$$= 2\omega_m \left(\frac{\|x-y\|}{2}\right)^m \geq \frac{\omega_m(\delta(R)-\epsilon)^m}{2^{m-1}} \tag{34}$$

And since $\epsilon > 0$ was arbitrary this completes the proof. □

Corollary 2. *Let R be a fixed region in $\mathbb{R}^m$. We have*

$$\frac{1}{2^{2m-1}} \leq \frac{\min_{p\in R}|F(R,p)|}{\max_{p\in R}|F(R,p)|} \leq 1. \tag{35}$$

We note that both the upper and lower bounds established above could use considerable improvement. This is especially true for the upper bound. Based on limited experimental evidence, we suggest the following tighter bound:

Conjecture 1

$$\frac{1}{2^{m-1}} \leq \frac{\min_{p\in R}|F(R,p)|}{\max_{p\in R}|F(R,p)|} \tag{36}$$

5 Arbitrary Regions

Up until now our formulas have been generally restricted to regions whose convex hull is a polytope. However we can extend these results to arbitrary regions by taking limits of polytopes. To define our limits we recall the following definition:

Definition 3. *The Hausdorff distance between two sets $X, Y \subseteq \mathbb{R}^m$ is defined to be*

$$d_H(X,Y) \equiv \max\left\{\sup_{x\in X} d(x,Y),\ \ \sup_{y\in Y} d(y,X)\right\}. \tag{37}$$

We will say that a sequence of sets $X_n \subseteq X$ converges *to a set X if*

$$\lim_{n\to\infty} d_H(X_n, X) = 0. \tag{38}$$

We denote this by $X_n \to X$.

A brief word of notation: if S is a set, we will use χ_S to denote its characteristic function. We now use this notion of distance to prove the following result on the convergence of forbidden zones when a convergent sequence of sites is considered.

Theorem 6. *Let $X_n, X \subseteq \mathbb{R}^m$ such that the sequence $X_n \to X$ and $p_n \to p \in \mathbb{R}^m$. We consider the associated sequence of forbidden zones $F_n = F(X_n, p_n)$. The following hold:*

1. *$F_n \to F(X,p) = F$*
2. *$\chi_{F_n} \to \chi_F$ pointwise.*

Proof. For a given $\epsilon > 0$, pick N large enough so that for $n \geq N$ we have $d_H(X_n, X) < \epsilon$ and $\|p_n - p\| < \epsilon$. Fix such an $n \geq N$ and let $y \in F$. Because $F = \bigcup_{x \in X} B(x, \|x - p\|)$, we know that there must be some $x \in X$ such that $\|y - x\| < \|x - p\|$. Because $d_H(X_n, X) < \epsilon$, we can pick some $x' \in X_n$ such that $\|x - x'\| < \epsilon$. We use the triangle inequality repeatedly to note

$$\begin{aligned} \|y - x'\| &\leq \|y - x\| + \|x - x'\| < \|x - p\| + \epsilon \\ &< \|x' - p_n\| + 3\epsilon. \end{aligned} \tag{39}$$

Combining the observation that $B(x', \|x' - p_n\|) \subseteq F_n$ with the result above that $y \in B(x', \|x' - p_n\| + 3\epsilon)$, we have that $d(y, F_n) < 3\epsilon$. Because $y \in F$ was arbitrary, this completes the proof that $\sup_{y \in F} d(y, F_n) < 3\epsilon$.

A similar computation will show that $\sup_{y' \in F_n} d(y', F) < 3\epsilon$. Therefore, we have shown that $d_H(F_n, F) < 3\epsilon$, and so we have $F_n \to F$.

For the proof of the second statement, fix some $y \in F$ and again find $x \in X$ such that $\|y - x\| < \|x - p\|$. Because this inequality is strict, we can find a $\delta > 0$ such that $\|y - x\| < \|x - p\| - \delta$, and therefore we will have that $y \in B(x, \|x - p\| + \delta)$. Following the argument above, let n be large enough for $\epsilon = \frac{\delta}{3}$ and pick $x' \in X_n$ such that $\|x' - x\| < \epsilon$. We then obtain, in the same fashion, that

$$\begin{aligned} \|y - x'\| &\leq \|y - x\| + \|x - x'\| \\ &< \|x - p\| - \delta + \epsilon \\ &< \|x' - p_n\| + 3\epsilon - \delta \\ &= \|x' - p_n\|. \end{aligned} \tag{40}$$

So we have $y \in B(x', p_n) \subseteq F_n$ and therefore $y \in F_n$ for n large enough, which yields $\chi_{F_n} \to \chi_F$ pointwise. □

A few immediate corollaries follow.

