On the Construction of Generalized Voronoi Inverse of a Rectangular Tessellation

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Abstract. We introduce a new concept of constructing a generalized Voronoi inverse (GVI) of a given tessellation $\mathcal{T}$ of the plane. Our objective is to place a set S_i of one or more sites in each convex region (cell) $t_i \in \mathcal{T}$, such that all edges of $\mathcal{T}$ coincide with edges of Voronoi diagram $V(S)$, where $S = \bigcup_i S_i$, and $\forall i, j, i \neq j, S_i \bigcap S_j = \emptyset$. Computation of GVI in general, is a difficult problem. In this paper, we study properties of GVI for the case when $\mathcal{T}$ is a rectangular tessellation and propose an algorithm that finds a minimal set of sites S. We also show that for a general tessellation, a solution of GVI always exists.

1 Introduction

1.1 Motivation and Problem Definition

In the design of Integrated Circuits (IC), the placement of modules is often guided by thermal constraints, which have become important because of high amount of power consumption per unit area and low thermal conductivities [12]. During the design phase it is thus required to estimate the thermal profile of each module and identify the locations for placement of heat sinks. A suitable geometry of heat sinks increases the dependability of the chip as the hot spots and the subsequent thermal gradient across the chip have a direct impact on its performance. The thermal environment around a cell depends on the thermal resistance between its location and the heat sink and the thermal contributions from other neighboring cells [24], [25]. The thermal resistance varies directly on the distance to the heat sink and inversely proportional to the thermal conductivity of the material on the way to heat sink [11]. Chen and Sapatnekar [13] studied partitioning based thermal placement methods to determine the location of heat sinks in each partition. In order to drain off the heat efficiently from a hot spot, dedicated heat sinks should be placed in the concerned partition so as to facilitate heat dissipation predominantly for the components belonging to that partition. Similar problems may arise in placing reservoir wells on a digital microfluidic biochip [14], where one or more reagents ought to be supplied independently to fluidic modules with least transportation cost from a source placed within the same block. These engineering design issues mandate a formal analysis and motivate us to address the following problem

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in rectangular tessellations. Before that, we introduce the concept generalized Voronoi inverse problem described below.

The Voronoi diagram $V(S)$ of a point set (site) $S = \{s_1, s_2, \ldots, s_n\}$ is defined as the partitioning of a plane into n convex regions (Voronoi regions). Any point in each Voronoi region $V(s_i)$, will be closer to $s_i \in S$ than to any other site of $S \setminus s_i$. The edges of the Voronoi diagram are the set of points in the plane that are equidistant to two nearest sites. The Voronoi vertices are the points equidistant to three (or more) sites.

Assume that we are given a tessellation $\mathcal{T}$ of the plane where each of its cell is convex. In the *inverse Voronoi diagram problem*, the following question is asked: Given $\mathcal{T}$, does there exist a set of sites S, one site for each cell of $\mathcal{T}$, such that the following holds. If $s_i \in S$ is placed in the cell $t_i \in \mathcal{T}$, then the Voronoi cell of s_i in the Voronoi diagram of S (denoted by $V(S)$) is t_i. Obviously, an inverse Voronoi diagram for any $\mathcal{T}$ may not exist. Therefore, we consider the problem of constructing a *generalized Voronoi inverse* (GVI) of a given tessellation $\mathcal{T}$ of the plane. Our objective is to place a set S_i of one or more sites in each of the cells of $\mathcal{T}$, such that each edge of $\mathcal{T}$ coincides with edges of $V(S)$, where $S = \bigcup_i S_i$, and $\forall i, j, i \neq j, S_i \bigcap S_j = \emptyset$. Observe that any cell of $V(S)$ must not lie in more than one cell of $\mathcal{T}$. Furthermore in GVI, such placement of sites ought to satisfy the following property: for any point $x \in t_i$, if $y \in S$ is the closest site of x then $y \in S_i$. Our objective is to identify such a set S of minimum cardinality. We define the cell t_i as the *Voronoi cell* of set S_i and denote it by $t(S_i)$.

The layout of an integrated circuit can be viewed as a tessellation of a rectangular region R. Given a layout the heat sinks in each block should be placed in such a fashion that the heat generated in one block is drained off locally without affecting the thermal load of adjacent blocks. Moreover, the number of heat sinks placed in a block should be minimum. It turns out that for a rectangular tessellation, such an exclusive heat sink placement problem is equivalent to finding a GVI.

1.2 Notations and Definitions

Given a rectangular bounding region R, a *rectangular tessellation* of R is a partitioning of the region R into isothetic rectangles. As in Voronoi diagrams we refer to each rectangle in the tessellation as a cell whose boundary is defined by axis-parallel segments. These segments are defined as the *edges* of the tessellation. The intersection of two orthogonal segments (edges) of a cell is defined as follows. A *T- junction* is a point where two orthogonal segments form a T-like structure (do not intersect), and a *cross junction* is an intersection point where two orthogonal segments cross each other. Here any intersection with the outermost boundary of the tessellation $\mathcal{T}$ is not considered as junction. We define a rectangular tessellation without any cross or T junction as *linear rectangular tessellation*.

In this paper we consider the GVI problem for a given rectangular tessellation $\mathcal{T}$ of a rectangular region R. We need to locate a set S consisting of minimum number of points inside $\mathcal{T}$ such that for each rectangular cell $t_i \in \mathcal{T}$, the set of sites $S_i \in S$ that lie inside t_i satisfies the following:

a) for any point $x \in t_i$ (x is not on a boundary) if y is its nearest neighbor in S, then $y \in S_i$ and the closest neighbor y need not be unique.

b) If point x lies on the edge adjacent to the cells t_i and t_j, then it is the perpendicular bisector between the two sites x and y such that $x \in S_i$ and $y \in S_j$ for $i \neq j$. It is assumed that there is no point on the boundary of the cells.

1.3 New Results

We study the generalized Voronoi inverse problem (GVI), i.e. we compute a set of points S of minimum cardinality, so that the given rectangular tessellation $\mathcal{T}$ is a subgraph of the 1-skeleton of the Voronoi diagram of S. We present the following results:

1. In Section 3 we present a linear-time algorithm for computing a GVI of optimum size for a linear rectangular tessellation.
2. In Section 4 we establish a combinatorial bound of $O(n^2)$ on the required number of sites for a general rectangular tessellation where n is the number of rectangles in the given rectangular tessellation. Here we propose an algorithm fore generating point set S for any general rectangular tessellation $\mathcal{T}$ which will provide a minimal solution for GVI. Later, in this section we establish lower bounds on the required number of sites for some special cases of rectangular tessellation.
3. In Section 5 we show that there always exists a feasible placement of sites that will correspond to any given arbitrary tessellation $\mathcal{T}$.