Corollary 3. *Using the notation introduced above, if we have $X_n \to X$ where X is bounded, then* $\lim_{n \to \infty} |F_n| = |F|$.

Proof. Because X is bounded, we have $X \subseteq B(0, r)$ for some $r > 0$. Using our above bounds on the forbidden zone, we therefore have that $F \subseteq B(0, 2r)$ as well. By Theorem 6, we will also have that, for n large enough, $d_H(F, F_n) < 1.$ and therefore $F_n \subseteq B(0, r + 1)$. So after throwing away finitely many terms we have that $\chi_{F_n} \leq \chi_{B(0,2r+1)}$ is integrable. We also know from the above theorem that $\chi_{F_n} \to \chi_F$ pointwise, so we can apply the Lebesgue Dominated Convergence Theorem to get

$$\lim_{n \to \infty} |F_n| = \lim_{n \to \infty} \int \chi_{F_n} = \int \chi_F = |F|. \tag{41}$$

□

Corollary 4. *The forbidden zone minimizer of a circle is its center, and the maximizer lies on the boundary.*

Proof. Let P_n denote the regular n-gon with vertices on the unit circle, S^1. We have shown previously that the forbidden zone maximizer of the polygon P_n occurs at a vertex, while the minimizer occurs at the center of P_n. Letting $n \to \infty$ we see that $P_n \to S^1$ and therefore applying our theorem above on the convergence of forbidden zones we will have as a consequence that the forbidden zone of S^1 is minimized at the origin, and maximized at a point on its boundary. □

6 Union and Intersection of Balls Having a Common Boundary Point

We note that the forbidden zone provides a new way to compute the volume of the union of m-balls. For two balls with nonempty intersection, take R to be a triangle with the following vertices: the centers of the balls, and an arbitrary point in the intersection of the boundaries of the balls. Placing the site at this intersection vertex (see Figure 8) gives us a forbidden zone equal to the union of the two balls, since the disc at the vertex of the intersection will have radius zero.

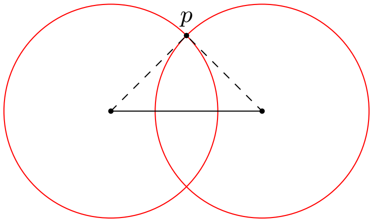


Fig. 8. Finding the area of two intersecting circles as a forbidden zone

For arbitrarily many balls, so long as all of the boundaries of the balls have a common point of intersection p we can essentially repeat the above construction. We create a forbidden zone with its region the convex hull of the centers of the balls along with the intersection point p. We also take the site to be p. The volume of this forbidden zone will be exactly that of the union of the balls. The proof of this uses the characterization of the forbidden zone as a union of balls about the vertices of the convex hull. We state this as a theorem.

Theorem 7. *Let $\{S_1, S_2, \ldots, S_n\}$ be a set of m-balls with centers $\{v_1, \ldots, v_n\} = V$. Assume that there is some point $p \in \bigcap_{i=1}^n \partial S_i$. Then we have $F(\{p\} \cup V, p) = \bigcup S_i$.*

We can use our formula for unions of balls, combined with the principle of inclusion-exclusion to obtain formulas for the intersections of the balls in terms of forbidden zones.

Theorem 8. *Let $\{S_1, S_2, \ldots, S_n\}$ be a set of m-balls with centers $\{v_1, \ldots, v_n\} = V$ such that the intersection of their boundaries contains the point $p \in \bigcap \partial S_i$. The volume of their intersection can then be expressed as a sum of the volumes of the associated forbidden zones. Specifically,*

$$\left|\bigcap_{i=1}^{n} S_i\right| = \sum_{T \subseteq V} (-1)^{\#T+1} |F(\{p\} \cup T, p)|, \tag{42}$$

where $\#T$ is the cardinality of T.

Proof. We will proceed inductively. Let S_1, S_2 be two balls with centers at v_1, v_2 respectively, whose boundaries intersect at a point p. By Theorem 7, we can write $S_1 \cup S_2 = F(\{v_1, v_2, p\}, p)$. It is also easy to see that $F(\{v_i, p\}, p) = S_i$. So by the principle of inclusion-exclusion we have

$$|S_1 \cap S_2| = |F(\{v_1, p\}, p)| + |F(\{v_2, p\}, p)| - |F(\{v_1, v_2, p\}, p)|. \tag{43}$$

We now extend this inductively to arbitrarily many balls.