1.4 Related Works

Balzer et al. [3,4] had worked on inverse Voronoi diagram with capacity constraints in each Voronoi region. A close correlation of inverse Voronoi diagram with facility location problem had been shown earlier [1,2]. Hartvigsen [6] showed that the construction of an inverse Voronoi diagram problem can be mapped to a linear programming problem. Discussions and literature survey on Generalized Voronoi diagrams have been presented by Gavrilova [5]. GVI has also applications in biological growth model [7], GIS system [8], and competitive facility location [9].

2 Preliminaries

Let $\mathcal{T}$ be a tessellation of rectangular regions, where each cell is a rectangle (Fig. 1(a)). Let the vertices of a tessellation $\mathcal{T}$ be the junction points of segments either of type T or *cross* type junction. A *Hanan grid* [10] is generated by constructing vertical and horizontal lines through each junction point. The length of the vertical boundary and the horizontal boundary of a cell (r) will be denoted by *width*(r) and *breadth* (r), respectively. Let $\mathcal{N}(\mathcal{T})$ denote the minimum number of sites in the GVI for a tessellation $\mathcal{T}$. The existence of GVI for any rectangular tessellation follows from the following theorem:

Theorem 1. *For any rectangular tessellation $\mathcal{T}$, there exists a point set S of size $O(n^2)$ such that any cell $V(s)(\subseteq V(S), s \in S)$ lies in exactly one cell of $\mathcal{T}$, where n is the number of cells in $\mathcal{T}$.*

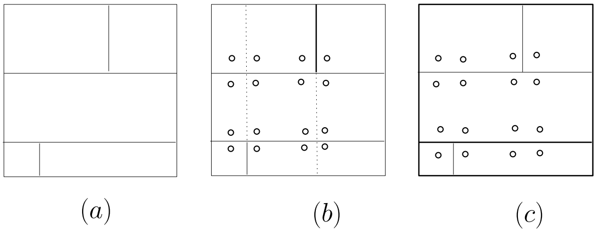

Fig. 1. Proof of Theorem 1. (a) Original tessellation (b) site placement on Hanan tessellation and (c) site placement in original tessellation.

Proof. A Hanan grid can be constructed from the given rectangular tessellation $\mathcal{T}$ (See Fig. 1a) that produces a *Hanan tessellation* (Fig. 1b). Here each cell of $\mathcal{T}$ is partitioned into one or more parts. Moreover, each cell of Hanan tessellation must lie inside exactly one cell of $\mathcal{T}$. Fix a small positive constant ϵ whose value is smaller than half of the smallest length among breadths and widths of all rectangular cells of $\mathcal{T}$. For each grid point of co-ordinate (a, b) of Hanan tessellation, place sites at positions $(a+\epsilon, b+\epsilon)$, $(a-\epsilon, b+\epsilon)$, $(a-\epsilon, b-\epsilon)$, $(a+\epsilon, b-\epsilon)$ as in (Fig. 1b) Observe that the junction points of the Hanan tessellation are Voronoi vertices and the edges of the tessellation emanating from the junction points are Voronoi edges. However, all Voronoi vertices and edges may not coincide with junction points and grid edges respectively. The site placement in the original tessellation can be obtained as in (Fig. 1c) just by ignoring these extra vertices or edges shown by dotted lines. For a point in a cell of Hanan tessellation, its nearest sites must be one of the sites in the cell. Hence, in this Voronoi diagram, the Voronoi region of a site must lie inside a cell of the tessellation. Thus, $\mathcal{T}$ is a subgraph of Voronoi diagram of S. Now we can bound the maximum number of sites required for any given rectangular tessellation containing n cells. There are at most $O(n^2)$ junction points in the respective Hanan grid. So, we can place four sites around each junction points in order to construct the GVI. Hence, the maximum number of sites required is of $O(n^2)$. In fact, $\mathcal{N}(\mathcal{T}) \leq 4n^2$. □

The above placement strategy may yield to a large number of sites compared to the optimal solution. We will discuss some special cases where the number of sites can be reduced significantly. In the next section, we consider only the linear rectangular tessellation.

3 Locating Sites in a Linear Rectangular Tessellation

Here we consider the placement of sites S in rectangular tessellation that is devoid of cross- and T- junctions.

Observation 1. *If the widths of all the cells in $\mathcal{T}$ are equal then $\mathcal{N}(\mathcal{T}) = n$, where n is the number of cells in the tessellation.*

Proof. Placing the sites S in the intersection points of the diagonals of the rectangles are alternately increasing and decreasing and widths of the rectangles are in such order. □

Observation 2. *For a rectangular tessellation* $\mathcal{T}$ *containing two rectangles of different widths,* $\mathcal{N}(\mathcal{T}) = 2$.

Proof. Let l be the line containing the segment common to both the rectangles. Place two sites on opposite halves of l at ϵ distance from l (Fig. 2(b)). □

Observation 3. *For a rectangular tessellation consisting of three rectangles of widths* w_1, w_2, *and* w_3 *respectively,* $\mathcal{N}(\mathcal{T}) \leq 4$.

Proof. Let the three rectangles be of widths w_1, w_2, and w_3 respectively such that (a) $w_1 < w_2$, (b), $w_3 < w_2$, (c) $(w_2/2 < w_1)$ and (d) $(w_2/2 < w_3)$. Observe that placing one site in each rectangle is necessary and sufficient if all the above conditions hold simultaneously. Suppose, all the above conditions do not hold simultaneously; in this case, placing two sites in the rectangle of width w_2 and one site in each of the rectangles corresponding to w_1 and w_3, is necessary and sufficient (Fig. 2(b)). □

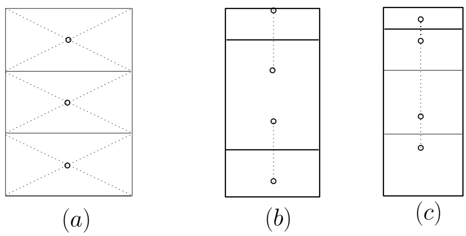

Fig. 2. Placement strategy of a linear rectangular tessellation having (a) equal width rectangles (b) 3 cells of different widths (c) arrangement of cells in non-decreasing width

Observation 4. *Given a linear rectangular tessellation* $\mathcal{T}$ *where the widths of the rectangles are in non-decreasing (or non-increasing) order, then* $\mathcal{N}(\mathcal{T}) = n$ *where* n *is the number of rectangles in* $\mathcal{T}$.

Proof. The placement of sites has to be made in sequential order starting from the rectangle with smallest width. Place a site anywhere in the rectangle of smaller width and then placement can be carried out for the next site in the adjacent rectangle, which is the reflective image of the previous one with respect to adjacent boundary (Fig. 2(c)). □

Observation 5. *For any given linear rectangular tessellation* $\mathcal{T}$, $n \leq \mathcal{N}(\mathcal{T}) \leq \lfloor \frac{3n}{2} \rfloor$.

Proof. The number of sites required will be maximum when the widths of the rectangles are alternately increasing and decreasing and widths of the rectangles are such order that for any 3 consecutive rectangles the reflective images of the outer boundary of two rectangles do not overlap at the middle rectangle. In a tessellation consisting of n rectangles there are $\lfloor \frac{n}{2} \rfloor$ rectangles where two sites are enough in each rectangle (refer Observation 3). Hence the total number of sites required will be $(2 * \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor = \lfloor \frac{3n}{2} \rfloor)$. □

Following observation leads to the tightness results of the bound of $\mathcal{N}(\mathcal{T})$.

Observation 6. *There are tessellations where* $\mathcal{N}(\mathcal{T}) = \lfloor \frac{3n}{2} \rfloor$.

Proof. Consider an arrangement formed by concatenation of linear tessellation of the basic pattern as shown in (Fig. 2(b)) where n is odd and there are $\lfloor \frac{n}{2} \rfloor$ rectangles of width w interspersed with $\lfloor \frac{n}{2} \rfloor$ rectangles of width less than $\frac{w}{2}$. Therefore, $\mathcal{N}(\mathcal{T})$ will be $(2 * \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor = \lfloor \frac{3n}{2} \rfloor)$.

Observation 7. *For any k between n and $\lfloor \frac{3n}{2} \rfloor$, there exists a linear rectangular tessellation of size n such that $\mathcal{N}(\mathcal{T}) = k$.*

Proof. The minimum number of sites required for any given tessellation is n and the maximum number of sites required is $\lfloor \frac{3n}{2} \rfloor$. Here it is shown that the required number of (sites can be any positive integer (it is obvious as the number of sites cannot be fractional)) between n and $\lfloor \frac{3n}{2} \rfloor$. Suppose there exists a partition where the optimal number of sites required is t where t is any number between n and $\lfloor \frac{3n}{2} \rfloor$. Now, the claim is: there exists a linear tessellation with n rectangles where the obtained number of sites will be $t+1$ or $t-1$. This can be done by increasing the width of one particular rectangle such that the reflective images of adjacent rectangles do not overlap, and as a result we have to increase the number of sites in that rectangle. Similarly, a tessellation with $t-1$ sites can be obtained by decreasing the width of one particular rectangle (where previously two sites are required because of non-overlapping of reflective images of the adjacent rectangles). □

Theorem 2. *The optimal placement of sites corresponding to any given linear rectangular tessellation $\mathcal{T}$ can be found in $O(n)$ time where $\mathcal{T}$ is a partition of size n.*

Proof. Consider a linear rectangular tessellation $\mathcal{T}$ that consists of a sequence of contiguous rectangles $\{A[0], A[1], \ldots, A[n]\}$ of equal length but of arbitrary widths. Now divide the sequence of rectangles into overlapping subsequences such that each one is a maximal subsequence of rectangles with either non-decreasing widths or with non-increasing widths. By maximal is meant that the subsequence is not contained in any larger increasing or decreasing subsequence. Note that, if the tessellation consists of rectangles of distinct widths, then the subsequences overlap at one rectangle. This division can be done in $O(n)$ time.

Consider the left-to-right order sequence of rectangles $A[0], A[1], ..., A[k]$ to be in non-decreasing order of their widths followed by $A[k+1], A[k+2], ..., A[t]$ in non-increasing order of the widths. Let l_i and r_i denote the left and right boundary of rectangle $A[i]$ respectively. We denote $I(A[i], l_i)$ as the image of rectangle $A[i]$ with respect to line segment l_i on the neighbor rectangle $A[i-1]$. If the image $I(A[i], l_i) \subset A[i-1]$, then for every possible location of a site in $A[i]$ there is a location for another site in $A[i-1]$ that defines the line l_i.

Now, consider the image $R_1 = I(I(\ldots(I(A[0], l_1), l_2), \ldots), l_k)$ of $A[0]$ on $A[k]$. Observe that R_1 is a rectangle of the same width as $A[0]$ lying inside the region of $A[k]$. Now similar process can be done for the region $A[t]$. Thus, we have the image $R_2 = I(I(\ldots(I(A[t], r_{t-1}), r_{t-2}), \ldots), r_k)$ inside $A[k]$.

Now we define $S_k = R_1 \cap R_2 \neq \emptyset$ as *safe region* of the k^{th} rectangle. If $R_1 \cap R_2 = \emptyset$, we need two sites at $A[k]$ and one site for $A[i]$, $i = 1, 2, \ldots, k-1, k+1, \ldots, t$. Otherwise we have a placement of one site per rectangle in the GVI partition $\{A[0], A[1], \ldots, A[t]\}$. Let $F_0 = I(I(\ldots I(I(S_k, l_k), l_{k-1}), \ldots), l_1)$ and $F_t = I(I(\ldots I(I(S_k, r_k), r_{k+1}), \ldots), r_{t-1})$ be the propagated image of S_k on $A[0]$ and $A[t]$ respectively. Therefore F_0 and F_t are the feasible regions for placing sites such that each rectangle requires at most one site. Note that, there may exist many such non-decreasing followed by non-increasing maximal subsequences. Consider another sequence of rectangles $\{A[t], A[t+1], \ldots, A[x]\}$ in non-decreasing order followed by $\{A[x], A[x+1], \ldots, A[z]\}$ in decreasing order of the widths. Now there will be a safe region S_x at $A[x]$ resulting from the sequence of rectangles $\{A[t], A[t+1], \ldots, A[x], \ldots, A[t]\}$. Corresponding to S_x there will be feasible region F'_t on $A[t]$. If $F_t \cap F'_t \neq \emptyset$ then it is always possible to satisfy the tessellation with one site for each rectangle. For each rectangle $\{A[t]\}$ whose width is less than both of its neighboring rectangles $A[t-1]$ and $A[t+1]$, compute the feasible regions $F = \{F_t\}$. If $F_t \neq \emptyset$ for all $F_t \in F$, then the tessellation $\mathcal{T}$ can be realized by n sites. These feasible regions can be generated in $O(n)$ time by traversing the partition once. Hence the theorem. □