For ease of notation, we write $F(X, p) = F(X)$ in this computation and assume all forbidden zones are with respect to the site p. We use inclusion-exclusion and the inductive hypothesis to compute

$$\begin{aligned}(-1)^{n-1}\left|\bigcap_{i=1}^{n} S_i\right| &= \left|\bigcup_{i=1}^{n} S_i\right| + \sum_{T \subsetneq S} (-1)^{\#T} \left|\bigcap_{i \in T} S_i\right| \\ &= |F(V)| + \sum_{T \subsetneq V} \sum_{U \subseteq T} (-1)^{\#T+\#U+1} |F(U)|. \end{aligned} \tag{44}$$

Reversing summation and gathering terms we obtain

$$\begin{aligned}(-1)^{n-1}\left|\bigcap_{i=1}^{n} S_i\right| &= |F(V)| - \sum_{U \subsetneq V} |F(U)| \sum_{U \subseteq T \subsetneq V} (-1)^{\#T+\#U} \\ &= |F(V)| - \sum_{U \subsetneq V} |F(U)| \sum_{m=0}^{n-\#U-1} (-1)^m \binom{n-\#U}{m} \\ &= |F(V)| - \sum_{U \subsetneq V} |F(U)| \left[(1-1)^{n-\#U} - (-1)^{n-\#U}\right] \\ &= \sum_{U \subseteq V} |F(U)| (-1)^{n+\#U}. \end{aligned} \tag{45}$$

□

7 General Forbidden Zones

Until now, the forbidden zone has been defined by a region and a distinguished point, called the site. We can easily generalize that definition to sites which are arbitrary subsets of the region.

Definition 4. *The forbidden zone for a region $R \subseteq \mathbb{R}^m$ and site $S \subseteq R$ is the set of all points that are closer to some point $y \in R$ than y is to any point in S. That is:*

$$F(R, S) \equiv \{z : d(z, y) < d(y, S) \text{ for some } y \in R\}. \quad (46)$$

Note that our definition refers to the entire set S as the site. However, in the case where S has multiple, disconnected parts it is reasonable to speak of a forbidden zone generated by a region and multiple sites.

Many properties of forbidden zones still apply in this more general setting, but not all. For example, in general $F(R, S) \neq F(conv(R), S)$ (see Figure 9). The work of determining which properties apply under what circumstances is still ongoing.

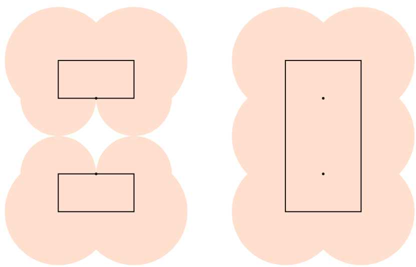


Fig. 9. Two forbidden zones with respect to two-point sites. The region on the right is the convex hull of the region on the left. Note that the forbidden zones are not the same.

In between the cases of an arbitrary subset $S \subseteq R$ and a single point $p \in R$, there are several interesting types of site, including convex polytopes, general polytopes, and closed sets. A site may also comprise several disconnected points, polytopes, or closed sets. In the case where S has multiple disconnected sub-sites, we can divide R into subregions dominated by each sub-site and treat each subregion and sub-site separately. This turns out to be a special case of a more general theorem about dividing the site into (possibly overlapping) sub-sites.

Definition 5. *Given a region $R \subseteq \mathbb{R}^m$ and site $S \subseteq R$ and sub-site $T \subseteq S$, we write $R_S(T)$ for the subset of R which is as close to T as it is to S. That is,*

$$R_S(T) = \{z \in R : d(z, T) = d(z, S)\}. \quad (47)$$

Note that $R_S(T)$ can be considered the intersection of R with the closure of the Voronoi cell for T.

Theorem 9. *For a region $R \subseteq \mathbb{R}^m$, site $S \subseteq R$, and sets T_1, T_2 such that $T_1 \cup T_2 = S$,*

$$F(R, S) = F(R_S(T_1), T_1) \cup F(R_S(T_2), T_2). \quad (48)$$

Proof. For an arbitrary $x \in F(R, S)$, there must be a $y \in R$ such that $d(x, y) < d(y, S)$. Since $R_1 \cup R_2 = R$, y must be in R_1, R_2, or both. If $y \in R_S(T_1)$, then $d(y, T_1) = d(y, S)$ and therefore $x \in F(R_S(T_1), T_1)$. Similarly, if $y \in R_S(T_2)$, then $x \in F(R_S(T_2), T_2)$. Therefore $F(R, S) \subseteq F(R_S(T_1), T_1) \cup F(R_S(T_2), T_2)$.

For an arbitrary $x \in F(R_S(T_1), T_1)$, there must be a $y \in R_S(T_1)$ such that $d(x, y) < d(y, T_1)$. Since $y \in R_S(T_1)$, $d(y, T_1) = d(y, S)$ and therefore $x \in F(R, S)$. Similarly, $F(R_S(T_2), T_2) \subseteq F(R, S)$. Therefore $F(R_S(T_1), T_1) \cup F(R_S(T_2), T_2) \subseteq F(R, S)$. □

Corollary 5. *For a region $R \subseteq \mathbb{R}^m$, site $S \subseteq R$, and sets $T_1, \ldots, T_k$ such that $T_1 \cup \ldots \cup T_k = S$,*

$$F(R, S) = \bigcup_{i=1}^{k} F(R_S(T_i), T_i). \tag{49}$$

Proof. By induction. □

Corollary 6. *For a region $R \subseteq \mathbb{R}^m$ and site $\{p_1, \ldots, p_k\} \subseteq R$,*

$$F(R, \{p_1, \ldots, p_k\}) = \bigcup_{i=1}^{k} F(R \cap \overline{V}(p_i), p_i) \tag{50}$$

where $\overline{V}(p_i)$ is the closure of the Voronoi cell for p_i.

Note that in Corollary 6, the sub-regions R_i are the intersection of R and the closure of the Voronoi cell for p_i. Since the sub-sites are now individual points, this means that we can find the forbidden zone by dividing R along the Voronoi boundaries and then finding the forbidden zones for each sub-region and point. In the case where R is a polytope, the resulting sub-regions will also be polytopes, so the forbidden zone will be the union of balls centered on each vertex, as seen in Figures 10 and 11.

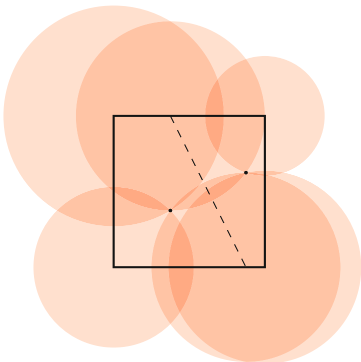


Fig. 10. The forbidden zone for a square and a two-point site depicted as the union of overlapping disks. The Voronoi boundary is shown as a dashed line.

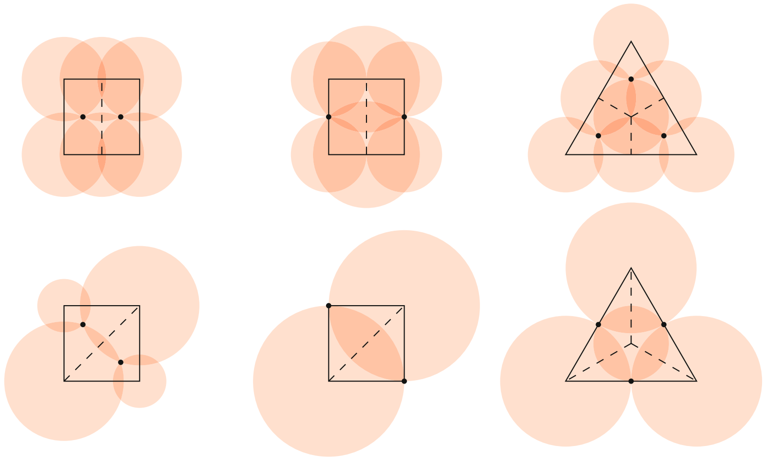


Fig. 11. Forbidden zones for various polygons and two- or three-point sites depicted as overlapping disks. The Voronoi boundaries are shown as dashed lines.

Since each subregion contains a single-point site, we can take the convex hull of the subregions without changing the forbidden zone. That is,

$$F(R, \{p_1, \ldots, p_k\}) = \bigcup_{i=1}^{k} F(conv(R \cap \overline{V}(p_i)), p_i). \tag{51}$$

In particular, disconnected portions of the region may become connected when the convex hull of each sub-region is taken, as seen in Figure 12.