4 Locating Sites in a General Rectangular Tessellation

Observation 8. *If the tessellation consists of only identical square cells, then placing one site in each of the cell is necessary and sufficient.*

Proof. Place one site in each of the cell at the intersection point of the diagonals of each cell (see Fig. 3). □

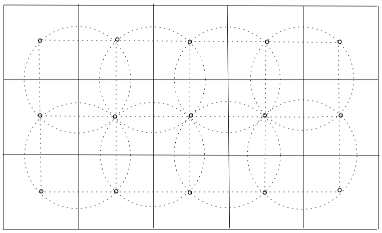

Fig. 3. One site is enough per rectangle

Theorem 3. $\mathcal{N}(\mathcal{T}) \leq (\lfloor \frac{3n}{2} \rfloor)^2$, *where $\mathcal{T}$ is a tessellation generated from a Hanan grid of size $n \times n$.*

Proof. Consider a column of a Hanan tessellation. From Observation 5, we can generate this part of the tessellation by placing at most $\lfloor \frac{3n}{2} \rfloor$ sites. Let A represent this placement of sites along a column. Similarly, we may consider a row of the tessellation. Here, we can generate this part of the tessellation by placing at most $\lfloor \frac{3n}{2} \rfloor$ sites and let B represent such a placement of sites along a row. Now replicate arrangement A in each

x-coordinate location of the arrangement B. Now we claim that this way of placement generates the given Hanan tessellation.

For justification, consider a site s of the placement and look at its four neighboring sites and their neighbors. The Voronoi edges of $V(s)$ form a rectangular cell and the cell must lie inside a cell of Hanan tessellation. Again observe that at least one vertical and one horizontal edge of the Hanan cell must coincide with the edges of $V(s)$. Hence, the placement is optimal. □

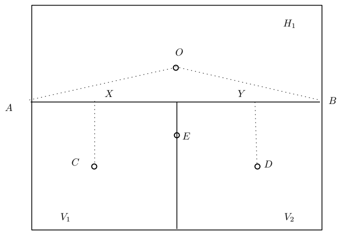

Fig. 4. Illustration of observation 9

Observation 9. *Consider a tessellation $\mathcal{T}(3)$ that consists of rectangular cells, H and V with a common horizontal edge between them. The cell V is further partitioned into two rectangles by a vertical segment. Then $\mathcal{N}(\mathcal{T}(3))$ is at least* 4.

Proof. Consider Fig. 4, where the given rectangular tessellation $\mathcal{T}(3)$ consists of three rectangles H_1, V_1 and V_2. Suppose the sites corresponding to V_1 and V_2 are placed at C and D respectively. We claim that we require at least two sites for H_1 to make AB a Voronoi edge. We will prove this by contradiction. Assume that there is only one site placed at O for the rectangle H_1. Therefore C and D should be at an infinite distance from O to make AB a Voronoi edge. But the given tessellation is bounded. Hence the contradiction. □

Now we describe an algorithm for generating a point set S for any general rectangular tessellation $\mathcal{T}$ which will provide a minimal solution for GVI. A minimal solution is a solution where the size of the solution set cannot be further reduced without repositioning the site locations in the solution.

Example: For the tessellation of Fig. 5(a), the steps of Algorithm 1, are shown in Fig. 5(b) and 5(c). Theorem 3 concludes that for any rectangular tessellation of size n, there always exists a feasible solution and that the upper bound on the size of sites is $O(n^2)$. Algorithm 1 produces a valid minimal solution of GVI with rectangular tessellation. From Observation 9 we can conclude that 4 sites are required around each T-junction. These sites will form a square and a circle centering at the T-junction will pass through all these 4 sites. Initially one can place 4 such sites around each T-junction; however, Algorithm 1 reduce the number of sites by sharing such placement in the adjacent squares. Such a solution is shown in Fig. 5(c). It may be noted that the solution

Algorithm 1. GVI for any rectangular tessellation $\mathcal{T}$

Input: A rectangular tessellation $\mathcal{T}$
Output: GVI of $\mathcal{T}$
foreach *cross- and T-junctions* j **do**
  Extend the vertical and horizontal lines through j up to the boundary of $\mathcal{T}$ to produce the Hanan Grid $\mathcal{G}$
end
$minr$ =the row of minimum width in $\mathcal{G}$;
$minc$ =the column of minimum width in $\mathcal{G}$;
Following the placement strategy of Theorem 2, place sites in the $minr^{th}$ row and $minc^{th}$ column of $\mathcal{G}$;
Using the reflection principle[a], place sites in the adjacent columns or rows starting from either $minr^{th}$ row or $minc^{th}$ column;
foreach *site* s_i **do**
  Check its cell boundaries $\{b_{ik}|k = 1, 2, 3, 4\}$;
  if b_{ik} *is a virtual segment*[b] *for all* $k = 1, \ldots, 4$ **then**
    remove the site s_i;
  end
end
Remove all virtual segments of the Hanan tessellation;

[a] Consider two adjacent rectangles and let one of which contains a site. Then there must exist a site in the other rectangle such that the euclidean distance of these two sites are equal from the common edge of the two adjacent rectangles; thus the second site is the reflective image of the first site.

[b] The segments which do not appear in the original tessellation.

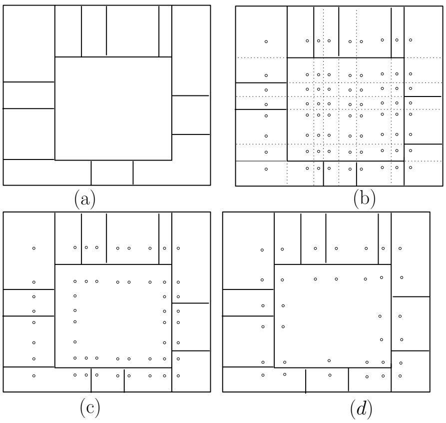

Fig. 5. (a) Original tessellation $\mathcal{T}$ (b-c) Steps of Algorithm 1 (d) tessellation with fewer number of sites

produced by Algorithm 1 may not be optimum. An solution with fewer number of sites is shown in Fig. 5(d). Determination of a minimum solution for GVI is posed as an open problem.