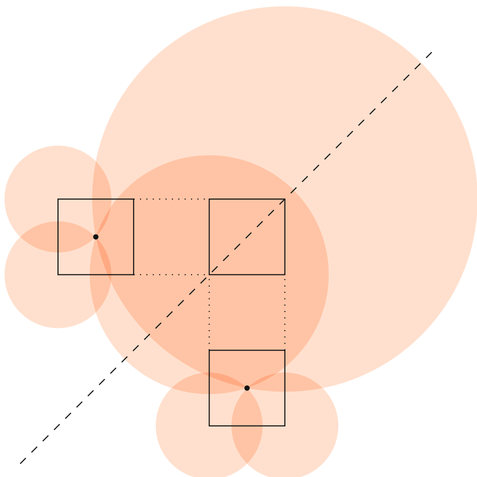


Fig. 12. The forbidden zone for a region comprising three disconnected squares and a two-point site. The Voronoi boundary is shown as a dashed line. The dotted lines show the additional boundaries of the convex hulls of the sub-regions. Note that the union of the convex hulls of the subregions is not itself convex.

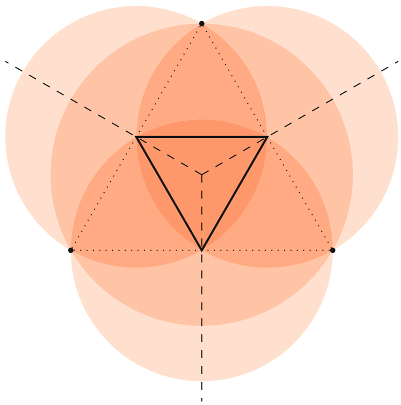


Fig. 13. The forbidden zone for a triangular region with a three-point site located outside the region, depicted as four overlapping disks. The dashed lines are the Voronoi boundaries, and the dotted lines indicate the convex hull of the sub-regions.

This also suggests that the definition of forbidden zones with respect to a region and a site not in the region from Section 2.2 can also be extended to the case where the site may contain several points not-necessarily inside the region (See Figure 13). As before, we first take the union of the region and the site as a new region, and then find the forbidden zone with respect to that. Equivalently,

$$F(R, \{p_1, \ldots, p_k\}) = \bigcup_{i=1}^{k} F(conv((R \cap \overline{V}(p_i)) \cup \{p_i\}), p_i). \tag{52}$$

7.1 Sites Which Generate Similar Forbidden Zones

For a fixed region $R \subseteq \mathbb{R}^m$ and points $p_1, p_2 \in R$, it is clear that $F(R, p_1) = F(R, p_2)$ if and only if $p_1 = p_2$. When the site is allowed to be an arbitrary subset of R, we can ask whether there are multiple sites which will produce "the same" forbidden zone. Naturally, since the forbidden zone excludes the site itself, it is trivially true that any change to the site will change the forbidden zone, but we can instead consider the union of the forbidden zone and the region and ask when that union does not change.

Theorem 10. *For a region $R \subseteq \mathbb{R}^m$ and site $S \subseteq R$ where $\partial S \subseteq R$, the portion of the forbidden zone with respect to R and S excluding R does not depend on any point in the interior of S. That is, for a set $T \subseteq S$,*

$$F(R, \partial S) \setminus R = F(R, \partial S \cup T) \setminus R. \tag{53}$$

Proof. Consider a point $z \in F(R, \partial S) \setminus R$. There must be a point $y \in R$ such that $d(z, y) < d(y, \partial S)$. We must have $y \notin S$, because $y \in S$ and $d(z, y) <$

$d(y, \partial S)$ can only hold if $z \in S$, which contradicts our assumptions. Therefore, $d(y, \partial S) = d(y, \partial S \cup T)$, since $T \subseteq S$, and $z \in F(R, \partial S \cup T)$. Thus, $F(R, \partial S) \setminus R \subseteq F(R, \partial S \cup T) \setminus R$.

By a similar argument, $F(R, \partial S \cup T) \setminus R = F(R, \partial S) \setminus R$. □

The implication of Theorem 10 is that any point in the interior of a site can be removed without changing the portion of the forbidden zone which extends beyond the region. Conversely, any gaps or holes in the site can be filled in without changing the exterior shape of the forbidden zone. Thus, the only differences between the forbidden zones shown in Figure 14 are the points in the interior of the site, which are part of the forbidden zone if and only if they are not part of the site.

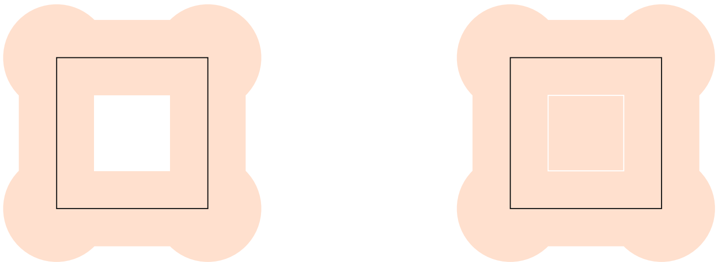


Fig. 14. Forbidden zones for square regions with square sites: filled on the left, and hollow on the right. The sites are shown in white, as they are excluded from the forbidden zones.

It may also be possible to expand the site outside its boundary without changing the forbidden zone outside the region. For example, in a U-shaped, it is possible to expand the site slightly into the bottom of the U without significantly changing the forbidden zone (see Figure 15). It remains to be seen how large the site may grow before it significantly changes the forbidden zone.

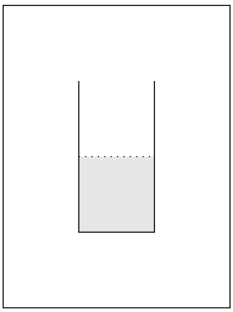


Fig. 15. A region and U-shaped site. If the area bounded by the dotted line is added to the site, the forbidden zone will not change outside the region.

7.2 Minimizing the Volume of Forbidden Zones

Allowing sites to be arbitrary subsets of their region greatly simplifies the task of finding the site which yields the smallest forbidden zone for a particular region: simply set the site equal to the region, making the forbidden zone empty. A more interesting problem is to find a k-point site which minimizes the forbidden zone for some region. As expected, even the area of the forbidden zone with respect to a polygonal region and k-point site in $\mathbb{R}^2$ has proven difficult to express in terms of the site locations.

There is however one important, if simple, case which we have been able to resolve.

Theorem 11. *Let e_1 be the first standard unit basis vector $(1, 0, 0, \ldots, 0) \in \mathbb{R}^m$. The problem of minimizing the area of the forbidden zone of the line between 0 and e_1 in $\mathbb{R}^m$, $m \geq 2$, with k sites is solved by placing the sites at $p_i = r^*(1 + 2^{\frac{m}{m-1}}(i-1))e_1$ for $r^* = \frac{1}{2}(1 + (k-1)2^{\frac{1}{m-1}})^{-1}$*

Proof. First we note that choosing the k sites at points $p_1 = t_1 e_1$, $p_2 = t_2 e_1, \ldots$, $p_k = t_k e_1$ with $t_1 \leq t_2 \leq \ldots, \leq t_k$ yields a forbidden zone given by the union of $k+1$ balls: Two centered at the endpoints $0, e_1$ with radii $r_0 = t_1$ and $r_k = 1 - t_k$ and the rest centered in the middle of two consecutive sites at $\frac{1}{2}(t_i + t_{i+1})$ with radii $r_i = \frac{1}{2}(t_{i+1} - t_i)$ for $i = 1, 2, \ldots k-1$ respectively. We note that choosing these k points on the line is equivalent to choosing the radii of these $k+1$ balls with the restriction that $r_0 + r_k + \sum_{i=1}^{k-1} 2r_i = 1$.

Given this restriction the volume we are minimizing is $\sum \omega_m r_i^m$ where ω_m is the volume of the unit ball in $\mathbb{R}^m$. Applying the method of Lagrange multipliers we find that our minima must be critical points of

$$\Lambda(r_0, \ldots r_k, \lambda) = \left(\sum_{i=1}^{k} \omega_m r_i^m \right) - \lambda \left(-1 + r_0 + r_k + \sum_{i=1}^{k-1} 2r_i \right) \tag{54}$$