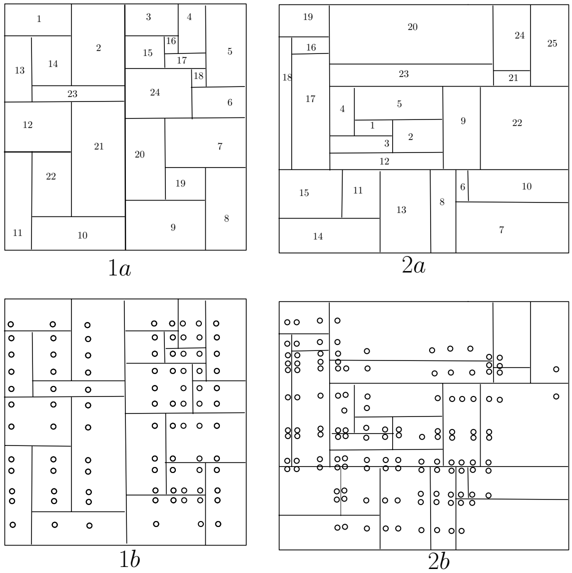

Fig. 6. (a) Original tessellations $\mathcal{T}$ corresponding to VLSI floorplans (b) Site placement by Algorithm 1

Results obtained by the proposed method for some well known VLSI benchmark floorplans [22], [23] are shown in Fig. 6.

4.1 Optimum Placement for Some Special Cases

In general, when the length and breadth of each of the rectangles are not equal, then from Theorem 1 we can derive an upper bound on the number of sites. In the next few observations, we will discuss some cases where we can have tighter results.

Consider a stack of n rectangles (all congruent to the rectangle H_1) on H_1 such that the length of the new tessellation remains the same as $\mathcal{T}(3)$. In such a case, two sites are sufficient in each of the stacked rectangles, which are congruent to H_1 because of the two base rectangles V_1 and V_2. The requirement of the base rectangle may *propagate* to the other rectangles. The impact of propagation would be enormous if there are m base rectangles (say, $\mathcal{T}(m)$) on the top of which there are n stacked rectangles (say, $\mathcal{T}(n)$). For the tessellation $\mathcal{T}(m,n)$ (n rectangles stacked above m rectangles), $\mathcal{N}(\mathcal{T}(m,n)) \leq n \cdot \mathcal{N}(\mathcal{T}(m)) + \mathcal{N}(\mathcal{T}(m))$. However, this provides only upper bound. For some special instances, the required number of sites may not be much less. There is a scope to reduce this *propagation effect* as discussed below.

Consider Fig. 7a, where the given rectangular tessellation $\mathcal{T}(2,3)$ consists of three stacked rectangles H_1, H_2, H_3 and two rectangles V_1 and V_2 at the base. Eight sites (refer Fig. 7c) are sufficient for the tessellation to follow Voronoi properties, i.e, $\mathcal{N}(\mathcal{T}(2,3)) \leq 8$. Note that two sites placed at R and S in the rectangle H_3 are necessary (Observation 9). But if we place one extra site at Q in the rectangle H_3, then rectangles H_2 and H_1

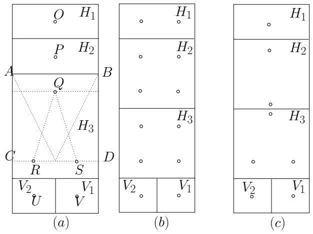

Fig. 7. Propagation depends on the aspect ratio

require 1 site each. This is possible since the perpendicular bisectors of the segments RQ and SQ do not intersect with the line segment AB, which is the common boundary between the rectangles H_2 and H_3. Therefore, placing one extra site in one rectangle may reduce the number of sites in subsequent rectangles. The placing of extra site depends upon the aspect ratio of the corresponding rectangle. This way of lowering down the required number of sites in the tessellation is called *blocking* of the *propagation effect*, which is illustrated next.

Observation 10. *For any rectangular tessellation of type $\mathcal{T}(m,n)$, $\mathcal{N}(\mathcal{T}(m,n))$ depends on the aspect ratios of the corresponding rectangles.*

Proof. Consider H_i to be one of the stacked rectangles in the tessellation $\mathcal{T}(m,n)$ where $H_1, H_2, \ldots, H_n$ are stacked from top to bottom respectively. Suppose the rectangle H_i requires a set S_i of m sites to make the separating edge between H_i and H_{i+1} a Voronoi edge, and those m sites are placed on a line, say l. As for instance, consider Fig.7 where H_3 requires 2 sites that are placed on R and S to take care of the common edge between H_3 and base Voronoi. Let s_l and s_r be the extreme left and the extreme right sites among S_i. Let d be the distance between s_l and s_r. We place another site s in H_i such that $d(s, s_l) = d(s, s_r)$ and $d(s, l) = d(s, s_l) = d(s, s_r)$, the site s will be placed on the perpendicular bisector of the segment (s_l, s_r) and the perpendicular distance of s from the line will be $\frac{d}{2}$. Therefore, if such a placement of site s is possible in a rectangle H_i then the neighboring rectangles $H_{i-1}, H_{i-2}, \ldots, H_1$ will require fewer sites in each of them. Thus the propagation effect in all the above rectangles is blocked. It might happen that if we can place extra $\frac{n}{2}$ sites in H_i such that in each of the stacked rectangles (from $H_{i-1}, \ldots, H_1$) $\frac{n}{2}$ sites are enough instead of n sites. For instance, in Fig. 7 where one extra site is placed in the rectangle H_3 at Q and as a result one site is enough both at H_2 and H_1. If such a placement is not possible, try to place two sites considering half of the sites of S for each new site. Recursively following the above procedure, we can reduce the required number of sites. □

By Observation 3, if the stacked and base tessellations $\mathcal{T}(n)$ and $\mathcal{T}(m)$ are considered separately then at most $\lfloor \frac{3m}{2} \rfloor$ and $\lfloor \frac{3n}{2} \rfloor$ sites will be required. Now we have the following observation:

Observation 11. *For all tessellations of type $\mathcal{T}(m,n)$, $(2m+n) \leq \mathcal{N}(\mathcal{T}(m,n)) \leq (\lfloor \frac{9mn}{4} \rfloor + \lfloor \frac{3n}{2} \rfloor)$. There exists an instance where $\mathcal{N}(\mathcal{T}(m,n)$ is exactly $(\lfloor \frac{9mn}{4} \rfloor + \lfloor \frac{3n}{2} \rfloor)$, and there also exists an instance where $\mathcal{N}(\mathcal{T}(m,n))$ is exactly $(2m+n)$.*

Proof. First we will show that $\lfloor\frac{3n}{2}\rfloor + \lfloor\frac{9mn}{4}\rfloor$ sites are sufficient in the tessellation of type $\mathcal{N}(\mathcal{T}(m,n))$ (See Fig.7). Initially we will place sites on the rectangles V_i's (here i varies from 1 to n) ignoring the H_i's (here i varies from 1 to m). Now $\lfloor\frac{3n}{2}\rfloor$ sites are sufficient to make all the edges between V_i's Voronoi edges (refer to Observation 6). We require exactly $\lfloor\frac{3n}{2}\rfloor$ sites to take care of the edge between H_m and V_i's (refer to Observation 9). Now there exists an instance (the worst case scenario) where among n stacked rectangles placed vertically above (See Fig. 7) there are $\frac{n}{2}$ rectangles where in each of them $2 \times (\lfloor\frac{3m}{2}\rfloor)$ sites are sufficient and in each of the remaining $\frac{n}{2}$ rectangles $\lfloor\frac{3m}{2}\rfloor$ sites are sufficient (follows from Observation 9). Hence, it follows that there exists an instance where $\mathcal{N}(\mathcal{T}(m,n))$ is exactly $(\frac{3m}{2} + 2 \times (\frac{n}{2}) \times (\lfloor\frac{3m}{2}\rfloor) + \frac{n}{2}(\lfloor\frac{3m}{2}\rfloor))$ on simplification, we get $\mathcal{N}(\mathcal{T}(m,n)) = \lfloor\frac{3m}{2}\rfloor + \lfloor\frac{9mn}{4}\rfloor$.

Now in proving the lower bound, at least one site is required in each of the m rectangles located in the base that is in $\mathcal{T}(m)$ to make the edges between each of the rectangle in $\mathcal{T}(m)$ a Voronoi edge. Now to make the edge located in between H_n and $\mathcal{T}(m)$ Voronoi m sites are enough in the rectangle H_n (follows from Observation 9). In the best case the propagation effect can be stopped with the inclusion of one site in H_n. There exists an instance where in each of the $n-1$ stacked rectangles $(H_{n-1} \ldots H_1)$, one site is sufficient. Hence, it follows that there exists an instance where $\mathcal{T}(m,n)$ is exactly $(m + (m+1) + (n-1))$, simplifying we get $\mathcal{N}(\mathcal{T}(m,n)) = (2m+n)$. □

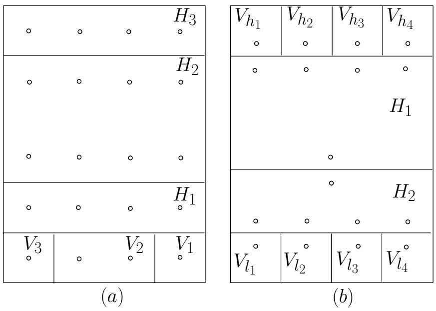

Fig. 8. (a) Observation 11 (b) Observation 12

Consider another variant of rectangular tessellation where the tessellation has another set of base rectangles on top of the stacked rectangles, i.e, the stacked rectangles are sandwiched between two set of base rectangles (See Fig. 8a). Denote this variant as $\mathcal{T}(m_1, m_2, n)$ where m_1, m_2 and n are the number of the top base rectangles, bottom base rectangles and stacked rectangles respectively. Now we have the following observation.

Observation 12. *For all tessellations of type* $\mathcal{T}(m_1, m_2, n)$*,* $(2(m_1+m_2)+n) \leq \mathcal{N}(\mathcal{T}(m_1,m_2,n)) \leq (\lfloor\frac{9n \times max\{m_1,m_2\}}{4}\rfloor + \lfloor\frac{3n}{2}\rfloor)$*. There exists an instance where* $\mathcal{N}(\mathcal{T}(m_1,m_2,n))$ *is exactly* $(\lfloor\frac{9n \times max\{m_1,m_2\}}{4}\rfloor + \lfloor\frac{3n}{2}\rfloor)$ *and there also exists an instance where* $\mathcal{N}(\mathcal{T}(m_1,m_2,n))$ *is exactly* $(2(m_1+m_2)+n)$.

Proof. First we will prove that the lower bound on the number of sites required for a partitioning of $\mathcal{T}$ into H, V_h and V_l (See Fig. 8b)) Suppose V_h needs m_1 sites and V_l needs m_2 sites; then we have to place m_1 (m_2) sites in the rectangle which is adjacent to V_h (V_l). In the best case we can *block the propagation effect* in the next adjacent rectangle such that we require 1 site in all other horizontal rectangles. Hence, we need at least $(2(m_1 + m_2) + n)$ sites.

Now we will prove the upper bound on the number of sites required for partitioning of $\mathcal{T}$ into H, V_l and V_h. The number of sites required is at most $(\lfloor \frac{9n \times max\{m_1, m_2\}}{4} \rfloor + \lfloor \frac{3n}{2} \rfloor)$ (Observation 11). If V_l and V_h are partitioned into m_1 and m_2 rectangles respectively then at most $\lfloor \frac{3m_1}{2} \rfloor$ and $\lfloor \frac{3m_2}{2} \rfloor$ sites are required in V_l and V_h respectively (Observation 5). Hence, the maximum number of sites required in the worst case is $(\lfloor \frac{9n \times max\{m_1, m_2\}}{4} \rfloor + \lfloor \frac{3n}{2} \rfloor)$. □

Observation 13. *For a tessellation $\mathcal{T}$ consisting of n rectangles with* cross- *and* ⊤ *junctions, there exists an instance where the minimum number of sites required will be of $\Omega(n^2)$.*