Taking partial derivatives with respect to the r_i we find that the only critical point of Λ occurs where $r_0 = r_k = (\frac{1}{2})^{\frac{1}{m-1}} r_i$ for $i = 1, 2, \ldots k-1$. From this we find $r_0 = \frac{1}{2}(1 + (k-1)2^{\frac{1}{m-1}})^{-1}$. We can check that this point is a minimum, and because our objective function is convex this local minimum must in fact be a global one. Solving back for $p_i = t_i e_1$ we note that

$$t_i = r_0 + \sum_{j=1}^{i-1} 2r_j = r_0 \left(1 + (2i-2)2^{\frac{1}{m-1}} \right) \tag{55}$$

and so noting $r_0 = r^*$ we have our result. □

Applying this result, we can calculate that the two-point site which minimizes the forbidden zone for the line segment $[0, 1] \times \{0\}$ in $\mathbb{R}^2$ will be $\{\frac{1}{6}e_1, \frac{5}{6}e_1\}$, the minimizing three-point site is $\left\{\frac{1}{10}e_1, \frac{1}{2}e_1, \frac{9}{10}e_1\right\}$, the minimizing four-point site is $\{\frac{1}{14}e_1, \frac{5}{14}e_1, \frac{9}{14}e_1, \frac{13}{14}e_1\}$, and so forth (see Figure 16). As we move to

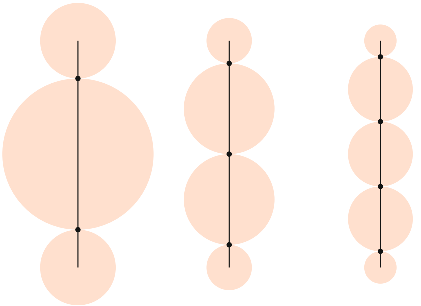


Fig. 16. Minimal-area forbidden zones with respect to line segments and 2-, 3-, and 4-point sites, in $\mathbb{R}^2$.

higher dimensions, the optimal sites. For $\mathbb{R}^3$, the minimizing two-point site is at $\{\frac{\sqrt{2}-1}{2}e_1, \frac{3-\sqrt{2}}{2}e_1\}$.

We can also consider a given region and site and determine which point will produce the smallest forbidden zone when added to the site. First, we will note that adding one or more points to the site will never add points to the forbidden zone.

Theorem 12. *Given a region $R \subseteq \mathbb{R}^m$ and site $S \subseteq R$, the forbidden zone with respect to R and S will not increase if $T \subseteq R$ is added to the site. That is,*

$$F(R, S \cup T) \subseteq F(R, S). \tag{56}$$

Proof. Consider a point $z \in F(R, S \cup T)$. There must be a $y \in R$ such that $d(z, y) < d(y, S \cup T)$. Since $d(x, S \cup T) \leq d(x, S)$ for any $x \in \mathbb{R}^m$, we have $d(z, y) < d(y, S)$ and $z \in F(R, S)$. □

In the case of a line segment and a k-point site, adding an additional point to the site has the effect of replacing one of the balls with two smaller ones. We can simply examine each ball to determine how much the area can be reduced by replacing it, and then replacing the ball with the largest reduction. For an interior ball with radius r, the minimal replacement would be two balls of radius $\frac{r}{2}$, giving a reduction of $\frac{1}{2}r^2$. For an edge ball of radius r, the minimal replacement is an interior ball of radius $r(1 + 2^{\frac{m}{m-1}})^{-1}$ and an edge ball of radius $r(2 + 2^{\frac{-1}{m-1}})^{-1}$.

7.3 General Bounds on the Volume of Generalized Forbidden Zones

In Theorem 5 we were able to put a lower bound on the volume of the forbidden zone of an arbitrary region with diameter δ. We did this by noting that the line segment corresponding to the diameter of the set was, by convexity, contained in our region. We then applied our lower bound on the forbidden zone of a line. Here we have an appealing solution to minimizing the forbidden zone of a line segment with k sites, however we cannot use this directly to achieve a similar lower bound because we can no longer assume the convexity of our region R.

An easy counterexample would be the case of minimizing the forbidden zone of $R = \{0, 1\}$ with respect to placement of 2 sites. Of course we can see that letting our sites be 0 and 1 themselves yields an empty forbidden zone, far less than the minimal forbidden zone of the line segment $[0, 1]$.

However, we are able to extend several previous results concerning the stability of forbidden zones to our consideration of more general sites. Recalling our notation that χ_R denotes the characteristic function of a set R, and $R_n \to R$ denotes R_n converges to R with respect to the Hausdorff distance we have the following generalization of Theorem 6.