Proof. Consider Fig. 8(a) where the widths of H_1, H_2, H_3 and V_1, V_2, are such that the reflective images of H_1 and H_3 do not overlap at H_2, similarly the reflective image of V_1 and V_3 do not overlap at V_2. Now from Observation 6, we require four sites altogether in V_1, V_2, and V_3. In order to make the common boundary between V_i (where $i = 1, 2, 3$) and H_1 an Voronoi edge, four sites are required. From the discussion made in Observation 9, it follows that four sites placed in H_1 will replicate in H_2 and H_3. The optimality is proved by the fact that if we replace any one site from V_i it will strictly contradict the Observation 6, and if we replace any one site from H_1 it will contradict Observation 9. We cannot further reduce the number of sites in H_2 and H_3 because of the patterns of the aspect ratios of the corresponding rectangles. □

5 Locating Sites in a General Tessellation

In this section we will provide a feasible solution of GVI for any arbitrary tessellation within a bounded region. In other words, $\mathcal{T}$ may be viewed as a partition of a bounded two-dimensional space into polygons, or a plane graph with no pendant vertices (See Fig. 9). Without loss of generality, assume that the bounded region is rectangular in shape. This imposition does not restrict us in deriving general theoretical results.

Theorem 4. *Given an arbitrary tessellation $\mathcal{T}$ there always exists a feasible placement of sites, which is a GVI of the tessellation $\mathcal{T}$.*

Proof. To prove the theorem we will use some earlier results on acute and non-obtuse triangulation of polygons and planar straight line graphs [15] [16] [17] [18] [19] [20] [21]. Consider a tessellation $\mathcal{T}$ that consists of n vertices. It is known that a tessellation, which is a planar straight line graph, admits an conforming non-obtuse triangulation if additional vertices and edges are included in $\mathcal{T}$ [15]. A conforming triangulation is defined as follows. Let V be the set of vertices in $\mathcal{T}$ and suppose V' is a point set containing V. We say a triangulation of V' conforms to $\mathcal{T}$ if the edges of the triangulation cover the edges of $\mathcal{T}$. A conforming non-obtuse triangulation of the tessellation $\mathcal{T}$ can be obtained by adding $O(n^{2.5})$ new vertices [15]. Once a non-obtuse triangulation of $\mathcal{T}$ is obtained, we fix a small positive quantity ϵ which is strictly smaller than half the length of the smallest edge of the triangulation, and an angle δ which is smaller than half of the smallest angle between any two edges of the triangles. Construct a circle of radius ϵ around each of the vertices of the triangles. Place sites on the circumference of the above circle such that each edge emanating from a vertex is a perpendicular bisector of the two neighboring sites (one in clockwise and another in anticlockwise) placed on the circle. See Figs. 10a, 10b. These two sites are so placed as the circle that they subtend an angle 2δ at the center. These sites are called $\epsilon-$ neighbor of the corresponding vertex. We repeat this site placement procedure for every vertex of the triangulated graph.

We will prove the fact that the above procedure will return a feasible solution of GVI by the following argument. Let ABC be one of the acute-angled triangle. Three circles C_A, C_B, C_C centering at the vertices A, B, C respectively each of radius ϵ, are drawn. Let D be any arbitrary point inside the triangle ABC which lies outside the circles C_A, C_B, C_C (See Fig. 11a). Suppose D is unable to fulfill its Voronoi requirement from any of the sites placed on the circumference of the circles C_A, C_B, C_C. On the contrary D finds its nearest neighbor from outside the triangle ABC, say at site P; note that by construction, P must be an ϵ- neighbor of a vertex of an acute-angled triangle. Then there will be two cases

1) Q is a vertex of the triangle whose one of its edge is BC (See Fig. 11a);

   Proof. A circle C_D is drawn centering at D and radius DQ. Now there will be 2 sub-cases.

   a) If the circle C_D intersects any of the circles C_A, C_B, C_C then it is obvious that D will satisfy its Voronoi requirement from a site placed on the circumference of the corresponding circles. This contradicts that P is the nearest neighbor of D.
   b) Suppose the circle C_D intersects the triangle at the points S, R. The segment EF subtends a right angle at Q since EF is the diameter. Note that $\angle SQR$ is greater than $\angle EQF$, and as a result $\angle SPR$ is obtuse. Therefore, the face BQC is not a non-obtuse triangle. Hence, contradiction.

2) Q is not the vertex of the triangle whose one of its edge is BC;

   Proof. Let XY be the edge of the triangle whose one of the vertex is M. The line segment DM is joined. Consider a point M' on the line segment DM such that M' is just outside the face XYM (See Fig. 11b). Note that there must be at least one edge crossing the line segment DM otherwise triangulation will be contradicted. It is easy to see that (Fig. 11c) the nearest neighbor of M' will be M as the circle C_D (with center at D and radius DM) includes the circle $C_{M'}$ (with center at M' and radius $M'M$). Hence, with respect to the face XYM and the point M', we land up with a situation which can be contradicted as in Case 1. Thus M cannot be nearest neighbor of M'. □

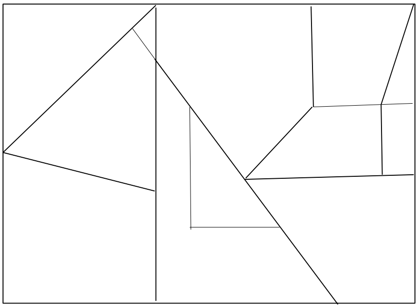

Fig. 9. Example of an arbitrary tessellation

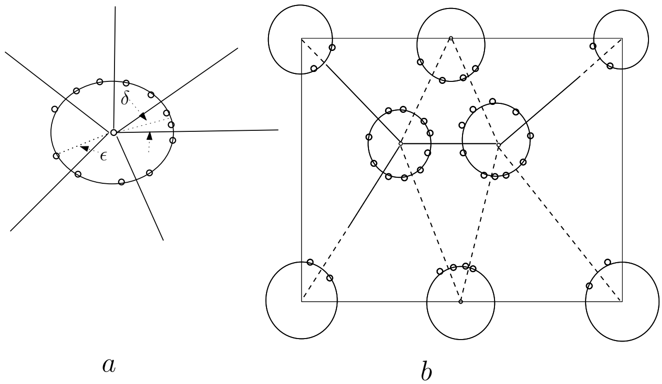

Fig. 10. (a) Applying theorem 4 in star (b) Applying theorem 4 in the given tessellation (dotted lines show extension of a segment in order to triangulate)

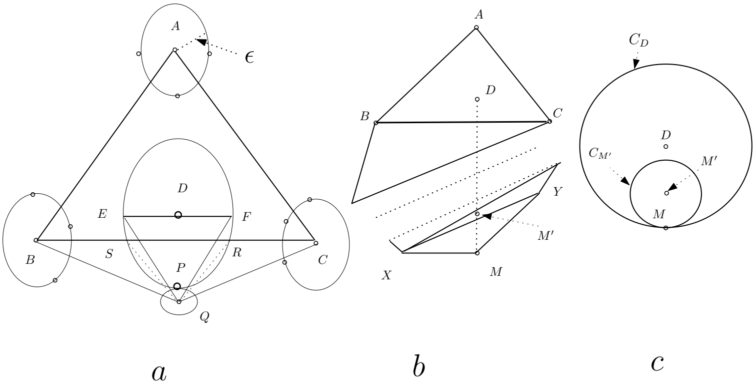

Fig. 11. Proof of Theorem 4

Now, we bound the number of sites required for a given arbitrary tessellation. The total number of vertices formed after obtaining conforming non-obtuse triangulation will be of $O(n^{2.5})$, where n is the number of vertices in the tessellation $\mathcal{T}$. Since, a triangulated graph is a plane graph, the number of edges will also be of order $O(n^{2.5})$. The number of sites placed around each vertex can be bounded by the number of edges as the number of sites placed is constant across each edge. Hence, the total number of sites required in the given tessellation $\mathcal{T}$ will be $O(n^{2.5})$.

6 Conclusion

We have introduced a new concept of Generalized Voronoi Diagram (GVI) of a given tessellation, which was motivated from several engineering design problems of VLSI and microfluidics. For a rectangular tessellation $\mathcal{T}$ we derive several interesting properties of GVI and proposed an algorithm that constructs a solution of minimal size for $\mathcal{T}$. Finding an optimum solution of a GVI problem for rectangular tessellation in polynomial time seems to be a challenging problem. We have also studied the GVI problem for the general case and suggested a method of constructing a feasible solution. Finding a minimal solution of GVI problem for an arbitrary tessellation is posed as an open problem.

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References

1. Aurenhammer, F., Klein, R.: Voronoi diagrams. In: Sack, V.J., Urrutia, G. (eds.) Handbook of Computational Geometry, pp. 201–290. Elsevier Science Publishing (2000)
2. Ash, P., Bolker, E., Crapo, H., Whiteley, W.: Convex polyhedra, Dirichlet tessellations, and spider webs. In: Senechal, M., Fleck, G. (eds.) Shaping Space: A Polyhedral Approach, ch. 17, pp. 231–250. Birkhauser, Basel (1988)
3. Balzer, M., Heck, D.: Capacity-constrained Voronoi diagrams in finite spaces. In: Proceedings of the 4th International Symposium on Voronoi Diagrams in Science and Engineering, pp. 44–56 (2008)
4. Balzer, M.: Capacity-constrained Voronoi diagrams in continuous spaces. In: Proceedings of the 5th International Symposium on Voronoi Diagrams in Science and Engineering, pp. 79–88 (2009)
5. Gavrilova, M.L.: Generalized Voronoi Diagram: A Geometry-Based approach to computational intelligence. SCI, vol. 15 (2008)
6. Hartvigsen, D.: Recognizing Voronoi diagrams with linear programming. ORSA J. Comput. 4(4), 369–374 (1992)
7. Suzuki, A., Iri, M.: Approximation of a tessellation of the plane by a Voronoi diagram. J. Oper. Res. Soc. Japan 29, 69–96 (1986)
8. Yuksek, K., Cezayirli, A.: Linking image zones to database by using inverse Voronoi diagrams: A Novel Liz-Ivd Method. In: IEEE International Symposium on Intelligent Control, Saint Petersburg, Russia, July 8-10, pp. 423–427 (2009)

9. Drezner, Z., Hamacher, H.W. (eds.): Facility location: applications and theory. Springer (2002)
10. Hanan, M.: On Steiners problem with rectilinear distance. SIAM Journal Appl. Math 14, 255–265 (1966)
11. Goplen, B.: Advanced placement techniques for future VLSI circuits: A short term longitudinal study, University of Minnesota (2006)
12. Tsai, C.H., Kang, S.M.: Cell-Level placement for improving substrate thermal distribution. IEEE Trans. CAD 19(2), 253–266 (2000)
13. Chen, G., Sapatnekar, S.S.: Partition-driven standard cell thermal placement. In: Proceedings of the International Symposium on Physical Design, pp. 75–80 (2003)
14. Chakrabarty, K., Xu, T.: Digital Microfluidic Biochips: Design and Optimization. CRC Press, Boca Raton (2010)
15. Bishop, C.J.: Non obtuse triangulations of PSLGS (2010) (manuscript)
16. Hangan, T., Itoh, J., Zamfirescu, T.: Acute triangulations. Bull. Math. Soc. Sci. Math. Roumanie 43, 279–286 (2000)
17. Yuan, L.: Acute triangulations of polygons. Discrete and Computational Geometry 34(4), 697–706 (2005)
18. Edelsbrunner, H.: Triangulations and meshes in computational geometry. Acta Numerica 9, 133–213 (2000)
19. Zamfirescu, C.T.: Survey of two-dimensional acute triangulations. Discrete Mathematics 313(1), 35–49 (2013)
20. Earten, H., Ungor, A.: Computing acute and non obtuse triangulations. In: Canadian Conference on Computational Geometry, Ottawa, Canada (2007)
21. Du, D.Z., Hwang, F.: Mesh generation and optimal triangulation. In: Bern, M., Eppstein, D. (eds.) Computing in Euclidean Geometry, pp. 23–80. World Scientific (1995)
22. Wimer, S., Koren, I., Cederbaum, I.: Optimal aspect ratios of building blocks in VLSI. IEEE Trans. CAD 8(2), 139–145 (1989)
23. Wang, T.C., Wong, D.F.: Optimal floorplan area optimization. IEEE Trans. CAD 11(8), 992–1002 (1992)
24. Majumder, S., Sur-Kolay, S., Nandy, S.C., Bhattacharya, B.B., Chakraborty, B.: Hot spots and zones in a chip: A geometrician's view. In: Poc. Int. Conf. VLSI Design, pp. 691–696 (2005)
25. Majumder, S., Bhattacharya, B.B.: Solving thermal problems of hot chips using Voronoi diagrams. In: Poc. Int. Conf. VLSI Design, pp. 545–548 (2006)