Theorem 13. *Let $R_n, R, S_n, S \subset \mathbb{R}^m$ such that $R_n \to R$ and $S_n \to S$. We consider the associated sequence of forbidden zones $F_n = F(R_n, S_n)$. The following hold*

1. $F_n \to F(R, S) = F$
2. $\chi_{F_n} \to \chi_F$ *pointwise.*

We note that the only difference here is that our convergent sequence of sites has been replaced by a sequence of sets which converge with respect to the Hausdorff distance. The proof is only a very minor variation of the argument from the original result. We present only the proof of the first statement, but the modifications needed for the second half are the same.

Proof. For a given $\epsilon > 0$, pick N large enough so that for $n \geq N$ we have $d_H(R_n, R) < \epsilon$ and $d_H(S_n, S) < \epsilon$. Fix such an $n \geq N$ and let $y \in F$. By the definition of the forbidden zone we know that there must be some $r \in R$ such that $\|y - r\| < d(r, S)$. Because $d_H(R_n, R) < \epsilon$, we can pick some $r' \in R_n$ such that $\|r - r'\| < \epsilon$. Additionally, because $d_H(S_n, S) < \epsilon$ we see that for any points x, x' we have $d(x, S) < d(x, S_n) + \epsilon < d(x', S_n) + 2\epsilon$ and so we can compute

$$\begin{aligned} \|y - r'\| &\leq \|y - r\| + \|r - r'\| \\ &< d(r, S) + \epsilon \\ &< d(r', S_n) + 3\epsilon. \end{aligned} \tag{57}$$

Combining the observation that $B(r', d(r', S_n)) \subseteq F_n$ with the result above that $y \in B(x', d(r', S_n) + 3\epsilon)$, we have that $d(y, F_n) < 3\epsilon$. Because $y \in F$ was arbitrary, this completes the proof that $\sup_{y \in F} d(y, F_n) < 3\epsilon$.

A similar computation will show that $\sup_{y' \in F_n} d(y', F) < 3\epsilon$. Therefore, we have shown that $d_H(F_n, F) < 3\epsilon$, and so we have $F_n \to F$ with respect to the Hausdorff distance.

□

From this result we also obtain the analogue of Corollary 3 about the volumes of forbidden zones with respect to multiple sites

Corollary 7. *Using the notation introduced above, if we have $R_n \to R$ where R is bounded and $S_n \to S$, then* $\lim_{n\to\infty} |F_n| = |F|$.

8 Conclusion

In this article, we have developed many properties of the forbidden zone of a given region R in a Euclidean space with respect to a specified site p. First, we assumed that R is a closed, convex polygon and the site p belongs to R. In this special case we developed formulas for computing the area of the forbidden zone, for the area of the overlapping of circles, for the circumference of the forbidden zone, as well as for optimal cases of these as the site is allowed to range in R. These optimization problems, aside from their theoretical interest, associate interesting geometric "centers" to a polygon, even in the case of a triangle.

We extended our formulas for the computation of the forbidden zone's area to the case when p is outside of the convex hull. In other words, our formula allows computing the area of the intersection of a set of circles having a common boundary point.

Aside from geometric interest, practical applications could also be described. For instance, the minimization of the area or the circumference of the forbidden zone could be considered as a problem of computing optimal locations of sensors for communication or security purposes.

It is also possible to define problems with multiple forbidden zones. For instance, consider the case of two given non-overlapping triangles where the objective is to place two sites, one in each, so that the corresponding forbidden zones do not intersect. One may even define games of strategy based on forbidden zones. We will consider these in future work.

In this article, we have also extended some of the above results and optimizations to arbitrary polytopes and bounded convex sets. From the computational point of view, or in terms of computing closed formulas, even simple cases in the three dimensional space become more challenging. For instance, consider the case of the forbidden zone where our region R is a tetrahedron. Our partitioning scheme for a triangle is extendable to a tetrahedron, however the computational formulas need to be examined. We will study these in future work as well.

Finally, in this article we have given a considerable generalization and characterization of the forbidden zones by allowing a singleton site to be replaced with an arbitrary subset of points. In particular, we considered the case when the site consists of a finite set of points. The corresponding optimization problems, even for two-point sites, and their characterizations result in many new and challenging open problems that are interesting from the theoretical and practical points of view. The results in this article is testimonial to the richness of the notion of forbidden zone. We anticipate that the results will lead to many new lines of research.

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